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Technical change in a neoclassical two-sector model of optimal growth

Mehdi Senouci*

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Abstract

This paper investigates into the asymmetrical consequences of consumption-specific and investment-specific technical change in a two-sector growth framework. Hicks-neutral technological shocks of the former type increase steady-state consumption one-for-one, and Hicks-neutral technological shocks of the latter type increase steady-state consumption by a factor equal to the ratio of the macroeconomic capital share to labor share. As this ratio does not necessarily tend to zero as investment-specific total factor productivity goes to infinity, we conclude that consumption can grow asymptotically even in the absence of gains of productivity in the consumption sector, through capital accumulation within the consumption sector alone – this accumulation itself being driven by technical progress within the investment sector. We illustrate this result when the production functions are of Cobb-Douglas type in both sectors, and present in this case a steady growth theorem.

Keywords: Productivity, optimal growth, golden rule, two-sector models.
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Introduction

History of mankind for the last thousands of years has been one of continuous evolution, ponctuated by a few turning points. As the last Ice Age came to an end, some 15,000 years ago, the Neolithic Revolution witnessed the birth of agriculture, of domestication of animals as well as of metal tools, writing, etc. Not so long before that, around 40 or 50,000 years ago, had already occured what Jared Diamond (1997) calls the "Great Leap Forward," the birth of the most primitive known tools and jewelry. And not so long after – and even less – eighteenth century England made the Industrial Revolution, realized probably the first breakthrough of the Malthusian barrier and opened the modern era of continuous growth of income per capita.

No doubt, all these economic transitions have first had some *investment* at their center: the passage from hunting and gathering to domestication of species and agriculture necessarily involved some foregone consumption in the first times, as some part of resources must have been employed to clear the land, raise the cattles, etc., instead of keeping on to produce immediate consumption. Escape from the Malthusian trap does not make an exception: as is well-known, growth take-offs traditionally (if not always) come together with high investment rates[1].

This short theoretical note compares the consequences of consumption-specific and investment-specific technical change in a neoclassical optimal growth framework. To take this perspective obviously requires to distinguish explicitly between consumption and capital goods and production technologies. Two-sector neoclassical growth models were first developed in the 1960's[2], bringing deep insights as an extension of their one-sector parents, thanks to the additional degree of freedom of these models have: the relative price of capital to the consumption good. Still, this litterature never made part of the standard tools of growth theory, probably in part because of the inconvenient capital-intensity assumption that they often require[3].

Fortunately, for our purpose we never need to make such assumptions and the algebra remains very simple all along the paper[4]. We thus also hope to mitigate the frequent statement that two-sector models are of difficultly tractable, or unsuitable for interpretation.

This work rests on the results of a previous paper[5] which explored the exact Ramsey solution in a two-good, two-sector framework without technical progress. We showed that an economy governed by a social planner eventually converges towards a quasi-golden rule steady state, where the wage-rental ratio only depends on the production function of the *capital* good, and is independent of the consumption goods technology. On the basis of this property, we show that investment-specific and consumption-specific technical change induce quite distinct effects on the economy.

[1] See Allen (2005) on technical progress, investment, growth and the distribution of income during the British Industrial Revolution.

[2] The seminal contributions include Uzawa (1961, 1963, 1964), Meade (1961), Srinivasan (1964) and Shell (1967).

[3] See the discussion of this assumption by Solow (1961). Two-sector models, in general, behave better when it is the consumption-sector that is relatively more capital-intensive. In optimal models like Uzawa (1964), Srinivasan (1964), Shell (1967) and Senouci (2011), the optimal path always exists but it takes different forms according to which industry is the most capital-intensive. The absence of capital-intensity reversals for some wage-rental ratios is an important and necessary hypothesis.

[4] Some models found in the literature take the clever assumption that their exists one good, which can be either consumed or transformed into a certain amount of capital via a constant returns to scale technology; the trick allowing to avoid to manipulate two different production functions. Indeed, those models constitute the polar case of the two-sector framework which we use here, in the particular case where both industries share the same optimal capital per worker ratio for each wage-rental ratio. For our purpose, at the cost of little additional notation the concepts are made clearer and more rigorous by the original two-sector formulation we take here. Of course, the properties enounced here are valid for this class of models, as a limit case.

[5] Senouci (2011).

The first part of the paper presents a comparative statics analysis on the golden rule steady state when a Hicks-neutral technological shock hits any of the two industries. Consumption-specific shocks we dub 'neutral' as they do *not* alter the golden rule capital-labor nor wage-rental ratios, nor the distribution of inputs between sectors. Such shocks merely increase steady-state consumption one-for-one. In particular, they do not incite to capital accumulation.

