

# BERNSTEIN-VON MISES THEOREMS FOR GAUSSIAN REGRESSION WITH INCREASING NUMBER OF REGRESSORS

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This paper brings a contribution to the Bayesian theory of non-parametric and semiparametric estimation. We are interested in the asymptotic normality of the posterior distribution in Gaussian linear regression models when the number of regressors increases with the sample size. Two kinds of Bernstein-von Mises Theorems are obtained in this framework: nonparametric theorems for the parameter itself, and semiparametric theorems for functionals of the parameter. We apply them to the Gaussian sequence model and to the regression of functions in Sobolev and  $C^\alpha$  classes, in which we get the minimax convergence rates. Adaptivity is reached for the Bayesian estimators of functionals in our applications.

**1. Introduction.** To estimate a parameter of interest in a statistical model, a Bayesian puts a prior distribution on it and looks at the posterior distribution, given the observations. A Bernstein-von Mises Theorem is a result giving conditions under which the posterior distribution is asymptotically normal, centered at the maximum likelihood estimator (MLE) of the model used, with a variance equal to the asymptotic frequentist variance of the MLE. Other centering can be used, see for instance (van der Vaart, 1998, p. 144, after the proof of Lemma 10.3).

Such an asymptotic posterior normality is important because it allows the construction of approximate credible regions, based on the posterior distribution, which retain good frequentist properties. In particular the Monte-Carlo Markov chain algorithms (MCMC) make feasible the construction of Bayesian confidence regions in complex models, for which frequentist confidence regions are difficult to build; however Bernstein-von Mises Theorems are difficult to derive in complex models.

Note that the Bernstein-von Mises Theorem has also links with Information Theory (see Clarke and Barron (1990), and Clarke and Ghosal (2010)).

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For parametric models, the Bernstein-von Mises Theorem is a well-known result, for which we refer to van der Vaart (1998). In nonparametric models (where the parameter space is infinite-dimensional or growing), and semiparametric models (when the parameter of interest is a finite-dimensional functional of the complete infinite-dimensional parameter), there are still relatively few asymptotic normality results. Freedman (1999) gives negative results, and we recall some positive ones below. However many recent papers deal with the convergence rate of posterior distributions in various settings, which is linked with the model complexity: we refer to Ghosal, Ghosh and van der Vaart (2000); Shen and Wasserman (2001) as early representatives of this school.

Nonparametric Bernstein-von Mises Theorems have been developed for models based on a sieve approximation, where the dimension of the parameter grows with the sample size. In particular two situations have been studied: regression models in Ghosal (1999); exponential models in Ghosal (2000), Clarke and Ghosal (2010), and Boucheron and Gassiat (2009) (this last one deals with the discrete case, when the observations follow some unknown infinite multinomial distribution).

In semiparametric frameworks the asymptotic normality has been obtained in several situations. Kim and Lee (2004) and Kim (2006) study the nonparametric right-censoring model and the proportional hazard model. Castillo (2010) obtains Bernstein-von Mises Theorems for Gaussian process priors, in the semiparametric framework where the unknown quantity is  $(\theta, f)$ , with  $\theta$  the parameter of interest and  $f$  an infinite-dimensional nuisance parameter. See also Shen (2002). Rivoirard and Rousseau (2009) obtains the Bernstein-von Mises Theorem for linear functionals of the density of the observations, in the context of a sieve approximation: sequences of spaces with an increasing dimension  $k_n$  are used to approximate an infinite dimensional space. These authors achieve also the frequentist minimax estimation rate for densities in specific regularity classes with a deterministic (non-adaptive) value of the dimension  $k_n$ .

Here we obtain nonparametric and semiparametric Bernstein-von Mises Theorems in a Gaussian regression framework with an increasing number of regressors. We address two challenging problems. First, we try to understand better when the Bernstein-von Mises Theorem holds and when it does not. In the latter case the Bayesian credible sets no longer preserve their frequentist asymptotic properties. Second, we look for adaptive Bayesian estimators in our semiparametric settings.

Our nonparametric results cover the case of a specific Gaussian prior, and the case of more generic smooth priors. They are said nonparametric because

we use sieve priors, i.e. the dimension of the parameter grows. These results improve on the preceding ones by Ghosal (1999) which did not suppose the normality of the errors but imposed other conditions, in particular on the growth rate of the number of regressors. We apply our results to the Gaussian sequence model, as well as to periodic Sobolev classes and to regularity classes  $C^\alpha[0, 1]$  in the context of the regression model (using respectively trigonometric polynomials and splines as regressors). In all these situations we get the asymptotic normality of the posterior in addition to the minimax convergence rates, with appropriate (non-adaptive) choices of the prior. We also show that for some priors known to reach this convergence rate, the Bernstein-von Mises Theorem does not hold.

We derive also semiparametric Bernstein-von Mises Theorems for linear and nonlinear functionals of the parameter. The linear case is an immediate corollary of the nonparametric theorems and does not need any additional conditions. We apply these results to the periodic Sobolev classes to estimate a linear functional and the  $L^2$  norm of the regression function  $f$  when it is smooth enough, and in both cases we are able to build an adaptive Bayesian estimator which achieves the minimax convergence rate in all classes of the collection, in addition to the asymptotic normality.

The paper is organized as follows. We present the framework in section 2. Section 3 states the nonparametric Bernstein-von Mises Theorems, for Gaussian and non-Gaussian priors. In section 4 we derive the semiparametric Bernstein-von Mises Theorems for linear and non-linear functionals of the parameter. Then in section 5 we give applications to the Gaussian sequence model, and to the regression of a function in a Sobolev and  $C^\alpha[0, 1]$  class. In section 6, the nonparametric and semiparametric Bernstein-von Mises Theorems are proved. The appendices contain various technical tools used in the main analysis; the appendices are available online as supplementary material.

**2. Framework.** We consider a Gaussian linear regression framework. For any  $n \geq 1$ , our observation  $Y = (Y_1, \dots, Y_n) \in \mathbb{R}^n$  is a Gaussian random vector

$$(1) \quad Y = F + \varepsilon$$

where the vector of errors  $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \sim \mathcal{N}(0, \sigma_n^2 I_n)$ , with  $I_n$  the  $n \times n$  identity matrix, and the mean vector  $F$  belongs to  $\mathbb{R}^n$ . Note that the dimension of  $Y$  is the sample size  $n$ , and that  $\sigma_n^2$  is known but may depend on  $n$ . Let  $F_0$  be the true mean vector of  $Y$  with distribution  $\mathcal{N}(F_0, \sigma_n^2 I_n)$ . Probability expectations under  $F_0$  are denoted  $P_{F_0}$  and  $E$ .

Let  $\phi_1, \dots, \phi_{k_n}$  a collection of  $k_n$  linearly independent regressors in  $\mathbb{R}^n$ , where  $k_n \leq n$  grows with  $n$ . We gather these regressors in the  $n \times k_n$ -matrix  $\Phi$  of rank  $k_n$ , and  $\langle \phi \rangle = \{ \Phi \theta : \theta = (\theta_1, \dots, \theta_{k_n}) \in \mathbb{R}^{k_n} \}$  denotes their linear span. The Bernstein-von Mises Theorems will be stated in association with  $\langle \phi \rangle$ , the vector space of possible mean vectors in the model, which is possibly misspecified. We denote by  $P_\theta$  the probability distribution of a random variable following  $\mathcal{N}(\Phi \theta, \sigma_n^2 I_n)$ , and  $E_\theta$  the associated expectation.

As examples, we present three different settings, each with its own collection of regressors. In section 5 the Bernstein-von Mises Theorems are applied to each of these frameworks.

### 1. The Gaussian sequence model.

Our first application concerns the Gaussian sequence model, which is also equivalent to the white noise model (see (Massart, 2007, ch. 4) for instance). We consider the infinite dimensional setting

$$(2) \quad Y_j = \theta_j^0 + \frac{1}{\sqrt{n}} \xi_j, \quad j \geq 1$$

where the random variables  $\xi_j, j \geq 1$  are independent and have distribution  $\mathcal{N}(0, 1)$ . Projecting on the first  $k_n$  coordinates with  $k_n \leq n$ , we retrieve our model (1) with  $\theta_0 = (\theta_j^0)_{1 \leq j \leq k_n}$ ,  $\sigma_n = 1/\sqrt{n}$ , and  $\Phi^T \Phi = I_{k_n}$ .

### 2. Regression of a function in a Sobolev class.

Let  $f : [0, 1] \rightarrow \mathbb{R}$  be a function in  $\mathbb{L}^2([0, 1])$ . We observe realizations of random variables

$$(3) \quad Y_i = f(i/n) + \varepsilon_i$$

for  $1 \leq i \leq n$ , where the errors  $\varepsilon_i$  are iid  $\mathcal{N}(0, \sigma_n^2)$  and  $\sigma_n$  does not depend on  $n$ .

We denote by  $(\varphi_j)_{j \geq 1}$  the Fourier basis

$$(4) \quad \begin{aligned} \varphi_1 &\equiv 1 \\ \varphi_{2m}(x) &= \sqrt{2} \cos(2\pi m x) \quad \forall m \geq 1 \\ \varphi_{2m+1}(x) &= \sqrt{2} \sin(2\pi m x) \quad \forall m \geq 1 \end{aligned}$$

In conjunction with the regular design  $x_i = i/n$  for  $1 \leq i \leq n$ , this gives the collection of regressors

$$\phi_j = (\varphi_j(i/n))_{1 \leq i \leq n}, \quad 1 \leq j \leq k_n.$$

In practice we suppose that  $f$  belongs to one of the periodic Sobolev classes:

DEFINITION 1. Let  $\alpha > 0$  and  $L > 0$ . Let  $(\varphi_j)_{j \geq 1}$  denote the Fourier basis (4). We define the Sobolev class  $\mathcal{W}(\alpha, L)$  as the collection of all functions  $f = \sum_{j=1}^{\infty} \theta_j \varphi_j$  in  $\mathbb{L}^2([0, 1])$  such that  $\theta = (\theta_j)_{j \geq 1}$  is an element of the ellipsoid of  $\ell^2(\mathbb{N})$

$$\Theta(\alpha, L) = \left\{ \theta \in \ell^2(\mathbb{N}) : \sum_{j=1}^{\infty} a_j^2 \theta_j^2 \leq \frac{L^2}{\pi^{2\alpha}} \right\}$$

where

$$(5) \quad a_j = \begin{cases} j^\alpha & \text{if } j \text{ is even;} \\ (j-1)^\alpha & \text{if } j \text{ is odd.} \end{cases}$$

### 3. Regression of a function in $C^\alpha[0, 1]$ .

Let  $\alpha > 0$ , and  $f \in C^\alpha[0, 1]$ . This means that  $f$  is  $\alpha_0$  times continuously differentiable with  $\|f\|_\alpha < \infty$ ,  $\alpha_0$  being the greatest integer less than  $\alpha$  and the seminorm  $\|\cdot\|_\alpha$  being defined by

$$\|f\|_\alpha = \sup_{x \neq x'} \frac{|f^{(\alpha_0)}(x) - f^{(\alpha_0)}(x')|}{|x - x'|^{\alpha - \alpha_0}}.$$

Consider a design  $(x_i^{(n)})_{n \geq 1, 1 \leq i \leq n}$ , not necessarily uniform. Here  $F_0$  is the vector  $(f(x_i^{(n)}))_{1 \leq i \leq n}$ . Once again we suppose that  $\sigma_n = \sigma$  does not depend on  $n$ .

Fix an integer  $q \geq \alpha$ , and let  $K = k_n + 1 - q$ . Partition the interval  $(0, 1]$  into  $K$  subintervals  $((j-1)/K, j/K]$  for  $1 \leq j \leq K$ . We want to perform the regression of  $f$  in the space of splines of order  $q$  defined on that partition, and use the B-splines basis  $(B_j)_{1 \leq j \leq k_n}$  (see for instance de Boor (1978)). Our collection of regressors is  $\phi_j = (B_j(x_i^{(n)}))_{1 \leq i \leq n}$ , for  $1 \leq j \leq k_n$ .

For any value of  $n \geq 1$ , let  $\widetilde{W}$  be a prior distribution on  $\mathbb{R}^{k_n}$  and, for  $F = \Phi\theta$ , let  $W$  be the prior distribution on  $F \in \mathbb{R}^n$  obtained from  $\widetilde{W}$  on  $\theta$ . Its support is included in  $\langle \phi \rangle$ . Let  $P^W$  denote the marginal distribution of  $Y$  under prior  $W$ , and  $W(dG(F)|Y)$  denote the posterior distribution of a functional  $G(F)$ . Note that everything depends on  $n$  ( $W$  for instance is a distribution on  $\mathbb{R}^n$ ) even if we do not use  $n$  as index to simplify our notations.

Both the parametrization by  $\theta$  and the corresponding collection of regressors  $\phi_1, \dots, \phi_{k_n}$  are arbitrary: what matters is the posterior distribution of  $F$  and this depends on the space  $\langle \phi \rangle$ , not on the basis used to parametrize it. The

span  $\langle \phi \rangle$  is characterized by the matrix  $\Sigma = \Phi(\Phi^T \Phi)^{-1} \Phi^T$  of the orthogonal projection onto  $\langle \phi \rangle$ .

The prior  $W$  is a sieve prior: that is, its support comes from a finite dimensional model whose dimension  $k_n$  grows with  $n$ . The collection of growing models  $\langle \phi \rangle$  (the sieve) can be seen as an approximation framework, each model being possibly misspecified. There is no true parameter in our setting: the true mean vector  $F_0$  may fall outside  $\langle \phi \rangle$  and correspond to none of the possible values of  $\theta$ . There is then a bias which has to be dealt with, linked to the choice of the cutoff  $k_n$ .

When dealing with Bernstein-von Mises results, the question of the asymptotic centering point arises. In nonparametric models constructed on an infinite dimensional parameter, there is no definition of a MLE; what the natural centering for a Bernstein-von Mises Theorem should be in such situations is not clear. In the model  $\langle \phi \rangle$ , the orthogonal projection  $Y_{\langle \phi \rangle} = \Sigma Y$  of  $Y$  is also the MLE of  $F_0$ . We set  $\theta_Y = (\Phi^T \Phi)^{-1} \Phi^T Y$  its associated parameter. Let also  $F_{\langle \phi \rangle} = \Phi \theta_0$  be the projection of  $F_0$  on  $\langle \phi \rangle$ , with  $\theta_0 = (\Phi^T \Phi)^{-1} \Phi^T F_0$ . Now,  $F_0 - F_{\langle \phi \rangle}$  corresponds to the bias introduced by the use of the model  $\langle \phi \rangle$ , and  $F_{\langle \phi \rangle}$  is the centering point of the distribution of the MLE  $Y_{\langle \phi \rangle}$  under  $P_{F_0}$ :

$$Y_{\langle \phi \rangle} \sim \mathcal{N}(F_{\langle \phi \rangle}, \sigma_n^2 \Sigma).$$

Although the MLE is naturally defined *in the sieve*  $\langle \phi \rangle$ , it heavily depends on the choice of  $\langle \phi \rangle$ . Therefore the Bernstein-von Mises Theorems we establish depends on the choice of the sieve the prior distribution is built on.

