SMOOTHING EFFECT OF WEAK SOLUTIONS FOR THE SPATIALLY HOMOGENEOUS BOLTZMANN EQUATION WITHOUT ANGULAR CUTOFF

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Abstract. In this paper, we consider the spatially homogeneous Boltzmann equation without angular cutoff. We prove that every L^1 weak solution to the Cauchy problem with finite moments of all order acquires the C^∞ regularity in the velocity variable for the positive time.

1. Introduction

Consider the Cauchy problem for the spatially homogeneous Boltzmann equation,

$$\left\{ \begin{array}{l} f_t(t,v) = Q(f,f)(t,v), \quad t \in \mathbb{R}^+,\, v \in \mathbb{R}^3, \\ f(0,v) = f_0(v), \end{array} \right. \tag{1.1}$$

where $f = f(t,v)$ is the density distribution function of particles with velocity $v \in \mathbb{R}^3$ at time t. The right hand side of (1.1) is given by the Boltzmann bilinear collision operator

$$Q(g,f) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B\left(v - v_*, \sigma\right)\left\{g(v'_*)f(v') - g(v_*)f(v)\right\} d\sigma dv_*\,,$$

which is well-defined for suitable functions f and g specified later. Notice that the collision operator $Q(\cdot\,,\,\cdot)$ acts only on the velocity variable $v \in \mathbb{R}^3$. In the following discussion, we will use the $\sigma-$representation, that is, for $\sigma \in \mathbb{S}^2$,

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma,\;\; v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma,$$

which give the relations between the post and pre collisional velocities. For mono-atomic gas, the non-negative cross section $B(z,\sigma)$ depends only on $|z|$ and the scalar product $\frac{z}{|z|}\cdot\sigma$. As in [5, 6, 7], we assume that it takes the form

$$B(v-v_*,\cos\theta) = \Phi(|v-v_*|)b(\cos\theta), \quad \cos\theta = \frac{v-v_*}{|v-v_*|}\cdot\sigma\,,\;\; 0 \le \theta \le \frac{\pi}{2}, \tag{1.2}$$

in which it contains a kinetic factor given by

$$\Phi(|v-v_*|) = \Phi_\gamma(|v-v_*|) = |v-v_*|^\gamma, \tag{1.3}$$

with $\gamma > -3$ and a factor related to the collision angle with singularity,

$$b(\cos\theta)\theta^{2+2s} \to K, \quad \text{when} \;\; \theta \to 0+, \tag{1.4}$$

for some positive constant K and $0 < s < 1$.

Date: 29-April-2011.
2000 *Mathematics Subject Classification.* 35A05, 35B65, 35D10, 35H20, 76P05, 84C40.
Key words and phrases. Boltzmann equation, weak solution, smoothing effect.

The main purpose of this paper is to show the smoothing effect of the spatially homogeneous Boltzmann equation, that is, any weak solution to the Cauchy problem (1.1) acquires regularity as soon as $t > 0$. Let us recall the precise definition of weak solution for the Cauchy problem (1.1) given in [15], see also [16]. To this end, we introduce the standard notation,

$$\|f\|_{L^p_\ell} = \left(\int_{\mathbb{R}^3} |f(v)|^p (1+|v|)^{\ell p} dv\right)^{1/p}, \quad \text{for } p \ge 1, \ell \in \mathbb{R},$$
$$\|f\|_{L\log L} = \int_{\mathbb{R}^3} |f(v)| \log(1+|f(v)|) dv\,.$$

Definition 1.1. *Let $f_0 \ge 0$ be a function defined on $\mathbb{R}^3$ with finite mass, energy and entropy, that is,*

$$\int_{\mathbb{R}^3} f_0(v)[1+|v|^2 + \log(1+f_0(v))]dv < +\infty.$$

f is a weak solution of the Cauchy problem (1.1), *if it satisfies the following conditions:*

$$f \ge 0,\; f \in C(\mathbb{R}^+; \mathcal{D}'(\mathbb{R}^3)) \cap L^1([0,T]; L^1_{2+\gamma^+}(\mathbb{R}^3)),$$
$$f(0,\,\cdot\,) = f_0(\,\cdot\,),$$
$$\int_{\mathbb{R}^3} f(t,v)\psi(v)dv = \int_{\mathbb{R}^3} f_0(v)\psi(v)dv \;\; \textit{for } \psi = 1, v_1, v_2, v_3, |v|^2;$$
$$f(t,\,\cdot\,) \in L\log L, \quad \int_{\mathbb{R}^3} f(t,v)\log f(t,v)dv \le \int_{\mathbb{R}^3} f_0 \log f_0 dv, \quad \forall t \ge 0;$$
$$\int_{\mathbb{R}^3} f(t,v)\varphi(t,v)dv - \int_{\mathbb{R}^3} f_0(v)\varphi(0,v)dv - \int_0^t d\tau \int_{\mathbb{R}^3} f(\tau,v)\partial_\tau\varphi(\tau,v)dv$$
$$= \int_0^t d\tau \int_{\mathbb{R}^3} Q(f,f)(\tau,v)\varphi(\tau,v)dv,$$

where $\varphi \in C^1(\mathbb{R}^+; C_0^\infty(\mathbb{R}^3))$. Here, the right hand side of the last integral given above is defined by

$$\int_{\mathbb{R}^3} Q(f,f)(v)\varphi(v)dv$$
$$= \frac{1}{2}\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} B\, f(v_*)f(v)(\varphi(v') + \varphi(v'_*) - \varphi(v) - \varphi(v_*))dvdv_*d\sigma.$$

Hence, this integral is well defined for any test function $\varphi \in L^\infty([0,T]; W^{2,\infty}(\mathbb{R}^3))$ (see p. 291 of [15]*).*

To state the main theorem in this paper, we introduce the entropy dissipation functional by

$$D(g,f) = -\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B\left(g'_* f' - g_* f\right)\log f dv dv_* d\sigma\,,$$

where $f = f(v), f' = f(v'), g_* = g(v_*), g'_* = g(v'_*)$.

Theorem 1.2. *Let the cross section B in the form* (1.2) *satisfy* (1.3) *and* (1.4) *with $0 < s < 1$.*

1) *Suppose that* $\gamma > \max\{-2s, -1\}$*. Let* f *be a weak solution of the Cauchy problem* (1.1)*. For* $0 \le T_0 < T_1$*, if* f *satisfies*

$$|v|^\ell f \in L^\infty([T_0, T_1];\ L^1(\mathbb{R}^3)) \quad \textit{for any} \quad \ell \in \mathbb{N}, \tag{1.5}$$

then

$$f \in L^\infty([t_0, T_1];\ \mathcal{S}(\mathbb{R}^3)),$$

for any $t_0 \in]T_0, T_1[$.
2) *When* $-1 \ge \gamma > -2s$*, the same conclusion as above holds if we have the following entropy dissipation estimate*

$$\int_{T_0}^{T_1} D(f(t), f(t))dt < \infty\,. \tag{1.6}$$

The existence of weak solutions to the Cauchy problem (1.1) was proved by Villani [15] when $\gamma \ge -2$, assuming additionally in the case $\gamma > 0$ that $f_0 \in L^1_{2+\delta}$ for some $\delta > 0$. One important property of the weak solution for the hard potentials (namely when $\gamma > 0$) is, according to the work by Wennberg [17] (cf. also Bobylev[8]), the moment gain property. It means that f satisfies (1.5) for arbitrary $T_0 > 0$ when the initial data only satisfies finite mass, energy and entropy. However, without assuming the moment condition (1.5), we can still consider the smoothing effect in case with mild singularity ($0 < s < 1/2$) for the hard potential($\gamma > 0$), and the argument is similar to the one used in [12] (see Theorem 5.2 in Section 5).

This kind of regularization property has been studied by many authors, cf. [2, 3, 10, 12, 13, 14]. However, to our knowledge, it has not yet been completely established in the sense that the kinetic factor $\Phi(|z|)$ was modified to avoid the singularity at the origin except the Maxwellian molecule case in previous works, and moreover some extra conditions other than those in Definition 1.1 of weak solution were required in [3, 10].

We would like to emphasize that the result of Theorem 1.2 gives the full regularization property for any weak solution satisfying some natural boundedness condition in some weighted L^1 and $L \log L$ space, that requires no differentiation on the solution.

Recently in [11], it was proved that $W^{1,1}_p \cap H^3$ (strong) solutions gain full regularity in the case $0 < s < 1/2$. Their method is based on the a priori estimate of the smooth solution, together with results given in [9] about the propagation of the norm $W^{1,1}_p$ and the uniqueness of the solution. Different from [11], we start from the weak solution given in Definition 1.1 without any known uniqueness result. Therefore, a priori estimate for the smooth function is not enough to show the regularity for the weak solution in L^1 with moments. For the proof of Theorem 1.2, some suitable mollifier, acting to the weak solution, becomes necessary, so that its commutator with the collision operator requires some subtle analysis.

Throughout this paper, we will use the following notations: $f \lesssim g$ means that there exists a generic positive constant C such that $f \le Cg$; while $f \gtrsim g$ means $f \ge Cg$. And $f \sim g$ means that there exist two generic positive constant c_1 and c_2 such that $c_1 f \le g \le c_2 g$.

The rest of the paper will be organized as follows. In the next section, we will prove a uniform coercivity estimate that improves the one given in [1] which has its own interest. The mollifier and the commutator estimate will be given in Section

3. In Section 4 we prove the smoothing effect of weak solution with extra L^2 assumption. The last section is devoted to the proof of Theorem 1.2.

2. A uniform coercive estimate

In this section, we will improve the coercive estimate for the collision operator obtained in [1] by removing the restriction on v in a bounded domain.

In view of the definition of the weak solution, for $D_0, E_0 > 0$ we set

$$\mathcal{U}(D_0, E_0) = \{g \in L^1_2 \cap L\log L\ ;\ g \geq 0\,,\ \ \|g\|_{L^1} \geq D_0,\ \ \|g\|_{L^1_2} + \|g\|_{L\log L} \leq E_0\ \}\,.$$

Set $B(R) = \{v \in \mathbb{R}^3\,;\, |v| \leq R\}$ for $R > 0$ and set $B_0(R, r) = \{v \in B(R)\,;\, |v - v_0| \geq r\}$ for a $v_0 \in \mathbb{R}^3$ and $r \geq 0$. It follows from the definition of $\mathcal{U}(D_0, E_0)$ that there exist positive constants $R > 1 > r_0$ depending only on D_0, E_0 such that

$$g \in \mathcal{U}(D_0, E_0) \ \text{ implies } \ \chi_{B_0(R,r_0)}\, g \in \mathcal{U}(D_0/2, E_0)\ , \tag{2.1}$$

where χ_A denotes a characteristic function of the set $A \subset \mathbb{R}^3$. In fact, noting that for $R, M > 0$

$$R^2 \int_{\{|v|>R\}} g dv + \log(1+M) \int_{\{g>M\}} g dv \leq E_0\,.$$

We have

$$\int_{\{|v|\leq R\}\cap\{g\leq M\}} g dv \geq 3D_0/4$$

if $R \geq 2\sqrt{2E_0/D_0}$ and $\log(1 + M) \geq 8E_0/D_0$, moreover we have

$$\int_{\{|v-v_0|<r_0\}\cap\{g\leq M\}} g dv \leq D_0/4$$

if $r_0 \leq (3D_0/(16\pi \exp(8E_0/D_0))^{1/3}$.