Investment-specific shocks have more complex consequences. Positive shocks *increase* golden-rule sectoral and overall capital-labor ratios, as well as the wage-rental ratio. We show that, when the social planner has a rate of preference for the present equal to 1, the elasticity of steady-state consumption with respect to the total factor productivity (TFP) of the investment sector is equal to the ratio of the capital-labor ratio to the wage-rental ratio, or equivalently to the ration of the macroeconomic capital bill to the macroeconomic wage bill, or to $\frac{s}{1-s}$ where s denotes the steady-state saving rate. Because these ratios do *not* necessarily tend to zero when A tends to infinity – indeed, they remain constant if the technologies are of Cobb-Douglas type –, we suggest the following paradox: consumption can increase asymptotically only with investment-specific technical progress; then, in the absence of consumption-specific technical change, consumption increases merely as a result of capital accumulation within the consumption sector.

In the second part of the paper, we prove rigorously this property in the form of a steady growth theorem, in the case both production functions are of Cobb-Douglas type[6].

The rest of the paper is organized as follows. Section 1 presents the two-sector optimal growth problem and the golden rules in the absence of technical progress. Section 2 analyzes the consequences of consumption-specific technical shocks. Section 3 does the same for investment-specific technical shocks and illustrates the paradox by looking at the case where the two production functions are of Cobb-Douglas type. Section 4 addresses the case where productivity grows at a steady rate in both sectors and proves a steady growth theorem (always under the Cobb-Douglas assumption). Section 5 concludes.

1 The two-sector golden rules

1.1 Technology and optimization problem

We suppose in this section that there is no technical change. Time is discrete. Population is constant at L and the economy is initially endowed with a stock of capital $K_0 = Lk_0$. Capital depreciates by a factor δ from one period to another.

At each period, a benevolent social planner allocates available capital (K) and labor (L) resources in the consumption sector and the investment sector: $\left(K^C, L^C\right)$ goes to the former and $\left(K^I, L^I\right)$ to the latter. Inputs are costlessly transferable from one sector to another at each date. Consumption technology is represented by the production function $C_t = F\left(K_t^C, L_t^C\right)$ and investment technology by $I_t = G(K_t^I, L_t^I)$. Both production functions satisfy usual neoclassical properties[7].

[6] The issue of technical progress in two-sector models of growth has been analyzed previously by Meade (1961) and Shell (1967). Indeed, Meade reaches very similar equations in his (famous) appendix, not in an optimal framework, but under the comportamental assumption that wages are consumed and profits are saved. The present paper can be thought as a way to reach Meade's results via the golden rule. Shell's analysis, however useful, is merely qualitative and does not characterize the asymptotical optimal path.

[7] Namely, we suppose that F and G are $\mathcal{C}^2$, show constant returns to scale, decreasing marginal productivity with respect to both inputs and verify Inada conditions.

The planner's intertemporal preferences are summarized by a well-behaved instantaneous utility function $v(C) = Lu(C/L)$ and by the rate of preferences for the present $\beta = 1/(1+\theta)$, where $\theta > 0$ denotes the subjective rate of impatience. The problem he solves is:

$$\left| \begin{array}{lrl} & \max_{\left(K_t^C, K_t^I, L_t^C, L_t^I\right)_{t\geq 0}} & \sum_{t=0}^{\infty} \frac{1}{(1+\theta)^t} v\left(C_t\right) \\ \text{s.t.} & C_t & \leq F\left(K_t^C, L_t^C\right) \\ & I_t & \leq G\left(K_t^I, L_t^I\right) \\ & K_t^C + K_t^I & \leq K_t \\ & L_t^C + L_t^I & \leq L \\ & K_{t+1} & = (1-\delta)K_t + I_t \end{array} \right. \tag{1}$$

Let's denote by $(p_t^C, p_t^I, r_t, w_t, q_t)_{t\geq 0}$ the vector of lagrangian multipliers[8].

