# A new integrable system on the sphere and conformally equivariant quantization

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**Abstract**

Taking full advantage of two independent projectively equivalent metrics on the ellipsoid leading to Liouville integrability of the geodesic flow via the well-known Jacobi-Moser system, we disclose a novel integrable system on the sphere $S^n$, namely the *dual* Moser system. The latter falls, along with the Jacobi-Moser and Neumann-Uhlenbeck systems, into the category of (locally) Stäckel systems. Moreover, it is proved that quantum integrability of both Neumann-Uhlenbeck and dual Moser systems is insured by means of the conformally equivariant quantization procedure.

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# Contents



# 1 Introduction

In the wake of the celebrated results of Moser [19] concerning the classical integrability of the geodesic flow on the ellipsoid (first proved by Jacobi in the three-dimensional case), and of the Neumann-Uhlenbeck problem on the $n$-dimensional spheres, we will present a (to our knowledge) new integrable system which relies on a preferred conformally flat metric on $S^n$. This integrable system is actually dual (in the sense of projective equivalence [25, 18]) to the above-mentioned Jacobi-Moser system.

Among the techniques found in the literature, we will mention two different constructs that produce integrable systems together with their Poisson-commuting first integrals:

- Bihamiltonian systems initiated by Magri [17] and Benenti [3], and further elaborated by, e.g., Ibort, Magri, and Marmo [13], Falqui and Pedroni [12].
- Projectively equivalent systems discovered by Levi-Civita [16], and geometrically developed by Tabachnikov [25, 26], Topalov and Matveev [27].

The equivalence of these two theories has been proved by Bolsinov and Matveev [6].

We have chosen to work using Tabachnikov's approach. With the help of his general construction [25, 20] providing, e.g., the Jacobi-Moser first integrals, and, adopting a "dual" approach, we derive the Poisson-commuting first integrals of the new geodesic flow, which we call the dual Moser system. This enables us to provide a *global* expression for these new first integrals.

Our dual Moser system turns out to be locally Stäckel (in ellipsoidal coordinates); it shows up as a "mirror image" of Jacobi-Moser relatively to Neumann-Uhlenbeck (see Table 1).

In contradistinction to the Jacobi-Moser system, conformal flatness of the new system is a fundamental input at the classical, and at the quantum level as well.

To deal with quantum integrability of these systems, we will resort to its commonly accepted definition, namely that the quantized first integrals should still be in involution. This, of course, leaves open the choice of an adapted quantization procedure.

It has been shown [8] that Stäckel systems, using Carter's quantization prescription [7], do remain integrable at the quantum level provided Robertson's condition holds [23]. It is therefore worthwhile to study quantum integrability making use of a genuine quantization theory that takes into account the conformal geometry underlying the dual Moser system: the conformally equivariant quantization [10, 9]. One striking discovery is that the dual Moser system passes the quantum test using the conformally equivariant quantization. We note that the same is true for the Neumann-Uhlenbeck system.

## 1.1 Prolegomena

Let us recall that two *independent* metrics on a Riemannian manifold are said to be projectively equivalent if they have the same *unparametrized* geodesics. A shown in, e.g., [27, 6] this entails that the associated geodesic flows are Liouville-integrable. We will resort to this pathbreaking result in the specific, historical, yet fundamental example of the geodesic flow of the ellipsoid.

The key point of our approach to Liouville/quantum integrability of the geodesic flow of the ellipsoid, $\mathcal{E}^n$, lies in the fact that $\mathcal{E}^n = \{Q \in \mathbb{R}^{n+1} | \sum_{\alpha=0}^{n} (Q^\alpha)^2/a_\alpha = 1\}$ with metric $\mathrm{g}_1 = \sum_{\alpha=0}^{n} (dQ^\alpha)^2|_{\mathcal{E}^n}$ admits, as a matter of fact, another independent, and projectively equivalent Riemannian metric, $\mathrm{g}_2$, that we will introduce shortly.

Put $Q^\alpha = q^\alpha \sqrt{a_\alpha}$ for all $\alpha = 0, \ldots, n$, so that we have $q \cdot q = \sum_{\alpha=0}^{n} (q^\alpha)^2 = 1$. The mapping $Q \mapsto q : \mathcal{E}^n \to S^n$ is a diffeomorphism and the metric on $S^n$, induced from the Euclidean ambient metric, reads now[1]

$$\mathrm{g}_1 = \mathrm{g}|_{S^n} \qquad \text{where} \qquad \mathrm{g} = \sum_{\alpha=0}^{n} a_\alpha dq_\alpha^2 \tag{1.1}$$

where g is, at the moment, viewed as a (flat) metric on $\mathbb{R}^{n+1}$. The equations of the geodesics of the ellipsoid are well-known, and retain the form

$$\ddot{q}^\alpha + \Gamma^\alpha_{\beta\gamma} \dot{q}^\beta \dot{q}^\gamma = 0, \tag{1.2}$$

for all $\alpha = 0, \ldots, n$, where the Christoffel symbols of $(S^n, \mathrm{g}_1)$ are given by

$$\Gamma^\alpha_{\beta\gamma} = \frac{q^\alpha}{a_\alpha} \frac{\delta_{\beta\gamma}}{\sum q_\lambda^2 / a_\lambda} \tag{1.3}$$

for all $\alpha, \beta, \gamma = 0, \ldots, n$. (See also Equation (2.12) yielding the associated geodesic spray.) We note that the constraint equation

$$\dot{q}^2 + q \cdot \ddot{q} = 0 \tag{1.4}$$

is indeed satisfied by (1.2).

Following [25, 27], let us define, on $S^n$, the conformally flat metric

$$\mathrm{g}_2 = \overline{\mathrm{g}}|_{S^n} \qquad \text{with} \qquad \overline{\mathrm{g}} = \frac{1}{\sum q_\beta^2 / a_\beta} \sum_{\alpha=0}^{n} dq_\alpha^2 \tag{1.5}$$

where the conformally flat metric $\overline{\mathrm{g}}$ is defined on $\mathbb{R}^{n+1} \backslash \{0\}$.

The equations of the geodesics for the latter metric are readily found by using the expression of the Christoffel symbols of $\overline{\mathrm{g}}$, namely

$$\overline{\Gamma}^\alpha_{\beta\gamma} = \frac{1}{\sum q_\lambda^2 / a_\lambda} \left[ \delta_{\beta\gamma} \frac{q^\alpha}{a_\alpha} - \delta^\alpha_\beta \frac{q_\gamma}{a_\gamma} - \delta^\alpha_\gamma \frac{q_\beta}{a_\beta} \right] \tag{1.6}$$

for all $\alpha, \beta, \gamma = 0 \ldots, n$. One obtains the equations of the geodesic of $(\mathbb{R}^{n+1} \backslash \{0\}, \overline{\mathrm{g}})$, viz.,

$$\ddot{q}^\alpha + \frac{q^\alpha}{a_\alpha} \frac{\sum \dot{q}_\beta^2}{\sum q_\gamma^2 / a_\gamma} = 2 \dot{q}^\alpha \frac{\sum q_\beta \dot{q}^\beta / a_\beta}{\sum q_\gamma^2 / a_\gamma} \tag{1.7}$$

[1] To avoid clutter, we will oftentimes write $q_\alpha \equiv q^\alpha$, as no confusion occurs in Euclidean space.

for all $\alpha = 0, \ldots, n$. The latter, suitably restrained to $S^n$, is precisely Equation (1.2) with a different parametrization (again, the constraint (1.4) is duly preserved by Equation (1.7)); the metrics $\mathrm{g}_1$ and $\mathrm{g}_2$ are projectively equivalent.

We will put this fact in broader perspective within Section 2.

**Remark 1.1.** The $\Gamma^\alpha_{\beta\gamma}$ and $\overline{\Gamma}^\alpha_{\beta\gamma}$, given by (1.3) and (1.6) respectively, may be viewed as the components of two projectively equivalent linear connections $\nabla$ and $\overline{\nabla}$ on $\mathbb{R}^{n+1}\backslash\{0\}$. While $\overline{\nabla}$ is clearly the Levi-Civita connection of the metric $\overline{\mathrm{g}}$, the connection $\nabla$ is, instead, a *Newton-Cartan* connection (see, e.g., [14]). This means that $\nabla$ is a symmetric linear connection that parallel-transports a (spacelike) contravariant symmetric 2-tensor $\gamma = \sum_{\alpha,\beta=0}^n \gamma^{\alpha\beta}\partial_{q^\alpha} \otimes \partial_{q^\beta}$ and a (timelike) 1-form $\theta = \sum_{\alpha=0}^n \theta_\alpha dq^\alpha$ spanning $\ker(\gamma)$. Here, the degenerate "metric" is given by

$$\gamma^{\alpha\beta} = \frac{1}{a_\alpha}\delta^{\alpha\beta} - \frac{q^\alpha q^\beta}{a_\alpha a_\beta}\frac{1}{\sum q_\lambda^2/a_\lambda} \qquad \text{and} \qquad \theta_\alpha = q_\alpha. \tag{1.8}$$

## 1.2 Main results

The main results of our article can be summarized as follows.

**Theorem 1.2.** *The geodesic flow on $(T^*S^n, \sum_\alpha dp_\alpha \wedge dq^\alpha)$ above the conformally flat manifold $(S^n, \mathrm{g}_2)$ is Liouville-integrable, and admits the following set of Poisson-commuting first integrals*

$$F_\alpha = q_\alpha^2 \sum_{\beta=0}^n a_\beta p_\beta^2 + \sum_{\beta\neq\alpha} \frac{(a_\alpha p_\alpha q_\beta - a_\beta p_\beta q_\alpha)^2}{a_\alpha - a_\beta} \tag{1.9}$$

*with $\alpha = 0, \ldots, n$. We will call the system $(F_0, \ldots, F_n)$ the dual Moser system.*

This theorem follows directly from Propositions 2.7, 2.8, and 2.10.

Let us call $\mathcal{D}_{\frac12,\frac12}(S^n)$ the space of differential operators on $S^n$ with arguments and values in the space of $\frac12$-densities of the sphere $S^n$; we, likewise, denote by $\mathrm{Pol}(T^*S^n)$ the space of fiberwise polynomial functions on $T^*M$. It has been proved [10] that there exists a unique invertible linear mapping $\mathcal{Q}_{\frac12,\frac12} : \mathrm{Pol}(T^*S^n) \to \mathcal{D}_{\frac12,\frac12}(S^n)$ that (i) intertwines the action of the conformal group $\mathrm{O}(n+1,1)$ and (ii) preserves the principal symbol: we call it the *conformally equivariant quantization* mapping.

**Theorem 1.3.** *Quantum integrability of the dual Moser system holds true in terms of the conformally equivariant quantization $\mathcal{Q}_{\frac12,\frac12}$.*

This last result stems from Theorem 3.4, Propositions 3.6, and 3.12.

### 1.3 Plan of the article

The paper is organized as follows.

Section 2 gives us the opportunity to introduce three distinguished classically integrable systems on the sphere, namely, the Jacobi-Moser system, its dual counterpart, and the Neumann-Uhlenbeck system. The construction of the set of mutually Poisson-commuting first integrals is reviewed and specialized to the case of the dual Moser system. The resulting system is shown to be Stäckel, the ellipsoidal coordinates being the separating ones.

In Section 3 we address the quantum integrability issue of these systems, in terms of the conformally equivariant quantization. We prove that the Neumann-Uhlenbeck and the dual Moser systems are, indeed, quantum integrable using this quantization method, well adapted to the conformal flatness of configuration space.

Section 4, provides a conclusion to the present article, and gathers some perspectives for future work.