**3. Nonparametric Bernstein-von Mises Theorems.** The proofs of our nonparametric results are delayed to section 6.

3.1. *With Gaussian priors.* We consider here a centered, normal prior distribution  $W$  which is isotropic on  $\langle \phi \rangle$ , so that  $W = \mathcal{N}(0, \tau_n^2 \Sigma)$  for some sequence  $\tau_n$ .  $\tau_n$  is a scale parameter, and essentially the only assumption needed in this case is that  $\tau_n$  is large enough as  $n$  grows. Let  $\|Q - Q'\|_{\text{TV}}$  denote the total variation norm between two probability distributions  $Q$  and  $Q'$ .

**THEOREM 1.** *Assume that  $\sigma_n = o(\tau_n)$ ,  $\|F_0\| = o(\tau_n^2/\sigma_n)$  and  $k_n = o(\tau_n^4/\sigma_n^4)$ . Then*

$$E \left\| W(dF|Y) - \mathcal{N}(Y_{\langle \phi \rangle}, \sigma_n^2 \Sigma) \right\|_{\text{TV}} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

In terms of  $\theta$  instead of  $F$ , an equivalent statement is

$$E \left\| \widetilde{W}(d\theta|Y) - \mathcal{N}(\theta_Y, \sigma_n^2 (\Phi^T \Phi)^{-1}) \right\|_{\text{TV}} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Theorem 1 does not deal with the modeling bias introduced by taking a prior restricted to  $\langle \phi \rangle$ . This is an important question in nonparametric statistics, and  $k_n$  has to be chosen in order to achieve a satisfactory bias-variance tradeoff.

As an example, let us consider a typical regression framework with  $F_0 = (f_0(x_i))_{1 \leq i \leq n}$ , where  $f_0$  is some function and  $(x_i)_{1 \leq i \leq n}$  some design. If  $\sigma_n$  does not depend on  $n$ , both conditions  $\|F_0\| = o(\tau_n^2/\sigma_n)$  and  $k_n = o(\tau_n^4/\sigma_n^4)$  are satisfied if  $f_0$  is bounded and  $n^{1/4} = o(\tau_n)$ . These conditions can be read in another way:  $\tau_n^4$  must be large enough with respect to  $\|F_0\|$  and  $k_n$ .

**3.2. With smooth priors.** We consider now more general priors. To understand better the conditions we use, we need to look at the mechanics of the Bernstein-von Mises Theorem.

Behind a Bernstein-von Mises Theorem there is a LAN structure: the log-likelihood admits a quadratic expansion near the MLE. Since the posterior density is proportional to the product of the prior density and the likelihood, the prior has to be locally constant to let the likelihood alone influence the posterior and produce the Gaussian shape. To prove a Bernstein-von Mises Theorem we look for a subset which is simultaneously (1) large enough, so that the posterior will concentrate on it, and (2) small enough, so that we can find approximately constant priors on it. The larger the dimension of the model is, the more difficult it is to combine these two requirements, and the more difficult it is to obtain a Bernstein-von Mises Theorem.

The geometry of the subsets are naturally suggested by the normal distribution we are looking for. For  $M > 0$ , consider the ellipsoid

$$(6) \quad \mathcal{E}_{\theta_0, \Phi}(M) = \left\{ \theta \in \mathbb{R}^{k_n} : (\theta - \theta_0)^T \Phi^T \Phi (\theta - \theta_0) \leq \sigma_n^2 M \right\}.$$

**THEOREM 2.** *Suppose that  $W$  is induced by a distribution  $\widetilde{W}$  on  $\theta$  admitting a density  $w(\theta)$  with respect to Lebesgue measure. If there exists a sequence  $(M_n)_{n \geq 1}$  such that*

1.  $\sup_{\|\Phi h\|^2 \leq \sigma_n^2 M_n, \|\Phi g\|^2 \leq \sigma_n^2 M_n} \frac{w(\theta_0 + h)}{w(\theta_0 + g)} \rightarrow 1$  as  $n \rightarrow \infty$ .
2.  $k_n \ln k_n = o(M_n)$
3.  $\max \left( 0, \ln \left( \frac{\sqrt{\det(\Phi^T \Phi)}}{\sigma_n^{k_n} w(\theta_0)} \right) \right) = o(M_n)$

Then

$$E \left\| W(dF|Y) - \mathcal{N}(Y_{\langle \phi \rangle}, \sigma_n^2 \Sigma) \right\|_{\text{TV}} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

With Condition 1 below we ask for a sufficiently flat prior  $\widetilde{W}$  in an ellipsoid  $\mathcal{E}_{\theta_0, \Phi}(M_n)$ . Condition 2 ensures in particular that the weight the normal distribution puts on  $\mathcal{E}_{\theta_0, \Phi}(M_n)$  in the limit goes to 1. Condition 3 makes quantities linked to the volume of  $\mathcal{E}_{\theta_0, \Phi}(M_n)$  appear and guarantees that it has enough prior weight. This kind of assumption is common in the literature dealing with the concentration of posterior distributions, see for instance Ghosal, Ghosh and van der Vaart (2000).

Several of our applications illustrate that priors known to induce the posterior minimax convergence rate may not be flat enough to get the Gaussian shape with the asymptotic variance  $\sigma_n^2 \Sigma$ .

An important remark is the following: Condition 2 does not really limit the growth rate of  $k_n$ . Read in conjunction with the other two conditions, we see that a flatter prior distribution will permit us to take  $M_n$  larger. Thus the only condition on the growth rate of  $k_n$  is  $k_n \leq n$ .

Note that Theorem 2 is not a generalization of Theorem 1: Theorem 1 is more powerful for isotropic Gaussian priors. Consider again the regression framework with  $F_0 = (f_0(x_i))_{1 \leq i \leq n}$ , where  $f_0$  is a bounded function and  $(x_i)_{1 \leq i \leq n}$  is some design. Suppose that  $\sigma_n$  does not depend on  $n$ , and take  $k_n = n$  and  $W = \mathcal{N}(0, \tau_n^2 \Sigma)$ . Then the conditions of Theorem 1 are satisfied as soon as  $n^{1/4} = o(\tau_n)$ , but with Theorem 2 we need  $n \ln n = o(\tau_n^2)$ .

Our main applications, to the Gaussian sequence model and to the regression model using trigonometric polynomials and splines, are developed in section 5. We now present two remarks about the parametric case and the comparison with the pioneer work of Ghosal (1999).

*The parametric case.* Consider the regression of a function  $f$  defined on  $[0, 1]$ , with a fixed number  $k$  of regressors. Set a design  $(x_i^{(n)})_{n \geq 1, 1 \leq i \leq n}$ , with  $x_i^{(n)} \in [(i-1)/n, i/n]$  for any  $n \geq 1$ , and  $F_0 = (f(x_i^{(n)}))_{1 \leq i \leq n}$ . Choose a finite number of piecewise continuous and linearly independent regressors  $(\varphi_j)_{1 \leq j \leq k}$  on  $[0, 1]$ , and set  $\phi_j = (\varphi_j(x_i^{(n)}))_{1 \leq i \leq n}$  for  $1 \leq j \leq k$ . Assume that  $f$ ,  $k_n = k$ ,  $\sigma_n = \sigma$ , and  $\widetilde{W}$  do not depend on  $n$ .

We would like to compare Theorem 2 with the usual Bernstein-von Mises Theorem for parametric models, applied to such a regression framework. In that setting, let us suppose that  $w$  is continuous and positive, and that  $f$  is bounded. Then Condition 1 becomes  $M_n = o(n)$ , while Condition 3 reduces to  $\ln n = o(M_n)$ . Clearly, there exist such sequences  $(M_n)_{n \geq 1}$ , so Theorem 2 applies. Here the rescaling by  $\sqrt{n}$  of the Bernstein-von Mises Theorem for parametric models is hidden in the asymptotic posterior variance  $\sigma^2(\Phi^T \Phi)^{-1}$  of the parameter  $\theta$ . Indeed,  $(1/n) \Phi^T \Phi$  is a Riemann sum, and converges

towards the Gramian matrix of the collection  $(\varphi_j)_{1 \leq j \leq k}$  in  $\mathbb{L}^2([0, 1])$ .

PROOF. We have  $\|\Phi\theta_0\| \leq \|F_0\| \leq \sqrt{n}\|f\|_\infty$ , and  $\|\theta_0\|^2 \leq \|(\Phi^T\Phi)^{-1}\| \cdot \|\Phi\theta_0\|^2 \leq \|n(\Phi^T\Phi)^{-1}\| \|f\|_\infty^2$ .  $(1/n)\Phi^T\Phi$  converges towards the Gramian matrix of the collection  $(\varphi_j)_{1 \leq j \leq k}$  in  $\mathbb{L}^2([0, 1])$ , and its smallest eigenvalue is lower bounded for  $n$  large enough. Therefore  $\theta_0$  is bounded, and we can consider it lies in some compact set on which  $w$  is uniformly continuous and lower bounded by a positive constant. The rest follows.  $\square$

*Comparison with Ghosal's conditions.* The Bernstein-von Mises Theorem in a regression setting when the number of parameters goes to infinity has been first studied by Ghosal (1999) as an early step in the development of frequentist nonparametric Bayesian theory. In his paper the errors  $\varepsilon_i$  are not supposed to be Gaussian. Under the Gaussianity assumption we get improved results, which means that we have a non-trivial generalization of the Ghosal (1999) conditions in the case of Gaussian errors. In particular our condition for the prior smoothness is simpler, and the growth rate of the dimension  $k_n$  is much less constrained.

- Ghosal (1999) does not admit a modeling bias between  $F_0$  and  $\Phi\theta_0$ . In the present work the normality of the errors permits us to take  $F_0 \neq \Phi\theta_0$  without any cost, as it appears in the core of the proof (Lemma 7). The possibility of considering misspecified models is an important improvement.
- In Ghosal (1999)  $\sigma_n$  is constant, which does not allow the application to the Gaussian sequence model.
- Ghosal (1999) restricts the growth of the dimension  $k_n$  to  $k_n^4 \ln k_n = o(n)$  (see below). In our setting we only require  $k_n \leq n$ . With Ghosal's condition we could not have obtained the applications to the Gaussian sequence model or to the regression model for Sobolev or  $C^\alpha$  classes.

Let  $\delta_n^2 = \|(\Phi^T\Phi)^{-1}\|$  be the operator norm of  $(\Phi^T\Phi)^{-1}$  for the  $\ell^2$  metric, and let  $\eta_n^2$  be the maximal value on the diagonal of  $\Sigma$ . With our notations, the last two assumptions of Ghosal (1999) become

(A3) There exists  $\eta_0 > 0$  such that  $w(\theta_0) > \eta_0^{k_n}$ . Moreover

$$(7) \quad |\ln w(\theta) - \ln w(\theta_0)| \leq L_n(C)\|\theta - \theta_0\|,$$

whenever  $\|\theta - \theta_0\| \leq C\delta_n k_n \sqrt{\ln k_n}$ , where the Lipschitz constant  $L_n(C)$  is subject to some growth restriction (see assumption A4).

(A4)

$$(8) \quad \forall C > 0, L_n(C)\delta_n k_n \sqrt{\ln k_n} \rightarrow 0 \quad \text{and} \quad \eta_n k_n^{3/2} \sqrt{\ln k_n} \rightarrow 0.$$

Further the design satisfies a condition on the trace of  $\Phi^T \Phi$ :

$$(9) \quad \text{tr}(\Phi^T \Phi) = O(nk_n).$$

Since  $\Sigma$  is an orthogonal projection matrix on a  $k_n$ -dimensional space,  $\text{tr}(\Sigma) = k_n$  and  $\eta_n^2 \geq k_n/n$ . Thus the last part of (8) implies  $k_n^4 \ln k_n = o(n)$ .

If we add the normality of the errors and a slight technical condition  $\ln n = o(k_n \ln k_n)$ , these assumptions imply ours. Indeed, set  $M_n = C^2 k_n^2 \ln k_n$  for some arbitrary value of  $C$ . Our condition 2 is immediate. Condition 1 is got from (7) and the first part of (8). The beginning of (A3) implies  $-\ln w(\theta_0) = O(k_n) = o(M_n)$ . Using the concavity of the  $\ln$  function and (9), we get  $\ln \det(\Phi^T \Phi) \leq k_n \ln \text{tr}(\Phi^T \Phi) - k_n \ln k_n = O(k_n \ln n) = o(M_n)$ . Therefore our condition 3 holds.

**4. Semiparametric Bernstein-von Mises Theorems.** We consider two kinds of functionals of  $F$ : linear and non-linear ones. These results can be easily adapted to functionals of  $\theta$ , using the maps  $\theta \mapsto \Phi\theta$  and  $F \mapsto (\Phi^T \Phi)^{-1} \Phi^T F$ .

4.1. *The linear case.* For linear functionals of  $F$ , we have the following corollary:

**COROLLARY 1.** *Let  $p \geq 1$  fixed, and  $G$  be a  $\mathbb{R}^p \times \mathbb{R}^n$ -matrix. Suppose that the conditions of either Theorem 1 or Theorem 2 are satisfied. Then*

$$E \left\| W(d(GF)|Y) - \mathcal{N}(GY_{\langle \phi \rangle}, \sigma_n^2 G \Sigma G^T) \right\|_{TV} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

*Further, the distribution of  $GY_{\langle \phi \rangle}$  is  $\mathcal{N}(GF_{\langle \phi \rangle}, \sigma_n^2 G \Sigma G^T)$ .*

Corollary 1 is just a linear transform of the preceding Theorems, and of the distribution of  $Y_{\langle \phi \rangle}$ .

An example of application is given in subsection 5.2, in the context of the regression on Fourier's basis.

4.2. *The nonlinear case.* The Bernstein-von Mises Theorem which is presented here for non-linear functionals is derived from the nonparametric theorems thanks to Taylor expansions. In the Taylor expansion of a functional, the first order term naturally leads to the posterior normality, as in the case of linear functionals. We do not want that the second order term interfere with this phenomenon: it has to be controlled. The conditions of Theorem 2 below are stated to permit this control of the second order term.

