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ON THE RELATION BETWEEN THE EQUATIONS FOR LARGE-EDDY SIMULATION OF TURBULENT FLOW AND FOR WEAKLY NONLINEAR EVOLUTION OF DISTURBANCES FOR FLOWS IN TRANSITION

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1 Summary

The work submitted herewith is based on the author's attempt to unveil the relationship between the equations for large-eddy simulation of turbulent flow (LES) and those for weakly nonlinear evolution of disturbances on the route to transition according to the nonlinear stability theory of Stewartson and Stuart [1]. Two flow examples in which three-dimensionality and rotation exercise significant effects on transition have been chosen for the purpose of illustrating the nature of this relationship. These are:

- The swirling flow between concentric circular cylinders with a rotating inner cylinder and an imposed axial pressure gradient, and
- The plane channel flow on a rotating system undergoing transition under the influence of the Coriolis force.

The relationship brought out by the study may be summarised in words as follows:

The amplitude evolution equation may be interpreted as one governing the slow modulation of the amplitude of a wave packet of disturbances under the influence of **Reynolds-stress like terms**. The **profiles of the Reynolds-stress like terms** in the wall-normal direction are given by those of the products of the fluctuating velocity components on the surface of neutral stability according to the classical linear stability theory.

2 Basic ideas of the two approaches

- **Large-eddy simulation** The basic idea in large-eddy simulation is a division of the turbulent motion in the flow into one with widely differing scales, viz. **large eddies**, and **small-scale motion** or **the rest**. Averaging of the equations of motion is performed only over scales of the latter, a procedure that filters out details of the **small-scale motion** leaving equations for the **filtered quantities** in which, however a **sub-grid scale stress tensor** appears, see eg. [2], [3]. There arises a necessity for a closure hypothesis.
- **Nonlinear stability Theory** The nonlinear stability theory of Stewartson and Stuart [1] seeks the **bifurcating solution** of the complete equations of motion when the flow parameter assumes values **just beyond transition**. The formalism of their method can be understood in terms of **multiple scales**, see eg. [5]. At values of the flow parameters just beyond transition, the classical linearized equations of motion for small-amplitude disturbances yield exponentially growing disturbances, however with a growth rate that remains small when the departure of the flow parameters from those on the surface of neutral stability is small. The slow growth rate for amplification of disturbances from linear theory may be associated with further scales of time and length entering the problem and suited to account for the effects from the nonlinearities (**Reynolds stres like terms**) that were ignored in the linear theory. The solution of the **nonlinear problem** is sought in a form closely related to the wave-form of disturbances according to linear stability theory, but with an amplitude that gets modulated by the nonlinearities in the problem. The approach results in a nonlinear matrix equation that describes the **amplitude evolution in terms of the newly introduced scales**.

3 Outline of the work

- **Step 1: Solution of the classical linearized problem for small-amplitude disturbances**

  1. Starting from the set of nonlinear equations of motion for the disturbance, solutions of the equations for infinitesimally small disturbances in the flow in question are obtained by a suitable numerical method, see eg. [4]. The classical eigenvalue problem thus posed has nontrivial solutions only when a certain **dispersion relation** of the problem is fulfilled which we write in conventional notation symbolically as follows:

- For the swirling annular flow with rotation of the inner cylinder and small gap width this is $\omega = \omega(\lambda_x, n_\varphi; Ta, S_i)$, where Ta is the **Taylornumber** and S_i is a **swirl parameter**.
- For the plane channel flow on a rotating system this is $\omega = \omega(\lambda_x, \lambda_z; Re, Ro)$, where Re and Ro are the **Reynoldsnumber** and the **Rotation number** respectively.

The dispersion relations for infinitesimally small disturbances yield the **surface of neutral stability** which is a surface in the space of the flow parameters, i.e. (Ta, S_i) or (Re, Ro) as the case may be, on which the **imaginary part of the frequency**, denoted ω_i, is zero.

2. From the solution of the classical eigenvalue problem, the solution for the propagation of a disturbance for arbitrary initial conditions is obtained as a Fourier-superposition of wave-modes for which the velocity disturbance $\mathbf{u}$ may be written in the form $\mathbf{u} = \mathbf{a}(y) \exp[i\Theta]+$ c.c.. Here, y is the co-ordinate normal to the walls, and the column vector $\mathbf{u}$ and the phase Θ are as follows:
   - In the swirling annular flow: $\mathbf{u} = (u_r, u_\varphi, u_x)^T$; $\Theta(x, \varphi, t) = (\lambda_x x + n_\varphi \varphi - \omega t)$.
   - In the plane channel flow: $\mathbf{u} = (u_y, u_z, u_x)^T$; $\Theta(x, z, t) = (\lambda_x x + \lambda_z z - \omega t)$;

   Here the wave-number pair (λ_x, n_φ), or (λ_x, λ_z) or as the case may be, are real but the frequency ω is permitted to be complex. Values on the entire real axis are permissible for λ_x, λ_z whereas n_φ may assume only real integer values.
3. For initial conditions that generate waves that remain close to the surface of neutral stabilty, the classical linear theory leads to the result that the form of the wave-packet for long times and at large distances from the source assumes a form similar to that of a single mode. The components of this (column) vector, written as a diagonal matrix $\mathbf{U}$ may be written in a coordinate system travelling with the wave-packet as $\mathbf{U}_N = \mathbf{A}_N(y) \exp[i\Theta_N] \exp[\omega_{Ni} t] + c.c.$. Here the subscript N denotes values on the surface of neutral stability and i the imaginary part. It is seen that a wave-packet has a growth rate given by ω_{Ni}, which depends upon the departure of the flow parameters from those on the surface of neutral stability, (Ta_N, S_{iN}) or (Re_N, Ro_N).

- **Step 2: Extension of the solution to the classical linearized problem to weakly nonlinear evolution of disturbances according to the method of Stewartson and Stuart**
  1. The method of Stewartson and Stuart examines solutions for flow parameters in the neighbourhood of the surface of neutral stability when ω_{Ni} is small but > 0. Following Stewartson and Stuart, [1], we introduce an **amplitude parameter**, ϵ_A, that is small and defined for the two flows as a linear combination of the departures of the flow parameters from their values on the surface of neutral stability. The **amplitude parameter** ϵ_A is defined through:
     - for the swirling annular flow: $\epsilon_A = \omega_{Ni} = d_{Ta}|Ta - Ta_N| + d_{Si}|S_i - S_{iN}|$, and
     - for the plane channel flow: $\epsilon_A = \omega_{Ni} = d_{Re}|Re - Re_N| + d_{Ro}|Ro - Ro_N|$.
  2. A solution for the nonlinear problem is sought in the form $\mathbf{U} \simeq \mathbf{B}\mathbf{A}_N \exp[i\Theta_N] + c.c$ where the diagonal matrix $\mathbf{B}$ is a function of the set of newly introduced scales. The elements of the diagonal matrix $\mathbf{U}$ - these are (B_r, B_φ, B_x), or (B_y, B_z, B_x) as the case may be - may be regarded as the components of the **filtered velocity disturbance**.
  3. A **solvability condition** yields the **amplitude evolution equations** (nonlinear matrix equations) for $\mathbf{B}$ for the flow problems in question. Their relationship to LES will be discussed at EC525.

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