Proposition 2.1. *Suppose that the cross section B of the form* (1.2) *satisfies* (1.3) *and* (1.4) *with $0 < s < 1$ and $\gamma > -3$. If $D_0, E_0 > 0$ and if $g \in \mathcal{U}(D_0, E_0)$ then there exist positive constants c_0, C depending only on D_0, E_0 such that for any $f \in \mathcal{S}(\mathbb{R}^3)$,*

$$-\Big(Q(g, f)\,,\, f\Big)_{L^2} \geq c_0 \|\langle v\rangle^{\gamma/2} f\|^2_{H^s} - C\|\langle v\rangle^{\gamma/2} f\|^2_{H^{(-\gamma/2)^+}}\,, \tag{2.2}$$

where $a^+ = \max\{a, 0\}$ for $a \in \mathbb{R}$. Furthermore, if $\gamma + 2s \leq 0$, $0 < s' < s$ and if g belongs to $L^{3/(3+\gamma+2s')}_{-\gamma}$ then there exists a $C_1 > 0$ independent of g such that for any $f \in \mathcal{S}(\mathbb{R}^3)$,

$$-\Big(Q(g, f)\,,\, f\Big)_{L^2} \geq c_0 \|\langle v\rangle^{\gamma/2} f\|^2_{H^s} - \big(C + C_1\|g\|_{L^{3/(3+\gamma+2s')}_{-\gamma}}\big)\|\langle v\rangle^{\gamma/2} f\|^2_{H^{s'}}\,. \tag{2.3}$$

Remark 2.2. *It should be noted that the above coercive estimate is more precise than Theorem 1.2 of* [11] *and more adaptable to prove the regularity of weak solutions. In fact, the coercive estimate* (2.2) *is uniform with respect to g. If $\gamma + 4s > 0$ and $D(g, g) < \infty$ then g belongs to $L^{3/(3+\gamma+2s')}_{-\gamma}$, provided that $g \in L^1_\ell$ for a sufficiently large ℓ. In fact, it follows from the proof of Corollary 2.4 below that $D(g, g) < \infty$ implies $\sqrt{g} \in H^s_{\gamma/2}$ and hence $\langle v\rangle^\gamma g \in L^{3/(3-2s)}$ by means of the Sobolev embedding theorem, which together with Lemma 3.8 below lead us to this conclusion.*

Proof. Put

$$\mathcal{C}_\gamma(g,\, f) = \iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} b(.)|v-v_*|^\gamma g_*(f'-f)^2 dv dv_* d\sigma,$$

and note that

$$\big(Q(g,\, f),\, f\big) = -\frac{1}{2}\mathcal{C}_\gamma(g,\, f) + \frac{1}{2}\iiint \Phi\, b\; g_*(f'^2 - f^2) dv dv_* d\sigma\,.$$

It follows from the Cancellation Lemma and Remark 6 in [1] that

$$\begin{aligned}\left|\iiint b|v-v_*|^\gamma\; g_*(f^2-f'^2)dvdv_*d\sigma\right| &\lesssim \left|\iint |v-v_*|^\gamma\; g_* f^2\right| dv dv_* \\ &\lesssim \|g\|_{L^1_{|\gamma|}} \|f\|^2_{H^{(-\gamma/2)+}_{\gamma/2}}\ ,\end{aligned}$$

where the last inequality in the case $\gamma \geq 0$ is trivial. While $\gamma < 0$, this follows from the fact that

$$|v-v_*|^\gamma \lesssim \langle v\rangle^\gamma\{\mathbf{1}_{|v-v_*|\geq\langle v\rangle/2} + \mathbf{1}_{|v-v_*|<\langle v\rangle/2}\langle v_*\rangle^{-\gamma}|v-v_*|^\gamma\},$$

and the Hardy inequality $\sup_{v_*}\int |v-v_*|^\gamma |F(v)|^2 dv \lesssim \|F\|^2_{H^{-\gamma/2}}$ for $F = \langle v\rangle^{\gamma/2} f$. Furthermore, it follows from the Hardy-Littlewood-Sobolev inequality that

$$\begin{aligned}&\left|\iint |v-v_*|^\gamma\; g_* f^2\right| dv dv_* \lesssim \|g\|_{L^1}\|F\|^2_{L^2} + \iint \frac{\langle v_*\rangle^{|\gamma|} g(v_*)F(v)^2}{|v-v_*|^{-\gamma}} dv dv_* \\ &\lesssim \|g\|_{L^1}\|F\|^2_{L^2} + \|\langle v\rangle^{|\gamma|} g\|_{L^{3/(3+\gamma+2s')}} \|F^2\|_{L^{3/(3-2s')}} \lesssim \|g\|_{L^{3/(3+\gamma+2s')}_{|\gamma|}} \|f\|^2_{H^{s'}_{\gamma/2}}\ ,\end{aligned}$$

where we have used the Sobolev embedding in the last inequality.

For the proof of the proposition, it now suffices to consider only the quantity $\mathcal{C}_\gamma(g,\, f)$. The case $\gamma = 0$ is obvious. In fact, by Corollary 3 and Proposition 2 in [1], there exists a $c_0 = c_0(D_0, E_0) > 0$ depending only on $D_0, E_0 > 0$ such that

$$\mathcal{C}_0(g,\, f) \geq c_0 \int_{\{|\xi|\geq 1\}} \Big||\xi|^s \hat{f}(\xi)\Big|^2 d\xi\,,\ \ \forall f \in \mathcal{S}(\mathbb{R}^3)\,, \tag{2.4}$$

where $\hat{f}(\xi)$ is the Fourier transform of f with respect to the variable $v \in \mathbb{R}^3$. From the proof in [1], it should be noted that (2.4) holds for any $f \in L^2$ such that the left hand side is finite.

We consider the case $\gamma \neq 0$, following the argument used in the proof of Lemma 2 of [1]. Choose R, r_0 such that (2.1) holds. Let φ_R be a non-negative smooth function not greater than one, which is 1 for $|v| \geq 4R$ and 0 for $|v| \leq 2R$. In view of

$$\frac{\langle v\rangle}{4} \leq |v-v_*| \leq 2\langle v\rangle \ \text{ on supp } (\chi_{B(R)})_*\varphi_R\,,$$

we have

$$\begin{aligned}&4^{|\gamma|}\Phi(|v-v_*|)g_*(f'-f)^2 \geq \big(g\chi_{B(R)}\big)_*\big(\langle v\rangle^{\gamma/2}\varphi_R\big)^2(f'-f)^2 \\ &\geq \big(g\chi_{B(R)}\big)_*\Big[\frac{1}{2}\Big(\big(\langle v\rangle^{\gamma/2}\varphi_R f\big)' - \langle v\rangle^{\gamma/2}\varphi_R f\Big)^2 - \Big(\big(\langle v\rangle^{\gamma/2}\varphi_R\big)' - \langle v\rangle^{\gamma/2}\varphi_R\Big)^2 f'^2\Big]\,.\end{aligned}$$

It follows from the mean value theorem that for a $\tau \in (0,1)$

$$\begin{aligned}\Big|\big(\langle v\rangle^{\gamma/2}\varphi_R\big)' - \langle v\rangle^{\gamma/2}\varphi_R\Big| &\lesssim \langle v+\tau(v'-v)\rangle^{\gamma/2-1}|v-v_*|\sin\frac{\theta}{2}\\ &\lesssim \langle v_*\rangle^{|\gamma/2-1|}\langle v'-v_*\rangle^{\gamma/2}\sin\frac{\theta}{2}\\ &\lesssim \langle v_*\rangle^{|\gamma/2|+|\gamma/2-1|}\langle v'\rangle^{\gamma/2}\sin\frac{\theta}{2},\end{aligned}$$

because $|v-v_*|/\sqrt{2} \le |v'-v_*| \le |v+\tau(v'-v)-v_*| \le |v-v_*|$ for $\theta \in [0,\pi/2]$. Therefore, we have

$$\mathcal{C}_\gamma(g,\, f) \ge 2^{-1-2|\gamma|}\mathcal{C}_0(g\chi_{B(R)},\, \varphi_R\langle v\rangle^{\gamma/2}f) - C_R\|g\|_{L^1}\|f\|^2_{L^2_{\gamma/2}}, \tag{2.5}$$

for a positive constant $C_R \sim R^{|\gamma|+|\gamma-2|}$. For a set $B(4R)$ we take a finite covering

$$B(4R) \subset \bigcup_{v_j\in B(4R)} A_j\,,\ \ A_j = \{v \in \mathbb{R}^3\,;\, |v-v_j| \le \frac{r_0}{4}\}\,.$$

For each A_j we choose a non-negative smooth function φ_{A_j} which is 1 on A_j and 0 on $\{|v-v_j| \ge r_0/2\}$. Note that

$$\frac{r_0}{2} \le |v-v_*| \le 6R \ \text{ on supp } (\chi_{B_j(R,r_0)})_*\varphi_{A_j}\,.$$

Then we have

$$\begin{aligned}&\Phi(|v-v_*|)g_*(f'-f)^2 \gtrsim \min\{r_0^{\gamma^+}, R^{-(-\gamma)^+}\}\big(g\chi_{B_j(R,r_0)}\big)_*\varphi^2_{A_j}(f'-f)^2\\ &\gtrsim R^{-\gamma^+}\min\{r_0^{\gamma^+}, R^{-(-\gamma)^+}\}\big(g\chi_{B_j(R,r_0)}\big)_*\\ &\quad\times\left[\frac{1}{2}\Big(\big(\langle v\rangle^{\gamma/2}\varphi_{A_j}f\big)' - \langle v\rangle^{\gamma/2}\varphi_{A_j}f\Big)^2 - \Big(\big(\langle v\rangle^{\gamma/2}\varphi_{A_j}\big)' - \langle v\rangle^{\gamma/2}\varphi_{A_j}\Big)^2 f'^2\right].\end{aligned}$$

Since $\Big|\big(\langle v\rangle^{\gamma/2}\varphi_{A_j}\big)' - \langle v\rangle^{\gamma/2}\varphi_{A_j}\Big| \lesssim R^{|\gamma|+1}\langle v'\rangle^{\gamma}\sin\theta/2$ if $|v_*| \le R$, we obtain

$$\begin{aligned}\mathcal{C}_\gamma(g,\, f) \gtrsim \min\{(r_0/R)^{\gamma^+}, R^{-(-\gamma)^+}\}&\mathcal{C}_0(g\chi_{B_j(R,r_0)},\, \varphi_{A_j}\langle v\rangle^{\gamma/2}f)\\ &- C'_{R,r_0}\|g\|_{L^1}\|f\|^2_{L^2_{\gamma/2}},\end{aligned}\tag{2.6}$$

for a positive constant $C'_{R,r_0} \sim R^{2+2|\gamma|}$. It follows from (2.4), (2.5) and (2.6) that there exist $c'_0, C, C' > 0$ depending only on D_0, E_0 such that

$$\begin{aligned}\mathcal{C}_\gamma(g,\, f) &\ge c'_0\Big(\|\langle D\rangle^s\varphi_R\langle v\rangle^{\gamma/2}f\|^2 + \sum_j\|\langle D\rangle^s\varphi_{A_j}\langle v\rangle^{\gamma/2}f\|^2\Big) - C\|f\|^2_{L^2_{\gamma/2}}\\ &\ge c'_0\|\langle v\rangle^{\gamma/2}f\|^2_{H^s} - C'\|f\|^2_{L^2_{\gamma/2}},\end{aligned}\tag{2.7}$$

because $\varphi_R^2 + \sum_j\varphi^2_{A_j} \ge 1$ and commutators $[\langle D\rangle^s, \varphi_R]$, $[\langle D\rangle^s, \varphi_{A_j}]$ are L^2 bounded operators. □

Remark 2.3. (2.7) *holds for any* $f \in L^2_{\gamma/2}$ *such that* $\mathcal{C}_\gamma(g,f)$ *is finite, because of the remark after* (2.4)*. Similarly,* (2.2) *holds for any* $f \in L^2_{\gamma/2}$ *if* $\gamma \ge 0$ *and if its left hand side is finite.*

Corollary 2.4. *Let $f(t) \in L^1_{\max\{2,\gamma\}} \cap L\log L$ be a weak solution. Suppose that the cross section B is the same as in Propostion 2.1. Assume that for a $T > 0$ we have*

$$\int_0^T D(f(\tau), f(\tau))d\tau < \infty\,. \tag{2.8}$$

Then there exist positive constants c_f and $C_f > 0$ such that

$$c_f \int_0^T \|\sqrt{f(\tau)}\|^2_{H^s_{\gamma/2}} d\tau \le \int_0^T D(f(\tau), f(\tau))d\tau + C_f \int_0^T \|f(\tau)\|_{L^1_{\gamma+}} d\tau\,. \tag{2.9}$$

Proof. We first consider the case $\gamma < 0$. Note

$$\begin{aligned}
D(f,f) &= -\iiint B\big(f'_*f' - f_*f\big)\log f dvdv_*d\sigma \\
&= \frac{1}{4}\iiint B\left(f'f'_* - ff_*\right)\big(\log f'f'_* - \log ff_*\big)dvdv_*d\sigma \\
&\ge \frac{1}{4}\iiint b(\cdot)\langle v - v_*\rangle^\gamma\left(f'f'_* - ff_*\right)\big(\log f'f'_* - \log ff_*\big)dvdv_*d\sigma,
\end{aligned}$$

because $(x-y)(\log x - \log y) \ge 0$ and $\Phi(|v - v_*|) \ge \langle v - v_*\rangle^\gamma$. Then we have

$$\begin{aligned}
D(f,f) &\ge -\iiint b(\cdot)\langle v - v_*\rangle^\gamma\big(f'_*f' - f_*f\big)\log f dvdv_*d\sigma \\
&= \iiint b(\cdot)\langle v - v_*\rangle^\gamma f_*\left(f\log\frac{f}{f'} - f + f'\right)dvdv_*d\sigma \\
&\quad + \iiint b(\cdot)\langle v - v_*\rangle^\gamma f_*\left(f - f'\right)dvdv_*d\sigma \\
&\ge \iiint b(\cdot)\langle v - v_*\rangle^\gamma f_*\left(\sqrt{f'} - \sqrt{f}\right)^2 dvdv_*d\sigma - C\|f\|^2_{L^1}\,,
\end{aligned}$$

where we have used $x\log(x/y) - x + y \ge (\sqrt{x} - \sqrt{y})^2$ and the Cancellation Lemma in the last inequality, as the same as in the proof of Theorem 1 in [1]. Since the proof of Propostion 2.1 still works with Φ replaced by $\langle v - v_*\rangle^\gamma$, we obtain the desired estimate in view of Remark 2.3. The case $\gamma \ge 0$ is easier because we do not need to replace Φ by $\langle v - v_*\rangle^\gamma$ when Cancellation Lemma is applied. □

3. Mollifier and commutator estimate

Since the weak solution is only in L^1, we can not use it directly as a test function in the definition of weak solution to get the energy estimate. To overcome this, we need to mollify it by some suitable mollifiers so that to consider the commutators between the mollifiers and the collision operator becomes necessary.