Let  $p \geq 1$  fixed, and  $G : \mathbb{R}^n \mapsto \mathbb{R}^p$  be a twice continuously differentiable function. For  $F \in \mathbb{R}^n$ , let  $\dot{G}_F$  denote the Jacobian matrix of  $G$  at  $F$ , and  $D_F^2 G(\cdot, \cdot)$  the second derivative of  $G$ , as a bilinear function on  $\mathbb{R}^n$ . For any  $F \in \langle \phi \rangle$  and  $a > 0$ , let

$$(10) \quad B_F(a) = \sup_{h \in \langle \phi \rangle : \|h\|^2 \leq \sigma_n^2 a} \sup_{0 \leq t \leq 1} \|D_{F+th}^2 G(h, h)\|.$$

where  $\|\cdot\|$  denotes the Euclidean norm of  $\mathbb{R}^p$ .

We also consider the following nonnegative symmetric matrix

$$(11) \quad \Gamma_F = \sigma_n^2 \dot{G}_F \Sigma \dot{G}_F^T.$$

In the following,  $\|\Gamma_F^{-1}\|$  denotes the Euclidean operator norm of  $\Gamma_F^{-1}$ , which is also the inverse of the smallest eigenvalue of  $\Gamma_F$ .

Let  $\mathcal{I}$  be the collection of all intervals in  $\mathbb{R}$ , and for any  $I \in \mathcal{I}$ , let  $\psi(I) = P(Z \in I)$ , where  $Z$  is a  $\mathcal{N}(0, 1)$  random variable. Recall that  $Y_{\langle \phi \rangle}$  is the MLE and the orthogonal projection of  $Y$  on  $\langle \phi \rangle$ .

**THEOREM 3.** *Let  $G : \mathbb{R}^n \mapsto \mathbb{R}^p$  be a twice continuously differentiable function, and let  $\Gamma_F$  be as just defined. Suppose that  $\Gamma_{F_{\langle \phi \rangle}}$  is nonsingular, and that there exists a sequence  $(M_n)_{n \geq 1}$  such that  $k_n = o(M_n)$  and*

$$(12) \quad B_{F_{\langle \phi \rangle}}^2(M_n) = o\left(\|\Gamma_{F_{\langle \phi \rangle}}^{-1}\|^{-1}\right).$$

*Suppose further that the conditions of either Theorem 1 or Theorem 2 are satisfied. Then, for any  $b \in \mathbb{R}^p$ ,*

$$(13) \quad E \left[ \sup_{I \in \mathcal{I}} \left| W \left( \frac{b^T (G(F) - G(Y_{\langle \phi \rangle}))}{\sqrt{b^T \Gamma_{F_{\langle \phi \rangle}} b}} \in I \mid Y \right) - \psi(I) \right| \right] \rightarrow 0 \text{ as } n \rightarrow \infty.$$

*Under the same conditions,*

$$(14) \quad \sup_{I \in \mathcal{I}} \left| P \left( \frac{b^T (G(Y_{\langle \phi \rangle}) - G(F_{\langle \phi \rangle}))}{\sqrt{b^T \Gamma_{F_{\langle \phi \rangle}} b}} \in I \right) - \psi(I) \right| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Note that  $\sup_{I \in \mathcal{I}} |Q(I) - Q'(I)|$  is the Levy-Prokhorov distance between two distributions  $Q$  and  $Q'$  on  $\mathbb{R}$ . The Levy-Prokhorov distance metrizes the convergence in distribution. So, when  $p = 1$  the Levy-Prokhorov distance between the distribution  $W(dG(F)|Y)$  and  $\mathcal{N}(G(Y_{\langle \phi \rangle}), \Gamma_{F_{\langle \phi \rangle}})$  goes to 0 in mean, while  $G(Y_{\langle \phi \rangle})$  goes to  $\mathcal{N}(G(F_{\langle \phi \rangle}), \Gamma_{F_{\langle \phi \rangle}})$  in distribution.

An application of Theorem 3 is given in subsection 5.2, in the context of the regression on Fourier's basis. The proof is delayed to subsection 6.3.

**5. Applications.** Here we give the three applications described in section 2. The models studied and the collections of regressors used have already been defined there.

5.1. *The Gaussian sequence model.* We consider the model (2). Here the MLE is the projection  $\theta_Y = (Y_j)_{1 \leq j \leq k_n}$ .

The nonparametric case corresponds to the estimation of  $\theta^0$ . Under the assumption that  $\theta^0$  is in some regularity class, we will obtain a Bernstein-von Mises Theorem with the posterior convergence rate already obtained in previous works, in particular Ghosal and van der Vaart (2007). On the other hand, for some priors known to achieve this rate, it will be seen that the centering point and the asymptotic variance of the posterior distribution do not fit with the ones expected in a Bernstein-von Mises Theorem. We also look at the semiparametric estimation of the squared  $\ell^2$  norm of  $\theta^0$ .

5.1.1. *The nonparametric estimation of  $\theta^0$ .*

PROPOSITION 1. *Suppose that  $\sum_{j=1}^{k_n} (\theta_j^0)^2$  is bounded. This holds when  $\theta^0$  is an element of  $\ell^2(\mathbb{N})$  not depending on  $n$ . With a prior  $\widetilde{W} = \mathcal{N}(0, \tau_n^2 I_{k_n})$  such that  $n^{-1/4} = o(\tau_n)$ , we have for any sequence  $k_n \leq n$ ,*

$$E \left\| \widetilde{W}(d\theta|Y) - \mathcal{N}\left(\theta_Y, \frac{1}{n} I_{k_n}\right) \right\|_{\text{TV}} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

and the convergence rate of  $\theta$  towards  $\theta_0$  is  $\sqrt{\frac{k_n}{n}}$ : for every  $\lambda_n \rightarrow \infty$ ,

$$E \left[ \widetilde{W} \left( \|\theta - \theta_0\| \geq \lambda_n \sqrt{\frac{k_n}{n}} \mid Y \right) \right] \rightarrow 0.$$

Recall that  $\theta_0 = (\theta_j^0)_{1 \leq j \leq k_n}$  is the projection of  $\theta^0$ .

PROOF. The beginning is an immediate corollary of Theorem 1. For the convergence rate, let  $\lambda_n \rightarrow \infty$ . Since  $\theta_Y - \theta_0 \sim \mathcal{N}(0, \frac{1}{n} I_{k_n})$ ,

$$P \left( \|\theta_Y - \theta_0\| \geq \frac{\lambda_n}{2} \sqrt{\frac{k_n}{n}} \right) \rightarrow 0.$$

In the same way

$$\begin{aligned} E \left[ \widetilde{W} \left( \|\theta - \theta_Y\| \geq \frac{\lambda_n}{2} \sqrt{\frac{k_n}{n}} \right) \right] &\leq E \left\| \widetilde{W}(d\theta|Y) - \mathcal{N}\left(\theta_Y, \frac{1}{n} I_{k_n}\right) \right\|_{\text{TV}} \\ &\quad + \mathcal{N}\left(0, \frac{1}{n} I_{k_n}\right) \left( \left\{ \|\theta - \theta_Y\| \geq \frac{\lambda_n}{2} \sqrt{\frac{k_n}{n}} \right\} \right), \end{aligned}$$

which goes to 0. Therefore

$$E \left[ \widetilde{W} \left( \|\theta - \theta_0\| \geq \lambda_n \sqrt{\frac{k_n}{n}} \right) \right] \rightarrow 0.$$

□

However in such a general setting we have no information about the bias between  $\theta^0$  and its projection  $\theta_0$ . Several authors add the assumption that the true parameter belongs to a Sobolev class of regularity  $\alpha > 0$ , defined by the relation  $\sum_{j=1}^{\infty} |\theta_j^0|^2 j^{2\alpha} < \infty$ . In this setting we show that for some priors the induced posterior may achieve the nonparametric convergence rate but with a centering point and a variance different from what is expected in the Bernstein-von Mises Theorem. Then we exhibit priors for which both the Bernstein-von Mises Theorem and the nonparametric convergence rate hold.

From now on, we suppose that  $\sum_{j=1}^{\infty} |\theta_j^0|^2 j^{2\alpha} < \infty$ . In this setting (Ghosal and van der Vaart, 2007, §7.6) considers a prior  $\widetilde{W}$  such that  $\theta_1, \theta_2, \dots$  are independent, and  $\theta_j$  is normally distributed with variance  $\sigma_{j,k_n}^2$ . Further, the variances are supposed to satisfy

$$(15) \quad c/k_n \leq \min\{\sigma_{j,k_n}^2 j^{2\alpha} : 1 \leq j \leq k_n\} \leq C/k_n$$

for some positive constants  $c$  and  $C$ . Suppose that  $\alpha \geq 1/2$  and there exists constants  $C_1$  and  $C_2$  such that  $C_1 n^{1/(1+2\alpha)} \leq k_n \leq C_2 n^{1/(1+2\alpha)}$ . Then (Ghosal and van der Vaart, 2007, Theorem 11) proved that the posterior converges at the rate  $n^{-\alpha/(1+2\alpha)}$ .

In order to get  $n^{-1} I_{k_n}$  as asymptotic variance, we need more stringent conditions on  $k_n$ , or a flatter prior. To see this is necessary consider, for  $k_n \approx n^{1/(1+2\alpha)}$ , the following choice for  $\sigma_{j,k_n}$ :

$$\sigma_{j,k_n}^2 = \begin{cases} k_n^{-1} & \text{if } 1 \leq j \leq k_n/2, \\ 2^{2\alpha}/n & \text{if } j > k_n/2. \end{cases}$$

Then  $\min\{\sigma_{j,k_n}^2 j^{2\alpha} : 1 \leq j \leq k_n\} \approx k_n^{-1}$ , and the posterior converges at the rate  $n^{-\alpha/(1+2\alpha)}$ .

For this case we can explicitly calculate the posterior distribution. This is similar to the calculation made in the proof of Theorem 1. The coordinates are independent, and

$$\widetilde{W}(d\theta_j|Y) = \mathcal{N} \left( \frac{\sigma_{j,k_n}^2}{\sigma_n^2 + \sigma_{j,k_n}^2} Y_j, \frac{\sigma_n^2 \sigma_{j,k_n}^2}{\sigma_n^2 + \sigma_{j,k_n}^2} \right).$$

For  $j > k_n/2$ ,  $\frac{\sigma_{j,k_n}^2}{\sigma_n^2 + \sigma_{j,k_n}^2} = \frac{4^\alpha}{1+4^\alpha}$ , and therefore  $\left\| \widetilde{W}(d\theta_j|Y) - \mathcal{N}(Y_j, \sigma_n^2) \right\|_{\text{TV}}$  is bounded away from 0.

By contrast, with an isotropic and flat prior we obtain the centering point and the asymptotic variance we expected, and the same convergence rate as previously. We have the following.

**PROPOSITION 2.** *Suppose that  $\theta^0$  belongs to the Sobolev class of regularity  $\alpha > 0$ . Choose a prior  $\widetilde{W} = \mathcal{N}(0, \tau_n^2 I_{k_n})$  such that  $n^{-1/4} = o(\tau_n)$ , which ensures the asymptotic normality of the posterior distribution as in Proposition 1.*

*If further  $k_n \approx n^{1/(1+2\alpha)}$ , then the convergence rate of  $\theta$  towards  $\theta_0$  and towards  $\theta^0$  is  $n^{-\alpha/(1+2\alpha)}$ : for every  $\lambda_n \rightarrow \infty$ ,*

$$E \left[ \widetilde{W} \left( \|\theta - \theta^0\| \geq \lambda_n n^{-\alpha/(1+2\alpha)} \mid Y \right) \right] \rightarrow 0.$$

**PROOF.** We consider  $\theta$  and  $\theta_0$  as elements of  $\ell^2(\mathbb{N})$  by setting  $\theta_j = \theta_{0,j} = 0$  for  $j \geq k_n + 1$ . The convergence rate towards  $\theta_0$  has already been established in Proposition 1. Since  $\theta_{0,j} = \theta_j^0$  for  $1 \leq j \leq k_n$ ,  $\|\theta^0 - \theta_0\| \leq k_n^{-\alpha} \sqrt{\sum_{j=k_n+1}^{\infty} (\theta_j^0)^2 j^{2\alpha}} = O(k_n^{-\alpha})$ . Therefore the convergence rate of  $\theta$  towards  $\theta^0$  is also  $n^{-\alpha/(1+2\alpha)}$ .  $\square$

**5.1.2. Semiparametric theorem for the  $\ell^2$  norm of  $\theta^0$ .** We consider the prior distribution used in Proposition 2, but now we look at the posterior distribution of  $\|\theta\|^2$ . To get asymptotic normality with variance  $n^{-1/2}$ , we just need  $k_n = o(\sqrt{n})$ . To control the bias term we need  $\alpha > 1/2$ , and in this case we get an adaptive Bayesian estimator.

**PROPOSITION 3.** *Let  $\alpha > 1/2$  and suppose that  $\theta^0$  belongs to the Sobolev class of regularity  $\alpha$ . Choose a prior  $\widetilde{W} = \mathcal{N}(0, \tau_n^2 I_{k_n})$  such that  $n^{-1/4} = o(\tau_n)$ . Then, for any choice of  $k_n$  such that  $k_n = o(\sqrt{n})$  and  $\sqrt{n} = o(k_n^{2\alpha})$ ,*

$$E \left[ \sup_{I \in \mathcal{I}} \left| \widetilde{W} \left( \frac{\sqrt{n} (\|\theta\|^2 - \|\theta_Y\|^2)}{2\|\theta^0\|} \in I \mid Y \right) - \psi(I) \right| \right] \rightarrow 0 \text{ as } n \rightarrow \infty$$

and  $\frac{\sqrt{n} (\|\theta_Y\|^2 - \|\theta_0\|^2)}{2\|\theta^0\|} \rightarrow \mathcal{N}(0, 1)$  in distribution, as  $n \rightarrow \infty$ . Further, the bias is negligible with respect to the square root of the variance:

$$\frac{\sqrt{n} (\|\theta_0\|^2 - \|\theta^0\|^2)}{2\|\theta^0\|} = o(1).$$

In particular the choice  $k_n = \sqrt{n/\ln n}$  is adaptive in  $\alpha$ .

PROOF. We set up an application of Theorem 3. Since  $\sigma_n = n^{-1/2}$ , the conditions of Theorem 1 are fulfilled.