## 2 A novel integrable system on the sphere: the dual Moser system

### 2.1 Projectively equivalent metrics and conservation laws

Let us recall, almost *verbatim*, Tabachnikov's construction [25] of a maximal set of independent Poisson-commuting first integrals for a special Liouville-integrable system, namely a bi-Hamiltonian system associated with two projectively equivalent metrics, $\mathrm{g}_1$ and $\mathrm{g}_2$, on a configuration manifold $M$.[2] See also [18, 6] for an alternative construction.

We start with two Riemannian manifolds $(M, \mathrm{g}_1)$ and $(M, \mathrm{g}_2)$ of dimension $n$. The tangent bundle $TM$ is endowed with two distinguished 1-forms $\lambda_1$ and $\lambda_2$, namely $\lambda_N = \mathrm{g}_N^*\theta$, where $\theta$ is the canonical 1-form of $T^*M$, and $\mathrm{g}_N : TM \to T^*M$ is viewed as a bundle isomorphism. We then write, locally, $\lambda_N = \mathrm{g}^N_{ij} u^i dx^j$ where $\mathrm{g}_N = \mathrm{g}^N_{ij}(x) dx^i \otimes dx^j$ for $N = 1, 2$. Likewise, the Lagrangian functions to consider are the fiberwise quadratic polynomials $L_N = \frac{1}{2}\mathrm{g}^N_{ij}(x) u^i u^j$. Denote by $\omega_N = d\lambda_N$ the corresponding symplectic 2-forms of $TM$, and also by $X_N = X_{L_N}$ the associated geodesic sprays. We have

$$\lambda_N(X_N) = 2L_N \qquad \text{and} \qquad \omega_N(X_N) = -dL_N \tag{2.1}$$

for all $N = 1, 2$. The (maximal) integral curves of the vector fields $X_N$ on $TM$ project onto configuration space as the geodesics of $(M, \mathrm{g}_N)$.

[2]The formalism can be easily extended to the case of Finsler structures [25]; here, we will not need such a generality.

Introduce then the diffeomorphism $\phi : TM \to TM$ defined by $\phi(x,u) = (x,\tilde{u})$ where $\tilde{u} = u\sqrt{L_1(x,u)/L_2(x,u)}$.[3] This diffeomorphism is, indeed, designed to relate the two Lagrangians, viz.,

$$L_1 = \phi^* L_2. \tag{2.2}$$

Clearly, the two metrics $g_1$ and $g_2$ have the same unparametrized geodesics (we write $g_1 \sim g_2$) iff

$$\phi_*(X_1) \wedge X_2 = 0 \tag{2.3}$$

i.e., iff the the push-forward $\phi_*(X_1)$ and $X_2$ are functionally dependent.

The method, to obtain a generating function for the conserved quantities in involution, consists then in singling out, apart from $\omega_1$, a preferred $X_1$-invariant 2-form constructed in terms of $\omega_2$.

**Proposition 2.1.** *Suppose that* $g_1 \sim g_2$, *and define* $\omega_2' = d(L_2^{-\frac{1}{2}}\lambda_2)$, *then*

$$L_{X_1}(\phi^*\omega_2') = 0. \tag{2.4}$$

*Proof.* We have $L_{X_1}(\phi^*\omega_2') = d((\phi^*\omega_2')(X_1)) = \phi^* d(\omega_2'(\phi_* X_1)) = \phi^* d(h\,\omega_2'(X_2))$, for some function $h$ (see (2.3)). The definition of $\omega_2'$ then readily yields $L_{X_1}(\phi^*\omega_2') = \phi^* d(h\, d(L_2^{-\frac{1}{2}}\lambda_2)(X_2)) = \phi^* d(h(-\frac{1}{2}L_2^{-\frac{3}{2}}(dL_2 \wedge \lambda_2)(X_2) + L_2^{-\frac{1}{2}}\omega_2(X_2))) = 0$ in view of (2.1). □

So, the sought extra $X_1$-invariant 2-form is $\phi^*\omega_2'$. (Note that $L_{X_1}(\phi^*\omega_2) \neq 0$.) It enters naturally into the definition of a "generating function" $f_t \in C^\infty(TM,\mathbb{R})$ of first-integrals given below.

**Corollary 2.2.** *The function*

$$f_t = \frac{(t^{-1}\omega_1 + \phi^*\omega_2')^n}{\omega_1^n} \tag{2.5}$$

*is $X_1$-invariant whenever $t \neq 0$.*

## 2.2 The example of two projectively equivalent geodesic sprays for the $n$-sphere

We recall the construction of the first-integrals in involution yielding the Liouville-integrability of the geodesic flow on $T\mathcal{E}^n \cong TS^n$.

Let us parametrize $T\mathbb{R}^{n+1}$ by the couples $q, v \in \mathbb{R}^{n+1}$. The constraints defining the embedding $TS^n \hookrightarrow T\mathbb{R}^{n+1}$ are

$$q^2 := \sum_{\alpha=0}^{n} q_\alpha^2 = 1 \qquad \text{and} \qquad v \cdot q := \sum_{\alpha=0}^{n} v_\alpha q_\alpha = 0. \tag{2.6}$$

[3] Note that $\phi$ is the identity on the zero section of $TM$.

As already mentioned in Section 1, the (unparametrized) geodesics of the ellipsoid $\mathcal{E}^n$ with semi-axes[4] $a_0, a_1, \cdots, a_n$ are precisely given by those of the sphere $S^n = \{q \in \mathbb{R}^{n+1} | \sum_{\alpha=0}^n q_\alpha^2 = 1\}$ endowed with either projectively equivalent metrics

$$\mathrm{g}_1 = \sum_{\alpha=0}^n a_\alpha dq_\alpha^2 \Big|_{S^n} \qquad \& \qquad \mathrm{g}_2 = \frac{1}{B}\sum_{\alpha=0}^n dq_\alpha^2 \Big|_{S^n} \tag{2.7}$$

where

$$B = \sum_{\alpha=0}^n \frac{q_\alpha^2}{a_\alpha}. \tag{2.8}$$

The corresponding Lagrangians on $TS^n$ are respectively

$$L_1 = \frac{1}{2}A \qquad \& \qquad L_2 = \frac{1}{2B}\sum_{\alpha=0}^n v_\alpha^2 \tag{2.9}$$

where

$$A = \sum_{\alpha=0}^n a_\alpha v_\alpha^2. \tag{2.10}$$

The associated Cartan 1-forms then read in this case

$$\lambda_1 = \sum_{\alpha=0}^n a_\alpha v_\alpha dq_\alpha \qquad \& \qquad \lambda_2 = \frac{1}{B}\sum_{\alpha=0}^n v_\alpha dq_\alpha. \tag{2.11}$$

**Proposition 2.3.** *(i) The geodesic sprays for the metrics $\mathrm{g}_N$ are given by the Hamiltonian vector fields $X_N = X_{L_N}$, for $N = 1, 2$, namely*

$$X_1 = \sum_{\alpha=0}^n v_\alpha \frac{\partial}{\partial q_\alpha} - \frac{v^2}{B}\sum_{\alpha=0}^n \frac{q_\alpha}{a_\alpha}\frac{\partial}{\partial v_\alpha} \tag{2.12}$$

*and*

$$X_2 = \sum_{\alpha=0}^n v_\alpha \frac{\partial}{\partial q_\alpha} - \frac{1}{B}\sum_{\alpha=0}^n \Big(2v_\alpha \sum_{\beta=0}^n \frac{v_\beta q_\beta}{a_\beta} - v^2 \frac{q_\alpha}{a_\alpha}\Big)\frac{\partial}{\partial v_\alpha} \tag{2.13}$$

*respectively.*

*(ii) Condition (2.3) holds true, implying $\mathrm{g}_1 \sim \mathrm{g}_2$.*

*Proof.* Using (2.1) together with the constraints (2.6), we thus have to solve for $X_1$, resp. $X_2$, the equation $\omega_N(X_N) + dL_N + \lambda\, d(q^2 - 1) + \mu\, d(v \cdot q) = 0$ where $\lambda$ and $\mu$ are Lagrange multipliers. The latter are, *in fine*, completely determined and readily yield (2.12), resp. (2.13).

Now, the diffeomorphism $\phi : (q, v) \mapsto (q, \tilde{v})$ introduced in Section 2.1 is given by $\tilde{v} = v\sqrt{AB/v^2}$; routine calculation yields $\phi_*(\partial_{q_\alpha}) = \partial_{q_\alpha} + q_\alpha/(a_\alpha B)\,\mathcal{E}$ with $\mathcal{E} = \sum v_\alpha \partial_{v_\alpha}$ the Euler vector field; also $\phi_*(\partial_{v_\alpha}) = \sqrt{AB/v^2}(\partial_{v_\alpha} - v_\alpha(v^{-2} - a_\alpha/A)\,\mathcal{E})$. This, along with the constraint $\sum v_\alpha q_\alpha = 0$, helps us prove Equation (2.3). ∎

[4]We will, later on, deal with the choice $0 < a_0 < a_1 < \cdots < a_n$.

## 2.3 The Jacobi-Moser system

Let us review here, and in some detail, the main result obtained by Tabachnikov [25] via the general procedure of Section 2.1, starting with the geodesic flow on $(S^n, \mathrm{g}_1)$.

The diffeomorphism $\phi$ of $TS^n$, namely $\phi(q,v) = (q, \tilde{v} = v\sqrt{L_1/L_2})$, is such that $\tilde{v} = v\sqrt{AB/v^2}$ where $v^2 = \sum v_\alpha^2$; whence $\phi^*\lambda_2 = C\sqrt{A}\sum v_\alpha dq_\alpha$, where

$$C = \frac{1}{\sqrt{Bv^2}}. \tag{2.14}$$

Now, since $\phi^*\omega_2' = d(L_1^{-\frac{1}{2}}\phi^*\lambda_2)$, easy computation then leads to

$$\phi^*\omega_2' = C\sqrt{2}\,\Omega_1 \quad \text{where} \quad \Omega_1 = \sum_{\alpha=0}^{n} dv_\alpha \wedge dq_\alpha + \frac{dC}{C} \wedge \sum_{\alpha=0}^{n} v_\alpha dq_\alpha. \tag{2.15}$$

The function $C$ defined by (2.14) is a first integral of the system, namely

$$X_1 C = 0. \tag{2.16}$$

This function $C$ is the *Joachimsthal* first-integral of the geodesic flow of $(S^n, \mathrm{g}_1)$.

The next step consists in lifting the 1-parameter family of first-integrals (2.5) to $T\mathbb{R}^{n+1}$ by taking advantage of the constraints (2.6), and to put

$$f_t = \frac{(t^{-1}\omega_1 + \Omega_1)^n \wedge d(v.q) \wedge q \cdot dq}{\omega_1^n \wedge d(v.q) \wedge q \cdot dq} \tag{2.17}$$

with a slight abuse of notation using the constancy of $C$ (see (2.16)) in (2.15). Elementary calculation yields

$$f_t = \frac{\big(\omega_*^n - (n/v^2)\,\omega_*^{n-1} \wedge v \cdot dv \wedge v \cdot dq\big) \wedge q \cdot dv \wedge q \cdot dq}{\omega_1^n \wedge (dv.q + v \cdot dq) \wedge q \cdot dq} \tag{2.18}$$

where $\omega_* = \sum b_\alpha dv_\alpha \wedge dq_\alpha - (1/v^2)\, v \cdot dv \wedge v \cdot dq$, together with $b_\alpha = t^{-1}a_\alpha + 1$, for all $\alpha = 0, \ldots, n$. The following lemma [25] will be used to complete the calculation.