Let $\lambda, N_0 \in \mathbb{R}$, $\delta > 0$ and put

$$M^\delta_\lambda(\xi) = \frac{\langle\xi\rangle^\lambda}{(1 + \delta\langle\xi\rangle)^{N_0}}\,, \quad \langle\xi\rangle = (1 + |\xi|^2)^{1/2}\,. \tag{3.1}$$

Then $M^\delta_\lambda(\xi)$ belongs to the symbol class $S^{\lambda - N_0}_{1,0}$ of pseudo-differential operators and belongs to $S^\lambda_{1,0}$ uniformly with respect to $\delta \in]0,1]$. $M^\delta_\lambda(D_v)$ denotes the associated pseudo-differential operator. By direct calculation we see that for any α there exists a $C_\alpha > 0$ independent of δ such that

$$\left|\partial^\alpha_\xi M^\delta_\lambda(\xi)\right| \le C_\alpha M^\delta_\lambda(\xi)\langle\xi\rangle^{-|\alpha|}\,. \tag{3.2}$$

Lemma 3.1. *There exists a constant $C>0$ independent of δ such that*

$$
\begin{aligned}
(3.3)\qquad & \big|M^\delta_\lambda(\xi)-M^\delta_\lambda(\xi-\xi_*)\big| \\
&\le C\langle\xi\rangle^\lambda \mathbf{1}_{\langle\xi_*\rangle\ge\sqrt2|\xi|}+CM^\delta_\lambda(\xi-\xi_*)\left\{\mathbf{1}_{\langle\xi_*\rangle\ge|\xi|/2}+\frac{\langle\xi_*\rangle}{\langle\xi\rangle}\mathbf{1}_{|\xi|/2>\langle\xi_*\rangle}\right\}\\
&+CM^\delta_\lambda(\xi-\xi_*)\left(\frac{M^\delta_\lambda(\xi_*)\big(1+\delta\langle\xi-\xi_*\rangle\big)^{N_0}}{\langle\xi-\xi_*\rangle^\lambda}\right)\mathbf{1}_{\sqrt2|\xi|>\langle\xi_*\rangle\ge|\xi|/2}\,.
\end{aligned}
$$

And if $p\ge N_0-\lambda$

$$
\begin{aligned}
(3.4)\quad \big|M^\delta_\lambda(\xi)-M^\delta_\lambda(\xi-\xi_*)\big| &\le CM^\delta_\lambda(\xi-\xi_*)\left\{\Big(\frac{\langle\xi_*\rangle}{\langle\xi\rangle}\Big)^p\mathbf{1}_{\langle\xi_*\rangle\ge\sqrt2|\xi|}\right.\\
&\left.+\left(\frac{M^\delta_\lambda(\xi_*)\big(1+\delta\langle\xi-\xi_*\rangle\big)^{N_0}}{\langle\xi-\xi_*\rangle^\lambda}+1\right)\mathbf{1}_{\sqrt2|\xi|>\langle\xi_*\rangle\ge|\xi|/2}+\frac{\langle\xi_*\rangle}{\langle\xi\rangle}\mathbf{1}_{|\xi|/2>\langle\xi_*\rangle}\right\}.
\end{aligned}
$$

Proof. We first note

$$
(3.5)\qquad \begin{cases}\langle\xi\rangle\lesssim\langle\xi_*\rangle\sim\langle\xi-\xi_*\rangle, & \text{on supp }\mathbf{1}_{\langle\xi_*\rangle\ge\sqrt2|\xi|},\\ \langle\xi\rangle\sim\langle\xi-\xi_*\rangle, & \text{on supp }\mathbf{1}_{\langle\xi_*\rangle\le|\xi|/2},\\ \langle\xi\rangle\sim\langle\xi_*\rangle\gtrsim\langle\xi-\xi_*\rangle, & \text{on supp }\mathbf{1}_{\sqrt2|\xi|\ge\langle\xi_*\rangle\ge|\xi|/2}\,.\end{cases}
$$

Since $\langle\xi\rangle^pM^\delta_\lambda(\xi)$ is increasing function of $\langle\xi\rangle$, we have

$$
\langle\xi\rangle^pM^\delta_\lambda(\xi)\lesssim\langle\xi_*\rangle^pM^\delta_\lambda(\xi_*)\sim\langle\xi_*\rangle^pM^\delta_\lambda(\xi-\xi_*)\ \text{ on supp }\mathbf{1}_{\langle\xi_*\rangle\ge\sqrt2|\xi|}\,,
$$

and trivially,

$$
M^\delta_\lambda(\xi)\le\langle\xi\rangle^\lambda.
$$

Note that

$$
M^\delta_\lambda(\xi)\sim M^\delta_\lambda(\xi_*)\sim M^\delta_\lambda(\xi-\xi_*)\frac{M^\delta_\lambda(\xi_*)\big(1+\delta\langle\xi-\xi_*\rangle\big)^{N_0}}{\langle\xi-\xi_*\rangle^\lambda}\ \text{ on supp }\mathbf{1}_{\sqrt2|\xi|\ge\langle\xi_*\rangle\ge|\xi|/2}.
$$

By the mean value theorem, we have

$$
\begin{aligned}
\big|M^\delta_\lambda(\xi)-M^\delta_\lambda(\xi-\xi_*)\big| &\le\int_0^1|\big(\nabla_\xi M^\delta_\lambda\big)(\xi+\tau(\xi-\xi_*))|d\tau|\xi_*|\\
&\lesssim M^\delta_\lambda(\xi-\xi_*)\frac{\langle\xi_*\rangle}{\langle\xi\rangle}\quad\text{on supp }\mathbf{1}_{\langle\xi_*\rangle\le|\xi|/2}.
\end{aligned}
$$

Here we have used (3.2) and the second formula of (3.5). The above estimates imply (3.4) and (3.3). $\square$

For the kinetic factor $|v-v_*|^\gamma$, we need to take into account the singular behavior close to $|v-v_*|=0$ except $\gamma=0$. Therefore, we decompose the kinetic factor in two parts. Let $0\le\phi(z)\le1$ be a smooth radial function with value 1 for z close to 0, and 0 for large values of z. Set

$$
\Phi_\gamma(z)=\Phi_\gamma(z)\phi(z)+\Phi_\gamma(z)(1-\phi(z))=\Phi_c(z)+\Phi_{\bar c}(z).
$$

And then correspondingly we can write

$$
Q(f,g)=Q_c(f,g)+Q_{\bar c}(f,g),
$$

where the kinetic factor in the collision operator is defined according to the decomposition respectively. Since $\Phi_{\bar c}(z)$ is smooth, and $\Phi_{\bar c}(z)\lesssim\tilde\Phi_\gamma(z)$, where $\tilde\Phi_\gamma(|\,z\,|)=(1+|z|^2)^{\gamma/2}$ is the regular kinetic factor studied in [4]. Then $Q_{\bar c}(f,g)$ has similar properties as for $Q_{\tilde\Phi_\gamma}(f,g)$ as regard to the upper bound and commutator estimations. We recall the Proposition 2.9 of [4].

Proposition 3.2. *Let $\lambda \in \mathbb{R}$ and $M(\xi)$ be a positive symbol of pseudo-differential operator in $S^\lambda_{1,0}$ in the form of $M(\xi) = \tilde M(|\xi|^2)$. Assume that, there exist constants $c, C > 0$ such that for any $s, \tau > 0$*

$$c^{-1} \le \frac{s}{\tau} \le c \quad \textit{implies} \quad C^{-1} \le \frac{\tilde M(s)}{\tilde M(\tau)} \le C,$$

and $M(\xi)$ satisfies

$$|M^{(\alpha)}(\xi)| = |\partial^\alpha_\xi M(\xi)| \le C_\alpha M(\xi)\langle \xi\rangle^{-|\alpha|}\,,$$

for any $\alpha \in \mathbb{N}^3$. Then, if $0 < s < 1/2$, for any $N > 0$ there exists a $C_N > 0$ such that

$$\begin{aligned}(3.6)\ &|(M(D_v)Q_{\bar c}(f,\, g) - Q_{\bar c}(f,\, M(D_v)g),\ h)_{L^2}| \\ &\qquad \le C_N \|f\|_{L^1_{\gamma+}}\Big(\|M(D_v)\,g\|_{L^2_{\gamma+}} + \|g\|_{H^{\lambda-N}_{\gamma+}}\Big)\|h\|_{L^2}.\end{aligned}$$

Furthermore, if $1/2 < s < 1$, for any $N > 0$ and any $\varepsilon > 0$, there exists a $C_{N,\varepsilon} > 0$ such that

$$\begin{aligned}(3.7)\qquad &|(M(D_v)Q_{\bar c}(f,\, g) - Q_{\bar c}(f,\, M(D_v)g),\ h)_{L^2}| \\ &\le C_{N,\varepsilon}\|f\|_{L^1_{(2s+\gamma-1)+}}\Big(\|M(D_v)g\|_{H^{2s-1+\varepsilon}_{(2s+\gamma-1)+}} + \|g\|_{H^{\lambda-N}_{\gamma+}}\Big)\|h\|_{L^2}\,.\end{aligned}$$

When $s = 1/2$ we have the same estimate as (3.7) with $(2s + \gamma - 1)$ replaced by $(\gamma + \kappa)$ for any small $\kappa > 0$.

Remark 3.3. *In the case $\gamma > 0$ and $0 < s < 1/2$, it follows from Lemma 3.1 of* [12] *and its proof that* (3.6) *can be replaced by*

$$\begin{aligned}&|(M(D_v)Q_{\bar c}(f,\, g) - Q_{\bar c}(f,\, M(D_v)g),\ h)_{L^2}| \\ &\qquad \le C_N \|f\|_{L^1_\gamma}\Big(\|M(D_v)\,g\|_{L^2_{\gamma/2}} + \|g\|_{H^{\lambda-N}_{\gamma/2}}\Big)\|h\|_{L^2_{\gamma/2}}.\end{aligned}$$

From now on, we concentrate on the study for the singular part $Q_c(f,g)$.