The exposition is made easier with per-capita variables. Let $k = K/L$, $c = C/L$, $i = I/L$, $k^C = K^C/L^C$, $k^I = K^I/L^I$, $f\left(k^C\right) = F\left(k^C, 1\right)$, $g\left(k^I\right) = G\left(k^I, 1\right)$ and let $\omega = w/r$ denote the wage-rental ratio[9]. Static efficiency requires that:

$$\omega = \frac{f\left(k^C\right)}{f'\left(k^C\right)} - k^C = \frac{g\left(k^I\right)}{g'\left(k^I\right)} - k^I \tag{3}$$

Because functions $\frac{f(x)}{f'(x)} - x$ and $\frac{g(x)}{g'(x)} - x$ are both strictly increasing, equation (3) determines each sector's capital per worker ratio as two strictly increasing functions of the wage-rental ratio, $k^C(\omega)$ and $k^I(\omega)$.

Let's define $l^C = L^C/L$ and $l^I = L^I/L$ the fractions of the labor force respectively employed in the consumption and in the investment sector. Since resources are fully employed, it holds that:

$$\left\{ \begin{array}{rl} l^C k^C + l^I k^I & = k \\ l^C + l^I & = 1 \end{array} \right. \tag{4}$$

Thus, if $k^C \neq k^I$, we have that:

$$\begin{array}{lll} l^C & = \frac{k-k^I}{k^C-k^I} & = l^C(k,\omega) \\ l^I & = \frac{k^C-k}{k^C-k^I} & = l^I(k,\omega). \end{array} \tag{5}$$

[8]This optimization problem we treated extensively in Senouci (2011). When both goods are produced, first-order equations are:

$$\left\{ \begin{array}{rcl} u'\left(c_t\right) & = & p_t^C \\ q_t & = & p_t^I \\ p_t^C \frac{\partial F}{\partial K_t^C} & = & r_t \\ p_t^I \frac{\partial G}{\partial K_t^I} & = & r_t \\ p_t^C \frac{\partial F}{\partial L_t^C} & = & w_t \\ p_t^I \frac{\partial G}{\partial L_t^I} & = & w_t \\ q_t & = & \frac{1}{1+\theta}(r_{t+1} + (1-\delta)q_{t+1}) \end{array} \right. \tag{2}$$

[9]As both inputs are assumed to be perfectly mobile, the (shadow) rate of profit r and wage rate w are equalized across both sectors.

1.2 The quasi-golden rule

Because of decreasing marginal productivity of capital, consumption cannot grow asymptotically in the absence of technical progress[10]. We have shown in Senouci (2011) that the economy governed by the social planner converged to a "quasi-golden rule" steady state independent of the initial conditions and of the instantaneous utility function, and characterized by[11]:

$$\left\{\begin{array}{rcl} g'\left(k^I\right) & = & \delta+\theta \\ \frac{f\left(k^C\right)}{f'\left(k^C\right)}-k^C & = & \frac{g\left(k^I\right)}{g'\left(k^I\right)}-k^I=\omega \\ l^I g\left(k^I\right) & = & \delta k \quad \Leftrightarrow \quad \frac{k^C-k}{k^C-k^I} g\left(k^I\right)=\delta k \end{array}\right. \tag{6}$$

We call ω_θ^* the quasi-golden rule wage-rental ratio, and k_θ^* the quasi-golden rule capital-labor ratio, for each $\theta>0$.

The first equation in (6) has the following interpretation. Suppose that the optimal steady state is reached, so that the social planner plans to set k^C, k^I, l^C, l^I (and, so, k) constant at their quasi-golden rule value for the infinite future. Suppose further that the social planner unexpectedly receives a free unit of capital at date $t=\tau$. If he installs this marginal unit in the consumption sector from $t=\tau$ on, the resulting intertemporal benefit is of:

$$\sum_{t=\tau}^{\infty}\left(\frac{1}{1+\theta}\right)^{t-\tau}(1-\delta)^{t-\tau} f'\left(k^C\left(\omega_\theta^*\right)\right) u'\left(c_\theta^*\right)=\frac{1+\theta}{\delta+\theta} f'\left(k^C\left(\omega_\theta^*\right)\right) u'\left(c_\theta^*\right) \tag{7}$$

Another strategy is to put this additional unit of capital in the investment sector at $t=\tau$ and, then, to install the windfall quantity of capital $\left(1-\delta+g'\left(k^I\right)\right)$ in the consumption sector for $t \geq \tau+1$. The increase in the objective function of the social planner seen from $t=\tau$ is:

$$\begin{array}{r} \left(1-\delta+g'\left(k^I\left(\omega_\theta^*\right)\right)\right) \sum_{t=\tau+1}^{\infty}\left(\frac{1}{1+\theta}\right)^{t-\tau}(1-\delta)^{t-\tau-1} f'\left(k^C\left(\omega_\theta^*\right)\right) u'\left(c_\theta^*\right) \\ =\frac{1-\delta+g'\left(k^I\left(\omega_\theta^*\right)\right)}{\delta+\theta} f'\left(k^C\left(\omega_\theta^*\right)\right) u'\left(c_\theta^*\right) \end{array} \tag{8}$$

Because of intertemporal optimality, these two strategies must yield the same benefits. Equality of (7) and (8) then reduces to the first equation in (6). The quasi-golden rule is thus centered on the technology of the investment sector. Notably, ω_θ^* is independent of the production function for consumption f. The second equation in (6) reflects equalization of technical marginal rates of substitution across the two sectors, and the third equation in (6) reflects the condition that net investment must be zero at steady state.