**Lemma 2.4.** *Let $\omega = \sum c_\alpha dv_\alpha \wedge dq_\alpha$ and $\omega_0 = \sum dv_\alpha \wedge dq_\alpha$, then*

$$\begin{aligned} \frac{\omega^n \wedge q \cdot dv \wedge q \cdot dq}{\omega_0^{n+1}} &= n! \prod_{\alpha=0}^{n} c_\alpha \sum_{\alpha=0}^{n} \frac{q_\alpha^2}{c_\alpha} \\ \frac{\omega^{n-1} \wedge v \cdot dv \wedge v \cdot dq \wedge q \cdot dv \wedge q \cdot dq}{\omega_0^{n+1}} &= (n-1)! \prod_{\alpha=0}^{n} c_\alpha \sum_{\alpha<\beta} \frac{(v_\alpha q_\beta - v_\beta q_\alpha)^2}{c_\alpha c_\beta}. \end{aligned}$$

Using the two above formulæ, we find $f_t = N/D$ where

$$\begin{aligned} N &= n! \prod b_\alpha \sum \frac{q_\alpha^2}{b_\alpha} - \frac{n(n-1)!}{v^2} \prod b_\alpha \sum_{\alpha<\beta} \frac{(v_\alpha q_\beta - v_\beta q_\alpha)^2}{b_\alpha b_\beta} \\ D &= n! \prod a_\alpha \sum q_\alpha^2 / a_\alpha. \end{aligned}$$

This entails $f_t = g_t/C$ (up to a constant overall factor), where $C$ is the *Joachimsthal* first-integral, and

$$g_t = v^2 \sum_{\alpha=0}^{n} \frac{q_\alpha^2}{b_\alpha} - \frac{1}{2} \sum_{\alpha\neq\beta} \frac{(v_\alpha q_\beta - v_\beta q_\alpha)^2}{b_\alpha b_\beta}.$$

Taking into account the expression $b_\alpha = t^{-1} a_\alpha + 1$, and the constraints (2.6), we end up with

$$g_t = \sum_{\alpha=0}^{n} \frac{a_\alpha v_\alpha^2}{a_\alpha + t} - \sum_{\alpha=0}^{n} \frac{a_\alpha v_\alpha^2}{a_\alpha + t} \sum_{\alpha=0}^{n} \frac{a_\alpha q_\alpha^2}{a_\alpha + t} + \left( \sum_{\alpha=0}^{n} \frac{a_\alpha v_\alpha q_\alpha}{a_\alpha + t} \right)^2. \tag{2.19}$$

At last, the first-integrals $F_\alpha$ defined by

$$g_t = \sum_{\alpha=0}^{n} \frac{F_\alpha}{a_\alpha + t} \tag{2.20}$$

are easily found to be

$$F_\alpha = a_\alpha v_\alpha^2 + \sum_{\beta\neq\alpha} \frac{a_\alpha a_\beta (v_\alpha q_\beta - v_\beta q_\alpha)^2}{a_\alpha - a_\beta}. \tag{2.21}$$

These are the *Jacobi-Moser* first-integrals. In terms of the momenta $p_\alpha = a_\alpha v_\alpha$ (see (2.11)), they read

$$F_\alpha = \frac{p_\alpha^2}{a_\alpha} + \sum_{\beta\neq\alpha} \frac{(p_\alpha a_\beta q_\beta - p_\beta a_\alpha q_\alpha)^2}{a_\alpha a_\beta (a_\alpha - a_\beta)}. \tag{2.22}$$

**Remark 2.5.** We indeed recover the Moser first integrals

$$F_\alpha = P_\alpha^2 + \sum_{\beta\neq\alpha} \frac{(P_\alpha Q_\beta - P_\beta Q_\alpha)^2}{a_\alpha - a_\beta} \tag{2.23}$$

by means of the canonical transformation $(p, q) \mapsto (P, Q)$ where $P_\alpha = p_\alpha/\sqrt{a_\alpha}$, and $Q_\alpha = q_\alpha \sqrt{a_\alpha}$.

## 2.4 The dual Moser system

Let us now adopt a "dual" standpoint by exchanging the rôle of the two projectively equivalent metrics on the $n$-sphere, i.e., by letting $\mathrm{g}_1 \leftrightarrow \mathrm{g}_2$ in the above derivation. In doing so, we will work out a complete set of commuting first-integrals of the geodesic flow on the conformally flat manifold $(S^n, \mathrm{g}_2)$. This will turn out to provide a new integrable system on the $n$-sphere.

### 2.4.1 The general construction

Let us now apply the general procedure outlined in Section 2.1 starting with the geodesic flow on $(S^n, \mathrm{g}_2)$, and replacing *mutatis mutandis* all reference to $\mathrm{g}_1$ by that of $\mathrm{g}_2$.

We first need to work out the expression of the 2-form $\phi^*\omega_1'$ in Proposition 2.1. The diffeomorphism $\phi$ of $TS^n$, viz., $\phi(q,v) = (q, \tilde{v} = v\sqrt{L_2/L_1})$, is $\tilde{v} = v\sqrt{v^2/(AB)}$ where, again, $v^2 = \sum v_\alpha^2$. We hence get $\phi^*\lambda_1 = \sqrt{v^2/(AB)}\sum a_\alpha v_\alpha dq_\alpha$. Now, since $\phi^*\omega_1' = d(L_2^{-\frac{1}{2}}\phi^*\lambda_1)$, thanks to $\phi^*L_1 = L_2$, we are led to

$$\phi^*\omega_1' = \sqrt{\frac{2}{A}}\,\widehat{\Omega}_2 \qquad \text{where} \qquad \widehat{\Omega}_2 = \sum_{\alpha=0}^{n} a_\alpha dv_\alpha \wedge dq_\alpha - \frac{dA}{2A}\wedge\lambda_1, \tag{2.24}$$

with $\lambda_1 = \sum a_\alpha v_\alpha dq_\alpha$ (see (2.11)).

We then find the *new Joachimsthal* first-integral, $J$, of the geodesic flow of $(S^n, \mathrm{g}_2)$ with the help of the expression (2.13) of the geodesic spray $X_2$. In fact, easy calculation yields $X_2 A = (2A/B)X_2(B)$, hence $X_2 J = 0$ where

$$J = \frac{A}{B^2}. \tag{2.25}$$

Utilizing this constant of the motion, and Equation (2.4), we obtain, see (2.24),

$$L_{X_2}\Omega_2 = 0 \qquad \text{where} \qquad \Omega_2 = \frac{\widehat{\Omega}_2}{B}. \tag{2.26}$$

Again, and to ease the calculation, we will lift the 1-parameter family of first-integrals (2.5) to $T\mathbb{R}^{n+1}$, using the constraints (2.6), and put this time

$$f_t = \frac{(t\,\omega_2 + \Omega_2)^n \wedge d(v\cdot q)\wedge q\cdot dq}{\omega_2^n \wedge d(v\cdot q)\wedge q\cdot dq}. \tag{2.27}$$

We trivially get

$$f_t = \frac{(t\,\widehat{\omega}_2 + \widehat{\Omega}_2)^n \wedge q\cdot dv\wedge q\cdot dq}{\widehat{\omega}_2^n \wedge q\cdot dv \wedge q\cdot dq} \tag{2.28}$$

where $\widehat{\omega}_2 = B\omega_2$, i.e., $\widehat{\omega}_2 = \sum dv_\alpha\wedge dq_\alpha - d\log B\wedge\sum v_\alpha dq_\alpha$ and $\widehat{\Omega}_2$ is as in (2.24).

Let us mention the following somewhat technical lemma.

**Lemma 2.6.** *Upon defining* $\widehat{\omega}_* = \sum b_\alpha dv_\alpha\wedge dq_\alpha$*, where* $b_\alpha = a_\alpha + t$*, for every* $\alpha = 0,1,\ldots,n$*, we have*

$$\begin{aligned}(t\,\widehat{\omega}_2 + \widehat{\Omega}_2)^k &= \widehat{\omega}_*^k - k\,\widehat{\omega}_*^{k-1}\wedge\Big(t\,d\log B\wedge v\cdot dq + \frac{1}{2}d\log A\wedge\lambda_1\Big)\\ &\quad + k(k-1)\,\widehat{\omega}_*^{k-2}\wedge t\,d\log B\wedge v\cdot dq\wedge\frac{1}{2}d\log A\wedge\lambda_1\end{aligned}$$

*for all* $k = 1, \ldots, n$, *and*

$$\widehat{\omega}_*^{n-1} = (n-1)! \sum_{\alpha<\beta} \prod_{\gamma\neq\alpha,\beta} b_\gamma dv_\gamma \wedge dq_\gamma$$

A somewhat demanding computation yields

$$f_t = \frac{\left(\widehat{\omega}_*^n - n\,\widehat{\omega}_*^{n-1} \wedge dA/(2A) \wedge \lambda_1\right) \wedge q\cdot dv \wedge q\cdot dq}{\widehat{\omega}_2^n \wedge q\cdot dv \wedge q\cdot dq}. \tag{2.29}$$

Resorting to Lemmas 2.4 and 2.6 in order to evaluate $f_t$, we obtain the partial result

$$\widehat{\omega}_*^n \wedge q\cdot dv \wedge q\cdot dq = n! \prod_{\beta=0}^n b_\beta \left(\sum_{\alpha=0}^n \frac{q_\alpha^2}{b_\alpha}\right) \omega_0^{n+1} \tag{2.30}$$

where $\omega_0 = \sum dv_\alpha \wedge dq_\alpha$. Likewise, some more effort is needed to find

$$\widehat{\omega}_*^{n-1} \wedge \frac{1}{2} d\log A \wedge \lambda_1 \wedge q\cdot dv \wedge q\cdot dq = (n-1)! \prod_{\gamma=0}^n b_\gamma \sum_{\alpha<\beta} \frac{(a_\alpha v_\alpha q_\beta - a_\beta v_\beta q_\alpha)^2}{b_\alpha b_\beta} \frac{\omega_0^{n+1}}{A}. \tag{2.31}$$

We just have to plug Equations (2.30) and (2.31) into the expression (2.28) to find[5]

$$f_t = \prod_{\gamma=0}^n b_\gamma \left(\sum_{\alpha=0}^n \frac{q_\alpha^2}{b_\alpha} - \frac{1}{A} \sum_{\alpha<\beta} \frac{(a_\alpha v_\alpha q_\beta - a_\beta v_\beta q_\alpha)^2}{b_\alpha b_\beta}\right). \tag{2.32}$$

We will, again, deal with the rescaled first-integral

$$g_t = \frac{J}{\prod b_\alpha} f_t \tag{2.33}$$

where $J$ is an (2.25), as a generating function of the sought conservative system $F_0, F_1, \ldots, F_n$. Using the definition $b_\alpha = a_\alpha + t$, and Equation (2.20), we readily prove that the geodesic flow on $TS^n$ above $(S^n, \mathrm{g}_2)$ admits the following first integrals, viz.,

$$F_\alpha = \frac{1}{B^2} \left(A q_\alpha^2 + \sum_{\beta\neq\alpha} \frac{(a_\alpha v_\alpha q_\beta - a_\beta v_\beta q_\alpha)^2}{a_\alpha - a_\beta}\right) \tag{2.34}$$

with $\alpha = 0, 1, \ldots, n$, where $A$ and $B$ are given by Equations (2.10) and (2.8), respectively.

With the help of the bundle isomorphism $\mathrm{g}_2^{-1} : T^*S^n \to TS^n$ provided by the metric $\mathrm{g}_2$, we can pull-back the previous first integrals to the cotangent bundle of $S^n$. Whence the following result.