Here  $G(\theta) = \theta^T \theta$ ,  $\dot{G}_\theta = 2\theta^T$  and  $\ddot{G}_\theta = 2I_{k_n}$ . Therefore  $B_{\theta_0}(M_n) = 2M_n/n$ , while  $\Gamma_{\theta_0} = 4\|\theta_0\|^2/n$ .

Let us choose  $(M_n)_{n \geq 1}$  such that  $k_n = o(M_n)$  and  $M_n = o(\sqrt{n})$ . Such sequences exist and fulfill the conditions of Theorem 3.

Since  $\|\theta_0\|^2 \rightarrow \|\theta^0\|^2$ , we can substitute the variance  $\Gamma_{\theta_0}$  by  $4\|\theta^0\|^2/n$  and get the two asymptotic normality results, (13) and (14).

As  $n \rightarrow \infty$ ,  $\|\theta^0\|^2 - \|\theta_0\|^2 = \|\theta^0 - \theta_0\|^2 = O(k_n^{-2\alpha})$ , as in the proof of Proposition 2. If  $\sqrt{n} = o(k_n^{2\alpha})$ , we get  $\sqrt{n}(\|\theta_0\|^2 - \|\theta^0\|^2) = o(1)$ .  $\square$

5.2. *Regression on Fourier's basis.* Now we consider the regression model (3) with a function  $f$  in a Sobolev class  $\mathcal{W}(\alpha, L)$ , and use Fourier's basis (4). For any  $\theta \in \mathbb{R}^{k_n}$ , we define  $f_\theta = \sum_{j=1}^{k_n} \theta_j \varphi_j$ . We also denote by  $\theta^0 \in \ell^2(\mathbb{N})$  the sequence of Fourier's coefficients of  $f$ :  $f = \sum_{j=1}^{\infty} \theta_j^0 \varphi_j$ .

The following useful Lemma about our collection of regressors can be found for instance in Tsybakov (2004) (we slightly modified it to take into account the case  $n$  even):

LEMMA 1. *Suppose either that  $n$  is odd and  $k_n \leq n$ , or  $n$  is even and  $k_n \leq n - 1$ . Consider the collection  $(\phi_j)_{1 \leq j \leq k_n}$  defined before, and  $\Phi$  the associated matrix. Then*

$$\Phi^T \Phi = nI_{k_n}.$$

This makes the regression on Fourier's basis very close to the Gaussian sequence model, and the results we obtain are similar.

In this subsection we first consider the estimation of  $f$  in a Sobolev class, for which we get a Bernstein-von Mises Theorem and the frequentist minimax  $n^{-\alpha/(1+2\alpha)}$  posterior convergence rate for the  $L^2$  norm. Then we consider two semiparametric settings: the estimation of a linear functional of  $f$ , and the estimation of the  $L^2$  norm of  $f$ . We get the adaptive  $\sqrt{n}$  convergence rate for any  $\alpha > 1/2$ .

### 5.2.1. Nonparametric Bernstein-von Mises Theorem in Sobolev classes.

PROPOSITION 4. *Suppose that  $f$  belongs to some Sobolev class  $\mathcal{W}(\alpha, L)$  for  $L > 0$  and  $\alpha > 1/2$ . Let  $k_n \approx n^{1/(1+2\alpha)}$  and  $\widetilde{W} = \mathcal{N}(0, \gamma_n I_{k_n})$  be the prior on  $\theta$ , for a sequence  $(\gamma_n)_{n \geq 1}$  such that  $1/\sqrt{\gamma_n} = o(\gamma_n)$ . Then*

$$E \left\| \widetilde{W}(d\theta|Y) - \mathcal{N} \left( \theta_Y, \frac{\sigma^2}{n} I_{k_n} \right) \right\|_{\text{TV}} \rightarrow 0 \text{ as } n \rightarrow \infty$$

and the convergence rate relative to the Euclidean norm for  $f_\theta$  is  $n^{-\alpha/(1+2\alpha)}$ : for every  $\lambda_n \rightarrow \infty$ ,

$$E \left[ \widetilde{W} \left( \|f_\theta - f\| \geq \lambda_n n^{-\alpha/(1+2\alpha)} \middle| Y \right) \right] \rightarrow 0.$$

PROOF. The conditions of Theorem 1 are fulfilled: with  $\tau_n^2 = n\gamma_n$ , we have  $n = o(\tau_n^4)$ . The first assertion follows.

Because of the orthogonal nature of Fourier's basis,  $\|f_\theta - f\| = \|\theta - \theta^0\|$  in  $\ell^2(\mathbb{N})$ . We use the decomposition  $\|\theta - \theta^0\|^2 \leq \|\theta - \theta_0\|^2 + \|\theta_0 - \theta^0\|^2$ . In the same way as in the proof of Proposition 1, for any  $\lambda_n \rightarrow \infty$ ,

$$E \left[ \widetilde{W} \left( \|\theta - \theta_0\| \geq \lambda_n \sqrt{\frac{k_n}{n}} \right) \right] \rightarrow 0.$$

Going back to Definition 1, we have

$$\|\theta_0 - \theta^0\|^2 = \sum_{j=k_n+1}^{\infty} (\theta_j^0)^2 \leq k_n^{-2\alpha} \sum_{j=k_n+1}^{\infty} a_j^{2\alpha} (\theta_j^0)^2 = O(k_n^{-2\alpha}).$$

This permits to get

$$E \left[ \widetilde{W} \left( \|\theta - \theta^0\| \geq \lambda_n n^{-\alpha/(1+2\alpha)} \middle| Y \right) \right] \rightarrow 0.$$

□

5.2.2. *Linear functionals of  $f$ .* Let  $g : [0, 1] \rightarrow \mathbb{R}$  be a function in  $\mathbb{L}^2([0, 1])$ . We want to estimate  $\mathcal{F}(f) = \int_0^1 fg$ , and we approximate it by

$$\frac{1}{n} \sum_{i=1}^n g(i/n) f(i/n) = GF_0$$

where  $G = (g(i/n)/n)_{1 \leq i \leq n}^T$ . The plug-in MLE estimator of  $GF_0$  in the misspecified model  $\langle \phi \rangle$  is  $GY_{\langle \phi \rangle}$ . More generally, we consider the functional  $F \mapsto GF$ . The following result is adaptive, in the sense that the same choice  $k_n = \lfloor n/\ln n \rfloor$  entails the convergence rate  $n^{-1/2}$  for all values of  $\alpha > 1/2$ .

PROPOSITION 5. *Suppose  $f$  is bounded, and let  $W$  be the prior induced by the  $\mathcal{N}(0, \gamma_n I_{k_n})$  distribution on  $\theta$ , for a sequence  $(\gamma_n)_{n \geq 1}$  such that  $1/\sqrt{\gamma_n} = o(\gamma_n)$ . Then*

1.

$$E \left\| W(d(GF)|Y) - \mathcal{N}(GY_{\langle \phi \rangle}, \sigma^2 G \Sigma G^T) \right\|_{\text{TV}} \rightarrow 0$$

and the distribution of  $GY_{\langle \phi \rangle}$  is  $\mathcal{N}(GF_{\langle \phi \rangle}, \sigma^2 G \Sigma G^T)$ .

2. Suppose further that  $f$  and  $g$  belong to some Sobolev class  $\mathcal{W}(\alpha, L)$  for  $L > 0$  and  $\alpha > 1/2$ . Then  $G\Sigma G^T \sim \frac{1}{n} \int_0^1 g^2$ ,

$$E \left\| W \left( d \frac{\sqrt{n}(GF - GY_{\langle \phi \rangle})}{\sigma \sqrt{\int_0^1 g^2}} \middle| Y \right) - \mathcal{N}(0, 1) \right\|_{\text{TV}} \rightarrow 0,$$

and  $\frac{\sqrt{n}(GY_{\langle \phi \rangle} - GF_{\langle \phi \rangle})}{\sigma \sqrt{\int_0^1 g^2}} \rightarrow \mathcal{N}(0, 1)$  in distribution, as  $n \rightarrow \infty$ .

3. Suppose that  $f$  and  $g$  belong to some Sobolev class  $\mathcal{W}(\alpha, L)$  for  $L > 0$  and  $\alpha > 1/2$ , and suppose further that  $k_n$  is large enough so that  $n = o(k_n^{2\alpha})$ . Then the bias is negligible with respect to the square root of the variance:

$$\frac{\sqrt{n}(GF_{\langle \phi \rangle} - \mathcal{F}(f))}{\sigma \sqrt{\int_0^1 g^2}} = o(1).$$

Before the proof we give two lemmas, proved in Appendix B, about the error terms of the approximation of a Sobolev class by a sieve build on Fourier's basis, and of the approximation of an integral by a Riemann sum.

LEMMA 2. Let  $\alpha > 1/2$  and  $L > 0$ . We suppose  $n$  odd or  $k_n < n$ . If  $f \in \mathcal{W}(\alpha, L)$ ,

$$\|F_0 - F_{\langle \phi \rangle}\| \leq (1 + o(1)) \frac{\sqrt{2}L}{\pi^\alpha} \frac{\sqrt{n}}{k_n^\alpha}.$$

Further,  $\|F_0\| \sim \sqrt{n \int_0^1 f^2}$  and  $\|F_0 - F_{\langle \phi \rangle}\| = O(k_n^{-\alpha} \|F_0\|)$ .

LEMMA 3. Let two functions  $f \in \mathcal{W}(\alpha, L)$  and  $g \in \mathcal{W}(\alpha', L')$  for some  $\alpha, \alpha' > 1/2$  and two positive numbers  $L$  and  $L'$ . Then

$$\left| \frac{1}{n} \sum_{i=1}^n f(i/n)g(i/n) - \int_0^1 fg \right| = O\left(n^{-\inf(\alpha, \alpha')}\right).$$

PROOF OF PROPOSITION 5. 1. The first assertion is just Corollary 1.

The conditions of Theorem 1 are fulfilled, as in the proof of Proposition 4.

2. If  $g \in \mathcal{W}(\alpha, L)$  for  $L > 0$  and  $\alpha > 1/2$ ,  $G\Sigma G^T = \|\Sigma G^T\|^2 \sim \|G^T\|^2$  by Lemma 2. In the meantime  $\|G^T\|^2 = \frac{1}{n^2} \sum_{i=1}^n g^2(x_i) \sim \frac{1}{n} \int_0^1 g^2$  by Lemma 3. So  $G\Sigma G^T \sim \frac{1}{n} \int_0^1 g^2$ , and the variance in the formulas of Corollary 1 can be substituted with  $\frac{1}{n} \int_0^1 g^2$ .

3. We decompose the bias into two terms,  $|GF_0 - \mathcal{F}(f)|$  and  $|GF_{\langle\phi\rangle} - GF_0|$ , and show that both are  $o(n^{-1/2})$ . The first term is controlled by Lemma 3. For the last one,  $|GF_{\langle\phi\rangle} - GF_0| \leq \|G^T\| \|F_{\langle\phi\rangle} - F_0\|$ . But  $\|G^T\| = O(n^{-1/2})$ ,  $\|F_{\langle\phi\rangle} - F_0\| = O(k_n^{-\alpha} \|F_0\|)$  by Lemma 2, and  $\|F_0\| = O(\sqrt{n})$ . We conclude thanks to the assumption  $n = o(k_n^{2\alpha})$ .  $\square$

5.2.3.  $L^2$  norm of  $f$ . Suppose that we want to estimate  $\mathcal{F}(f) = \int_0^1 f^2$ . We can consider the plug-in MLE estimator

$$G(Y_{\langle\phi\rangle}) = \frac{1}{n} \|Y_{\langle\phi\rangle}\|^2 = \frac{1}{n} \sum_{i=1}^n \left( \sum_{j=1}^{k_n} \theta_{Y,j} \varphi_j(i/n) \right)^2.$$

More generally we define, for any  $F \in \mathbb{R}^n$ ,

$$(16) \quad G(F) = \frac{1}{n} \|F\|^2.$$

With a Gaussian prior, we obtain the following result, which is also adaptive: the same  $k_n = \lfloor \sqrt{n} / \ln n \rfloor$  is suitable whatever  $\alpha > 1/2$ .

PROPOSITION 6. *Let  $G(F) = \|F\|^2/n$ . Suppose that  $f \in \mathcal{W}(\alpha, L)$  for some  $L > 0$  and  $\alpha > 1/2$ . Let  $W$  be the prior induced by the  $\mathcal{N}(0, \gamma_n I_{k_n})$  distribution on  $\theta$ , for a sequence  $(\gamma_n)_{n \geq 1}$  such that  $1/\sqrt{n} = o(\gamma_n)$ . The sequence  $(k_n)_{n \geq 1}$  can be chosen such that  $k_n = o(\sqrt{n})$  and  $\sqrt{n} = o(k_n^{2\alpha})$ , and with such a choice,*

$$E \left[ \sup_{I \in \mathcal{I}} \left| W \left( \frac{\sqrt{n} (G(F) - G(Y_{\langle\phi\rangle}))}{2\sigma \sqrt{\mathcal{F}(f)}} \in I \mid Y \right) - \psi(I) \right| \right] \rightarrow 0 \text{ as } n \rightarrow \infty$$

and  $\frac{\sqrt{n} (G(Y_{\langle\phi\rangle}) - G(F_{\langle\phi\rangle}))}{2\sigma \sqrt{\mathcal{F}(f)}} \rightarrow \mathcal{N}(0, 1)$  in distribution, as  $n \rightarrow \infty$ . Further, the bias is negligible with respect to the square root of the variance:

$$\frac{\sqrt{n} (G(F_{\langle\phi\rangle}) - \mathcal{F}(f))}{2\sigma \sqrt{\mathcal{F}(f)}} = o(1).$$

A similar corollary could be stated for a non-Gaussian prior.

PROOF. First, let us note that the conditions of Theorem 1 are fulfilled, as in the proof of Proposition 4. Lemma 10 in Appendix B insures that  $f$  is bounded.

In this setting  $\dot{G}_F = (2/n) F^T$  and  $D_F^2 G(h, h) = (2/n) \|h\|^2$  for any  $F \in \mathbb{R}^n$  and any  $h \in \mathbb{R}^n$ . Therefore  $B_F(a) = 2\sigma^2 a/n$ , and  $\Gamma_F = 4(\sigma^2/n^2) \|F\|^2$ . By Lemma 2,  $\|F_{\langle\phi\rangle}\|^2 \sim \|F_0\|^2 \sim n \mathcal{F}(f)$ . Thus  $\Gamma_{F_{\langle\phi\rangle}} = 4(1 + o(1)) \mathcal{F}(f)/n$ .