Proposition 3.4. *Assume that $0 < s < 1, \gamma + 2s > 0$. Let $0 < s' < s$ satisfy $\gamma + 2s' > 0$ and $2s' \ge (2s-1)^+$. If*

$$5 + \gamma \ge 2(N_0 - \lambda)\,, \tag{3.8}$$

then we have

1) If $s' + \lambda < 3/2$, then

$$\Big|\Big(M^\delta_\lambda(D_v)\,Q_c(f,g) - Q_c(f, M^\delta_\lambda(D_v)\,g), h\Big)\Big| \lesssim \|f\|_{L^1}||M^\delta_\lambda(D_v)g||_{H^{s'}}\,||h||_{H^{s'}}\,.$$

2) If $s' + \lambda \ge 3/2$, then

$$\begin{aligned}&\Big|\Big(M^\delta_\lambda(D_v)\,Q_c(f,g) - Q_c(f, M^\delta_\lambda(D_v)\,g), h\Big)\Big| \\ &\qquad \lesssim \Big(\|f\|_{L^1} + \|f\|_{H^{(\lambda+s'-3)+}}\Big)||M^\delta_\lambda(D_v)g||_{H^{s'}}\,||h||_{H^{s'}}\,.\end{aligned}$$

Furthermore, if $s > 1/2$ and $\gamma > -1$, then the assumption (3.8) *can be relaxed to*

$$4 + \gamma + 2s > 2(N_0 - \lambda)\,. \tag{3.9}$$

Proof. For the proof we shall follow some of arguments from [5]. By using the formula from the Appendix of [1], we have

$$
\begin{aligned}
(Q_c(f,g),h) = \iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} & b\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)[\hat\Phi_c(\xi_*-\xi^-)-\hat\Phi_c(\xi_*)] \\
& \times \hat f(\xi_*)\hat g(\xi-\xi_*)\overline{\hat h(\xi)}d\xi d\xi_* d\sigma\,,
\end{aligned}
$$

where $\xi^- = \frac{1}{2}(\xi-|\xi|\sigma)$. Therefore

$$
\begin{aligned}
&\Big(M^\delta_\lambda(D)\,Q_c(f,g)-Q_c(f,M^\delta_\lambda(D)\,g),h\Big) \\
&\qquad= \iiint b\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)[\hat\Phi_c(\xi_*-\xi^-)-\hat\Phi_c(\xi_*)] \\
&\qquad\qquad\times\Big(M^\delta_\lambda(\xi)-M^\delta_\lambda(\xi-\xi_*)\Big)\hat f(\xi_*)\hat g(\xi-\xi_*)\overline{\hat h(\xi)}d\xi d\xi_* d\sigma \\
&\qquad= \iiint_{|\xi^-|\le\frac12\langle\xi_*\rangle}\cdots\ d\xi d\xi_* d\sigma + \iiint_{|\xi^-|\ge\frac12\langle\xi_*\rangle}\cdots\ d\xi d\xi_* d\sigma \\
&\qquad=\mathcal{A}_1(f,g,h)+\mathcal{A}_2(f,g,h)\ .
\end{aligned}
$$

Then, we write $\mathcal{A}_2(f,g,h)$ as

$$
\begin{aligned}
\mathcal{A}_2 &= \iiint b\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\mathbf{1}_{|\xi^-|\ge\frac12\langle\xi_*\rangle}\hat\Phi_c(\xi_*-\xi^-)\cdots d\xi d\xi_* d\sigma \\
&\quad - \iiint b\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\mathbf{1}_{|\xi^-|\ge\frac12\langle\xi_*\rangle}\hat\Phi_c(\xi_*)\cdots d\xi d\xi_* d\sigma \\
&= \mathcal{A}_{2,1}(f,g,h)-\mathcal{A}_{2,2}(f,g,h)\,.
\end{aligned}
$$

On the other hand, for $\mathcal{A}_1$ we use the Taylor expansion of $\hat\Phi_c$ of order 2 to have

$$
\mathcal{A}_1 = \mathcal{A}_{1,1}(f,g,h)+\mathcal{A}_{1,2}(f,g,h),
$$

where

$$
\begin{aligned}
\mathcal{A}_{1,1} = \iiint & b\ \xi^-\cdot(\nabla\hat\Phi_c)(\xi_*)\mathbf{1}_{|\xi^-|\le\frac12\langle\xi_*\rangle}\Big(M^\delta_\lambda(\xi)-M^\delta_\lambda(\xi-\xi_*)\Big) \\
& \times\hat f(\xi_*)\hat g(\xi-\xi_*)\bar{\hat h}(\xi)d\xi d\xi_* d\sigma,
\end{aligned}
$$

and $\mathcal{A}_{1,2}(f,g,h)$ is the remaining term corresponding to the second order term in the Taylor expansion of $\hat\Phi_c$.

We first consider $\mathcal{A}_{1,1}$. By writing

$$
\xi^- = \frac{|\xi|}{2}\left(\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\frac{\xi}{|\xi|}-\sigma\right)+\left(1-\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\right)\frac{\xi}{2},
$$

we see that the integral corresponding to the first term on the right hand side vanishes because of the symmetry on $\mathbb{S}^2$. Hence, we have

$$
\mathcal{A}_{1,1} = \iint_{\mathbb{R}^6} K(\xi,\xi_*)\Big(M^\delta_\lambda(\xi)-M^\delta_\lambda(\xi-\xi_*)\Big)\hat f(\xi_*)\hat g(\xi-\xi_*)\bar{\hat h}(\xi)d\xi d\xi_*\,,
$$

where

$$
K(\xi,\xi_*) = \int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\left(1-\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\right)\frac{\xi}{2}\cdot(\nabla\hat\Phi_c)(\xi_*)\mathbf{1}_{|\xi^-|\le\frac12\langle\xi_*\rangle}d\sigma\,.
$$

Note that $|\nabla\hat{\Phi}_c(\xi_*)| \lesssim \frac{1}{\langle\xi_*\rangle^{3+\gamma+1}}$, from the Appendix of [5]. If $\sqrt{2}|\xi| \le \langle\xi_*\rangle$, then $\sin(\theta/2)\ |\xi| = |\xi^-| \le \langle\xi_*\rangle/2$ because $0 \le \theta \le \pi/2$, and we have

$$|K(\xi,\xi_*)| \lesssim \int_0^{\pi/2} \theta^{1-2s} d\theta \frac{\langle\xi\rangle}{\langle\xi_*\rangle^{3+\gamma+1}} \lesssim \frac{1}{\langle\xi_*\rangle^{3+\gamma}} \left(\frac{\langle\xi\rangle}{\langle\xi_*\rangle}\right).$$

On the other hand, if $\sqrt{2}|\xi| \ge \langle\xi_*\rangle$, then

$$|K(\xi,\xi_*)| \lesssim \int_0^{\pi\langle\xi_*\rangle/(2|\xi|)} \theta^{1-2s} d\theta \frac{\langle\xi\rangle}{\langle\xi_*\rangle^{3+\gamma+1}} \lesssim \frac{1}{\langle\xi_*\rangle^{3+\gamma}} \left(\frac{\langle\xi\rangle}{\langle\xi_*\rangle}\right)^{2s-1}.$$

Hence we obtain

$$\begin{aligned}|K(\xi,\xi_*)| \lesssim \frac{1}{\langle\xi_*\rangle^{3+\gamma}} \Bigg\{ &\left(\frac{\langle\xi\rangle}{\langle\xi_*\rangle}\right) \mathbf{1}_{\langle\xi_*\rangle \ge \sqrt{2}|\xi|} \\ &+ \mathbf{1}_{\sqrt{2}|\xi| \ge \langle\xi_*\rangle \ge |\xi|/2} + \left(\frac{\langle\xi\rangle}{\langle\xi_*\rangle}\right)^{2s-1} \mathbf{1}_{|\xi|/2 \ge \langle\xi_*\rangle} \Bigg\}.\end{aligned} \tag{3.10}$$

Similar to $\mathcal{A}_{1,1}$, we can also write

$$\mathcal{A}_{1,2} = \iint_{\mathbb{R}^6} \tilde{K}(\xi,\xi_*) \Big(M_\lambda^\delta(\xi) - M_\lambda^\delta(\xi-\xi_*) \Big) \hat{f}(\xi_*) \hat{g}(\xi-\xi_*) \bar{\hat{h}}(\xi) d\xi d\xi_*,$$

where

$$\tilde{K}(\xi,\xi_*) = \int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big) \int_0^1 (1-\tau)(\nabla^2\hat{\Phi}_c)(\xi_* - \tau\xi^-)\cdot\xi^-\cdot\xi^- \mathbf{1}_{|\xi^-| \le \frac{1}{2}\langle\xi_*\rangle} d\tau d\sigma.$$

Again from the Appendix of [5], we have

$$|(\nabla^2\hat{\Phi}_c)(\xi_* - \tau\xi^-)| \lesssim \frac{1}{\langle\xi_* - \tau\xi^-\rangle^{3+\gamma+2}} \lesssim \frac{1}{\langle\xi_*\rangle^{3+\gamma+2}},$$

because $|\xi^-| \le \langle\xi_*\rangle/2$, which leads to

$$\begin{aligned}|\tilde{K}(\xi,\xi_*)| \lesssim \frac{1}{\langle\xi_*\rangle^{3+\gamma}} \Bigg\{ &\left(\frac{\langle\xi\rangle}{\langle\xi_*\rangle}\right)^2 \mathbf{1}_{\langle\xi_*\rangle \ge \sqrt{2}|\xi|} \\ &+ \mathbf{1}_{\sqrt{2}|\xi| \ge \langle\xi_*\rangle \ge |\xi|/2} + \left(\frac{\langle\xi\rangle}{\langle\xi_*\rangle}\right)^{2s} \mathbf{1}_{|\xi|/2 \ge \langle\xi_*\rangle} \Bigg\}.\end{aligned} \tag{3.11}$$

It follows from (3.4) of Lemma 3.1, (3.10) and (3.11) that if $p = N_0 - \lambda$, then

$$|\mathcal{A}_1| \lesssim |\mathcal{A}_{1,1}| + |\mathcal{A}_{1,2}| \lesssim A_1 + A_2 + A_3,$$

where

$$A_1 = \iint_{\mathbb{R}^6} \left| \frac{\hat{f}(\xi_*)}{\langle\xi_*\rangle^{3+\gamma}} \right| \left| M_\lambda^\delta(\xi-\xi_*) \hat{g}(\xi-\xi_*) \right| |\hat{h}(\xi)| \left(\frac{\langle\xi_*\rangle}{\langle\xi\rangle}\right)^{p-1} \mathbf{1}_{\langle\xi_*\rangle \ge \sqrt{2}|\xi|} d\xi_* d\xi, \tag{3.12}$$

and

$$A_2=\iint_{\mathbb{R}^6}\left|\frac{\hat f(\xi_*)}{\langle\xi_*\rangle^{3+\gamma}}\right|\left|M^\delta_\lambda(\xi-\xi_*)\hat g(\xi-\xi_*)\right||\hat h(\xi)|$$
$$\times\left(\frac{M^\delta_\lambda(\xi_*)\Big(1+\big(\delta\langle\xi-\xi_*\rangle\big)^{N_0}\Big)}{\langle\xi-\xi_*\rangle^\lambda}+1\right)\mathbf{1}_{\sqrt2|\xi|>\langle\xi_*\rangle\ge|\xi|/2}d\xi_*d\xi\,;$$
$$A_3=\iint_{\mathbb{R}^6}\left|\frac{\hat f(\xi_*)}{\langle\xi_*\rangle^{3+\gamma}}\right|\left|M^\delta_\lambda(\xi-\xi_*)\hat g(\xi-\xi_*)\right||\hat h(\xi)|\left(\frac{\langle\xi\rangle}{\langle\xi_*\rangle}\right)^{2s-1}\mathbf{1}_{|\xi|/2>\langle\xi_*\rangle}d\xi_*d\xi\,.$$

Putting $\hat G(\xi)=\langle\xi\rangle^{s'}M^\delta_\lambda(\xi)\hat g(\xi)$ and $\hat H(\xi)=\langle\xi\rangle^{s'}\hat h(\xi)$, then we have

$$|A_1|^2\lesssim\|\hat f\|^2_{L^\infty}\left(\int_{\mathbb{R}^3}\frac{d\xi_*}{\langle\xi_*\rangle^{3+\gamma+2s'}}\int_{\mathbb{R}^3_\xi}|\hat H(\xi)|^2d\xi\right)$$
$$\times\left(\int_{\mathbb{R}^3}\frac{d\xi}{\langle\xi\rangle^{3+\gamma+2s'}}\int_{\mathbb{R}^3}\left(\frac{\langle\xi\rangle}{\langle\xi_*\rangle}\right)^{3+\gamma-2(p-1)}\mathbf{1}_{\langle\xi_*\rangle\ge\sqrt2|\xi|}|\hat G(\xi-\xi_*)|^2d\xi_*\right)$$
$$\lesssim\|f\|^2_{L^1}\|M^\delta_\lambda g\|^2_{H^{s'}}\|h\|^2_{H^{s'}}\,,$$

because $\gamma+2s'>0$, and $3+\gamma-2(p-1)\ge0$ from (3.8). Here we have used the fact that $\langle\xi_*\rangle\sim\langle\xi-\xi_*\rangle$ if $\langle\xi_*\rangle\ge\sqrt2|\xi|$.