[5]We have $\widehat{\omega}_2^n \wedge q\cdot dv \wedge q\cdot dq = n!\, \omega_0^{n+1}$.

**Proposition 2.7.** *The geodesic flow on the cotangent bundle of* $(S^n, \mathrm{g}_2)$ *admits the following set of first integrals, viz.,*

$$F_\alpha = q_\alpha^2 \sum_{\beta=0}^{n} a_\beta p_\beta^2 + \sum_{\beta\neq\alpha} \frac{(a_\alpha p_\alpha q_\beta - a_\beta p_\beta q_\alpha)^2}{a_\alpha - a_\beta} \tag{2.35}$$

*with* $\alpha = 0, \ldots, n$.

In the next section we will prove that these first integrals are actually independent and are mutually Poisson commuting.

### 2.4.2 Liouville integrability of the unconstrained system

From now on, we choose to work in a purely Hamiltonian framework which will turn out to be well-suited to the quantization procedure that we will examine in the next section.

Let us recall that the Hamiltonian of the system is given by

$$H = \frac{1}{2} \sum_{\alpha,\beta=0}^{n} \mathrm{g}_2^{\alpha\beta} p_\alpha p_\beta = \frac{1}{2} B \sum_{\alpha=0}^{n} p_\alpha^2 \tag{2.36}$$

where $B = \sum_{\alpha=0}^{n} q_\alpha^2/a_\alpha$ (see (2.8)).

We will show that the new set (2.35) of first-integrals of motion indeed turns the geodesic flow on the sphere $(S^n, \mathrm{g}_2)$ into an integrable system dual to the Jacobi-Moser geodesic flow on the ellipsoid.

**Proposition 2.8.** *The functions* $F_0, \ldots, F_n$ *of* $(T^*\mathbb{R}^{n+1}, \sum_{\alpha=0}^{n} dp_\alpha \wedge dq^\alpha)$ *given by (2.35) are in involution, namely*

$$\{F_\alpha,\, F_\beta\} = 0 \tag{2.37}$$

*for all* $\alpha, \beta = 0, \ldots, n$. *Moreover the following holds true*

$$\{H, F_\alpha\} = 2\Big(B\, a_\alpha p_\alpha^2 - q_\alpha^2 \sum_\beta p_\beta^2\Big) \sum_\gamma p_\gamma q_\gamma. \tag{2.38}$$

*Proof.* Write $F_\alpha = \mathcal{A}_\alpha + \mathcal{B}_\alpha$ with

$$\mathcal{A}_\alpha = q_\alpha^2 J, \qquad J = \sum_{\beta=0}^{n} a_\beta p_\beta^2, \tag{2.39}$$

where $J$ is the Joachimsthal first integral (2.25), and

$$\mathcal{B}_\alpha = \sum_{\beta\neq\alpha} \frac{\mathcal{M}_{\alpha\beta}^2}{a_\alpha - a_\beta}, \qquad \mathcal{M}_{\alpha\beta} = a_\alpha p_\alpha q_\beta - a_\beta p_\beta q_\alpha. \tag{2.40}$$

One can check the following relationships

$$\{\mathcal{A}_\alpha, \mathcal{A}_\beta\} = -4Jq_\alpha q_\beta \mathcal{M}_{\alpha\beta}, \qquad \{\mathcal{A}_\alpha, \mathcal{B}_\beta\} = 4Ja_\alpha q_\alpha q_\beta \frac{\mathcal{M}_{\alpha\beta}}{a_\alpha - a_\beta}, \qquad \{\mathcal{B}_\alpha, \mathcal{B}_\beta\} = 0,$$

for all $\alpha, \beta = 0, \ldots, n$, which readily imply Equation (2.37).

Let us furthermore observe that we have the following Poisson brackets, viz.,

$$\{H, \mathcal{A}_\alpha\} = 2JBq_\alpha p_\alpha - 4q_\alpha^2 \frac{H}{B} \sum_{\beta=0}^{n} p_\beta q_\beta$$

and

$$\{H, \mathcal{B}_\alpha\} = -2JBq_\alpha p_\alpha + 2Ba_\alpha p_\alpha^2 \sum_{\beta=0}^{n} p_\beta q_\beta,$$

which proves Equation (2.38). The proof is complete. □

**Remark 2.9.** Notice in contradistinction to the Jacobi-Moser case, that (i) the metric $\overline{\mathrm{g}}$ given by (1.5) on the ambient space $\mathbb{R}^{n+1}\backslash\{0\}$ is no longer flat, and (ii) the conservation relations $\{H, F_\alpha\} = 0$ for $\alpha = 0, \ldots, n$ are only valid for the constrained system, see (2.38), where $p \cdot q = 0$.

Let us also mention the interesting relations

$$\sum_{\alpha=0}^{n} F_\alpha = \sum_{\alpha=0}^{n} q_\alpha^2 \sum_{\beta=0}^{n} a_\beta p_\beta^2 \tag{2.41}$$

$$\sum_{\alpha=0}^{n} \frac{F_\alpha}{a_\alpha} = \left(\sum_{\alpha=0}^{n} p_\alpha q_\alpha\right)^2 \tag{2.42}$$

$$\sum_{\alpha=0}^{n} \frac{F_\alpha}{a_\alpha^2} = -2H + 2\sum_{\alpha=0}^{n} p_\alpha q_\alpha \sum_{\beta=0}^{n} \frac{p_\beta q_\beta}{a_\beta}, \tag{2.43}$$

of which the last one leads to another proof of (2.38).

### 2.4.3 The Dirac brackets

Our goal is now to deduce from the knowledge of (2.35) independent quantities in involution $I_1, \ldots, I_n$ on $(T^*S^n, \sum_{i=1}^n d\xi_i \wedge dx^i)$ from the symplectic embedding $\iota : T^*S^n \hookrightarrow T^*\mathbb{R}^{n+1}$ defined by the constraints

$$Z_1(p,q) = \sum_{\alpha=0}^{n} q_\alpha^2 - 1 = 0, \qquad Z_2(p,q) = \sum_{\alpha=0}^{n} p_\alpha q_\alpha = 0. \tag{2.44}$$

**Proposition 2.10.** *The restrictions $F_\alpha|_{T^*S^n} = F_\alpha \circ \iota$ of the functions (2.35) do Poisson-commute on $T^*S^n$.*

*Proof.* We get, using the Dirac brackets [1, 21],

$$\begin{aligned} \{F_\alpha|_{T^*S^n}, F_\beta|_{T^*S^n}\} &= \{F_\alpha, F_\beta\}|_{T^*S^n} \\ &\quad - \frac{1}{\{Z_1, Z_2\}} \left[\{Z_1, F_\alpha\}\{Z_2, F_\beta\} - \{Z_1, F_\beta\}\{Z_2, F_\alpha\}\right]|_{T^*S^n} \end{aligned} \tag{2.45}$$

for second-class constraints. The denominator $\{Z_1, Z_2\} = -2\sum_{\alpha=0}^n q_\alpha^2$ does not vanish; one can also check that $\{Z_1, F_\alpha\} = -4\, a_\alpha p_\alpha q_\alpha (1 + Z_1)$ and that $\{Z_2, F_\alpha\} = 0$ for all $\alpha = 0, \ldots, n$. The fact that $\{F_\alpha, F_\beta\} = 0$ completes the proof. □

### 2.4.4 The constrained integrable system as a Stäckel system

In order to provide explicit expressions for the sought functions in involution $I_1, \ldots, I_n$, we resort to Jacobi ellipsoidal coordinates $x^1, \ldots, x^n$ on $S^n$. Those are defined by

$$Q_\lambda(q, q) = \sum_{\alpha=0}^n \frac{q_\alpha^2}{a_\alpha - \lambda} = -\frac{U_x(\lambda)}{V(\lambda)} \tag{2.46}$$

where

$$U_x(\lambda) = \prod_{i=1}^n (\lambda - x^i) \qquad \text{and} \qquad V(\lambda) = \prod_{\alpha=0}^n (\lambda - a_\alpha) \tag{2.47}$$

and are such that

$$a_0 < x^1 < a_1 < x^2 < \ldots < x^n < a_n. \tag{2.48}$$

Notice that Equation (2.46) yields the local expressions

$$q_\alpha^2(x) = \frac{\displaystyle\prod_{i=1}^n (a_\alpha - x^i)}{\displaystyle\prod_{\beta \neq \alpha} (a_\alpha - a_\beta)} \tag{2.49}$$

for all $\alpha = 0, 1, \ldots, n$.

Let us mention the following identity, deduced from (2.49), viz.,

$$\frac{\partial q_\alpha}{\partial x^i} = -\frac{1}{2} \frac{q_\alpha}{a_\alpha - x^i} \tag{2.50}$$

for all $i = 1, \ldots, n$ and $\alpha = 0, 1, \ldots, n$

It is easy to show that the induced metric $\mathrm{g}_2 = (1/B) \sum_{\alpha=0}^n dq_\alpha^2|_{S^n}$, see (2.7), is indeed given by $\mathrm{g}_2 = \sum_{i,j=1}^n \mathrm{g}_{ij}(x) dx^i dx^j$ with

$$\mathrm{g}_{ij}(x) = \frac{1}{4B} \sum_{\alpha=0}^n \frac{q_\alpha^2}{(a_\alpha - x^i)(a_\alpha - x^j)} = \mathrm{g}_i(x)\, \delta_{ij} \tag{2.51}$$

where, using a result taken from [19], we have

$$\mathrm{g}_i(x) = -\frac{1}{4B}\frac{U_x'(x^i)}{V(x^i)} = -\frac{1}{4B}\frac{\prod_{j\neq i}(x^i - x^j)}{\prod_\alpha (x^i - a_\alpha)}. \tag{2.52}$$

This metric is actually positive-definite because of the inequalities (2.48). In these ellipsoidal coordinates, we obtain

$$B = \sum_{\alpha=0}^{n} \frac{q_\alpha^2}{a_\alpha} = \frac{1}{a_0}\frac{x^1\cdots x^n}{a_1\cdots a_n}.$$

Upon defining the constrained "momenta" $\xi_i$ (for $i = 1, \ldots, n$), via the induced canonical 1-form $\lambda|_{T^*S^n} = \sum_{i=1}^n \xi_i\, dx^i = \iota^* \sum_{\alpha=0}^n p_\alpha dq^\alpha$, we find

$$p_\alpha(\xi, x) = -\frac{q_\alpha(x)}{2B}\sum_{i=1}^{n}\frac{\mathrm{g}^i(x)\xi_i}{a_\alpha - x^i}. \tag{2.53}$$

We express, for convenience, the Hamiltonian (2.36) on $(T^*S^n, \sum_{i=1}^n d\xi_i \wedge dx^i)$, which is then found to be

$$H = \frac{1}{2}\sum_{i=1}^{n}\mathrm{g}^i(x)\xi_i^2 \tag{2.54}$$

where $\mathrm{g}^i(x) = 1/\mathrm{g}_i(x)$.

Let us now compute the expression of the conserved quantities (2.35) on $T^*S^n$.

**Proposition 2.11.** *The dual Moser conserved quantities* $(F_\alpha|_{T^*S^n})_{\alpha=0,\ldots,n}$ *retain the form*

$$F_\alpha|_{T^*S^n} = \frac{a_\alpha q_\alpha^2(x)}{B}\sum_{i=1}^{n}\frac{x^i\mathrm{g}^i(x)\xi_i^2}{a_\alpha - x^i}. \tag{2.55}$$

*Proof.* On the one hand, in view of Equations (2.49) and (2.53), one gets, see (2.39),

$$\mathcal{A}_\alpha|_{T^*S^n} = \frac{q_\alpha^2(x)}{B}\sum_{i=1}^{n} x^i g^i(x)\xi_i^2,$$

using the identities

$$\sum_{\alpha=0}^{n}\frac{q_\alpha^2}{a_\alpha - x^i} = 0$$

for all $i = 1, \ldots, n$.