Let us choose  $(M_n)_{n \geq 1}$  such that  $k_n = o(M_n)$  and  $M_n = o(\sqrt{n})$ . Such sequences exist and fulfill the conditions of Theorem 3. We can substitute the variance  $\Gamma_{F_{\langle\phi\rangle}}$  by  $4\mathcal{F}(f)/n$  and get the two asymptotic normality results.

Let us now consider the bias term.

$$\mathcal{F}(f) - G(F_{\langle\phi\rangle}) \leq \frac{\|F_0\|^2 - \|F_{\langle\phi\rangle}\|^2}{n} + \left( \int_0^1 f^2 - \frac{1}{n} \sum_{i=1}^n f^2(i/n) \right)$$

We use Lemma 2 to control  $\|F_0\|^2 - \|F_{\langle\phi\rangle}\|^2$ , and Lemma 3 for the other term:

$$|\mathcal{F}(f) - G(F_{\langle\phi\rangle})| = O(k_n^{-2\alpha}) + O(n^{-\alpha}).$$

This is a  $o(1/\sqrt{n})$  under the assumptions of Corollary 6.  $\square$

**5.3. Regression on splines.** Here we consider the regression model for functions in  $C^\alpha[0, 1]$  with  $\alpha > 0$ , using splines, set up in section 2. We first develop further the framework and the assumptions used here, and recall the previous result of (Ghosal and van der Vaart, 2007, §7.7.1) which obtains the posterior concentration at the frequentist minimax rate. Then we present two Bernstein-von Mises Theorems: the first one with the same prior as Ghosal and van der Vaart (2007) but a stronger condition on  $k_n$  (or equivalently on  $\alpha$ ); the second one with a flatter prior, for which we obtain the minimax convergence rate in addition to the asymptotic Gaussianity of the posterior distribution.

To see this, we begin with some preliminaries. For any  $\theta \in \mathbb{R}^{k_n}$ , define  $f_\theta = \sum_{j=1}^{k_n} \theta_j B_j$ . The B-splines basis has the following approximation property: for any  $\alpha > 0$ , there exist  $C_\alpha > 0$  such that, if  $f \in C^\alpha[0, 1]$ , there exists  $\theta^\infty \in \mathbb{R}^{k_n}$  satisfying

$$(17) \quad \|f - f_{\theta^\infty}\|_\infty \leq C_\alpha k_n^{-\alpha} \|f\|_\alpha.$$

We need the design  $(x_i^{(n)})_{n \geq 1, 1 \leq i \leq n}$  to be sufficiently regular and, as stressed in Ghosal and van der Vaart (2007), the spatial separation property of B-splines permits us to express the precise condition in terms of the covariance matrix  $\Phi^T \Phi$ . We suppose that there exist positive constants  $C_1$  and  $C_2$  such that, as  $n$  increases, for any  $\theta \in \mathbb{R}^{k_n}$ ,

$$(18) \quad C_1 \frac{n}{k_n} \|\theta\|^2 \leq \theta^T \Phi^T \Phi \theta \leq C_2 \frac{n}{k_n} \|\theta\|^2.$$

Let us associate the norm  $\|f\|_n = \sqrt{\frac{1}{n} \sum_{i=1}^n |f(x_i)|^2}$  to the design. Note that  $\sqrt{n}\|f_\theta\|_n = \|\Phi\theta\|$  if  $\theta \in \mathbb{R}^{k_n}$ . Under (18) we have a relation between  $\|\cdot\|_n$  and the Euclidean norm on the parameter space: for every  $\theta_1$  and  $\theta_2$

$$C_1\|\theta_1 - \theta_2\| \leq \sqrt{k_n} \|f_{\theta_1} - f_{\theta_2}\|_n \leq C_2\|\theta_1 - \theta_2\|.$$

With these conditions (Ghosal and van der Vaart, 2007, Theorem 12) gets the posterior concentration at the minimax rate. Take  $\alpha \geq 1/2$ , let  $\widetilde{W} = \mathcal{N}(0, I_{k_n})$  be the prior on the spline coefficients, and suppose there exists constants  $C_3$  and  $C_4$  such that  $C_3 n^{1/(1+2\alpha)} \leq k_n \leq C_4 n^{1/(1+2\alpha)}$ . Then the posterior concentrates at the minimax rate  $n^{-\alpha/(1+2\alpha)}$  relative to  $\|\cdot\|_n$ : for every  $\lambda_n \rightarrow \infty$ ,

$$E \left[ \widetilde{W} \left( \|f_\theta - f\|_n \geq \lambda_n n^{-\alpha/(1+2\alpha)} \mid Y \right) \right] \rightarrow 0.$$

This is equivalent to a convergence rate  $n^{\frac{1-2\alpha}{2(1+2\alpha)}}$  relative to the Euclidean norm for  $\theta$ :

$$E \left[ \widetilde{W} \left( \|\theta - \theta_0\| \geq \lambda_n n^{\frac{1-2\alpha}{2(1+2\alpha)}} \mid Y \right) \right] \rightarrow 0.$$

Indeed (17) and the projection property imply

$$\|f_{\theta_0} - f\|_n \leq \|f_{\theta^\infty} - f\|_n \leq \|f_{\theta^\infty} - f\|_\infty \leq C_\alpha \|f\|_\alpha k_n^{-\alpha}.$$

Now, with modified assumptions we get the Bernstein-von Mises Theorem in two different settings. First, with the same prior as Ghosal and van der Vaart (2007):

PROPOSITION 7. *Assume that  $f$  is bounded,  $k_n = o\left(\left(\frac{n}{\ln n}\right)^{1/3}\right)$ , and (18) holds. Let  $\widetilde{W} = \mathcal{N}(0, I_{k_n})$  be the prior on the spline coefficients. Then*

$$(19) \quad E \left\| \widetilde{W}(d\theta|Y) - \mathcal{N}(\theta_Y, \sigma^2(\Phi^T \Phi)^{-1}) \right\|_{\text{TV}} \rightarrow 0 \text{ as } n \rightarrow \infty$$

and the convergence rate relative to the Euclidean norm for  $\theta$  is  $\frac{k_n}{\sqrt{n}}$ .

*Remarks.* We need  $\alpha > 1$  to get the Gaussian shape with the same convergence rate as in Ghosal and van der Vaart (2007). The conditions of Proposition 7 are satisfied in particular if there exists constants  $C_3$  and  $C_4$  such that  $C_3 n^{1/(1+2\alpha)} \leq k_n \leq C_4 n^{1/(1+2\alpha)}$ . In this case the convergence rate for  $\theta$  is  $n^{\frac{1-2\alpha}{2(1+2\alpha)}}$ .

PROOF. We set up an application of Theorem 2. We can choose  $M_n$  such that  $k_n \ln n = o(M_n)$  and  $M_n = o\left(\frac{n}{k_n^2}\right)$ . Assumption 2 is then trivially satisfied.

From (18) we get  $\|\Phi^T \Phi\| \leq C_2 \frac{n}{k_n}$  and  $\|(\Phi^T \Phi)^{-1}\| \leq C_1^{-1} \frac{k_n}{n}$ . We have also  $\ln \det(\Phi^T \Phi) \leq k_n \ln C_2 + k_n \ln\left(\frac{n}{k_n}\right) = O(k_n \ln n) = o(M_n)$ . Since  $\theta_0 = \Phi(\Phi^T \Phi)^{-1} F_0$ ,

$$\|\theta_0\|^2 \leq \frac{k_n}{C_1 n} \|F_0\|^2 \leq \frac{\|f\|_\infty}{C_1} k_n.$$

Therefore  $-\ln w(\theta_0) = O(1) + \frac{1}{2} \|\theta_0\|^2 = O(k_n) = o(M_n)$ , and assumption 3 holds.

Let  $h \in \mathbb{R}^{k_n}$  such that  $\|\Phi h\|^2 \leq \sigma^2 M_n$ . We have  $\|h\|^2 \leq \|(\Phi^T \Phi)^{-1}\| \cdot \|\Phi h\|^2 \leq \frac{\sigma^2 k_n M_n}{C_1 n} = o(k_n^{-1})$ . Therefore

$$(20) \quad \sup_{\|\Phi h\|^2 \leq \sigma^2 M_n} \left| \ln \frac{w(\theta_0 + h)}{w(\theta_0)} \right| \leq \sup_{\|\Phi h\|^2 \leq \sigma^2 M_n} \frac{\|h\|^2 + 2\|h\|\|\theta_0\|}{2} = o(1)$$

and assumption 1 follows.

Let us now prove the convergence rate. Let  $\lambda_n \rightarrow \infty$ . Then

$$P\left(\|\theta_Y - \theta_0\| \geq \frac{\lambda_n k_n}{2\sqrt{n}}\right) \leq P\left(\|\Phi(\theta_Y - \theta_0)\|^2 \geq \frac{C_1 \lambda_n^2 k_n}{4}\right) \rightarrow 0$$

since  $\|\Phi(\theta_Y - \theta_0)\|^2 \sim \sigma^2 \chi^2(k_n)$ . In the same way

$$\begin{aligned} E\left[\widetilde{W}\left(\|\theta - \theta_Y\| \geq \frac{\lambda_n k_n}{2\sqrt{n}}\right)\right] &\leq E\left\|\widetilde{W}(d\theta|Y) - \mathcal{N}(\theta_Y, \sigma^2(\Phi^T \Phi)^{-1})\right\|_{\text{TV}} \\ &\quad + \mathcal{N}(0, \sigma^2(\Phi^T \Phi)^{-1})\left(\left\{h : \|h\| \leq \frac{\lambda_n k_n}{2\sqrt{n}}\right\}\right) \\ &\rightarrow 0 \end{aligned}$$

where Theorem 2 controls the first term in the right. Therefore assumption 3 holds:

$$E\left[\widetilde{W}\left(\|\theta - \theta_0\| \geq \frac{\lambda_n k_n}{\sqrt{n}}\right)\right] \rightarrow 0.$$

Now, (19) is the same as Theorem 2 in terms of  $\widetilde{W}$ .  $\square$

The situation is similar to the one we encountered with the Gaussian sequence model. To get the Bernstein-von Mises Theorem with the same convergence rate as Ghosal and van der Vaart (2007) for  $\alpha \leq 1$ , we need a flatter prior:

PROPOSITION 8. Assume that  $f$  is bounded and (18) holds. Let  $\widetilde{W} = \mathcal{N}(0, \tau_n^2 I_{k_n})$  be the prior on the spline coefficients, with the sequence  $\tau_n$  satisfying

$$\frac{k_n^2 \ln n}{n} = o(\tau_n^2) \quad \text{and} \quad \frac{k_n^3 \ln n}{n} = o(\tau_n^4).$$

Then

$$E \left\| \widetilde{W}(d\theta|Y) - \mathcal{N}(\theta_Y, \sigma^2(\Phi^T \Phi)^{-1}) \right\|_{\text{TV}} \rightarrow 0 \text{ as } n \rightarrow \infty$$

and the convergence rate relative to the Euclidean norm for  $\theta$  is  $\frac{k_n}{\sqrt{n}}$ .

When  $\alpha > 0$  and  $k_n$  is of order  $n^{1/(1+2\alpha)}$ , the conditions reduce to  $n^{\frac{2-2\alpha}{1+2\alpha}} \ln n = o(\tau_n^4)$ . So we obtain the convergence rate of Ghosal and van der Vaart (2007) in addition to the Gaussian shape with the same  $k_n$ , even for  $\alpha \leq 1$ , but with a different prior.

PROOF. The proof is essentially the same as for Proposition 7.  $M_n$  can be chosen so that  $k_n \ln n = o(M_n)$ ,  $M_n = o\left(\frac{n\tau_n^2}{k_n}\right)$ , and  $M_n = o\left(\frac{n\tau_n^4}{k_n^2}\right)$ . These last two conditions are the ones needed to obtain the same upper bounds as in (20).  $\square$

## 6. Proofs.

6.1. *Proof of Theorem 1.* In the present setting all distributions are explicit and admit known densities with respect to the corresponding Lebesgue measure. We decompose any  $y \in \mathbb{R}^n$  in two orthogonal components  $y = \Phi\theta_y + y'$ , with  $\Phi^T y' = 0$ . Then

$$\begin{aligned} dP_\theta(y) &= c_1 \exp \left\{ -\frac{1}{2\sigma_n^2} (\|\Phi\theta\|^2 + \|\Phi\theta_y\|^2 + \|y'\|^2 - 2\theta^T \Phi^T \Phi \theta_y) \right\} \\ d\widetilde{W}(\theta) &= c_2 \exp \left\{ -\frac{1}{2\tau_n^2} \|\Phi\theta\|^2 \right\} \\ dP_\theta(y) d\widetilde{W}(\theta) &= c_1 c_2 \exp \left\{ -\frac{\sigma_n^2 + \tau_n^2}{2\sigma_n^2 \tau_n^2} \left\| \Phi \left( \theta - \frac{\tau_n^2}{\sigma_n^2 + \tau_n^2} \theta_y \right) \right\|^2 \right. \\ &\quad \left. - \frac{1}{2(\sigma_n^2 + \tau_n^2)} \|\Phi\theta_y\|^2 - \frac{1}{2\sigma_n^2} \|y'\|^2 \right\} \end{aligned}$$

where  $c_1 = (2\pi)^{-n/2} \sigma_n^{-n}$  and  $c_2 = (2\pi)^{-k_n/2} \tau_n^{-k_n} \det(\Phi^T \Phi)^{-1}$ .

Using the Bayes rule, we get the density of  $\widetilde{W}(d\theta|Y)$ , in which we recognize the normal distribution

$$(21) \quad \widetilde{W}(d\theta|Y) = \mathcal{N} \left( \frac{\tau_n^2}{\sigma_n^2 + \tau_n^2} \theta_Y, \frac{\sigma_n^2 \tau_n^2}{\sigma_n^2 + \tau_n^2} (\Phi^T \Phi)^{-1} \right).$$

So we have an exact expression for  $\widetilde{W}(d\theta|Y)$ , but the centering and the variance do not correspond to the limit distribution given in Theorem 1. Therefore we make use of the triangle inequality, with intermediate distribution  $Q = \mathcal{N} \left( \frac{\tau_n^2}{\sigma_n^2 + \tau_n^2} \theta_Y, \sigma_n^2 (\Phi^T \Phi)^{-1} \right)$ :

$$(22) \quad \left\| \widetilde{W}(d\theta|Y) - \mathcal{N}(\theta_Y, \sigma_n^2 (\Phi^T \Phi)^{-1}) \right\|_{\text{TV}} \leq \left\| \widetilde{W}(d\theta|Y) - Q \right\|_{\text{TV}} + \left\| Q - \mathcal{N}(\theta_Y, \sigma_n^2 (\Phi^T \Phi)^{-1}) \right\|_{\text{TV}}.$$

We first deal with the change in the variance, i.e. the first term on the right in (22).