1.3 The pure golden rule

When the rate of impatience θ tends to zero, steady-state approaches the pure golden rule defined by:

$$\left\{\begin{array}{rcl} g'\left(k^I\right) & = & \delta \\ \frac{f\left(k^C\right)}{f'\left(k^C\right)}-k^C & = & \frac{g\left(k^I\right)}{g'\left(k^I\right)}-k^I=\omega \\ \frac{k^C-k}{k^C-k^I} g\left(k^I\right) & = & \delta k \end{array}\right. \tag{9}$$

[10] Notice that because production function G satisfies the Inada conditions, there exists a maximum sustainable capital stock defined by $G\left(\overline{K}, L\right)=\delta\overline{K}$.

[11] See Phelps (1961, 1965) and Cass (1965) for the original one-sector formulation of the golden rules. Srinivasan (1964), Uzawa (1965), Shell (1967), Haque (1970), Galor (1992) and Cremers (2006) among others present the golden rule in two-sector frameworks.

which also corresponds to the maximization of steady-state consumption problem. Let's call ω° and k° the pure golden rule values of ω and k. Clearly, for all $\theta > 0$, $\omega^\circ > \omega^*_\theta$ and $k^\circ > k^*_\theta$.

Some remarkable identities, already presented in Cremers (2006), hold at the golden rule steady state:

Theorem 1.1. *At the golden rule allocation:*

$$k = \frac{l^I g\left(k^I\right)}{g'\left(k^I\right)} \tag{10}$$

$$l^C = \frac{f\left(k^C\right) - k^C f'\left(k^C\right)}{f\left(k^C\right)} \tag{11}$$

$$l^I = \frac{k^C f'\left(k^C\right)}{f\left(k^C\right)} \tag{12}$$

$$\frac{l^C k^C}{k} = \frac{g\left(k^I\right) - k^I g'\left(k^I\right)}{g\left(k^I\right)} \tag{13}$$

$$\frac{l^I k^I}{k} = \frac{k^I g'\left(k^I\right)}{g\left(k^I\right)} \tag{14}$$

where all variables are at their golden rule level – $k = k^\circ$, $\omega = \omega^\circ$, $k^I = k^I\left(\omega^\circ\right)$, $k^C = k^C\left(\omega^\circ\right)$, $l^I = l^I\left(k^\circ, \omega^\circ\right)$ *and* $l^C = l^C\left(k^\circ, \omega^\circ\right)$.

Proof. Proof is straightforward by combining equations (3), (4) and (9). □

That is, the fraction of total labor supply employed in the consumption sector is equal to the labor share in the consumption sector; the fraction of total labor supply employed in the investment sector is equal to the capital share in the consumption sector; the fraction of total capital supply employed in the consumption sector is equal to the labor share in the investment sector; and the fraction of total capital supply employed in the investment sector is equal to the capital share in the investment sector.

Further, it is immediate from first-order conditions (see equations (2) in footnote 8) and from equations (11) to (14) that consumption is equal to the macroeconomic labor share of income and that investment is equal to the macroeconomic capital share of income at the golden rule steady state[12]:

$$c = l^C f\left(k^C\right) = \frac{w}{p^C} \tag{15}$$

$$i = l^I g\left(k^I\right) = \frac{rk}{p^I} \tag{16}$$

And so:

$$(y \equiv)\ p^C c + p^I i = w + rk \tag{17}$$

[12]This proposition notably makes the model that suppose that wages are consumed and profits are invested not so absurd, even if it is unpleasant to take the shortcut to make arbitriraly this kind of assumption. It is not really clear why this double duality (wages–consumption, capital–investment) prevails, but it is a tenacious conclusion of the golden rule. Srinivasan (1964) shows that this property is no longer true when the social planner has a rate of preference for the present strictly less than unity; in that case, savings are *less* than non-wage income.