On the other hand, a similar computation gives

$$\mathcal{M}_{\alpha\beta}|_{T^*S^n} = \frac{a_\alpha - a_\beta}{2B} q_\alpha(x) q_\beta(x) \sum_{i=1}^{n}\frac{x^i g^i(x)\xi_i}{(a_\alpha - x^i)(a_\beta - x^i)},$$

so that

$$\mathcal{B}_\alpha|_{T^*S^n} = \frac{q_\alpha^2(x)}{B}\sum_{i=1}^n \frac{(x^i)^2 g^i(x)\xi_i^2}{a_\alpha - x^i},$$

proving that $F_\alpha|_{T^*S^n}$ (where $F_\alpha = \mathcal{A}_\alpha + \mathcal{B}_\alpha$) is, indeed, as in (2.55). □

**Proposition 2.12.** *The following holds on $T^*S^n$, viz*

$$\sum_{\alpha=0}^n F_\alpha|_{T^*S^n} = \frac{1}{B}\sum_{i=1}^n x^i \mathrm{g}^i(x)\xi_i^2 = J|_{T^*S^n} \tag{2.56}$$

$$\sum_{\alpha=0}^n \frac{F_\alpha|_{T^*S^n}}{a_\alpha} = 0 \tag{2.57}$$

$$\sum_{\alpha=0}^n \frac{F_\alpha|_{T^*S^n}}{a_\alpha^2} = -2H. \tag{2.58}$$

*Proof.* The proof is a direct consequence of Equations (2.41), (2.42), and (2.43), together with the constraints (2.44). □

As a preparation to the proof that our system is, indeed Stäckel, let us introduce, for convenience, the symmetric functions, $\sigma_k(x)$ and $\sigma_k^i(x)$ with $x = (x^1, \ldots, x^n)$, that will be useful in the sequel, namely,

$$U_x(\lambda) \equiv \prod_{j=1}^n (\lambda - x^j) = \sum_{k=0}^n (-1)^k \lambda^{n-k}\sigma_k(x) \tag{2.59}$$

$$\frac{U_x(\lambda)}{\lambda - x^i} \equiv \prod_{j\neq i}(\lambda - x^j) = \sum_{k=1}^n (-1)^{k-1}\lambda^{n-k}\sigma_{k-1}^i(x). \tag{2.60}$$

We will also use $\sigma_k(a)$ and $\sigma_k^\alpha(a)$ with $a = (a_0, a_1, \ldots a_n)$, which are defined similarly.

Let us notice that the previous conserved quantities (2.55) can be written as

$$F_\alpha|_{T^*S^n} = \frac{a_\alpha\, G_{a_\alpha}(\xi, x)}{\displaystyle\prod_{\beta\neq\alpha}(a_\alpha - a_\beta)}$$

where

$$G_\lambda(\xi, x) = \frac{1}{B}\sum_{i=1}^n x^i \mathrm{g}^i(x)\prod_{j\neq i}(\lambda - x^j)\,\xi_i^2. \tag{2.61}$$

**Proposition 2.13.** *Let the functions $I_1, \ldots, I_n$ of $T^*S^n$ be defined by*

$$G_\lambda(\xi, x) = \sum_{k=1}^{n} (-1)^{k-1} \lambda^{n-k} I_k(\xi, x). \tag{2.62}$$

*Then*

$$I_k(\xi, x) = \sum_{i=1}^{n} A_k^i(x) \xi_i^2 \qquad \text{with} \qquad A_k^i(x) = \frac{1}{B} x^i \mathrm{g}^i(x) \sigma_{k-1}^i(x). \tag{2.63}$$

*Proof.* By plugging the definition (2.60) of the symmetric functions $\sigma_k^i(x)$ of order $k = 0, 1, \ldots, n-1$ (in the variables $(x^1, \ldots, x^n)$, with the exclusion of index $i$) into (2.61), one gets the desired result. □

**Theorem 2.14.** *The dual Moser system $I_1, \ldots, I_n$ on $T^*S^n$, given by (2.63), defines a Stäckel system, with Stäckel matrix $B = A^{-1}$ of the form*

$$B_k^i(x^k) = (-1)^i \, \frac{(x^k)^{n-i-1}}{4V(x^k)} \tag{2.64}$$

*for $i, k = 1, \ldots, n$. This implies [22] that the functions $I_1, \ldots, I_n$ are independent and in involution, which entails that the Stäckel coordinates $x^1, \ldots, x^n$ are separating for the Hamilton-Jacobi equation.*

*Proof.* It is obvious from its expression (2.64) that $B$ is a Stäckel matrix [22]. We just need to prove that $A$ is the inverse matrix of $B$. To this aim we first prove a useful identity. Let us consider the integral in the complex plane

$$\frac{1}{2\mathrm{i}\pi} \int_{|z|=R} \frac{z^{n-i}}{(z-\lambda)} \frac{U_x(\lambda)}{U_x(z)} \, dz.$$

When $R \to \infty$ the previous integral vanishes because the integrand decreases as $1/R^2$ for large $R$ (let us recall that $i \geq 1$). We then compute this integral using the theorem of residues and get the identity

$$\sum_{k=1}^{n} \frac{(x^k)^{n-i}}{U_x'(x^k)} \prod_{j \neq k} (\lambda - x^j) = \lambda^{n-i}. \tag{2.65}$$

Equipped with this identity let us now prove that $\sum_{k=1}^{n} B_k^i \, A_j^k = \delta_j^i$. Multiplying this relation by $(-1)^{j-1} \lambda^{n-j}$ and summing over $j$ from 1 to $n$, we get the equivalent relation

$$\sum_{k=1}^{n} B_k^i \sum_{j=1}^{n} (-1)^{j-1} \lambda^{n-j} A_j^k = (-1)^{i-1} \lambda^{n-i},$$

which becomes, using (2.63) and (2.60):

$$\sum_{k=1}^{n} B^i_k\, g^k(x) \prod_{j\neq k}(\lambda - x^j) = (-1)^{i-1}\lambda^{n-i}.$$

Using the explicit form of $g^k(x)$ given via (2.52) and of the matrix $B$, this last relation reduces to the identity (2.65), which completes the derivation of (2.64). □

**Remark 2.15.** A few remarks are in order.

1. The first integral defined by (2.56) is precisely the Joachimsthal invariant (2.25) of the dual Moser system.

2. It should be emphasized that the bihamiltonian character of our system is obvious with our choice of ellipsoidal coordinates since, from their very definition, the $x^i$ are the eigenvalues of the Benenti $(1,1)$-tensor field $(L^j_i)$ associated with a special conformal Killing tensor.

3. One can give some simple potentials for dual Moser. Denoting by $J_k$ the new first integrals, we have

$$J_k = I_k - v_k, \qquad v_k = \mu\sigma_k(x) + \nu(\sigma_1(x)\,\sigma_k(x) - \sigma_{k+1}(x)) \tag{2.66}$$

with $k = 1, \ldots, n$. Those will pairwise Poisson commute (see [22], p. 101) if the potential terms can be written in the form $v_k = \sum_{i=1}^n A^i_k(x)\, f_i(x^i)$, implying $f_i(x^i) = \sum_{k=1}^n B^k_i\, v_k$. A short computation, using the explicit form (2.64) of the matrix $B$ and the relation (2.59), indeed gives

$$f_i(x^i) = \frac{(x^i)^{n-1}}{4V(x^i)}(\mu + \nu\, x^i).$$

## 2.5 Three Stäckel systems

### 2.5.1 The Neumann-Uhlenbeck system

In addition to the two previously studied integrable systems, it may be useful to consider the well-known Neumann-Uhlenbeck system [29, 30, 19] on the cotangent bundle of the round sphere $S^n$. It is initially defined on $(T^*\mathbb{R}^{n+1}, \sum_{\alpha=0}^n dp_\alpha \wedge dq_\alpha)$ by the Hamiltonian

$$H = \frac{1}{2}\sum_{\alpha=0}^{n}\left(p_\alpha^2 + a_\alpha q_\alpha^2\right) \tag{2.67}$$

with the parameters $0 < a_0 < a_1 < \ldots < a_n$. This system is classically integrable, with the following commuting first integrals of the Hamiltonian flow in $T^*\mathbb{R}^{n+1}$:

$$F_\alpha(p,q) = q_\alpha^2 + \sum_{\beta\neq\alpha}\frac{\left(p_\alpha q_\beta - p_\beta q_\alpha\right)^2}{a_\alpha - a_\beta} \qquad \text{with} \qquad \alpha = 0, 1, \ldots, n. \tag{2.68}$$

Under symplectic reduction, with the second class constraints (2.44), it becomes an integrable system on $(T^*S^n, \sum_{i=1}^n d\xi_i \wedge dx^i)$. Writing $\widetilde{\mathrm{g}} = \sum_{i=1}^n \widetilde{\mathrm{g}}_i(x)(dx^i)^2$ the induced Euclidean metric on $S^n$ with $\widetilde{\mathrm{g}}_i(x) = -\frac{1}{4}U'_x(x^i)/V(x^i)$, the independent Poisson-commuting functions $I_k$ $(k = 1, \ldots, n)$ are

$$I_k(\xi, x) = \sum_{i=1}^n \widetilde{\mathrm{g}}^i(x)\sigma^i_{k-1}(x)\xi_i^2 - \sigma_k(x) \qquad \text{with} \qquad H = \frac{1}{2}I_1, \tag{2.69}$$

where $\widetilde{\mathrm{g}}^i = 1/\widetilde{\mathrm{g}}_i$.

### 2.5.2 A synthetic presentation

Let us observe that the previous calculation enables us to have a synthetic viewpoint unifying the Jacobi-Moser, Neumann-Uhlenbeck, and dual Moser systems. This highlights the novelty of the dual Moser system spelled out in this article.

In Table 1, we display in each row, and for each system, the metric, the first integrals in involution, the Hamiltonian, and the Stäckel matrix. (See, e.g., [8] for a derivation of the formulæ in the first two columns of this table.). Let us emphasize that in all three cases, the metric in the first row is indeed the Stäckel metric coming from $I_1$, and which will be, later on, involved in the quantization procedures.

| Jacobi-Moser | Neumann-Uhlenbeck | dual Moser |
|---|---|---|
| $\mathrm{g}_i = x^i\, \widetilde{\mathrm{g}}_i$ | $\widetilde{\mathrm{g}}_i = -\frac{U'_x(x^i)}{4V(x^i)}$ | $\mathrm{g}_i = \frac{1}{x^i}\, \widetilde{\mathrm{g}}_i$ |
| $I_k = \sum_i \frac{\widetilde{\mathrm{g}}^i}{x^i}\, \sigma^i_{k-1}\xi_i^2$ | $I_k = \sum_i \widetilde{\mathrm{g}}^i\, \sigma^i_{k-1}\xi_i^2 - \sigma_k(x)$ | $I_k = \sum_i x^i\, \widetilde{\mathrm{g}}^i\, \sigma^i_{k-1}\xi_i^2$ |
| $H = \frac{1}{2}\, I_1$ | $H = \frac{1}{2}\, I_1$ | $H = \frac{1}{2\sigma_{n+1}(a)}\, I_n$ |
| $x^k\, \widetilde{B}^i_k(x^k)$ | $\widetilde{B}^i_k(x^k) = (-1)^i \frac{(x^k)^{n-i}}{4V(x^k)}$ | $\frac{1}{x^k}\, \widetilde{B}^i_k(x^k)$ |

Table 1: Three Stäckel systems

# 3 Quantum integrability

Start with a configuration manifold $M$ of dimension $n$, and consider the space, $\mathcal{S}(M)$, of Hamiltonians on $T^*M$ that are fiberwise polynomial. A quantization prescription is a linear isomorphism $\mathcal{Q}$ between this space of *symbols*, $\mathcal{S}(M)$, and the space, $\mathcal{D}(M)$, of linear differential operators on $M$; this identification is, in addition, assumed to preserve the principal symbol.