We consider the case $s>1/2,\gamma>-1$. For $s>s'>1/2$ we have

$$|A_1|^2\lesssim\|\hat f\|^2_{L^\infty}\left(\int_{\mathbb{R}^3}\frac{d\xi_*}{\langle\xi_*\rangle^{3+\gamma+1}}\int_{\mathbb{R}^3_\xi}|\hat H(\xi)|^2d\xi\right)$$
$$\times\left(\int_{\mathbb{R}^3}\frac{d\xi}{\langle\xi\rangle^{3+\gamma+1}}\int_{\mathbb{R}^3}\left(\frac{\langle\xi\rangle}{\langle\xi_*\rangle}\right)^{3+\gamma+(2s'-1)-2(p-1)}\frac{\mathbf{1}_{\langle\xi_*\rangle\ge\sqrt2|\xi|}}{\langle\xi\rangle^{2(2s'-1)}}|\hat G(\xi-\xi_*)|^2d\xi_*\right)$$
$$\lesssim\|f\|^2_{L^1}\|M^\delta_\lambda g\|^2_{H^{s'}}\|h\|^2_{H^{s'}}\,,$$

if $3+\gamma+(2s'-1)-2(p-1)>0$. Thus (3.8) can be relaxed to (3.9) to get the desired estimate for A_1. Here we remark that (3.8) or (3.9) are only required to estimate the part A_1.

Noting the third formula of (3.5), we get

$$|A_2|^2\lesssim\left\{\int_{\mathbb{R}^3}\frac{|\hat f(\xi_*)|^2d\xi_*}{\langle\xi_*\rangle^{6+2\gamma+2s'}}\int_{\langle\xi-\xi_*\rangle\lesssim\langle\xi_*\rangle}\left(\frac{\langle\xi_*\rangle^{2\lambda}}{\langle\xi-\xi_*\rangle^{2(\lambda+s')}}\right.\right.$$
$$\left.\left.+\frac{\langle\xi_*\rangle^{2(\lambda-N_0)}}{\langle\xi-\xi_*\rangle^{2(\lambda-N_0+s')}}+\frac{1}{\langle\xi-\xi_*\rangle^{2s'}}\right)d\xi\right\}$$
$$\times\left(\iint_{\mathbb{R}^6}|\hat G(\xi-\xi_*)|^2|\hat H(\xi)|^2d\xi d\xi_*\right)\,.$$

If $\lambda+s'<3/2$, then

$$|A_2|^2\lesssim\int_{\mathbb{R}^3}\frac{|\hat f(\xi_*)|^2}{\langle\xi_*\rangle^{3+2(\gamma+2s')}}d\xi_*\|M^\delta_\lambda g\|^2_{H^{s'}}\|h\|^2_{H^{s'}}$$
$$\lesssim\|f\|^2_{L^1}\|M^\delta_\lambda g\|^2_{H^{s'}}\|h\|^2_{H^{s'}}\,.$$

If $\lambda + s' \geq 3/2$, then

$$|A_2|^2 \lesssim \int_{\mathbb{R}^3} \frac{|\hat{f}(\xi_*)|^2 \langle \xi_* \rangle^{2(\lambda+s'+\varepsilon)}}{\langle \xi_* \rangle^{6+2(\gamma+2s')}} d\xi_* \|M_\lambda^\delta g\|^2_{H^{s'}} \|h\|^2_{H^{s'}}$$
$$\lesssim \|f\|^2_{H^{\lambda+s'-3}} \|M_\lambda^\delta g\|^2_{H^{s'}} \|h\|^2_{H^{s'}} .$$

Since $2s' \geq 2s - 1$ and $\gamma + 2s' > 0$, we have

$$|A_3|^2 \lesssim \|\hat{f}\|^2_{L^\infty} \left(\int_{\mathbb{R}^3} \frac{d\xi_*}{\langle \xi_* \rangle^{3+\gamma+2s'}} \int_{\mathbb{R}^3} |\hat{H}(\xi)|^2 d\xi \right)$$
$$\times \left(\int_{\mathbb{R}^3} \frac{d\xi_*}{\langle \xi_* \rangle^{3+\gamma+2s'}} \int_{\mathbb{R}^3} \left(\frac{\langle \xi_* \rangle}{\langle \xi \rangle} \right)^{2\{2s'-(2s-1)\}} \mathbf{1}_{|\xi|/2 \geq \langle \xi_* \rangle} |\hat{G}(\xi - \xi_*)|^2 d\xi \right)$$
$$\lesssim \|f\|^2_{L^1} \|M_\lambda^\delta g\|^2_{H^{s'}} \|h\|^2_{H^{s'}} .$$

The above four estimates yield the desired estimate for $\mathcal{A}_1(f, g, h)$.

Next we consider $\mathcal{A}_2(f, g, h) = \mathcal{A}_{2,1}(f, g, h) - \mathcal{A}_{2,2}(f, g, h)$. The fact that $|\xi^-| = |\xi| \sin(\theta/2) \geq \langle \xi_* \rangle/2$ and $\theta \in [0, \pi/2]$ imply $\sqrt{2}|\xi| \geq \langle \xi_* \rangle$. Write

$$\mathcal{A}_{2,j} = \iint_{\mathbb{R}^6} K_j(\xi, \xi_*) \Big(M_\lambda^\delta(\xi) - M_\lambda^\delta(\xi - \xi_*) \Big) \hat{f}(\xi_*) \hat{g}(\xi - \xi_*) \bar{\hat{h}}(\xi) d\xi d\xi_* .$$

Then we have

$$|K_2(\xi, \xi_*)| = \left| \int b\Big(\frac{\xi}{|\xi|} \cdot \sigma\Big) \hat{\Phi}_c(\xi_*) \mathbf{1}_{|\xi^-| \geq \frac{1}{2}\langle \xi_* \rangle} d\sigma \right|$$
$$\lesssim \frac{1}{\langle \xi_* \rangle^{3+\gamma}} \frac{\langle \xi \rangle^{2s}}{\langle \xi_* \rangle^{2s}} \mathbf{1}_{\sqrt{2}|\xi| \geq \langle \xi_* \rangle}$$
$$\lesssim \frac{1}{\langle \xi_* \rangle^{3+\gamma}} \left\{ \mathbf{1}_{\sqrt{2}|\xi| \geq \langle \xi_* \rangle \geq |\xi|/2} + \left(\frac{\langle \xi \rangle}{\langle \xi_* \rangle} \right)^{2s} \mathbf{1}_{|\xi|/2 \geq \langle \xi_* \rangle} \right\} ,$$

which shows the desired estimate for $\mathcal{A}_{2,2}$, by exactly the same way as the estimation on A_2 and A_3.

As for $\mathcal{A}_{2,1}$, it suffices to work under the condition $|\xi_* \cdot \xi^-| \geq \frac{1}{2}|\xi^-|^2$. In fact, on the complement of this set, we have $|\xi_* - \xi^-| > |\xi_*|$, and $\hat{\Phi}_c(\xi_* - \xi^-)$ is the the same as $\hat{\Phi}_c(\xi_*)$. Therefore, we consider $\mathcal{A}_{2,1,p}$ which is defined by replacing $K_1(\xi, \xi_*)$ by

$$K_{1,p}(\xi, \xi_*) = \int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|} \cdot \sigma\Big) \hat{\Phi}_c(\xi_* - \xi^-) \mathbf{1}_{|\xi^-| \geq \frac{1}{2}\langle \xi_* \rangle} \mathbf{1}_{|\xi_* \cdot \xi^-| \geq \frac{1}{2}|\xi^-|^2} d\sigma .$$

By noting

$$\mathbf{1} = \mathbf{1}_{\langle \xi_* \rangle \geq |\xi|/2} \mathbf{1}_{\langle \xi - \xi_* \rangle \leq 2\langle \xi_* - \xi^- \rangle} + \mathbf{1}_{\langle \xi_* \rangle \geq |\xi|/2} \mathbf{1}_{\langle \xi - \xi_* \rangle > 2\langle \xi_* - \xi^- \rangle} + \mathbf{1}_{\langle \xi_* \rangle < |\xi|/2},$$

we decompose respectively

$$\mathcal{A}_{2,1,p} = B_1 + B_2 + B_3 .$$

On the sets for above integrals, we have $\langle \xi_* - \xi^- \rangle \lesssim \langle \xi_* \rangle$, because $|\xi^-| \lesssim |\xi_*|$ that follows from $|\xi^-|^2 \leq 2|\xi_* \cdot \xi^-| \lesssim |\xi^-| |\xi_*|$. Furthermore, on the sets for B_1 and B_2 we have $\langle \xi \rangle \sim \langle \xi_* \rangle$, so that $\langle \xi_* - \xi^- \rangle \lesssim \langle \xi \rangle$ and $b\ \mathbf{1}_{|\xi^-| \geq \frac{1}{2}\langle \xi_* \rangle} \mathbf{1}_{\langle \xi_* \rangle \geq |\xi|/2}$ is bounded.

Putting again $\hat G(\xi)=\langle\xi\rangle^{s'}M^\delta_\lambda(\xi)\hat g(\xi)$ and $\hat H(\xi)=\langle\xi\rangle^{s'}\hat h(\xi)$, by Lemma 3.1 we have

$$
\begin{aligned}
|B_1|^2 \lesssim \Bigg[\iiint & \left|\frac{\hat\Phi_c(\xi_*-\xi^-)}{\langle\xi_*-\xi^-\rangle^{s'}}\right|^2 |\hat f(\xi_*)|^2\\
&\times\Bigg\{M^\delta_\lambda(\xi_*)^2\left(\frac{\mathbf{1}_{\langle\xi-\xi_*\rangle\lesssim\langle\xi_*-\xi^-\rangle}}{\langle\xi-\xi_*\rangle^{2(s'+\lambda)}}+\frac{\delta^{2N_0}\mathbf{1}_{\langle\xi-\xi_*\rangle\lesssim\langle\xi_*-\xi^-\rangle}}{\langle\xi-\xi_*\rangle^{2(s'+\lambda-N_0)}}\right)\\
&+\frac{\mathbf{1}_{\langle\xi-\xi_*\rangle\lesssim\langle\xi_*-\xi^-\rangle}}{\langle\xi-\xi_*\rangle^{2s'}}\Bigg\}d\xi d\xi_* d\sigma\Bigg]\left(\iiint|\hat G(\xi-\xi_*)|^2|\hat H(\xi)|^2 d\sigma d\xi d\xi_*\right).
\end{aligned}
$$

Noting that $\langle\xi_*\rangle\sim\langle\xi\rangle\sim\langle\xi^+\rangle\lesssim\langle\xi^+-u\rangle+\langle u\rangle$ with $u=\xi_*-\xi^-$, and moreover $\langle u\rangle\lesssim\langle\xi_*\rangle$, we see that if $\lambda\ge 0$ then

$$
M^\delta_\lambda(\xi_*)^2\lesssim\frac{\langle\xi^+-u\rangle^{2\lambda}+\langle u\rangle^{2\lambda}}{(1+\delta\langle u\rangle)^{2N_0}}.
$$

This is true even if $\lambda<0$. Therefore, if $s'+\lambda<3/2$ we have

$$
\begin{aligned}
&|B_1|^2\lesssim\|f\|^2_{L^1}\int\langle u\rangle^{-(6+2\gamma+2s')}\\
&\times\Big\{\int_{\langle\xi^+-u\rangle\le\langle u\rangle}\frac{(\langle\xi^+-u\rangle^{2s'}+\langle u\rangle^{2\lambda})}{(1+\delta\langle u\rangle)^{2N_0}}\Big(\frac{1}{\langle\xi^+-u\rangle^{2(s'+\lambda)}}+\frac{\delta^{2N_0}}{\langle\xi^+-u\rangle^{2(s'+\lambda-N_0)}}\Big)d\xi^+\\
&+\int_{\langle\xi^+-u\rangle\le\langle u\rangle}\frac{d\xi^+}{\langle\xi^+-u\rangle^{2s'}}\Big\}du\ \|M^\delta_\lambda(D)g\|^2_{H^{s'}}\|h\|^2_{H^{s'}}\\
&\lesssim\|f\|^2_{L^1}\|M^\delta_\lambda(D)g\|^2_{H^{s'}}\|h\|^2_{H^{s'}}\int\frac{du}{\langle u\rangle^{3+2(\gamma+2s')}}.
\end{aligned}
$$

Here we have used the change of variables $(\xi,\xi_*)\to(\xi^+,u)$ whose Jacobian is

$$
\begin{aligned}
\Big|\frac{\partial(\xi^+,u)}{\partial(\xi,\xi_*)}\Big|&=\Big|\frac{\partial\xi^+}{\partial\xi}\Big|=\frac{\left|I+\frac{\xi}{|\xi|}\otimes\sigma\right|}{8}\\
&=\frac{|1+\frac{\xi}{|\xi|}\cdot\sigma|}{8}=\frac{\cos^2(\theta/2)}{4}\ge\frac18,\qquad\theta\in[0,\frac{\pi}{2}].
\end{aligned}
$$

If $s'+\lambda\ge3/2$, in view of $\gamma+2s'>0$ we have

$$
\begin{aligned}
|B_1|^2&\lesssim\int|\hat f(\xi_*)|^2\left\{\langle u\rangle^{2\lambda-(6+2\gamma+2s')}\log\langle u\rangle\right\}d\xi_*\ \|M^\delta_\lambda(D)g\|^2_{H^{s'}}\|h\|^2_{H^{s'}}\\
&\lesssim\|f\|^2_{H^{(\lambda+s'-3)^+}}\|M^\delta_\lambda(D)g\|^2_{H^{s'}}\|h\|^2_{H^{s'}},
\end{aligned}
$$

because $\langle u\rangle\lesssim\langle\xi_*\rangle$ on the set of the integral.