In extensive form, we can write (17) as:

$$(Y \equiv)\; p^C C + p^I I = wL + rK,$$

which reflects the fact that labor plus capital income equals product at the macroeconomic level.

Consequently, the golden rule property that the saving rate is equal to the macroeconomic relative share of capital[13] also holds:

$$\begin{aligned} s &\equiv \tfrac{p^I i}{p^C c + p^I i} \\ &= \tfrac{rk}{w + rk} \end{aligned} \tag{18}$$

$$s = \frac{k/\omega}{1 + k/\omega}. \tag{19}$$

2 The 'neutrality' of technological shocks in the consumption goods sector

Imagine that the economy is initially on its steady state characterized by (6) if $\theta \neq 0$ or by (9) if $\theta = 0$. What happens when the economy experiences a permanent technological shock in the consumption-sector? If this shock is Hicks-neutral – i.e. if it turns the technology $F\left(K^C, L^C\right)$ into $BF\left(K^C, L^C\right)$ – then basically nothing. In (6) and in (9), the quasi-golden rule and pure golden rule steady-state values of ω, k^C, k^I, l^C, l^I, and k only depend on the production function of the *investment* good, and not at all on that of the consumption good. Therefore, steady-state consumption increases (if the shock is positive) but the distribution of inputs, as well as the capital-balor ratio, remain unaltered. In particular, the economy does not react to consumption-specific technical progress by capital accumulation[14].

Therefore, we have the following neutrality theorem:

Theorem 2.1. *Let B denote total factor productivity in the consumption sector and c denote quasi-golden rule (or pure golden rule) steady-state consumption per capita level. Then:*

$$\frac{B}{c}\frac{\partial c}{\partial B} = 1. \tag{20}$$

3 The paradox: technological shocks in the investment goods sector

What happens to steady-state consumption after a (permanent) positive technological shock on the *investment* technology? Such a shock makes maintenance of the existing stock of capital less resource-demanding, and thus expands steady-state production possibilities and steady-state consumption.

So, final consumption increases only through capital accumulation in the consumption sector, and not because of consumption-specific technological advance. The quantitative response depends

[13] See Phelps (1961) and (1965) and Robinson (1962) for the original one-sector formulation, and Srinivasan (1964) on the two-sector version.

[14] Because steady-state ω is not altered by a shock on B, we also conclude that w and r increase proportionnally after the shock.

on the planner's subjective discount rate: the more patient the planner is, the more we can expect him to take advantage of investment-specific technical progress.

We escape from this difficulty by deriving comparative statics property only at the pure golden rule steady state. Thus, we show that consumption can *technically* grow asymptotically only with investment-specific – and no consumption-specific – technical progress. We can conjecture that, in the case where $\theta \neq 0$, optimal asymptotical growth rates are independent of the rate of preference for the present.

We first prove a general result, and then apply it to the case where the two production functions are of Cobb-Douglas type. In this case, we show that the elasticity of consumption with respect to investment-specific TFP is constant and strictly positive.

3.1 A general property

When the TFP of the investment sector is $A > 0$, the golden rule steady-state values of ω, k^I, k^C, and k are determined through the following system[15]:

$$\left\{\begin{array}{rcl} Ag'\left(k^I\right) & = & \delta \\ \frac{f\left(k^C\right)}{f'\left(k^C\right)} - k^C & = & \frac{g\left(k^I\right)}{g'\left(k^I\right)} - k^I = \omega \\ \frac{k^C - k}{k^C - k^I} Ag\left(k^I\right) & = & \delta k \end{array}\right. \tag{21}$$

Then, the effect of Hicks-neutral, investment-specific technical shock on golden rule steady-state level consumption is summarized by the following theorem:

Theorem 3.1. *Let A denote total factor productivity in the investment sector, so that investment technology is $i = Al^I g\left(k^I\right)$, and let c denote the pure golden rule steady-state consumption per capita level. Then:*

$$\frac{A}{c}\frac{\partial c}{\partial A} = \frac{k}{\omega} \tag{22}$$

where k is the macroeconomic golden rule capital-labor ratio and where ω is the golden rule wage-rental ratio.