Let  $\alpha_n = \frac{\tau_n}{\sigma_n} \sqrt{\ln \left( 1 + \frac{\sigma_n^2}{\tau_n^2} \right)}$ , and  $f$  and  $g$  be respectively the density functions of  $\mathcal{N}(0, I_{k_n})$  and  $\mathcal{N} \left( 0, \frac{\tau_n^2}{\sigma_n^2 + \tau_n^2} I_{k_n} \right)$ . Let  $U$  be a random variable following the chi-square distribution with  $k_n$  degrees of freedom  $\chi^2(k_n)$ . Let  $\sqrt{\Phi^T \Phi}$  be a square root of the matrix  $\Phi^T \Phi$ . The total variation norm is invariant under the bijective affine map  $\theta \mapsto \frac{1}{\sigma_n} \sqrt{\Phi^T \Phi} \left( \theta - \frac{\tau_n^2}{\sigma_n^2 + \tau_n^2} \theta_Y \right)$ , so

$$\begin{aligned} \left\| \widetilde{W}(d\theta|Y) - Q \right\|_{\text{TV}} &= \left\| \mathcal{N}(0, I_{k_n}) - \mathcal{N} \left( 0, \frac{\tau_n^2}{\sigma_n^2 + \tau_n^2} I_{k_n} \right) \right\|_{\text{TV}} \\ &= \int_{\mathbb{R}^{k_n}} (g - f)_+ = \int_{\|x\| \leq \sqrt{k_n} \alpha_n} (g(x) - f(x)) d^n x \\ &= P \left( U \leq \frac{\sigma_n^2 + \tau_n^2}{\tau_n^2} k_n \alpha_n^2 \right) - P(U \leq k_n \alpha_n^2) \\ &= P \left( \sqrt{k_n} (\alpha_n^2 - 1) \leq \frac{U - k_n}{\sqrt{k_n}} \leq \sqrt{k_n} \left( \frac{\sigma_n^2 + \tau_n^2}{\tau_n^2} \alpha_n^2 - 1 \right) \right). \end{aligned}$$

As  $n$  goes to infinity,  $\frac{U - k_n}{\sqrt{k_n}}$  converges towards  $\mathcal{N}(0, 1)$  in distribution. Using the Taylor expansion of  $\ln$ , we find

$$\alpha_n^2 = 1 - \frac{\sigma_n^2}{2\tau_n^2} + o \left( \frac{\sigma_n^2}{\tau_n^2} \right),$$

and therefore

$$\begin{aligned}\sqrt{k_n}(\alpha_n^2 - 1) &\sim -\sqrt{k_n}\frac{\sigma_n^2}{2\tau_n^2}, \\ \sqrt{k_n}\left(\frac{\sigma_n^2 + \tau_n^2}{\tau_n^2}\alpha_n^2 - 1\right) &\sim \sqrt{k_n}\frac{\sigma_n^2}{2\tau_n^2}.\end{aligned}$$

Since  $k_n = o(\tau_n^4/\sigma_n^4)$ , both these quantities go to 0. As a consequence,  $\left\|\widetilde{W}(d\theta|Y) - Q\right\|_{\text{TV}}$  goes to zero as  $n$  goes to infinity.

Let us now deal with the centering term, i.e. the second term on the right in (22).

LEMMA 4. *Let  $U$  be a standard normal random variable, let  $k \geq 1$ , and let  $Z \in \mathbb{R}^k$ . Then*

$$\|\mathcal{N}(0, I_k) - \mathcal{N}(Z, I_k)\|_{\text{TV}} = P(|U| \leq \|Z\|/2) \leq \|Z\|/\sqrt{2\pi}.$$

PROOF. Let  $g$  be the density of  $\mathcal{N}(0, I_k)$ . Then

$$\begin{aligned}\|\mathcal{N}(0, I_k) - \mathcal{N}(Z, I_k)\|_{\text{TV}} &= \int_{\mathbb{R}^k} (g(x) - g(x - Z))_+ d^k x \\ &= \int_{\{2x^T Z \leq \|Z\|^2\}} (g(x) - g(x - Z)) d^k x \\ &= P(U \leq \|Z\|/2) - P(U + \|Z\| \leq \|Z\|/2) \\ &\leq \|Z\|/\sqrt{2\pi}.\end{aligned}$$

The last line comes from the density of  $\mathcal{N}(0, 1)$  being bounded by  $1/\sqrt{2\pi}$ .  $\square$

Using again the invariance of the total variation norm under the bijective affine map  $\theta \mapsto \frac{1}{\sigma_n}\sqrt{\Phi^T\Phi}\left(\theta - \frac{\tau_n^2}{\sigma_n^2 + \tau_n^2}\theta_Y\right)$ ,

$$\begin{aligned}\|\mathcal{N}(\theta_Y, \sigma_n^2(\Phi^T\Phi)^{-1}) - Q\|_{\text{TV}} &= \left\|\mathcal{N}(0, I_{k_n}) - \mathcal{N}\left(\frac{\sigma_n\sqrt{\Phi^T\Phi}\theta_Y}{\tau_n^2 + \sigma_n^2}, I_{k_n}\right)\right\|_{\text{TV}} \\ &\leq \frac{1}{\sqrt{2\pi}}\frac{\sigma_n}{(\tau_n^2 + \sigma_n^2)}\|\Phi\theta_Y\| \\ &\leq \frac{1}{\sqrt{2\pi}}\frac{\sigma_n}{(\tau_n^2 + \sigma_n^2)}\left(\|F_0\| + \sqrt{\varepsilon^T\Sigma\varepsilon}\right).\end{aligned}$$

$\varepsilon^T \Sigma \varepsilon$  is a random variable following  $\sigma_n^2 \chi^2(k_n)$  distribution. By Jensen's inequality,  $E \left[ \sqrt{\varepsilon^T \Sigma \varepsilon} \right] \leq \sqrt{E[\varepsilon^T \Sigma \varepsilon]} = \sigma_n \sqrt{k_n}$ . Therefore

$$E \left\| \mathcal{N}(\theta_Y, \sigma_n^2 (\Phi^T \Phi)^{-1}) - Q \right\|_{\text{TV}} \leq \frac{1}{\sqrt{2\pi}} \frac{\sigma_n}{\tau_n^2 + \sigma_n^2} \left( \|F_0\| + \sigma_n \sqrt{k_n} \right),$$

which goes to zero under the assumptions of Theorem 1.

To conclude the proof, note that we deduce the results on  $W(dF|Y)$  from the ones on  $\widetilde{W}(d\theta|Y)$ , by the linear relation  $F = \Phi\theta$ .

6.2. *Proof of Theorem 2.* We make the proof for  $\widetilde{W}(d\theta|Y)$ . Then the result for  $W(dF|Y)$  is immediate. Our method is adapted from Boucheron and Gassiat (2009).

To any probability measure  $P$  on  $\mathbb{R}^{k_n}$ , we associate the probability

$$(23) \quad P^M = \frac{P(\cdot \cap \mathcal{E}_{\theta_0, \Phi}(M))}{P(\mathcal{E}_{\theta_0, \Phi}(M))}$$

with support in  $\mathcal{E}_{\theta_0, \Phi}(M)$ . It can be easily checked that

$$(24) \quad \|P - P^M\|_{\text{TV}} = P(\mathcal{E}_{\theta_0, \Phi}^c(M)).$$

The proof is divided into three steps based on the use of  $M_n$  as a threshold to truncate the probability distributions. Lemma 5 below controls  $E \left\| \mathcal{N}(\theta_Y, \sigma_n^2 (\Phi^T \Phi)^{-1}) - \mathcal{N}^{M_n}(\theta_Y, \sigma_n^2 (\Phi^T \Phi)^{-1}) \right\|_{\text{TV}}$ , Lemma 6 controls  $E \left\| \widetilde{W}^{M_n}(d\theta|Y) - \mathcal{N}^{M_n}(\theta_Y, \sigma_n^2 (\Phi^T \Phi)^{-1}) \right\|_{\text{TV}}$ , and Proposition 9 controls  $E \left\| \widetilde{W}(d\theta|Y) - \widetilde{W}^{M_n}(d\theta|Y) \right\|_{\text{TV}}$ . Taken together, these results give Theorem 2.

LEMMA 5. *If  $k_n < 4M_n$ , then*

$$E \left\| \mathcal{N}(\theta_Y, \sigma_n^2 (\Phi^T \Phi)^{-1}) - \mathcal{N}^{M_n}(\theta_Y, \sigma_n^2 (\Phi^T \Phi)^{-1}) \right\|_{\text{TV}} \leq 2e^{-\frac{(\sqrt{M_n} - 2\sqrt{k_n})^2}{8}}.$$

If  $k_n = o(M_n)$ , for  $n$  large enough, this bound can be replaced by  $e^{-M_n/9}$ .

PROOF. To control this quantity we consider two cases, depending on whether  $\theta_Y$  is near or far from  $\theta_0$ :

$$(25) \quad \begin{aligned} & \left\| \mathcal{N}(\theta_Y, \sigma_n^2 (\Phi^T \Phi)^{-1}) - \mathcal{N}^{M_n}(\theta_Y, \sigma_n^2 (\Phi^T \Phi)^{-1}) \right\|_{\text{TV}} \\ &= \mathcal{N}(\theta_Y, \sigma_n^2 (\Phi^T \Phi)^{-1}) (\mathcal{E}_{\theta_0, \Phi}^c(M_n)) \\ &\leq \mathbb{1}_{(\theta_Y - \theta_0)^T \Phi^T \Phi (\theta_Y - \theta_0) > \sigma_n^2 M_n / 4} \\ &\quad + \mathcal{N}(\theta_0, \sigma_n^2 (\Phi^T \Phi)^{-1}) (\mathcal{E}_{\theta_0, \Phi}^c(M_n/4)) \end{aligned}$$

Let  $U$  a random variable following a  $\chi^2(k_n)$  distribution. Taking the expectation on both sides of (25) gives

$$E \left\| \mathcal{N}(\theta_Y, \sigma_n^2(\Phi^T \Phi)^{-1}) - \mathcal{N}^{M_n}(\theta_Y, \sigma_n^2(\Phi^T \Phi)^{-1}) \right\|_{\text{TV}} \leq 2P(U > M_n/4).$$

Now, Cirelson's inequality (see for instance Massart (2007))

$$(26) \quad P(\sqrt{U} > \sqrt{k_n} + \sqrt{2x}) \leq \exp(-x)$$

used with  $x = \frac{(\sqrt{M_n} - 2\sqrt{k_n})^2}{8}$  implies Lemma 5.  $\square$

LEMMA 6. *If  $\sup_{\|\Phi h\|^2 \leq \sigma_n^2 M_n, \|\Phi g\|^2 \leq \sigma_n^2 M_n} \frac{w(\theta_0 + h)}{w(\theta_0 + g)} \rightarrow 1$  as  $n \rightarrow \infty$ , then*

$$E \left\| \widetilde{W}^{M_n}(d\theta|Y) - \mathcal{N}^{M_n}(\theta_Y, \sigma_n^2(\Phi^T \Phi)^{-1}) \right\|_{\text{TV}} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

PROOF. Let us first note that, for every  $\theta$  and  $\tau$  in  $\mathbb{R}^{k_n}$ , for every  $Y \in \mathbb{R}^n$ ,

$$(27) \quad \begin{aligned} \frac{dP_\theta(Y)}{dP_\tau(Y)} &= \exp \left\{ \frac{-\|\Phi\theta\|^2 + \|\Phi\tau\|^2 - 2Y^T \Phi(\tau - \theta)}{2\sigma_n^2} \right\} \\ &= \frac{d\mathcal{N}(\theta_Y, \sigma_n^2(\Phi^T \Phi)^{-1})(\theta)}{d\mathcal{N}(\theta_Y, \sigma_n^2(\Phi^T \Phi)^{-1})(\tau)}. \end{aligned}$$

This directly comes from the expressions for the Gaussian densities.

In the following the first lines are just rewriting. Then we use Jensen's inequality with the convex function  $x \mapsto (1-x)_+$ , and make use of (27). We abbreviate  $\mathcal{N}^{M_n}(\theta_Y, \sigma_n^2(\Phi^T \Phi)^{-1})$  into  $\mathcal{N}^{M_n}$ .

$$\begin{aligned} & \left\| \widetilde{W}^{M_n}(d\theta|Y) - \mathcal{N}^{M_n} \right\|_{\text{TV}} \\ &= \int \left( 1 - \frac{d\mathcal{N}^{M_n}(\theta)}{d\widetilde{W}^{M_n}(\theta|Y)} \right)_+ d\widetilde{W}^{M_n}(\theta|Y) \\ &= \int \left( 1 - \frac{d\mathcal{N}^{M_n}(\theta) \int \frac{w(\tau)}{d\mathcal{N}^{M_n}(\tau)} dP_\tau(Y) d\mathcal{N}^{M_n}(\tau)}{w(\theta) dP_\theta(Y)} \right)_+ d\widetilde{W}^{M_n}(\theta|Y) \\ &\leq \int \int \left( 1 - \frac{w(\tau) d\mathcal{N}^{M_n}(\theta) dP_\tau(Y)}{w(\theta) d\mathcal{N}^{M_n}(\tau) dP_\theta(Y)} \right)_+ d\mathcal{N}^{M_n}(\tau) d\widetilde{W}^{M_n}(\theta|Y) \\ &= \int \int \left( 1 - \frac{w(\tau)}{w(\theta)} \right)_+ d\mathcal{N}^{M_n}(\tau) d\widetilde{W}^{M_n}(\theta|Y) \\ &\leq 1 - \inf_{\|\Phi h\|^2 \leq \sigma_n^2 M_n, \|\Phi g\|^2 \leq \sigma_n^2 M_n} \frac{w(\theta_0 + h)}{w(\theta_0 + g)}. \end{aligned}$$

$\square$

PROPOSITION 9 (Posterior concentration). *Suppose that Condition 1, Condition 2, and Condition 3 of Theorem 2 hold. Then*

$$E \left\| \widetilde{W}(d\theta|Y) - \widetilde{W}^{M_n}(d\theta|Y) \right\|_{\text{TV}} = E \left[ \widetilde{W}(\mathcal{E}_{\theta_0, \Phi}^C(M_n) | Y) \right] \\ \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Proposition 9 is proved in Appendix A. However we state here the following important Lemma, because of its significance.