It is well-known that there is, in general, no uniquely defined quantization. However, no matter how the quantization is chosen, we will adhere to the following, usual, definition of quantum integrability; see, e.g., [28, 18, 4, 5].

**Definition 3.1.** *A classically integrable system with independent, and mutually Poisson-commuting observables $I_1, \ldots I_n$, is integrable at the quantum level iff*

$$[\mathcal{Q}(I_k), \mathcal{Q}(I_\ell)] = 0 \tag{3.1}$$

*for all $k, \ell = 1, \ldots, n$*

As a consequence, for a given integrable classical system, depending on the quantization procedure used, quantum integrability may be achieved or not. In what follows we will consider and use two quantization schemes for quadratic Hamiltonians: (i) the theory of conformally equivariant quantization, and (ii) Carter's minimal prescription.

## 3.1 Conformally equivariant quantization

Let us recall that there exists no quantization mapping that intertwines the action of $\mathrm{Diff}(M)$. To bypass this obstruction, equivariant quantization [15, 10] proposes to further endow $M$ with a $G$-structure, and to look under which conditions the existence and uniqueness of a $G$-equivariant quantization can be guaranteed (the proper subgroup $G \subset \mathrm{Diff}(M)$ only is assumed to intertwine the quantization mapping $\mathcal{Q}$).

We recall that the space $\mathcal{F}_\lambda(M)$ of $\lambda$-densities on $M$, where $\lambda$ is some complex-valued weight, is the space of sections of the complex line bundle $|\Lambda^n T^*M|^\lambda \otimes \mathbb{C}$. If $M$ is orientable, $(M, \mathrm{vol})$, such a $\lambda$-density can be, locally, cast into the form $\phi = f|\mathrm{vol}|^\lambda$ with $f \in C^\infty(M)$; this entails that $\phi$ transforms under the action of $a \in \mathrm{Diff}(M)$ according to $f \mapsto a_* f |(a_*\mathrm{vol})/\mathrm{vol}|^\lambda$, or infinitesimally as

$$L_X^\lambda(f) = X(f) + \lambda\,\mathrm{Div}(X)\,f \tag{3.2}$$

for all $X \in \mathrm{Vect}(M)$.

**Remark 3.2.** Note that the completion $\mathcal{H}(M)$ of the space of compactly supported half-densities, $\mathcal{F}^c_{\frac{1}{2}}(M)$, is a Hilbert space canonically attached to $M$ that will be used in the sequel. The scalar product of two half-densities reads

$$\langle \phi, \psi \rangle = \int_M \overline{\phi}\,\psi$$

where the bar stands for complex conjugation.

We will denote by $\mathcal{S}_\delta(M) = \mathcal{S}(M) \otimes \mathcal{F}_\delta(M)$ the graded space of symbols of weight $\delta$. This space is turned into a $\mathrm{Vect}(M)$-module using the definition (3.2) of the Lie derivative extended to the canonical lift of $\mathrm{Vect}(M)$ to $T^*M$.

Likewise, we will introduce the filtered space $\mathcal{D}_{\lambda,\mu}(M)$ of differential operators sending $\mathcal{F}_\lambda(M)$ to $\mathcal{F}_\mu(M)$. A differential operator of order $k$ is, locally, written as

$$A = A_k^{i_1\dots i_k}(x)\partial_{i_1}\dots\partial_{i_k} + \cdots + A_1^i(x)\partial_i + A_0(x) \tag{3.3}$$

where $A_\ell^{i_1\dots i_\ell} \in C^\infty(M)$ for $\ell = 0, 1, \dots, k$. It is clear that this space of weighted differential operators, $\mathcal{D}_{\lambda,\mu}(M)$, becomes a $\mathrm{Vect}(M)$-module via the following definition of the Lie derivative, namely,

$$L_X^{\lambda,\mu}(A) = L_X^\mu \circ A - A \circ L_X^\lambda \tag{3.4}$$

for all $X \in \mathrm{Vect}(M)$.

From now on, we will be dealing with the case of a conformal (Riemannian) structure, $G = \mathrm{SO}(n+1,1)$, with $n > 2$, dictated by the conformal flatness of our main example: the dual Moser system.

**Theorem 3.3** ([10])**.** *Given a conformally flat Riemannian manifold $(M, \mathrm{g})$, there exists (except for a discrete set of values of $\delta = \mu - \lambda$ called resonances) a unique conformally-equivariant quantization, i.e., a linear isomorphism*

$$\mathcal{Q}_{\lambda,\mu} : \mathcal{S}_\delta(M) \to \mathcal{D}_{\lambda,\mu}(M) \tag{3.5}$$

*that (i) preserves the principal symbol, and (ii) intertwines the actions of the Lie algebra $\mathrm{o}(n+1,1) \subset \mathrm{Vect}(M)$.*

In the particular and pivotal case of symbols of degree two, at the core of the present study, explicit formulæ are given by the following theorem.

**Theorem 3.4** ([9])**.** *(i) Let $(M, \mathrm{g})$ be a conformally flat Riemannian manifold of dimension $n \geq 3$. The conformally equivariant quantization mapping (3.5) restricted to symbols $P = P_2^{ij}(x)\xi_i\xi_j + P_1^i(x)\xi_i + P_0(x)$ of degree two is given, for non-resonant values of $\delta$, by*

$$\begin{aligned}
\mathcal{Q}_{\lambda,\mu}(P) \quad = \quad & -P_2^{ij} \circ \nabla_i \circ \nabla_j \\
& +\mathrm{i}\left(\beta_1 \nabla_i P_2^{ij} + \beta_2\, \mathrm{g}^{ij}\mathrm{g}_{k\ell}\nabla_i P_2^{k\ell} + P_1^j\right) \circ \nabla_j \\
& +\beta_3 \nabla_i \nabla_j (P_2^{ij}) + \beta_4\, \mathrm{g}^{ij}\mathrm{g}_{k\ell}\nabla_i\nabla_j(P_2^{k\ell}) + \beta_5 R_{ij} P_2^{ij} + \beta_6 R\, \mathrm{g}_{ij} P_2^{ij} \\
& +\alpha \nabla_i (P_1^i) + P_0
\end{aligned}$$

*where $\nabla$ denotes the levi-Civita connection,*[6] $R_{ij}$ *(resp.* $R$*) the components of the Ricci tensor in the chosen chart (resp. the scalar curvature) of the metric* $\mathrm{g}$*; the coefficients* $\alpha, \beta_1, \ldots, \beta_6$ *depend on* $\lambda, \mu$*, and* $n$ *in an explicit fashion.*[7]

*(ii) The quantization mapping* $\mathcal{Q}_{\lambda,\mu}$ *depends only on the conformal class of* $\mathrm{g}$.

The above formula can be specialized to the case of half-density quantization of quadratic symbols $P = P^{ij}(x)\xi_i\xi_j$; one finds[8]

$$\mathcal{Q}_{\frac{1}{2},\frac{1}{2}}(P) = \widehat{P} + \beta_3\,\nabla_i\nabla_j(P^{ij}) + \beta_4\,\mathrm{g}^{ij}\mathrm{g}_{k\ell}\nabla_i\nabla_j(P^{k\ell}) + \beta_5\,R_{ij}P^{ij} + \beta_6\,R\,\mathrm{g}_{ij}P^{ij} \quad (3.6)$$

where

$$\widehat{P} = -\nabla_i \circ P^{ij} \circ \nabla_j. \quad (3.7)$$

**Remark 3.5.** The quantization prescription (3.7), called "minimal" in [8], has been put forward by Carter [7], who dealt with polynomial symbols of degree at most two. A great many studies of the quantum spectrum for various integrable models use naturally Carter's quantization [24, 28, 18, 2]. Along with Equation (3.7), the formulæ for the minimal quantization of lower degree monomials are respectively

$$\widehat{P_0} = P_0 \quad (3.8)$$

$$\widehat{P_1} = \frac{\mathsf{i}}{2}\left(P_1^i \circ \nabla_i + \nabla_i \circ P_1^i\right) \quad (3.9)$$

so that

$$\widehat{P_k} = \mathcal{Q}_{\frac{1}{2},\frac{1}{2}}(P_k), \qquad \forall k = 0, 1. \quad (3.10)$$

Accordingly, a generalization to cubic monomials has been proposed in [8]:

$$\widehat{P_3} = -\frac{\mathsf{i}}{2}\left(\nabla_i \circ P_3^{ijk} \circ \nabla_j \circ \nabla_k + \nabla_i \circ \nabla_j \circ P_3^{ijk} \circ \nabla_k\right). \quad (3.11)$$

All previously defined operators are formally self-adjoint on $\mathcal{F}^c_{\frac{1}{2}}(M)$; see Remark 3.2.

In the case where the quadratic observable $P = P^{ij}(x)\xi_i\xi_j$ stems from a Killing tensor,[9] i.e., if $\nabla_{(i}P_{jk)} = 0$ for all $i, j, k = 1\ldots, n$, we can rewrite Equation (3.6) as

$$\mathcal{Q}_{\frac{1}{2},\frac{1}{2}}(P) = \widehat{P} + f(P) \quad (3.12)$$

where $\widehat{P}$ is as in (3.7), and the scalar term is given by

$$f(P) = c_1\,\Delta_{\mathrm{g}}\mathrm{Tr}(P) + c_2\,R_{ij}P^{ij} + c_3\,R\cdot\mathrm{Tr}(P) \quad (3.13)$$

[6]The covariant derivative of $\lambda$-densities $\phi = f|\mathrm{vol}_{\mathrm{g}}|^\lambda$, locally defined in terms of the Riemannian density, $|\mathrm{vol}_{\mathrm{g}}|$, reads $\nabla\phi = df|\mathrm{vol}_{\mathrm{g}}|^\lambda$.

[7]See Equations (3.3), (3.4), and (4.4) in [9].

[8]The value $\delta = 0$ is non-resonant [10].

[9]This is the case for the integrable systems of Stäckel type we are studying.

where $\Delta_\mathrm{g}$ is the Laplace operator of $(M, \mathrm{g})$, and $\mathrm{Tr}(P) = P^{ij}\mathrm{g}_{ij}$; the coefficients in (3.13) are respectively

$$c_1 = \frac{n^2}{8(n+1)(n+2)}, \quad c_2 = \frac{n^2}{4(n+1)(n-2)}, \quad c_3 = \frac{-n^2}{2(n^2-1)(n^2-4)}. \tag{3.14}$$

## 3.2 Quantum commutators

In order to implement Definition 3.1 of quantum integrability, we will need some preparation regarding the quantum commutators of Poisson-commuting symbols. In doing so, we will opt for the conformally equivariant quantization $\mathcal{Q} \equiv \mathcal{Q}_{\frac{1}{2},\frac{1}{2}}$.