Proof. In virtue of (11), we have that $c = l^C f\left(k^C\right) = f\left(k^C\right) - k^C f'\left(k^C\right)$ at golden rule steady state. Therefore:

$$\frac{\partial c}{\partial A} = -k^C \frac{\partial k^C}{\partial \omega}\frac{\partial \omega}{\partial A} f''\left(k^C\right) \tag{23}$$

By differenciating (3), efficiency implies that:

$$\frac{\partial k^C}{\partial \omega} = -\left(\frac{f'^2}{ff''}\right)\left(k^C\right) \tag{24}$$

And:

$$\frac{\partial k^I}{\partial \omega} = -\left(\frac{g'^2}{gg''}\right)\left(k^I\right) \tag{25}$$

Differenciation of the first equation in (21) with respect to A yields:

[15] Notice that $Ag/Ag' = g/g'$, which reflects the invariance of the distribution of income in one particular industry to a Hicks-neutral technological shock. The wage-rental ratio, here, increases because the overall capital-labor ratio increases, and the two sectoral capital intensity ratios increase as well.

$$\frac{\partial \omega}{\partial A}\frac{\partial k^I}{\partial \omega}\left(-g''\left(k^I\right)\right) = \frac{\delta}{A^2} \tag{26}$$

Combining (25) and again the first equation in (21), we can reduce (26) to:

$$\frac{\partial \omega}{\partial A} = \frac{1}{A}\frac{g\left(k^I\right)}{g'\left(k^I\right)} \tag{27}$$

In view of (24) and (27) and of the golden rule identities (10) and (12), we can successively re-write (23) as:

$$\begin{aligned}\frac{\partial c}{\partial A} &= \frac{1}{A}\left(k^C\frac{f'^2\left(k^C\right)}{f\left(k^C\right)}\right)\frac{g\left(k^I\right)}{g'\left(k^I\right)} \\ &= \frac{1}{A}\frac{l^I g\left(k^I\right)}{g'\left(k^I\right)}f'\left(k^C\right) \\ &= \frac{1}{A}kf'\left(k^C\right)\end{aligned}$$

Finally, remark that, in virtue of (3) and (11):

$$\frac{f'\left(k^C\right)}{l^C f\left(k^C\right)} = \frac{1}{\omega},$$

which gives the final result:

$$\frac{A}{c}\frac{\partial c}{\partial A} = \frac{k}{\omega}.$$

□

From equations (15) and (16), the ratio $\frac{k}{\omega}$ is also equal to the ratio between investment and consumption, when expressed in the same unit:

$$\begin{aligned}\frac{k}{\omega} &= \frac{rK}{wL} = \frac{p^I i}{p^C c} = \frac{p^I I}{p^C C} \\ &= \frac{s}{1-s}.\end{aligned}$$

Intuitively, maybe one would tend to think that, because of decreasing marginal returns to capital in the consumption sector, consumption cannot increase asymptotically solely as result of technical progress in the investment sector, and so the ratio $\frac{k}{\omega}$ should tend to zero as A tends to infinity. Under this scenario, investment-specific technical progress barely makes the existing stock of capital less costly to maintain, freeing resources for the consumption industry, until the consumption sector asymptotically employs the whole of capital and labor inputs. But this intuition is wrong. In principle, k/ω, can either increase, decrease, or remain constant when A tends to infinity. Indeed, when production functions are of Cobb-Douglas type, this ratio does not depend on A, and so elasticity of consumption with regards to investment-specific TFP is constant.

3.2 Application: the Cobb-Douglas case

We suppose that the production technologies are now:

$$c = Bl^C\left(k^C\right)^b \tag{28}$$

$$i = Al^I\left(k^I\right)^a \tag{29}$$

The effect of investment-specific technical progress on golden rule steady-state consumption is now characterized through a very simple equation.

Proposition 3.1. *At the golden rule steady state, with Cobb-Douglas production functions, one has that:*

$$\left(\frac{k}{\omega}=\right)\frac{A}{c}\frac{\partial c}{\partial A}=\frac{b}{1-a}. \tag{30}$$

Proof. Efficiency equation (3) is now an equation of proportionnality:

$$\omega=\frac{1-a}{a}k^I \tag{31}$$

From equation (12), we have that $l^I=b$ at the golden rule, and so equation (10) with Cobb-Douglas production functions yields:

$$k=l^I\frac{\left(k^I\right)^a}{a\left(k^I\right)^{a-1}}=\frac{b}{a}k^I=\frac{b}{1-a}\omega.$$

□

Unsurprisingly, this ratio is highest – and maybe greater than one if the macroeconomic capital bill is greater than the wage bill – when both production technologies F and G are very capital-intensive.

4 The case of steady technical progress with Cobb-Douglas technology

Previous results were based on comparative statics between different golden rule steady states, themselves defined in the *absence* of technical progress, like in (6) and (9). Still, we are not so far able to conclude on the asymptotic growth rate of consumption when *steady* technical progress is expected in any or both of the two industries.