As for B_2, we first note that, on the set of the integral, $\xi^+=\xi-\xi_*+u$ implies

$$
\frac{\langle\xi-\xi_*\rangle}{2}\le\langle\xi-\xi_*\rangle-|u|\le\langle\xi^+\rangle\le\langle\xi-\xi_*\rangle+|u|\lesssim\langle\xi-\xi_*\rangle,
$$

so that

$$
(\ M^\delta_\lambda(\xi)\sim\ )\ M^\delta_\lambda(\xi^+)\sim M^\delta_\lambda(\xi-\xi_*).
$$

and hence we have by the Cauchy-Schwarz inequality

$$\begin{aligned}|B_2|^2 \lesssim & \|f\|^2_{L^1} \iiint \frac{|\hat\Phi_c(\xi_* - \xi^-)|}{\langle \xi_* - \xi^-\rangle^{2s'}} |\hat G(\xi-\xi_*)|^2 d\sigma d\xi d\xi_* \\ & \times \iiint \frac{|\hat\Phi_c(\xi_* - \xi^-)|}{\langle \xi_* - \xi^-\rangle^{2s'}} |\hat H(\xi)|^2 d\sigma d\xi d\xi_* \\ \lesssim & \|f\|^2_{L^1} \|M^\delta_\lambda(D) g\|^2_{H^{s'}} \|h\|^2_{H^{s'}} \,,\end{aligned}$$

because $\gamma + 2s' > 0$.

On the set of integral for B_3 we recall $\langle \xi\rangle \sim \langle \xi - \xi_*\rangle$ and

$$|M^\delta_\lambda(\xi) - M^\delta_\lambda(\xi-\xi_*)| \lesssim \frac{\langle \xi_*\rangle}{\langle \xi\rangle} M^\delta_\lambda(\xi-\xi_*)\,,$$

so that

$$\begin{aligned}|B_3|^2 \lesssim & \|f\|^2_{L^1} \iiint b\, \mathbf{1}_{|\xi^-|\ge \frac12 \langle \xi_*\rangle} \frac{|\hat\Phi_c(\xi_* - \xi^-)|\langle \xi_*\rangle}{\langle \xi\rangle^{2s'+1}} |\hat G(\xi-\xi_*)|^2 d\sigma d\xi d\xi_* \\ & \times \iiint b\, \mathbf{1}_{|\xi^-|\ge \frac12 \langle \xi_*\rangle} \frac{|\hat\Phi_c(\xi_* - \xi^-)|\langle \xi_*\rangle}{\langle \xi\rangle^{2s'+1}} |\hat H(\xi)|^2 d\sigma d\xi d\xi_* \,.\end{aligned}$$

We use the change of variables $\xi_* \to u = \xi_* - \xi^-$. Note that $|\xi^-| \ge \frac12 \langle u + \xi^-\rangle$ implies $|\xi^-| \ge \langle u\rangle/\sqrt{10}$, and that

$$\langle \xi_*\rangle \lesssim \langle \xi_* - \xi^-\rangle + |\xi| \sin\theta/2\,.$$

Then we have

$$\begin{aligned}\iint b\, \mathbf{1}_{|\xi^-|\ge \frac12 \langle \xi_*\rangle} \frac{|\hat\Phi_c(\xi_* - \xi^-)|\langle \xi_*\rangle}{\langle \xi\rangle^{2s'+1}} d\sigma d\xi_* \lesssim & \int \frac{\mathbf{1}_{\langle u\rangle \lesssim |\xi|}}{\langle u\rangle^{3+\gamma+2s'}} \left(\frac{\langle u\rangle}{\langle \xi\rangle}\right)^{2s'} \\ \times \Big(\int b\, \mathbf{1}_{|\xi^-|\gtrsim \langle u\rangle} \frac{\langle u\rangle}{\langle \xi\rangle} d\sigma + & \int b \sin(\theta/2) \mathbf{1}_{|\xi^-|\gtrsim \langle u\rangle} d\sigma \Big) du\,,\end{aligned}$$

from which we also can obtain the desired bound for B_3 if $\gamma + 2s' > 0$. In fact, the first integral on the sphere is bounded above by $\langle u\rangle^{1-2s}/\langle \xi\rangle^{1-2s}$ and the second integral has the same bound when $s > 1/2$. On the other hand, the second integral is bounded by a constant when $s < 1/2$ and by $|\log(\langle \xi\rangle/\langle u\rangle)|$ when $s = 1/2$. The proof of 1) and 2) of the proposition is then completed. □

The combination of Proposition 3.4 and Proposition 3.2 together with its remark yield the following theorem.

Theorem 3.5. *Assume that* $0 < s < 1, \gamma + 2s > 0$. *Let* $0 < s' < s$ *satisfy* $\gamma + 2s' > 0$, $2s' \ge (2s-1)^+$. *If a pair* (N_0, λ) *satisfies* (3.8) *then we have*
1) *If* $s' + \lambda < 3/2$, *we have*

$$\begin{aligned}&\Big|\Big(M^\delta_\lambda(D_v)\, Q(f,g) - Q(f, M^\delta_\lambda(D_v)\, g), h\Big)\Big| \\ &\qquad \lesssim \|f\|_{L^1_{\gamma^+ + (2s-1)^+}} ||M^\delta_\lambda(D_v) g||_{H^{s'}_{\gamma^+ + (2s-1)^+}} ||h||_{H^{s'}}\,.\end{aligned}$$

2) *If* $s' + \lambda \ge 3/2$, *we have*

$$\begin{aligned}&\Big|\Big(M^\delta_\lambda(D_v)\, Q(f,g) - Q(f, M^\delta_\lambda(D_v)\, g), h\Big)\Big| \\ &\qquad \lesssim \Big(\|f\|_{L^1_{\gamma^+ + (2s-1)^+}} + \|f\|_{H^{(\lambda+s'-3)^+}}\Big) ||M^\delta_\lambda(D_v) g||_{H^{s'}_{\gamma^+ + (2s-1)^+}} ||h||_{H^{s'}}\,.\end{aligned}$$

Furthermore, if $s > 1/2$ *and* $\gamma > -1$ *then the same conclusion as above holds even when the condition* (3.8) *is replaced by* (3.9). *When* $0 < s < 1/2$ *and* $\gamma > 0$, *we can use* $\|M^\delta_\lambda(D_v)g\|_{H^{s'}_{\gamma/2}}\ \|h\|_{H^{s'}_{\gamma/2}}$ *for the corresponding terms in above estimates with smaller weight in the variable* v.

We recall also the following upper bound estimate, Proposition 2.1 of [7], where we need the assumption $\gamma + 2s > 0$ (see also Theorem 2.1 from [4]).

Proposition 3.6. *Let* $\gamma + 2s > 0$ *and* $0 < s < 1$. *For any* $r \in [2s-1, 2s]$ *and* $\ell \in [0, \gamma+2s]$ *we have*

$$\Big|\Big(Q(f,g),h\Big)_{L^2(\mathbb{R}^3)}\Big| \lesssim \|f\|_{L^1_{\gamma+2s}} \|g\|_{H^r_{\gamma+2s-\ell}} \|h\|_{H^{2s-r}_\ell}\,.$$

In the following analysis, we shall need an interpolation inequality concerning weighted type Sobolev spaces in v, see for instance [10, 12].

Lemma 3.7. *For any* $k \in \mathbb{R}, p \in \mathbb{R}_+, \delta > 0$,

$$\|f\|^2_{H^k_p(\mathbb{R}^3_v)} \le C_\delta \|f\|_{H^{k-\delta}_{2p}(\mathbb{R}^3_v)} \|f\|_{H^{k+\delta}_0(\mathbb{R}^3_v)}.$$

And also another interpolation in L^q is given by

Lemma 3.8. *Let* $1 < q < p$. *Assume that* $f \in L^p$ *and* $\langle v\rangle^\ell f \in L^1$ *for any* ℓ. *Then* $\langle v\rangle^\ell f \in L^q$ *for any* ℓ. *More precisely, we have*

$$\|f\|_{L^q_\ell} \le 2\|f\|^{\frac{p(q-1)}{q(p-1)}}_{L^p} \|f\|^{\frac{p-q}{q(p-1)}}_{L^1_{\frac{\ell q(p-1)}{p-q}}}\,.$$

Proof. Take $\lambda > 0$, we rewrite

$$\begin{aligned}\|f\|^q_{L^q_\ell} &= \int_{\langle v\rangle^{\ell q}|f(v)|^{q-p}\le\lambda} \langle v\rangle^{\ell q}|f(v)|^q dv + \int_{\langle v\rangle^{\ell q}|f(v)|^{q-p}>\lambda} \langle v\rangle^{\ell q}|f(v)|^q dv\\ &\le \lambda\|f\|^p_{L^p} + \lambda^{\frac{q-1}{q-p}}\|f\|_{L^1_{\frac{\ell q(p-1)}{p-q}}}\,.\end{aligned}$$

Taking

$$\lambda = \|f\|^{\frac{(p-q)}{(p-1)}}_{L^1_{\frac{\ell q(p-1)}{p-q}}}\ \|f\|^{-\frac{p(p-q)}{(p-1)}}_{L^p},$$

we obtain the desired estimate. □

4. Smoothing effect of L^2 weak solutions

We start from a weak solution in L^2 with bounded moments.

Theorem 4.1. *Assume that* $0 < s < 1$, $\gamma+2s > 0$. *If* f *belongs to* $L^\infty([t_0,T]; L^2_\ell(\mathbb{R}^3))$ *for any* $\ell \in \mathbb{N}$ *and is a non-negative weak solution of* (1.1) , *then for any* $t_0 < \tilde t_0 < T$, *we have*

$$f \in L^\infty([\tilde t_0, T]; \mathcal{S}(\mathbb{R}^3)).$$

Proof. Without loss of generality, take $t_0 = 0$. Assume that, for $a \ge 0$, we have

$$\sup_{[0,T]} \|f(t,\cdot)\|_{H^a_\ell} < \infty \quad \text{for any } \ell \in \mathbb{N}. \tag{4.1}$$

Take $\lambda(t) = Nt + a$ for $N > 0$. Choose $N_0 = a + (5+\gamma)/2$. Then the pair $(N_0, \lambda(t))$ satisfies (3.8). If we choose $N, T_1 > 0$ such that $NT_1 = (1-s)$, then

$$\lambda(T_1) - N_0 - a \le \lambda(T_1) - N_0 < -3/2\,,$$

from which we have, for $t, t' \in [0, T_1]$,

$$M^{\delta}_{\lambda(t)} f(t') \in L^{\infty}([0, T_1] \times [0, T_1]; H^{3/2}_{\ell}(\mathbb{R}^3) \cap L^{\infty}(\mathbb{R}^3)), \tag{4.2}$$

because of (4.1). By the same way as in (3.5) and (3.6) of [13], we have

$$M^{\delta}_{\lambda(t)} f(t) \in C([0, T_1]; L^2(\mathbb{R}^3)), \tag{4.3}$$

and for any $t \in]0, T_1]$, we have

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^3} \big(M^{\delta}_{\lambda(t)} f(t)\big)^2 dv - \frac{1}{2}\int_0^t \int_{\mathbb{R}^3} f(\tau)\Big(\partial_\tau (M^{\delta}_{\lambda(\tau)})^2\Big) f(\tau) dv d\tau \\
&= \frac{1}{2}\int_{\mathbb{R}^3} \big(M^{\delta}_{\lambda(0)} f_0\big)^2 dv \\
&+ \int_0^t \Big(Q\big(f(\tau), M^{\delta}_{\lambda(\tau)} f(\tau)\big),\, M^{\delta}_{\lambda(\tau)} f(\tau)\Big)_{L^2} d\tau \\
&+ \int_0^t \Big(M^{\delta}_{\lambda(\tau)} Q\big(f(\tau), f(\tau)\big) - Q\big(f(\tau), M^{\delta}_{\lambda(\tau)} f(\tau)\big),\, M^{\delta}_{\lambda(\tau)} f(\tau)\Big)_{L^2} d\tau,
\end{aligned} \tag{4.4}$$

by taking $(M^{\delta}_{\lambda(t)})^2 f(t)$ as a test function in the definition of the weak solution, though it does not belong to $L^{\infty}([0, T_1]; W^{2,\infty}(\mathbb{R}^3))$. In fact, we can show (4.3) and (4.4) under a weaker condition than (4.2), which will be given in Lemma 4.3 below.