**Proposition 3.6.** *Let $P$ and $Q$ be two, Poisson-commuting, quadratic symbols on $(T^*M, \omega = \sum_{i=1}^n d\xi_i \wedge dx^i)$. The commutator of the two operators $\mathcal{Q}(P)$ and $\mathcal{Q}(Q)$, given by (3.12), retains the form*

$$[\mathcal{Q}(P), \mathcal{Q}(Q)] = \mathrm{i}\, \mathcal{Q}(A_{P,Q} + V_{P,Q}), \tag{3.15}$$

*where*

$$A_{P,Q} = -\frac{2}{3}\left(\nabla_j B^{jk}_{P,Q}\right)\xi_k \tag{3.16}$$

*with*[10]

$$\begin{aligned} B^{jk}_{P,Q} &= P^{\ell[j}\nabla_\ell \nabla_m Q^{k]m} + P^{\ell[j}R^{k]}{}_{m,n\ell}Q^{mn} - (P \leftrightarrow Q) \\ &\quad -\nabla_\ell P^{m[j}\nabla_m Q^{k]\ell} - P^{\ell[j}R_{\ell m}Q^{k]m} \end{aligned} \tag{3.17}$$

*and*

$$V_{P,Q} = 2\left(P^{jk}\,\partial_j f(Q) - Q^{jk}\,\partial_j f(P)\right)\xi_k. \tag{3.18}$$

*Proof.* Start with two quadratic observables $P$ and $Q$. As shown in [8], we have $-\mathrm{i}[\widehat{P}, \widehat{Q}] = \widehat{\{P,Q\}} + \widehat{A}_{P,Q}$, where the monomial $A_{P,Q}$, and the skew-symmetric tensor $B_{P,Q}$ are as in (3.16), and (3.17), respectively. If it is then assumed that $\{P,Q\} = 0$, Equation (3.15) follows directly from the explicit expression (3.12) of the conformally equivariant quantization mapping, $\mathcal{Q}$, and from Equation (3.10). □

Now, for the Liouville-integrable systems considered below, all Poisson-commuting fiberwise polynomial symbols have the form $P = P_2 + P_0$, where the indices 0 and 2 refer to the homogeneity degree. In view of Equation (3.15), and of results obtained in [8], we find $[\mathcal{Q}(P_2+P_0), \mathcal{Q}(Q_2+Q_0)] = \mathrm{i}\,\mathcal{Q}(A_{P_2,Q_2} + V_{P_2,Q_2})$, which means that the zero degree terms $P_0$ and $Q_0$ produce no quantum corrections.

[10] We use the following convention for the Riemann and Ricci tensors, namely, $R^\ell{}_{i,jk} = \partial_j\Gamma^\ell_{ik} + \Gamma^\ell_{sj}\Gamma^s_{ik} - (j \leftrightarrow k)$, and $R_{ij} = R^s{}_{i,sj}$.

The structure of the quantum corrections (the right hand side of Equation (3.15)) is rather involved, because of the complexity of the tensor $B_{P,Q}$; see (3.17). Nevertheless, for Stäckel systems major simplifications occur. Indeed, the observables $I_k = I_{2,k} + I_{0,k}$ with $I_{2,k} = \sum_i g^i(x)\sigma^i_{k-1}(x)\xi_i^2$ generate diagonal Killing tensors. Using the separating coordinates $x^i$ and considering $H = \frac{1}{2}I_{2,1} = \frac{1}{2}\sum_i \mathrm{g}^i(x)\xi_i^2$ for the Hamiltonian fixes up the diagonal metric to be $\mathrm{g} = \sum_i \mathrm{g}_i(x)(dx^i)^2$, with $\mathrm{g}_i = 1/\mathrm{g}^i$ for all $i = 1, \ldots, n$. Under these assumptions, Proposition 3.9 in [8] gives

$$B^{k\ell}_{I_{2,i},I_{2,j}} = -2\, I^{s[k}_{2,i}\, R_{st}\, I^{\ell]t}_{2,j} \tag{3.19}$$

for all $i, j, k, \ell = 1, \ldots n$, which entails:

**Proposition 3.7.** *A sufficient condition for a Stäckel system to be integrable at the quantum level is*

$$R_{ij} = 0, \qquad \forall i \neq j \tag{3.20}$$

*where $i, j = 1, \ldots, n$, in the special separating coordinate system $(x^i)$.*

*Proof.* The Killing tensors $I_{2,i}$ are diagonal, for $i = 1, \ldots, n$, in the Stäckel coordinate system, and the result follows from (3.19). ☐

**Remark 3.8.**
1. Condition (3.20) is the well-known Robertson condition [23], which has to hold in the separating coordinates system. The relation (3.19) was also obtained in [5] by a direct computation of the commutator in separating coordinates; however the explicit form of the tensor $B_{P,Q}$ (3.17) was not given there.
2. In Corollary 3.10 of [8] the Robertson condition was misleadingly claimed to be also necessary.
3. It has been shown by Benenti et al. [4] that the Robertson condition (3.20) is necessary and sufficient for the separability of the Schrödinger equation, comforting the above definition of quantum integrability.

In the next subsections we will examine, successively, quantum integrability for the following Stäckel systems: the Neumann-Uhlenbeck, the dual Moser and the Jacobi-Moser systems. As previously explained, the potential, i.e., zero degree terms in the classical observables never induce quantum corrections; they will therefore be systematically omitted.

## 3.3 The quantum Neumann-Uhlenbeck system

Let us recall that, for the Neumann-Uhlenbeck system (see Table 1), the Stäckel metric $\widetilde{\mathrm{g}} = \sum_i \widetilde{\mathrm{g}}_i(x)(dx^i)^2$, is

$$\widetilde{\mathrm{g}}_i(x) = -\frac{1}{4}\frac{U'_x(x^i)}{V(x^i)} = -\frac{1}{4}\frac{\prod_{j\neq i}(x^i - x^j)}{\prod_\alpha (x^i - a_\alpha)} \tag{3.21}$$

for $i = 1, \ldots, n$. If we put $\widetilde{\mathrm{g}}^i = 1/\widetilde{\mathrm{g}}_i$, the independent and Poisson-commuting observables are given by

$$I_k = \sum_{i=1}^n \widetilde{\mathrm{g}}^i(x)\sigma^i_{k-1}(x)\,\xi_i^2$$

for $k = 1, 2, \ldots, n$, and the Stäckel Hamiltonian is $H = \frac{1}{2}I_1 = \frac{1}{2}\sum_i \widetilde{\mathrm{g}}^i(x)\,\xi_i^2$.

**Proposition 3.9.** *The conformally equivariant quantization does preserve quantum integrability of the Neumann-Uhlenbeck system.*

*Proof.* From the fact that $(S^n, \widetilde{\mathrm{g}})$ is the round sphere, we have

$$\widetilde{R}_{ij} = (n-1)\,\widetilde{\mathrm{g}}_i\,\delta_{ij} \qquad \& \qquad \widetilde{R} = n(n-1). \tag{3.22}$$

Straightforward computation then leads to

$$f(I_k) = (n-k+1)[c_4\,\sigma_{k-1}(x) + 2(n-k+2)c_1\,\sigma_{k-1}(a)] \tag{3.23}$$

for all $k = 1, \ldots, n$, where

$$c_4 = -2(n+1)c_1 + (n-1)c_2 + n(n-1)c_3. \tag{3.24}$$

Relations (3.14) readily imply the vanishing of $c_4$. As a consequence, the $f(I_k)$ are just constant, ensuring that $V_{I_k,I_\ell} = 0$ (see (3.18)). Equation (3.15) and the fact that $B_{I_k,I_\ell} = 0$ (since the Ricci tensor is diagonal in this coordinate system) entail that the conformally equivariant quantization (which coincides, up to a constant term, with Carter's) preserves integrability of the system at the quantum level. □

## 3.4 The quantum dual Moser system

In the basic geometrical construction of the Poisson-commuting conserved quantities $I_k$, we have been considering the conformally flat metric $\mathrm{g}_2$ given by (2.51) and (2.52). Now, in the quantum approach to integrability, we choose to use, again, the Stäckel metric, g, associated with $I_1$. One has

$$\mathrm{g} = \sum_{i=1}^n \mathrm{g}_i(x)(dx^i)^2, \qquad \text{with} \qquad \mathrm{g}_i(x) = \frac{1}{x^i}\,\widetilde{\mathrm{g}}_i(x), \tag{3.25}$$

where the Neumann-Uhlenbeck metric $\widetilde{\mathrm{g}}$ is given by (3.21), while the first integrals for $k = 1, \ldots, n$ are

$$I_k = \sum_{i=1}^{n} \mathrm{g}^i(x)\, \sigma^i_{k-1}(x)\xi_i^2, \qquad \mathrm{g}^i(x) = \frac{1}{\mathrm{g}_i(x)}. \tag{3.26}$$

**Lemma 3.10.** *The metric (3.25) has Ricci tensor*

$$R_{ij} = \Big((n-2)x^i + n\sum_{k=1}^{n} x^k - (n-1)\sum_{\alpha=0}^{n} a_\alpha\Big)\mathrm{g}_i\,\delta_{ij} \tag{3.27}$$

*and scalar curvature*

$$R = (n-1)\Big((n+2)\sum_{k=1}^{n} x^k - n\sum_{\alpha=0}^{n} a_\alpha\Big). \tag{3.28}$$

*It is conformally flat for $n = \dim(M) \geq 3$.*

*Proof.* The Ricci tensor can be computed with the help of classical formulæ for a diagonal metric (see for instance [11], p. 119). The only possibly non-vanishing components of the Riemann tensor are $R_{ik,kj}$, for $i \neq j \neq k$, and $R_{ij,ji}$, for $i \neq j$. Using the relations

$$\partial_i(\ln \mathrm{g}_j) = \frac{1}{x^i - x^j} \quad (i \neq j), \qquad\qquad \partial_{ij}(\ln \mathrm{g}_k) = 0 \quad (i \neq j \neq k)$$

one easily gets $R_{ik,kj} = 0$, implying

$$R_{ij} = -\sum_{k=1}^{n} \mathrm{g}^k R_{ik,kj} = 0, \qquad \forall i \neq j.$$

The computation of the remaining components involves a sum which is conveniently computed using the theorem of residues, giving

$$R_{ik,ik} = (x^i + x^k + \sum_{s=1}^{n} x^s - \sum_{\alpha=0}^{n} a_\alpha)\, \mathrm{g}_i \mathrm{g}_k$$

from which one deduces easily the diagonal part of the Ricci tensor, given by (3.27), and the scalar curvature (3.28). Some extra computation shows that the conformal Weyl tensor vanishes in dimension $n \geq 4$, and that the Cotton-York tensor vanishes as well for $n = 3$. □

**Remark 3.11.** Although the metric $\mathrm{g}_2$ given by (2.51) is clearly conformally flat, it is by no means trivial that the same is true for the Stäckel metric, g, given by (3.25) on $S^n$.

We are now in position to prove the following proposition.

**Proposition 3.12.** *The conformally equivariant quantization procedure (3.12) does preserve quantum integrability of the dual Moser system.*

*Proof.* Using the definition (3.13) of the scalar term in the formula (3.12) for the conformally equivariant quantization of the $I_k$, we find

$$\begin{aligned} f(I_k) &= [(n-2)c_2 + k(c_6 - c_4)]\sigma_1(x)\,\sigma_k(x) + [kc_5 - (n-2)c_2]\sigma_{k+1}(x) \\ &\quad -kc_4\sigma_1(a)\,\sigma_k(x) - 2k(n-k+1)c_1\,\sigma_{k+1}(a) \end{aligned}$$

where $c_4$ was already defined in (3.24) and shown to vanish in the proof of Proposition 3.9; we also have

$$c_5 = 2(n+2)c_1 - (n-2)c_2, \qquad c_6 = -2c_1 + c_2 + 2(n-1)c_3.$$

Taking into account the relations (3.14) one gets $c_5 = c_6 = 0$, and we are left, for $k = 1, \ldots, n$, with

$$f(I_k) = 2c_1\Big[(n+2)\,[\sigma_k(x)\sigma_1(x) - \sigma_{k+1}(x)] - k(n-k+1)\,\sigma_{k+1}(a)\Big] \tag{3.29}$$

where we posit $\sigma_{n+1}(x) = 0$.