LEMMA 7. *Let  $a \in \mathbb{R}^n$  such that  $\Phi^T a = 0$ . Then, for any  $y \in \mathbb{R}^n$ ,  $W(dF|Y = y) = W(dF|Y = y + a)$ .*

Lemma 7 states that the distribution  $W(dF|Y)$  is invariant under any translation of  $Y$  orthogonal to  $\langle \phi \rangle$ . Now, regard  $W(dF|Y)$  as a random variable. Then any statement on  $W(dF|Y)$  or  $\widetilde{W}(d\theta|Y)$  valid when  $Y \sim \mathcal{N}(F_0, \sigma_n^2 I_n)$  with  $F_0 \in \langle \phi \rangle$  can be extended at zero cost by Lemma 7 to the case  $F_0 \in \mathbb{R}^n$ . For instance, proving Proposition 9 in the case  $F_0 = \Phi \theta_0$  is enough.

6.3. *Proof of Theorem 3.* We begin with (13). Consider the following Taylor expansion

$$G(F) - G(Y_{\langle \phi \rangle}) = \dot{G}_{F_{\langle \phi \rangle}}(F - Y_{\langle \phi \rangle}) \\ + \frac{1}{2} \int_0^1 (1-t) D_{F_{\langle \phi \rangle} + t(F - F_{\langle \phi \rangle})}^2 G(F - F_{\langle \phi \rangle}, F - F_{\langle \phi \rangle}) dt \\ - \frac{1}{2} \int_0^1 (1-t) D_{F_{\langle \phi \rangle} + t(Y_{\langle \phi \rangle} - F_{\langle \phi \rangle})}^2 G(Y_{\langle \phi \rangle} - F_{\langle \phi \rangle}, Y_{\langle \phi \rangle} - F_{\langle \phi \rangle}) dt,$$

using the Lagrange form of the error term. Suppose that  $F \in \langle \phi \rangle$ ,  $\|F - F_{\langle \phi \rangle}\|^2 \leq \sigma_n^2 M_n$ , and  $\|Y_{\langle \phi \rangle} - F_{\langle \phi \rangle}\|^2 \leq \sigma_n^2 M_n$ . Then, for any  $b \in \mathbb{R}^p$ ,

$$\left| b^T \left( G(F) - G(Y_{\langle \phi \rangle}) - \dot{G}_{F_{\langle \phi \rangle}}(F - Y_{\langle \phi \rangle}) \right) \right| \leq \|b\| B_{F_{\langle \phi \rangle}}(M_n).$$

On the other hand,  $\sqrt{b^T \Gamma_{F_{\langle \phi \rangle}} b} \geq \sqrt{\left\| \Gamma_{F_{\langle \phi \rangle}}^{-1} \right\|^{-1}} \|b\|$ . Moreover

$$\left\| W \left( d \frac{b^T \dot{G}_{F_{\langle \phi \rangle}}(F - Y_{\langle \phi \rangle})}{\sqrt{b^T \Gamma_{F_{\langle \phi \rangle}} b}} \middle| Y \right) - \mathcal{N}(0, 1) \right\|_{\text{TV}} \\ \leq \|W(dF|Y) - \mathcal{N}(Y_{\langle \phi \rangle}, \sigma_n^2 \Sigma)\|_{\text{TV}}.$$

Let  $\eta_n = \sqrt{\|\Gamma_{F_{\langle\phi\rangle}}^{-1}\|} B_{F_{\langle\phi\rangle}}(M_n)$ , which tends to 0 by hypothesis. Let also

$$I_{\eta_n} = \{x \in \mathbb{R} : \exists x' \in I, |x - x'| \leq \eta_n\}.$$

Note that  $\psi(I_{\eta_n}) \leq \psi(I) + \sqrt{\frac{2}{\pi}}\eta_n$ .

Gathering all this information, we can get the upper bound

$$\begin{aligned} W\left(\frac{b^T(G(F) - G(Y_{\langle\phi\rangle}))}{\sqrt{b^T\Gamma_{F_{\langle\phi\rangle}}b}} \in I \middle| Y\right) & \\ & \leq W\left(\frac{b^T\dot{G}_{F_{\langle\phi\rangle}}(F - Y_{\langle\phi\rangle})}{\sqrt{b^T\Gamma_{F_{\langle\phi\rangle}}b}} \in I_{\eta_n} \middle| Y\right) \\ & \quad + \mathbb{1}_{\|Y_{\langle\phi\rangle} - F_{\langle\phi\rangle}\|^2 > \sigma_n^2 M_n} + W(\|F - F_{\langle\phi\rangle}\|^2 > \sigma_n^2 M_n | Y) \\ & \leq \psi(I) + \sqrt{\frac{2}{\pi}}\eta_n + \|W(dF|Y) - \mathcal{N}(Y_{\langle\phi\rangle}, \sigma_n^2 \Sigma)\|_{\text{TV}} \\ & \quad + \mathbb{1}_{\|Y_{\langle\phi\rangle} - F_{\langle\phi\rangle}\|^2 > \sigma_n^2 M_n} + W(\|F - F_{\langle\phi\rangle}\|^2 > \sigma_n^2 M_n | Y). \end{aligned}$$

A lower bound is obtained in the same way. Taking the expectation,

$$\begin{aligned} (28) \quad E\left|W\left(\frac{b^T(G(F) - G(Y_{\langle\phi\rangle}))}{\sqrt{b^T\Gamma_{F_{\langle\phi\rangle}}b}} \in I \middle| Y\right) - \psi(I)\right| & \\ & \leq o(1) + P(\|Y_{\langle\phi\rangle} - F_{\langle\phi\rangle}\|^2 > \sigma_n^2 M_n) \\ & \quad + E[W(\|F - F_{\langle\phi\rangle}\|^2 > \sigma_n^2 M_n | Y)]. \end{aligned}$$

But  $\|Y_{\langle\phi\rangle} - F_{\langle\phi\rangle}\|^2$  follows the  $\sigma_n^2 \chi^2(k_n)$  distribution, and since  $k_n = o(M_n)$ ,

$$P(\|Y_{\langle\phi\rangle} - F_{\langle\phi\rangle}\|^2 > \sigma_n^2 M_n) = o(1).$$

To bound (28), we use the following.

LEMMA 8. *Suppose that the conditions of either Theorem 1 or Theorem 2 are satisfied. Then*

$$E[W(\|F - F_{\langle\phi\rangle}\|^2 > \sigma_n^2 M_n | Y)] \rightarrow 0 \text{ as } n \rightarrow \infty.$$

PROOF. For smooth priors, this is an immediate corollary of Proposition 9. Let us suppose we are under the conditions of Theorem 1.

Let  $Z$  be a  $\mathcal{N}\left(0, \frac{\sigma_n^2 \tau_n^2}{\sigma_n^2 + \tau_n^2} \Sigma\right)$  random vector in  $\mathbb{R}^n$  independent on  $Y$ , and  $U$  a random variable following  $\chi^2(k_n)$ . From (21) we get

$$W(dF|Y) = \mathcal{N}\left(\frac{\tau_n^2}{\sigma_n^2 + \tau_n^2} Y_{\langle \phi \rangle}, \frac{\sigma_n^2 \tau_n^2}{\sigma_n^2 + \tau_n^2} \Sigma\right).$$

Therefore

$$\begin{aligned} & W(\|F - F_{\langle \phi \rangle}\|^2 > \sigma_n^2 M_n | Y) \\ &= P\left(\left\|Z + \frac{\tau_n^2}{\sigma_n^2 + \tau_n^2} Y_{\langle \phi \rangle} - F_{\langle \phi \rangle}\right\|^2 > \sigma_n^2 M_n\right) \\ &\leq P\left(\|Z\| > \sigma_n \sqrt{M_n} - \left\|\frac{\tau_n^2}{\sigma_n^2 + \tau_n^2} Y_{\langle \phi \rangle} - F_{\langle \phi \rangle}\right\|\right) \\ &\leq \begin{cases} 1 & \text{if } \left\|\frac{\tau_n^2}{\sigma_n^2 + \tau_n^2} Y_{\langle \phi \rangle} - F_{\langle \phi \rangle}\right\| > \frac{2\sigma_n \sqrt{M_n}}{3} \\ P(\|Z\|^2 > \sigma_n^2 \frac{M_n}{9}) = P\left(U > \frac{\sigma_n^2 + \tau_n^2}{\tau_n^2} \frac{M_n}{9}\right) & \text{otherwise.} \end{cases} \end{aligned}$$

Since  $k_n = o(M_n)$ ,  $P(U > M_n/9) = o(1)$ . On the other hand,

$$\begin{aligned} \left\|\frac{\tau_n^2}{\sigma_n^2 + \tau_n^2} Y_{\langle \phi \rangle} - F_{\langle \phi \rangle}\right\| &= \left\|\Sigma \left(\frac{\tau_n^2}{\sigma_n^2 + \tau_n^2} \varepsilon + \frac{\sigma_n^2}{\sigma_n^2 + \tau_n^2} F_0\right)\right\| \\ &\leq \|\Sigma \varepsilon\| + \frac{\sigma_n}{\sqrt{\sigma_n^2 + \tau_n^2}} \|F_0\| \end{aligned}$$

Since  $\|F_0\| = o(\tau_n^2/\sigma_n)$ ,  $\frac{\sigma_n^2 \|F_0\|^2}{\sigma_n^2 + \tau_n^2} = o(1) < \frac{M_n}{9}$  for  $n$  large enough.  $\|\Sigma \varepsilon\|^2$  is a  $\sigma_n^2 \chi^2(k_n)$  variable. Therefore, for  $n$  large enough,

$$E[W(\|F - F_{\langle \phi \rangle}\|^2 > \sigma_n^2 M_n | Y)] \leq 2P(U > M_n/9) = o(1).$$

□

Now, (28) gives (13).

The proof of the frequentist assertion (14) is similar and delayed to Appendix C.

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## APPENDIX A: POSTERIOR CONSISTENCY

PROOF OF LEMMA 7. We decompose any  $y \in \mathbb{R}^n$  in two orthogonal components  $y = \Phi\theta_y + y'$ , with  $\Phi^T y' = 0$ . The density of  $P_\theta$  is

$$\begin{aligned} dP_\theta(y) &= \frac{1}{(\sigma_n \sqrt{2\pi})^n} \exp \left\{ -\frac{\|\Phi\theta_y + y' - \Phi\theta\|^2}{2\sigma_n^2} \right\} \\ &= \frac{1}{(\sigma_n \sqrt{2\pi})^n} \exp \left\{ -\frac{\|y'\|^2}{2\sigma_n^2} \right\} \exp \left\{ -\frac{\|\Phi\theta_y - \Phi\theta\|^2}{2\sigma_n^2} \right\}. \end{aligned}$$

Since  $\Phi^T a = 0$ , we have also

$$\begin{aligned} dP_\theta(y + a) &= \frac{1}{(\sigma_n \sqrt{2\pi})^n} \exp \left\{ -\frac{\|y' + a\|^2}{2\sigma_n^2} \right\} \exp \left\{ -\frac{\|\Phi\theta_y - \Phi\theta\|^2}{2\sigma_n^2} \right\} \\ &= \exp \left\{ -\frac{\|y' + a\|^2 - \|y'\|^2}{2\sigma_n^2} \right\} dP_\theta(y). \end{aligned}$$

Therefore

$$\begin{aligned} dP^W(y + a) &= \int dP_\theta(y + a) w(\theta) d\theta \\ &= \exp \left\{ -\frac{\|y' + a\|^2 - \|y'\|^2}{2\sigma_n^2} \right\} dP^W(y) \end{aligned}$$

and

$$\begin{aligned} \widetilde{W}(d\theta|Y = y + a) &= \frac{dP_\theta(y + a) w(\theta) d\theta}{dP^W(y + a)} \\ &= \widetilde{W}(d\theta|Y = y). \end{aligned}$$

□

Let us now prove Proposition 9. As remarked in the comments after Lemma 7, we can suppose that  $F_0 = \Phi\theta_0$ . Let  $U$  a random variable following a  $\chi^2(k_n)$  distribution. Proceeding as in Boucheron and Gassiat (2009); van der Vaart (1998), we introduce a test

$$(29) \quad T_n = \mathbb{1}_{(\theta_Y - \theta_0)^T \Phi^T \Phi (\theta_Y - \theta_0) > \sigma_n^2 M_n / 4}.$$

Note that  $ET_n = P(U > M_n/4) = o(1)$ . Then

$$E \left[ \widetilde{W}(\mathcal{E}_{\theta_0, \Phi}^C(M_n) | Y) \right] \leq ET_n + E \left[ (1 - T_n) \widetilde{W}(\mathcal{E}_{\theta_0, \Phi}^C(M_n) | Y) \right].$$

Next, let  $(r_n)_{n \geq 1}$  be a sequence of positive numbers such that  $r_n$  goes to 0 and  $-\ln(r_n) = o(M_n/k_n)$  as  $n$  goes to infinity. Consider the mixture distribution  $P_{\theta_0, r_n}^W$  with density

$$(30) \quad dP_{\theta_0, r_n}^W(y) = \int_{\mathcal{E}_{\theta_0, \Phi}(r_n)} dP_{\theta}(y) \widetilde{W}^{r_n}(d\theta),$$

where  $\widetilde{W}^{r_n}$  is the rescaled restriction of  $\widetilde{W}$  to  $\mathcal{E}_{\theta_0, \Phi}(r_n)$ , as in (23). The following Lemma illustrates the link between  $P_{\theta_0}$  and  $P_{\theta_0, r_n}^W$ .