It is beyond the scope of this paper to give a complete characterization of the optimal path of allocation of resources for any initial conditions. Instead, we show a much more pleasant result that supports previous conclusions: if technologies are of Cobb-Douglas type in equations (28) and (29), then there exists a "golden rule of thumb" for the allocation of resources that, for *some* initial capital-labor ratio k_0, makes consumption grow at the gross rate of:

$$1+z_c=(1+z_B)(1+z_A)^{\frac{b}{1-a}}, \tag{32}$$

where $1+z_B=\frac{B_{t+1}}{B_t}$ and $1+z_A=\frac{A_{t+1}}{A_t}$. We thus prove the existence of a 'pseudo-balanced', steady growth path where investment, capital and consumption grow at constant rates.

The golden rule of thumb, unsurprisingly, we characterize through the golden rule identities from (11) to (14). Suppose that the social planner allocates resources according to the following rule, at all date[16]:

$$\begin{cases} l^C & =1-b \\ l^I & =b \\ \frac{l^Ck^C}{k} & =1-a \\ \frac{l^Ik^I}{k} & =a \end{cases} \tag{33}$$

which corresponds to sectoral capital intensities:

[16]Notice that this allocation is always feasible, as it obviously verifies (4).

$$\left\{\begin{array}{ll} k^C & = \frac{1-a}{1-b}k \\ k^I & = \frac{a}{b}k \end{array}\right. \tag{34}$$

The accumulation equation, when the golden rule of thumb is applied, becomes:

$$k_{t+1} = (1-\delta)k_t + bA_0(1+z_A)^t \left(\frac{a}{b}k_t\right)^a \tag{35}$$

Let's first prove the following lemma:

Lemma 4.1. *If the initial capital-labor ratio satisfies the following adjusted golden rule:*

$$A_0 g'\left(\frac{a}{b}k_0\right) = \delta + (1+z_A)^{\frac{1}{1-a}} - 1 \tag{36}$$

and if the allocation of resources follows the golden rule of thumb (33), a similar adjusted golden rule is verified at all dates:

$$\forall t, \quad A_t g'\left(\frac{a}{b}k_t\right) = \delta + (1+z_A)^{\frac{1}{1-a}} - 1 \tag{37}$$

Proof. If (37) is satisfied for some $t \geq 0$:

$$A_0(1+z_A)^t a\left(\frac{a}{b}k_t\right)^{a-1} = \delta + (1+z_A)^{\frac{1}{1-a}} - 1.$$

Then, from (35):

$$\begin{aligned} A_{t+1}g'\left(\tfrac{a}{b}k_{t+1}\right) &= A_0(1+z_A)^{t+1}a\left(\tfrac{a}{b}\right)^{a-1}\left\{(1-\delta)k_t + bA_0(1+z_A)^t\left(\tfrac{a}{b}k_t\right)^a\right\}^{a-1} \\ &= A_0(1+z_A)^{t+1}a\left(\tfrac{a}{b}\right)^{a-1}\left\{(1-\delta)k_t + A_0(1+z_A)^t a\left(\tfrac{a}{b}k_t\right)^{a-1}k_t\right\}^{a-1} \\ &= A_0(1+z_A)^{t+1}a\left(\tfrac{a}{b}\right)^{a-1}\left\{\left(1-\delta+\delta+(1+z_A)^{\frac{1}{1-a}}-1\right)k_t\right\}^{a-1} \\ &= A_0(1+z_A)^t a\left(\tfrac{a}{b}\right)^{a-1}(k_t)^{a-1} \\ &= \delta + (1+z_A)^{\frac{1}{1-a}} - 1. \end{aligned}$$

Therefore, if the property holds for $t = 0$, then it holds for all $t \geq 0$.

□

We now prove the following existence theorem:

Theorem 4.1. *If the production functions of the two sectors are of Cobb-Douglas type like in (28) and (29) and if technical progress in each sector is Hicks-neutral and increases at steady rate, with* $\frac{B_{t+1}}{B_t} = 1 + z_B$ *and* $\frac{A_{t+1}}{A_t} = 1 + z_A$ *for all* $t \geq 0$*, then there exists a growth path where investment and the capital stock grow at the same constant gross rate of* $(1+z_A)^{\frac{1}{1-a}}$*, and where consumption grows at the constant gross rate of* $(1+z_B)(1+z_A)^{\frac{b}{1-a}}$*.*