Let us now compute $V_{I_k,I_\ell}$ defined by (3.18) needed to check quantum integrability via the commutator (3.15).

In view of (3.26), one finds

$$V_{I_k,I_\ell} = 2\sum_{i=1}^{n} \mathrm{g}^i \Big(\sigma^i_{k-1}\partial_i f(I_\ell) - \sigma^i_{\ell-1}\partial_i f(I_k)\Big)\xi_i.$$

Now, using the relations [3]

$$\partial_i\sigma_k(x) = \sigma^i_{k-1}(x), \qquad \forall i,k = 1,\ldots,n,$$

and

$$\sigma_k(x) = \sigma^i_k(x) + x^i\sigma^i_{k-1}(x), \qquad \forall k = 1,\ldots,n-1,$$

as well as

$$\sigma_n(x) = x^i\,\sigma^i_{n-1}(x),$$

one gets $\partial_i f(I_k) = 2(n+2)c_1[x^i + \sigma_1(x)]\sigma^i_{k-1}(x)$ which obviously yields $V_{I_k,I_\ell} = 0$, implying, at last

$$[\mathcal{Q}(I_k), \mathcal{Q}(I_\ell)] = 0 \tag{3.30}$$

for all $k, \ell = 1, \ldots, n$. □

**Remark 3.13.** Carter's (minimal) prescription (3.7) also leads to quantum integrability of the system because of the diagonal form (3.27) of the Ricci tensor in the separating coordinates. Now, in contradistinction with the Neumann-Uhlenbeck quantum system, the scalar terms $f(I_k)$ given by (3.29) are no longer constant, yielding quite different quantum observables $\mathcal{Q}(I_k)$ and $\hat{I}_k$. So, the fact that quantum integrability is not only preserved by Carter's quantum prescription, but also by conformally equivariant quantization is a new and noteworthy phenomenon.

## 3.5 The quantum Jacobi-Moser system

The Stäckel metric, associated to $I_1$ is now (see Table 1):

$$\mathrm{g} = \sum_{i=1}^{n} \mathrm{g}_i(x)(dx^i)^2, \qquad \text{with} \qquad \mathrm{g}_i(x) = x^i\, \widetilde{\mathrm{g}}_i(x), \tag{3.31}$$

where the Neumann-Uhlenbeck metric $\widetilde{\mathrm{g}}_i(x)$ is given by (3.21), and the first integrals by

$$I_k = \sum_{i=1}^{n} \mathrm{g}^i(x)\sigma^i_{k-1}(x)\xi_i^2, \qquad \mathrm{g}^i(x) = \frac{1}{\mathrm{g}_i(x)}. \tag{3.32}$$

**Lemma 3.14.** *The Ricci tensor and the scalar curvature of the metric (3.31) are given by*

$$R_{ij} = \frac{\sigma_{n+1}(a)}{\sigma_n^2(x)}\, \sigma^i_{n-2}(x)\, \mathrm{g}_i\, \delta_{ij}, \qquad R = 2\, \frac{\sigma_{n+1}(a)}{\sigma_n^2(x)}\, \sigma_{n-2}(x). \tag{3.33}$$

*Proof.* It is completely similar to the proof of Lemma 3.10. □

This allows us to prove:

**Proposition 3.15.** *Carter's prescription (3.7) preserves quantum integrability of the Jacobi-Moser system, while the conformally equivariant quantization does not.*

*Proof.* Since the Ricci tensor is diagonal, quantum integrability is established for the prescription (3.7). For the conformally equivariant quantization (3.12), we will just give a counter-example. One has

$$f(I_1) = 2(c_2 + nc_3)\, \sigma_{n+1}(a)\, \frac{\sigma_{n-2}(x)}{\sigma_n^2(x)}$$

and

$$f(I_2) = (n-1)c_3\, \frac{\sigma_{n+1}(a)}{\sigma_n^2(x)} \Big[2(n-1)\, \sigma_{n-1}(x) - n\, \sigma_1(x)\, \sigma_{n-2}(x)\Big] - 2n(n-1)\, c_1.$$

A simple computation gives $V^i_{I_1,I_2} = \partial V_{I_1,I_2}/\partial\xi_i = \partial_i f(I_2) - (\sigma_1(x) - x^i)\partial_i f(I_1)$, hence the non-vanishing result

$$V^i_{I_1,I_2} = (-c_3)\frac{\sigma_{n+1}(a)}{x^i\,\sigma_n^2(x)}\Big[-2(\sigma_1^i(x)\sigma_{n-2}(x) + \sigma_1(x)\sigma^i_{n-2}(x)) + (n^2-3n+4)\sigma_{n-1}(x) + n(3n-5)\sigma^i_{n-1}(x)\Big],$$

showing that the system looses its quantum integrability via conformally equivariant quantization. □

**Remark 3.16.** Let us mention that quantum integrability of the Neumann-Uhlenbeck and Jacobi-Moser systems has first been established, in terms of Carter's quantum prescription (3.7), by Toth [28].

# 4 Conclusion and outlook

To sum up the main results of the article, let us mention that we have disclosed a new integrable system on $S^n$, in duality with the well-known Jacobi-Moser system in terms of projective equivalence. As opposed to that of the generic ellipsoid, the "dual" metric is conformally flat. This remarkable fact enables us to have naturally recourse to conformally equivariant quantization. The latter turns out to preserve integrability at the quantum level. It is, to our knowledge, the first instance of conformally driven quantum integrability.

This opens new perspectives related, e.g., to the determination of the conditions under which a classically integrable system, stemming from second-order Killing tensors on a conformally flat configuration manifold, remains quantum-integrable via the conformally equivariant quantization. Also, possible generalizations of the Jacobi-Moser system and its dual counterpart might conceivably be put to light in a similar manner.

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# References

[1] R. Abraham, and J.E. Marsden, *Foundations of Mechanics*, Second Edition, Addison-Wesley Publishing Company, Inc (1987).

[2] M. Bellon, M. Talon, *The quantum Neumann model: refined semiclassical results*, Phys. Lett. A, **356** (2006) 110–114, and references therein.

[3] S. Benenti, *Inertia tensors and Stäckel systems in the Euclidean spaces*, Rend. Sem. Mat. Univ. Pol. Torino **50** (1992) 315–341.

[4] S. Benenti, C. Chanu, and G. Rastelli, *Remarks on the connection between the additive separation of the Hamilton-Jacobi equation and the multiplicative separation of the Schrödinger equation. I. The completeness and Robertson conditions*, J. Math. Phys. **43** (2002) 5183–5222.

[5] S. Benenti, C. Chanu, and G. Rastelli, *Remarks on the connection between the additive separation of the Hamilton-Jacobi equation and the multiplicative separation of the Schrödinger equation. II. First integrals and symmetry operators*, J. Math. Phys. **43** (2002) 5223–5253.

[6] A.V. Bolsinov, V.S. Matveev, *Geometrical interpretation of Benenti systems*, J. Geom. Phys., **44**:4 (2003) 489–506.

[7] B. Carter, *Killing tensor quantum numbers and conserved currents in curved space*, Phys. Rev. D**16** (1977) 3395–3414.

[8] C. Duval, G. Valent, *Quantum Integrability of Quadratic Killing Tensors*, J. Math. Phys. **46** (2005) 053516(22).

[9] C. Duval, V. Ovsienko, *Conformally equivariant quantum Hamiltonians*, Selecta Math. (N.S.) **7**:3 (2001) 291–320.

[10] C. Duval, P. Lecomte, V. Ovsienko, *Conformally equivariant quantization: existence and uniqueness*, Ann. Inst. Fourier. **49**:6 (1999) 1999–2029.

[11] L. P. Eisenhart, *Riemannian geometry*, Princeton Landmarks in Mathematics and Physics, Princeton University Press (1997).

[12] G. Falqui, M. Pedroni, *Separation of Variables for Bi-Hamiltonian Systems*, Math. Phys. Anal. Geom. **6** (2003) 139–179.

[13] A. Ibort, F. Magri, G. Marmo, *Bihamiltonian structures and Stäckel separability*, J. Geom. Phys. **33** (2000) 210–228.

[14] H.P. Künzle, "Galilei and Lorentz structures on spacetime: Comparison of the corresponding geometry and physics," Ann. Inst. H. Poincaré, Phys. Théor **17** (1972), 337–362.

[15] P.B.A. Lecomte and V. Ovsienko, *Projectively invariant symbol calculus*, Lett. Math. Phys. **49**:3 (1999) 173–196.

[16] T. Levi-Civita, *Sulle trasformazioni delle equazioni dinamiche*, Ann. Mat. Ser. 2a **24** (1896) 255–300.

[17] F. Magri, *A simple model of the integrable Hamiltonian equation*, J. Math. Phys. **19**:5 (1978) 1156–1162.

[18] V.S. Matveev, P.J. Topalov, *Quantum integrability of Beltrami-Laplace operator as geodesic equivalence*, Math. Z. **238** (2001) 833–866.

[19] J. Moser, *Various aspects of the integrable Hamiltonian systems*, in Dynamical systems (C.I.M.E. Summer School Bressanone; 1978), Progress in Mathematics, Birkhäuser **8** (1981) 233–289.

[20] V. Ovsienko, and S. Tabachnikov, *Projective Differential Geometry Old And New: From The Schwarzian Derivative To The Cohomology Of Diffeomorphism Group*, Cambridge University Press (2005).

[21] J.-E. Marsden, T. Ratiu, *Introduction to Mechanics and Symmetry*, Springer, 1999.

[22] A.M. Perelomov, *Integrable Systems of Classical Mechanics and Lie Algebras*, Vol I, Birkhäuser (1990), and references therein.

[23] H. P. Robertson, *Bermerkung über separierbare Systeme in der Wellenmechanik*, Math. Annal. **98** (1927) 749–752.

[24] E.K. Sklyanin, *The quantum Toda chain*, in Non-linear equations in classical and quantum field theory. Springer Notes in Physics, vol. 226, (1985).

[25] S. Tabachnikov, *Projectively equivalent metrics, exact transverse line fields and the geodesic flow on the ellipsoid*, Comm. Math. Helv. **74**:1 (1999) 306–321.

[26] S. Tabachnikov, *Ellipsoids, complete integrability and hyperbolic geometry*, Moscow Mathematical Journal, **2**:1, (2002) 185–198.

[27] P.J. Topalov, V.S. Matveev, *Geodesic equivalence and integrability*, MPIM preprint series, no. 74 (1998), `arXiv:math/9911062v1 [math.DG]`.

[28] J.A. Toth, *Various quantum mechanical aspects of quadratic forms*, J. Funct. Anal. **130**:1 (1995) 1–42.

[29] K. Uhlenbeck, *Minimal* 2*-spheres and tori in* $S^k$, Preprint University of Illinois at Chicago Circle (1975).

[30] K. Uhlenbeck, *Equivariant harmonic maps into spheres*, in Proceedings of the 1980 NSF-CBMS Regional Conference on “Harmonic Maps”, R.J. Knill, M. Kalka, and H.C.J. Sealey (Eds), LNM 949 (1982) 146–158.