LEMMA 9. *Using the preceding notations,*

$$\|P_{\theta_0, r_n}^W - P_{\theta_0}\|_{\text{TV}} \leq \sqrt{\frac{r_n}{2\pi}} = o(1).$$

PROOF. We use convexity, and Lemma 4 since  $P_{\theta} = \mathcal{N}(\Phi\theta, \sigma_n^2 I_n)$ :

$$\|P_{\theta_0, r_n}^W - P_{\theta_0}\|_{\text{TV}} \leq \sup_{\|\Phi h\|^2 \leq \sigma_n^2 r_n} \|P_{\theta_0+h} - P_{\theta_0}\|_{\text{TV}} \leq \frac{\sqrt{r_n}}{\sqrt{2\pi}}.$$

□

Since  $\widetilde{W}(\mathcal{E}_{\theta_0, \Phi}^C(M_n) | Y) \leq 1$ , we have

$$\begin{aligned} E \left[ (1 - T_n) \widetilde{W}(\mathcal{E}_{\theta_0, \Phi}^C(M_n) | Y) \right] \\ \leq \|P_{\theta_0, r_n}^W - P_{\theta_0}\|_{\text{TV}} + E_{\theta_0, r_n}^W \left[ (1 - T_n) \widetilde{W}(\mathcal{E}_{\theta_0, \Phi}^C(M_n) | Y) \right]. \end{aligned}$$

At that point, the Bayes rule and Fubini's Theorem give

$$\begin{aligned}
 & E_{\theta_0, r_n}^W \left[ (1 - T_n) \widetilde{W}(\mathcal{E}_{\theta_0, \Phi}^C(M_n) | Y) \right] \\
 &= \frac{1}{\widetilde{W}(\mathcal{E}(r_n))} \int_{\mathcal{E}(r_n)} \left( \int_{\mathbb{R}^n} \left[ (1 - T_n) \right. \right. \\
 &\quad \left. \left. \cdot \int_{\mathcal{E}^C(M_n)} \frac{dP_\tau(Y) w(\tau) d\tau}{\int_{\mathbb{R}^{k_n}} dP_\eta(Y) w(\eta) d\eta} \right] dP_\theta(Y) \right) w(\theta) d\theta \\
 &= \frac{1}{\widetilde{W}(\mathcal{E}(r_n))} \int_{\mathcal{E}^C(M_n)} E_\tau \left[ (1 - T_n) \widetilde{W}(\mathcal{E}(r_n) | Y) \right] w(\tau) d\tau \\
 &\leq \frac{1}{\widetilde{W}(\mathcal{E}_{\theta_0, \Phi}(r_n))} \sup_{\|\Phi h\|^2 > \sigma_n^2 M_n} E_{\theta_0+h}(1 - T_n) \\
 &\leq \frac{1}{\widetilde{W}(\mathcal{E}_{\theta_0, \Phi}(r_n))} \sup_{\|\Phi h\|^2 > \sigma_n^2 M_n} P_{\theta_0+h} \left( \|\Phi(\theta_Y - \theta_0 - h)\|^2 > \sigma_n^2 M_n / 4 \right) \\
 &= \frac{P(U > M_n / 4)}{\widetilde{W}(\mathcal{E}_{\theta_0, \Phi}(r_n))}.
 \end{aligned}$$

Let  $B_k(0, 1)$  be the unit ball in  $\mathbb{R}^k$ . We control the volume of the ellipsoid

$$\widetilde{W}(\mathcal{E}_{\theta_0, \Phi}(r_n)) \geq \left( \inf_{\|\Phi h\|^2 \leq \sigma_n^2 r_n} \frac{w(\theta_0 + h)}{w(\theta_0)} \right) \frac{\sigma_n^{k_n} w(\theta_0)}{\sqrt{\det(\Phi^T \Phi)}} r_n^{k_n/2} \text{vol}(B_{k_n}(0, 1))$$

and make use of the following relation

$$-\ln \text{vol}(B_k(0, 1)) = \ln \frac{\Gamma(1 + k/2)}{\pi^{k/2}} \underset{k \rightarrow \infty}{\sim} \frac{k}{2} \ln k$$

which follows from Stirling's approximation.

Next we can use Cirelson's inequality (26) as in Lemma 5 and get, for  $n$  large enough,

$$\begin{aligned}
 & \ln \left( E_{\theta_0, r_n}^W \left[ (1 - T_n) \widetilde{W}(\mathcal{E}_{\theta_0, \Phi}(M_n) | Y) \right] \right) \\
 & \leq \ln \left( \frac{\sqrt{\det(\Phi^T \Phi)}}{\sigma_n^{k_n} w(\theta_0)} \right) - \frac{M_n}{9} - \frac{k_n}{2} \ln(r_n) - \ln \text{vol}(B_{k_n}(0, 1)) + o(1) \\
 & \sim -\frac{M_n}{9}
 \end{aligned}$$

which goes to minus infinity as  $n$  goes to infinity. This proves Proposition 9.

## APPENDIX B: SOBOLEV CLASSES

We begin with a simple lemma, then we prove Lemma 2 and Lemma 3

LEMMA 10. *Let  $\alpha > 1/2$ ,  $L > 0$ , and  $\theta \in \Theta(\alpha, L)$ . Then*

$$\sum_{j=1}^{\infty} |\theta_j| < \infty.$$

*As a consequence,  $f$  is the uniform limit of the series  $\sum \theta_j \varphi_j$  and  $f$  is continuous.*

PROOF OF LEMMA 10. We have a simple control on the sum of the coefficients

$$\sum_{j=2}^{\infty} |\theta_j| \leq \sqrt{\sum_{j \geq 2} a_j^{-2}} \sqrt{\sum_{j \geq 2} a_j^2 \theta_j^2} \leq \frac{L}{\pi^\alpha} \sqrt{\sum_{j \geq 1} j^{-2\alpha}} < \infty.$$

Since all functions  $\varphi_j$  are continuous and bounded by  $\sqrt{2}$ , the other points follow.  $\square$

PROOF OF LEMMA 2.  $F_{\langle \phi \rangle}$  is the orthogonal projection of  $F_0$  on the span of the first  $k_n$  vectors of the orthogonal basis  $(\phi_j)_{1 \leq j \leq n}$  of  $\mathbb{R}^n$ . So

$$\|F_0 - F_{\langle \phi \rangle}\|^2 = \sum_{j=k_n+1}^n (F_0^T \phi_j)^2 = n \sum_{j=k_n+1}^n \left( \frac{1}{n} \sum_{i=1}^n f(i/n) \varphi_j(i/n) \right)^2.$$

Following Tsybakov (2004), we set  $\zeta_j = \frac{1}{n} \sum_{i=1}^n f(i/n) \varphi_j(i/n) - \theta_j^0$  for  $1 \leq j \leq n$ . Then

$$\|F_0 - F_{\langle \phi \rangle}\|^2 = n \sum_{j=k_n+1}^n (\zeta_j + \theta_j^0)^2 \leq 2n \left( \sum_{j=1}^n \zeta_j^2 + \sum_{j=k_n+1}^n (\theta_j^0)^2 \right).$$

Using Lemma 1, for any  $1 \leq j \leq n$ ,

$$\begin{aligned} \zeta_j &= \frac{1}{n} \sum_{i=1}^n \left( \sum_{m=1}^{\infty} \theta_m^0 \varphi_m(i/n) \right) \varphi_j(i/n) - \theta_j^0 \\ &= \frac{1}{n} \sum_{i=1}^n \left( \sum_{m=n+1}^{\infty} \theta_m^0 \varphi_m(i/n) \right) \varphi_j(i/n). \end{aligned}$$

So, using Lemma 1 again,

$$\sum_{j=1}^n \zeta_j^2 \leq \frac{1}{n} \sum_{i=1}^n \left( \sum_{m=n+1}^{\infty} \theta_m^0 \varphi_m(i/n) \right)^2.$$

We recognize a Riemann sum of the function  $(\sum_{m=n+1}^{\infty} \theta_m^0 \varphi_m)^2$ , which is continuous according to Lemma 10. Therefore

$$\sum_{j=1}^n \zeta_j^2 \leq (1 + o(1)) \int_0^1 \left( \sum_{m=n+1}^{\infty} \theta_m^0 \varphi_m \right)^2 = \sum_{m=n+1}^{\infty} (\theta_m^0)^2$$

and

$$\begin{aligned} \|F_0 - F_{\langle \phi \rangle}\|^2 &\leq (2n + o(n)) \sum_{m=k_n+1}^{\infty} (\theta_m^0)^2 \\ &\leq \frac{2n + o(n)}{a_{k_n+1}^2} \sum_{m=k_n+1}^{\infty} a_m^2 (\theta_m^0)^2 \\ &\leq \frac{2L^2}{\pi^{2\alpha}} \frac{n + o(n)}{k_n^{2\alpha}}. \end{aligned}$$

On the other hand,  $f$  is continuous, and  $(1/n) \|F_0\|^2$  is a Riemann sum of  $f^2$ . Therefore  $(1/n) \|F_0\|^2$  goes to  $\int_0^1 f^2$  as  $n$  goes to infinity.  $\square$

**PROOF OF LEMMA 3.** Let  $(\theta'_j)_{j \geq 1}$  the Fourier coefficients of  $g$ . As in the previous proof, we set for  $1 \leq j \leq n$

$$\begin{aligned} \zeta_j &= \frac{1}{n} \sum_{i=1}^n f(i/n) \varphi_j(i/n) - \theta_j^0 \\ \zeta'_j &= \frac{1}{n} \sum_{i=1}^n g(i/n) \varphi_j(i/n) - \theta'_j. \end{aligned}$$

We have  $F_0 = \sum_{j=1}^n (\zeta_j + \theta_j^0) \phi_j$ , so

$$\frac{1}{n} \sum_{i=1}^n f(i/n) g(i/n) = \sum_{j=1}^n (\zeta_j + \theta_j^0) (\zeta'_j + \theta'_j).$$

In the meantime

$$\int_0^1 fg = \sum_{j=1}^{\infty} \theta_j^0 \theta'_j.$$

So

$$\begin{aligned}
& \left| \frac{1}{n} \sum_{i=1}^n f(i/n)g(i/n) - \int_0^1 fg \right| \\
&= \left| \sum_{j=1}^n \zeta_j \zeta'_j + \sum_{j=1}^n \zeta_j \theta'_j + \sum_{j=1}^n \theta_j^0 \zeta'_j - \sum_{j=n+1}^{\infty} \theta_j^0 \theta'_j \right| \\
&\leq \sqrt{\sum_{j=1}^n \zeta_j^2} \sqrt{\sum_{j=1}^n \zeta_j'^2} + \sqrt{\sum_{j=1}^n \zeta_j^2} \sqrt{\sum_{j=1}^n \theta_j'^2} \\
&\quad + \sqrt{\sum_{j=1}^n \zeta_j'^2} \sqrt{\sum_{j=1}^n (\theta_j^0)^2} + \sqrt{\sum_{j=n+1}^{\infty} (\theta_j^0)^2} \sqrt{\sum_{j=n+1}^{\infty} \theta_j'^2}.
\end{aligned}$$

As in the proof of Lemma 2, we have

$$\sum_{j=1}^n \zeta_j^2 \sim \sum_{j=n+1}^{\infty} (\theta_j^0)^2 \leq \frac{L^2}{\pi^{2\alpha} n^{2\alpha}}$$

and on the other hand,

$$\sum_{j=1}^n (\theta_j^0)^2 \leq \int_0^1 f^2 = \|f\|^2.$$

Thus

$$\begin{aligned}
\left| \frac{1}{n} \sum_{i=1}^n f(i/n)g(i/n) - \int_0^1 fg \right| &\leq (1 + o(1)) \left( \frac{LL'}{(\pi n)^{\alpha+\alpha'}} + \frac{L'\|f\|}{(\pi n)^{\alpha'}} + \frac{L\|g\|}{(\pi n)^{\alpha}} \right) \\
&= O\left(n^{-\inf(\alpha, \alpha')}\right).
\end{aligned}$$

□

### APPENDIX C: ASYMPTOTIC NORMALITY OF THE MLE FOR FUNCTIONALS

Here we prove (14), which is a part of Theorem 3. The proof is similar to the one of (13) in subsection 6.3. Taylor's expansion gives

$$\begin{aligned}
G(Y_{\langle \phi \rangle}) - G(F_{\langle \phi \rangle}) &= \dot{G}_{F_{\langle \phi \rangle}}(Y_{\langle \phi \rangle} - F_{\langle \phi \rangle}) \\
&\quad + \frac{1}{2} \int_0^1 (1-t) D_{F_{\langle \phi \rangle} + t(Y_{\langle \phi \rangle} - F_{\langle \phi \rangle})}^2 G(Y_{\langle \phi \rangle} - F_{\langle \phi \rangle}, Y_{\langle \phi \rangle} - F_{\langle \phi \rangle}) dt.
\end{aligned}$$

Therefore, for any  $b \in \mathbb{R}^p$ ,

$$\left| b^T \left( G(Y_{\langle \phi \rangle}) - G(F_{\langle \phi \rangle}) - \dot{G}_{F_{\langle \phi \rangle}}(Y_{\langle \phi \rangle} - F_{\langle \phi \rangle}) \right) \right| \leq \frac{1}{2} \|b\| B_{F_{\langle \phi \rangle}}(M_n).$$

Recall that  $\sqrt{b^T \Gamma_{F_{\langle \phi \rangle}} b} \geq \sqrt{\|\Gamma_{F_{\langle \phi \rangle}}^{-1}\|^{-1}} \|b\|$ . As before, set  $\eta_n = \sqrt{\|\Gamma_{F_{\langle \phi \rangle}}^{-1}\|} B_{F_{\langle \phi \rangle}}(M_n)$ , which tends to 0 by hypothesis. Then

$$\begin{aligned} & P \left( \frac{b^T (G(Y_{\langle \phi \rangle}) - G(F_{\langle \phi \rangle}))}{\sqrt{b^T \Gamma_{F_{\langle \phi \rangle}} b}} \in I \right) \\ & \leq P(\|Y_{\langle \phi \rangle} - F_{\langle \phi \rangle}\|^2 > \sigma_n^2 M_n) + P \left( \frac{b^T \dot{G}_{F_{\langle \phi \rangle}}(Y_{\langle \phi \rangle} - F_{\langle \phi \rangle})}{\sqrt{b^T \Gamma_{F_{\langle \phi \rangle}} b}} \in I_{\eta_n} \right). \end{aligned}$$

The first term in the right goes to 0 (see subsection 6.3), and the latter is exactly  $\psi(I_{\eta_n})$ . Indeed,  $Y_{\langle \phi \rangle} \sim \mathcal{N}(F_{\langle \phi \rangle}, \sigma_n^2 \Sigma)$  implies

$$\frac{b^T \dot{G}_{F_{\langle \phi \rangle}}(Y_{\langle \phi \rangle} - F_{\langle \phi \rangle})}{\sqrt{b^T \Gamma_{F_{\langle \phi \rangle}} b}} \sim \mathcal{N}(0, 1).$$

Recall that  $|\psi(I_{\eta_n}) - \psi(I)| \leq \sqrt{\frac{2}{\pi}} \eta_n$ . Therefore

$$\begin{aligned} & P \left( \frac{b^T (G(Y_{\langle \phi \rangle}) - G(F_{\langle \phi \rangle}))}{\sqrt{b^T \Gamma_{F_{\langle \phi \rangle}} b}} \in I \right) - \psi(I) \\ & \leq \sqrt{\frac{2}{\pi}} \eta_n + P(\|Y_{\langle \phi \rangle} - F_{\langle \phi \rangle}\|^2 > \sigma_n^2 M_n) \rightarrow 0. \end{aligned}$$

Note that this upper bound does not depend on  $I$ . A lower bound can be obtained in the same way. Therefore (14) holds.