Proof. If initial capital-labor ratio is at the level given by equation (36), and if the golden rule of thumb of allocation of resources is followed, (37) holds and so equation (35) can be reduced to:

$$\begin{aligned} k_{t+1} &= (1-\delta)k_t + aA_0(1+z_A)^t\left(\tfrac{a}{b}k_t\right)^{a-1}k_t \\ &= \left(1-\delta+\delta+(1+z_A)^{\frac{1}{1-a}}-1\right)k_t \\ &= (1+z_A)^{\frac{1}{1-a}}k_t. \end{aligned}$$

Consequently, for all t:

$$k_t = k_0\left((1+z_A)^{\frac{1}{1-a}}\right)^t. \tag{38}$$

Investment per capita, in this case, reduces to:

$$\begin{aligned} i_t &= A_t l_t^I \left(\tfrac{a}{b} k_t\right)^a \\ &= A_0 \left(1+z_A\right)^t \left(\tfrac{a}{b} k_0 \left(\left(1+z_A\right)^{\frac{1}{1-a}}\right)^t\right)^a \\ &= i_0 \left(\left(1+z_A\right)^{\frac{1}{1-a}}\right)^t . \end{aligned}$$

From (28), (33), (34) and (38), we can finally write that:

$$\begin{aligned} c_t &= (1-b) B_0 (1+z_B)^t \left\{ \tfrac{1-a}{1-b} k_0 \left(\left(\left(1+z_A\right)^{\frac{1}{1-a}}\right)^t\right)\right\}^b \\ &= \left\{(1-b) B_0 \left(\tfrac{1-a}{1-b} k_0\right)^b\right\} \left\{(1+z_B)(1+z_A)^{\frac{b}{1-a}}\right\}^t \\ &= c_0 (1+z_c)^t . \end{aligned}$$

□

Along this 'pseudo-balanced' path, with only investment-specific technical progress ($z_A > 0$ and $z_B = 0$), the ratio of real consumption to real investment and the ratio of real consumption to real capital-stock both tend to zero when as time goes on. With both types of technical progress, consumption and investment do not generally grow at the same rate. Indeed, the only balanced path eventually arises when:

$$\left(1+z_A\right)^{1-b} = \left(1+z_B\right)^{1-a} . \tag{39}$$

Perhaps surprisingly, if we let p denote the relative price of the investment good in terms of the consumption good, the quantities pi and pk grow at the same rate than c. Equality of technical marginal rates of substitution across sectors implies that the relative price can be written as[17]:

$$p = \frac{B f'\left(k^C\right)}{A g'\left(k^I\right)} . \tag{40}$$

Along the growth path of theorem 4.1, the gross rate of growth of p is constant and equal to:

$$1 + z_p = \left(1+z_B\right)\left(1+z_A\right)^{-\frac{1-b}{1-a}} , \tag{41}$$

which can be more or less than unity, and is exactly one if and only if technical progress in both sectors satisfy condition (39). One important conclusion is that the relative price of the investment good may rise without technical progress in the investment industry being steadier, and conversely.

5 Conclusion

This article put in regards the consequences of consumption- and investment-specific technical change in a neoclassical, optimal growth framework.

What are the real-world insights that one could gain from this results with regards to modern growth phenomenon? The results we have presented yield some predictions about the links between technology and input and output prices and quantities, which could lead to some interesting empirical view on current and past growth regimes. In the perspective given in the paper, one can speculate that China has grown so fastly for several decades because productivity growth was faster in the investment industry than in the consumption industry – and it seems that it has, at

[17] See (2), footnote 8.

the global level[18] – while concomitantly the Chinese economy has continuously showed a high ratio of domestic investment to domestic consumption, thus accentuating the effect of investment-biased technical change on product and income relatively to what occured in other countries.

The first-best results presented may also serve as a benchmark for further theoretical research. In particular, when it comes to the deep nature of investment in the growth process[19], it would constitute an interesting extension to analyze the dynamical consequences of investment-specific technical progress when some friction distorts the investment decisions.

[18]Greenwood, Hercowitz and Krusell (1997) and Gort, Greenwood and Rupert (1999) show that the share of GDP growth that is due to investment-specific technical change far outweights the share of investment in GDP in post-war United States. Other interesting empirical studies include Collins and Williamson (2001), who document that the relative price of investment goods (and especially the one of *equipment* goods) shows a downward secular tendency in the nineteenth and the twentieth centuries, and Jones (1994) who shows that the price of machinery is strongly negatively correlated with growth performances in cross-country regressions.

[19]See the discussion in the present introduction.

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