Noting

$$\partial_t M^{\delta}_{\lambda(t)} = N(\log\langle\xi\rangle) M^{\delta}_{\lambda(t)}\,,$$

by Theorem 3.5 we have

$$\begin{aligned}
&\frac{1}{2}\|\big(M^{\delta}_{\lambda} f\big)(t)\|^2_{L^2} \le \frac{1}{2}\|f(0)\|^2_{H^a} + \int_0^t \Big(Q(f(\tau), \big(M^{\delta}_{\lambda} f\big)(\tau)\,), \big(M^{\delta}_{\lambda} f\big)(\tau)\Big) d\tau \\
&\quad + C_f \int_0^t \|\big(M^{\delta}_{\lambda} f\big)(\tau)\|_{H^{s'}_{(2s+\gamma-1)^+}} \|\big(M^{\delta}_{\lambda} f\big)(\tau)\|_{H^{s'}} d\tau \\
&\quad + CN \int_0^t \|(\log\langle D\rangle)^{1/2}\big(M^{\delta}_{\lambda} f\big)(\tau)\|^2_{L^2} d\tau\,.
\end{aligned} \tag{4.5}$$

Since the uniform coercive estimate (2.2) together with the interpolation in the Sobolev space yields

$$\Big(Q(f(\tau), \big(M^{\delta}_{\lambda} f\big)(\tau)\,), \big(M^{\delta}_{\lambda} f\big)(\tau)\Big) \le -c_f \|\big(M^{\delta}_{\lambda} f\big)(\tau)\|^2_{H^s_{\gamma/2}} + C_f \|f(\tau)\|^2_{H^{-2}_{\gamma/2}}\,,$$

by means of Lemma 3.7 we have

(4.6)

$$\|\big(M^{\delta}_{\lambda} f\big)(t)\|^2_{L^2} + c_f \int_0^t \|\big(M^{\delta}_{\lambda} f\big)(\tau)\|^2_{H^s_{\gamma+/2}} d\tau \le \|f(0)\|^2_{H^a} + C_f \int_0^t \|f(\tau)\|^2_{H^a_{\ell}} d\tau\,.$$

Taking $\delta \to +0$ and $t = T_1$, we have $f(T_1) \in H^{\lambda(T_1)} = H^{NT_1+a}$. This is true for any $0 < T_1 \le T$. Choosing $N = (1-s)T_1^{-1}$, we have that for any $0 < T_1 \le T$,

$$f(T_1) \in H^{(1-s)+a}\,.$$

Fix $0 < s_0 < (1-s)$. Then, by using Lemma 3.7 and assumption (4.1), we see that for any $0 < t_1 < \tilde{t}_0$ and any ℓ,

$$\sup_{[t_1, T]} \|f(t, \cdot)\|_{H^{s_0+a}_{\ell}} < \infty\,.$$

We can restart by replacing a by $a+s_0=a_1$ and t_0 by t_1. By induction, for $a_0=0, a_k=k\,s_0$, and $t_k=\tilde t_0-(2k)^{-1}(\tilde t_0-t_0)$, we have for any $k\in\mathbb{N}$ and any ℓ,

$$f\in L^\infty([t_k,\,T];H^{a_k}_\ell(\mathbb{R}^3)),$$

which concludes the proof of Theorem 4.1. □

Remark 4.2. *When $0<s<1/2$ and $\gamma>0$ we can use $\int_0^t\|\big(M^\delta_\lambda f\big)(\tau)\|^2_{H^{s'}_{\gamma/2}}d\tau$ for the corresponding term in* (4.5)*. Hence, instead of* (4.6)*, we can obtain*

$$\|\big(M^\delta_\lambda f\big)(t)\|^2_{L^2}\le\|f(0)\|^2_{H^a}+C_f\int_0^t\|f(\tau)\|^2_{H^{-2}_{\gamma/2}}d\tau\,,$$

which shows that $f(t)\in L^\infty([0,T];L^2\cap L^1_2(\mathbb{R}^3))$ implies $f(t)\in H^\infty(\mathbb{R}^3)$ for $t>0$.

Lemma 4.3. *Let $T_1>0$ and let $M^\delta_{\lambda(t)}(\xi)$ be defined by* (3.1) *with $\lambda=\lambda(t)=Nt+a$ for $NT_1<1$ and $a\in\mathbb{R}$. Suppose that*

$$f\in L^1([0,T_1];L^1_{\max\{\gamma+2s,2\}}(\mathbb{R}^3))\cap L^\infty([0,T_1];H^a(\mathbb{R}^3)).$$

If there exists $s_1>s$ such that

$$M^\delta_{\lambda(t)}f(t',v)\in L^\infty([0,T_1]_t\times[0,T_1]_{t'};H^{s_1}_{\ell_0}(\mathbb{R}^3_v))$$

for $\ell_0=\max\{\gamma/2+s,\gamma^++(2s-1)^+\}$, then we have (4.3)*, and* (4.4) *for any $t\in[0,T_1]$. Furthermore, if $0<s<1/2$ and $\gamma>0$ we can take $\ell_0=\gamma/2+s$.*

Proof. In Definition 1.1, taking $\varphi(t,v)=\psi(v)\in C^\infty_0(\mathbb{R}^3)$, we get

$$\int_{\mathbb{R}^3}f(t)\psi dv-\int_{\mathbb{R}^3}f(t')\psi dv=\int_{t'}^t d\tau\int_{\mathbb{R}^3}Q(f(\tau),f(\tau))\psi dv\,,\ \ 0\le t'\le t\le T_0\,.$$

By taking a sequence $\{\psi_j(v)\}_{j=1}^\infty\subset C^\infty_0(\mathbb{R}^3_v)$ such that $(M^\delta_{\lambda(t)})^{-1}\psi_j\to M^\delta_{\lambda(t)}f(t)$ in $H^s_{\ell_0}$, we can set $\psi=(M^\delta_{\lambda(t)})^2f(t)$ for a fixed t because

$$\left|\int_{\mathbb{R}^3}f(t')(M^\delta_{\lambda(t)})^2f(t)dv\right|\le\|M^\delta_{\lambda(t)}f(t')\|_{L^2}\|M^\delta_{\lambda(t)}f(t)\|_{L^2}<\infty\,,$$

and by noting

$$(Q(f,f),(M^\delta_\lambda)^2f)=(Q(f,M^\delta_\lambda f),M^\delta_\lambda f)+(M^\delta_\lambda Q(f,f)-Q(f,M^\delta_\lambda f),M^\delta_\lambda f)\,,$$

we have

$$\begin{aligned}&\left|\int_{t'}^t d\tau\int_{\mathbb{R}^3}Q(f(\tau),f(\tau))(M^\delta_{\lambda(t)})^2f(t)dv\right|\\&\lesssim\int_{t'}^t\|f(\tau)\|_{L^1_{\gamma+2s}}d\tau\Big(\sup_{\tau,t\in[0,T_1]}\|M^\delta_{\lambda(t)}f(\tau)\|_{H^s_{\gamma/2+s}}\|M^\delta_{\lambda(t)}f(t)\|_{H^s_{\gamma/2+s}}\Big)\\&\quad+\Big(\int_{t'}^t\|f(\tau)\|_{L^1_{\gamma^++(2s-1)^+}}d\tau+|t-t'|\sup_{\tau\in[0,T_1]}\|f(\tau)\|_{H^a}\Big)\\&\qquad\times\Big(\sup_{\tau,t\in[0,T_1]}\|M^\delta_{\lambda(t)}f(\tau)\|_{H^s_{\gamma^++(2s-1)^+}}\|M^\delta_{\lambda(t)}f(t)\|_{H^s}\Big)\,,\end{aligned}$$

thanks to Proposition 3.6 and Theorem 3.5. Setting $\psi = (M^\delta_{\lambda(t')})^2 f(t')$ also and taking the sum, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \big(M^\delta_{\lambda(t)} f(t)\big)^2 dv - \int_{\mathbb{R}^3} \big(M^\delta_{\lambda(t')} f(t')\big)^2 dv \\
&\qquad = \int_{\mathbb{R}^3} f(t) \Big((M^\delta_{\lambda(t)})^2 - (M^\delta_{\lambda(t')})^2 \Big) f(t') dv \\
&\qquad\quad + \int_{t'}^{t} d\tau \int_{\mathbb{R}^3} Q(f(\tau), f(\tau)) \Big((M^\delta_{\lambda(t)})^2 f(t) + (M^\delta_{\lambda(t')})^2 f(t') \Big)\, dv\,.
\end{aligned}
\tag{4.7}
$$

Since it follows from the mean value theorem that the first term on the right hand side is estimated by

$$
|t-t'| \sup_{0\le t' < \tilde\tau < t \le T_1} \|M^\delta_{\lambda(\tilde\tau)} f(t)\|_{L^2} \|(\log\langle D\rangle) M^\delta_{\lambda(\tilde\tau)} f(t')\|_{L^2},
$$

we obtain (4.3), namely $M^\delta_{\lambda(t)} f(t) \in C([0,T_0]; L^2(\mathbb{R}^3))$.

Taking $\psi = (\log\langle D\rangle)^2 (M^\delta_{\lambda(t')})^2 f(t')$, we also have

$$
(\log\langle D\rangle) M^\delta_{\lambda(t)} f(t) \in C([0,T_0]; L^2(\mathbb{R}^3))\,.
$$

Taking the difference, instead of (4.7), we get

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \big(M^\delta_{\lambda(t)} f(t)\big)^2 dv + \int_{\mathbb{R}^3} \big(M^\delta_{\lambda(t')} f(t')\big)^2 dv \\
&\qquad = \int_{\mathbb{R}^3} f(t) \Big((M^\delta_{\lambda(t)})^2 + (M^\delta_{\lambda(t')})^2 \Big) f(t') dv \\
&\qquad\quad + \int_{t'}^{t} d\tau \int_{\mathbb{R}^3} Q(f(\tau), f(\tau)) \Big((M^\delta_{\lambda(t)})^2 f(t) - (M^\delta_{\lambda(t')})^2 f(t') \Big)\, dv\,,
\end{aligned}
$$

which shows

$$
\lim_{t'\to t} \int_{\mathbb{R}^3} f(t) \Big((M^\delta_{\lambda(t)})^2 + (M^\delta_{\lambda(t')})^2 \Big) f(t') dv = 2 \int_{\mathbb{R}^3} \big(M^\delta_{\lambda(t)} f(t)\big)^2 dv\,,
$$

and moreover

$$
\lim_{t'\to t} \int_{\mathbb{R}^3} \big(M^\delta_{\lambda(t)} f(t)\big)\big(M^\delta_{\lambda(t')} f(t')\big) dv = \int_{\mathbb{R}^3} \big(M^\delta_{\lambda(t)} f(t)\big)^2 dv\,. \tag{4.8}
$$

To prove (4.4) we introduce

$$
M^{\delta,\kappa}_{\lambda(t)} = \frac{M^\delta_{\lambda(t)}(\xi)}{1+\kappa\langle\xi\rangle}\,,
$$

with a new parameter $\kappa > 0$. Divide $[0,t]$ into k subintervals with the same length and put $t_j = jt/k$ for $j = 0, \cdots, k$. Similar to (4.7), we have

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \big(M^{\delta,\kappa}_{\lambda(t)} f(t_j)\big)^2 dv - \int_{\mathbb{R}^3} \big(M^{\delta,\kappa}_{\lambda(t_{j-1})} f(t_{j-1})\big)^2 dv \\
&\qquad = \int_{\mathbb{R}^3} f(t_j) \Big((M^{\delta,\kappa}_{\lambda(t_j)})^2 - (M^{\delta,\kappa}_{\lambda(t_{j-1})})^2 \Big) f(t_{j-1}) dv \\
&\qquad\quad + \int_{t_{j-1}}^{t_j} d\tau \int_{\mathbb{R}^3} Q(f(\tau), f(\tau)) \Big((M^{\delta,\kappa}_{\lambda(t_j)})^2 f(t_j) + (M^{\delta,\kappa}_{\lambda(t_{j-1})})^2 f(t_{j-1}) \Big)\, dv\,.
\end{aligned}
\tag{4.9}
$$

Since we have

$$
\begin{aligned}
&\int f(t_j)\Big((M^{\delta,\kappa}_{\lambda(t_j)})^2-(M^{\delta,\kappa}_{\lambda(t_{j-1})})^2\Big)f(t_{j-1})dv\\
&=\int 2Nf(t_j)(\log\langle D\rangle)(M^{\delta,\kappa}_{\lambda(\tau_j)})^2 f(t_{j-1})dv(t_j-t_{j-1})\quad \tau_j\in]t_{j-1},t_j[\\
&=2N\int\big(\sqrt{\log\langle D\rangle}M^{\delta,\kappa}_{\lambda(t_j)}f(t_j)\big)\big(\sqrt{\log\langle D\rangle}M^{\delta,\kappa}_{\lambda(t_{j-1})}f(t_{j-1})\big)dv(t_j-t_{j-1})\\
&+N^2\Big(\sup_{\tau',\tau''\in[0,T_1]}\|\log\langle D\rangle M^{\delta,\kappa}_{\lambda(\tau')}f(\tau'')\|_{L^2}\Big)^2 O\big(|t_j-t_{j-1}|^2\big)\,,
\end{aligned}
$$

it follows from a similar formula as (4.8) that

$$
\begin{aligned}
\lim_{k\to\infty}\sum_{j=1}^{k}&\int f(t_j)\Big((M^{\delta,\kappa}_{\lambda(t_j)})^2-(M^{\delta,\kappa}_{\lambda(t_{j-1})})^2\Big)f(t_{j-1})dv\\
&=N\int_0^t\int_{\mathbb{R}^3}\big(\sqrt{\log\langle D\rangle}M^{\delta,\kappa}_{\lambda(\tau)}f(\tau)\big)^2 dvd\tau\,.
\end{aligned}
$$

Summing up (4.9) with respect to $j=1,\cdots,k$ and making $k\to\infty$, we obtain

$$
\begin{aligned}
&\frac12\int_{\mathbb{R}^3}\big(M^{\delta,\kappa}_{\lambda(t)}f(t)\big)^2dv-\frac12\int_0^t\int_{\mathbb{R}^3}f(\tau)\Big(\partial_\tau(M^{\delta,\kappa}_{\lambda(\tau)})^2\Big)f(\tau)dvd\tau\\
&=\frac12\int_{\mathbb{R}^3}\big(M^{\delta,\kappa}_{\lambda(0)}f_0\big)^2dv\\
&+\int_0^t\Big(Q\big(f(\tau),M^{\delta}_{\lambda(\tau)}f(\tau)\big),\,\frac{M^{\delta}_{\lambda(\tau)}}{(1+\kappa\langle D\rangle)^2}f(\tau)\Big)_{L^2}d\tau\\
&+\int_0^t\Big(M^{\delta}_{\lambda(\tau)}Q\big(f(\tau),f(\tau)\big)-Q\big(f(\tau),M^{\delta}_{\lambda(\tau)}f(\tau)\big),\,\frac{M^{\delta}_{\lambda(\tau)}}{(1+\kappa\langle D\rangle)^2}f(\tau)\Big)_{L^2}d\tau,
\end{aligned}
\tag{4.10}
$$

thanks to Proposition 3.6 and Theorem 3.5. In fact, for example, we have

$$
\begin{aligned}
&\left|\Big(Q\big(f(\tau),M^{\delta}_{\lambda(t_j)}f(\tau)\big),\,\frac{M^{\delta}_{\lambda(t_j)}}{(1+\kappa\langle D\rangle)^2}f(t_j)\Big)_{L^2}\right|\\
&\lesssim\sup_{\tau,t_j\in[0,T_1]}\Big\{\|f(\tau)\|_{L^1_{\gamma+2s}}\|M^{\delta}_{\lambda(t_j)}f(\tau)\|_{H^s_{\gamma/2+s}}\|\frac{M^{\delta}_{\lambda(t_j)}}{(1+\kappa\langle D\rangle)^2}f(t_j)\|_{H^s_{\gamma/2+s}}\Big\}\,,
\end{aligned}
$$

and hence the Lebesgue convergence theorem yields (4.10) because,

$$
\|\frac{M^{\delta}_{\lambda(t_j)}}{(1+\kappa\langle D\rangle)^2}f(t_j)\|_{H^s_{\gamma/2+s}}\to\|\frac{M^{\delta}_{\lambda(\tau)}}{(1+\kappa\langle D\rangle)^2}f(\tau)\|_{H^s_{\gamma/2+s}}\ \text{ as } |t_j-\tau|\to 0.
$$

Taking $\kappa\to 0$ in (4.10) we obtain the desired formula. The last assertion of the lemma follows easily from the one of Theorem 3.5. □

5. Smoothing effect of L^1 weak solutions

We come back to the proof of Theorem 1.2 starting from the L^1 weak solution. The fist part of the theorem is restated as follows:

Theorem 5.1. *Assume that* $0 < s < 1, \gamma > \max\{-2s, -1\}$. *If* f *belongs to* $L^\infty([t_0, T]; L^1_\ell(\mathbb{R}^3))$ *for any* $\ell \in \mathbb{N}$ *and is a weak solution of* (1.1), *then for any* $t_0 < \tilde{t}_0 < T$, *we have*

$$f \in L^\infty([\tilde{t}_0, T]; \mathcal{S}(\mathbb{R}^3)).$$

Proof. By Theorem 4.1, it is sufficient to prove, for any $0 < t_1 \le T$, (take again $t_0 = 0$)

$$f \in L^\infty([t_1, T]; L^2_\ell(\mathbb{R}^3)). \tag{5.1}$$

Since $L^1(\mathbb{R}^3) \subset H^{-3/2-\varepsilon}$, we assume that for any ℓ and any $0 < \varepsilon << 1$

$$\sup_{[0,T]} \|f(t,\cdot)\|_{H_\ell^{-3/2-\varepsilon}} < \infty\,. \tag{5.2}$$

As in the proof of Theorem 4.1, we shall prove the theorem by induction. Assume that for $0 > a \ge -3/2 - \varepsilon$, we have

$$\sup_{[0,T]} \|f(t,\cdot)\|_{H^a_\ell} < \infty\,.$$

Take also $\lambda(t) = Nt + a$ for $N > 0$.

We first consider the case $0 < s \le 1/2$. Choose $N_0 = a + (5+\gamma)/2 \ge 1 - \varepsilon + (\gamma/2) > 0$ such that (3.8) is fulfilled. Put $\varepsilon_0 = (1-2s')/8 > 0$ and consider $\varepsilon = \varepsilon_0$, where $0 < s' < s$ is chosen to satisfy $\gamma + 2s' > 0$. If we choose $N, T_1 > 0$ such that $NT_1 = \varepsilon_0$ then

$$s + \lambda(T_1) - N_0 - a = s + \varepsilon_0 - N_0 \le s - 1 + 2\varepsilon_0 - (\gamma/2) < (s' - 1/2) + 2\varepsilon_0 < 0\,,$$

which shows

$$M^\delta_{\lambda(t)} f(t) \in L^\infty([0, T_1]; H^s_\ell(\mathbb{R}^3))\,. \tag{5.3}$$

This and Lemma 4.3 lead to (4.4), and hence we obtain (4.5) using Theorem 3.5, and (4.6) by means of (2.2) and Lemma 3.7. The same procedure as in the proof of Theorem 4.1 shows (5.1) by induction.

When $s > 1/2$ we choose $1/2 < s' < s$ such that $\gamma + 2s' > 0, 2s' \ge (2s-1)$. Choose $N_0 = a + (5+\gamma+2s'-1)/2$ such that (3.9) is satisfied. Put $\varepsilon_0 = (\gamma+1)/10 > 0$ and consider $\varepsilon = \varepsilon_0$. Then, we have

$$s + \lambda(T_1) - N_0 - a = s + \varepsilon_0 - N_0 \le s - s' + 2\varepsilon_0 - (1+\gamma)/2 = s - s' - 3\varepsilon_0\,. \tag{5.4}$$

Since we may assume $s - s' \le \varepsilon_0$, (5.4) also shows (5.3), which completes the proof of the theorem by the same way as in the case $0 < s \le 1/2$. □

In view of Remark 4.2 and the last assertion of Lemma 4.3, the proof of Theorem 5.1 in the case $0 < s < 1/2$ leads us easily to the following theorem where the assumption (1.5) can be removed.

Theorem 5.2. *Suppose that the cross section* B *of the form* (1.2) *satisfies* (1.3) *and* (1.4) *with* $0 < s < 1/2$ *and* $\gamma > 0$. *If*

$$f \in L^\infty([0,T]; L^1_{\max\{2,\gamma/2+s\}}(\mathbb{R}^3) \cap L\log L) \cap L^1([0,T]; L^1_{2+\gamma}(\mathbb{R}^3))$$

is a weak solution, then $f \in L^\infty([t_0, T];\, H^\infty(\mathbb{R}^3))$ *for any* $t_0 \in]0, T[$.

We consider now the second part of Theorem 1.2, which is stated as follows:

Theorem 5.3. *Assume that $-1 \geq \gamma > -2s$. Let $f \in L^\infty([t_0, T]; L^1_\ell(\mathbb{R}^3))$ for any $\ell \in \mathbb{N}$ be a weak solution of* (1.1) *satisfying the entropy dissipation estimate*

$$\int_{t_0}^{T} D(f(t), f(t))dt < +\infty.$$

Then for any $t_0 < \tilde{t}_0 < T$, we have

$$f \in L^\infty([\tilde{t}_0,\, T]; \mathcal{S}(\mathbb{R}^3)).$$

For the proof, we only need to reconsider the term A_1 defined in (3.12) under the hypothesis $-1 \geq \gamma > -2s$. Note that we can now choose arbitrarily large N_0 in (3.1) because neither (3.8) nor (3.9) is required. Hence $(M^\delta_{\lambda(t)})^2 f(t)$ belongs to $W^{2,\infty}$, which enable us to take $(M^\delta_{\lambda(t)})^2 f(t)$ as a test function. However $\lambda(t)$ can not be taken as large as we want, because it is also restricted to the small gain regularity coming from the dissipation estimate. Thanks to Theorem 4.1, it suffices to show $f \in L^\infty([T_0, T_1]; L^2_\ell)$ by induction, starting from (5.2) where we take again $t_0 = 0$.

It follows from (3.3) of Lemma 3.1, (3.10) and (3.11) that A_1 can be replaced by

$$\tilde{A}_{1,\lambda} = \iint_{\mathbb{R}^6} \frac{|\hat{f}(\xi_*)|}{\langle \xi_* \rangle^{3+\gamma}} |\hat{g}(\xi - \xi_*)|\, |\hat{h}(\xi)| \langle \xi \rangle^\lambda \mathbf{1}_{\langle \xi_* \rangle \geq \sqrt{2}|\xi|} \frac{\langle \xi \rangle}{\langle \xi_* \rangle} d\xi d\xi_* \tag{5.5}$$

We divide the proof in three steps.

1st step: Take $s' > 1/2$ such that $\gamma + 2s' > 0$ and $s' < s$. Put $s_0 = \frac{1}{4}(\gamma + 2s')$. For arbitrary $t > 0$ and $N > 0$ satisfying $Nt = s_0$, we set

$$\lambda_1(\tau) = N\tau - \frac{3}{2} - \varepsilon \ \text{ for } \ \tau \in [0, t]\,,$$

where $\varepsilon > 0$ is arbitrarily small. If we substitute $\lambda = \lambda_1(\tau)$ into (5.5) then, in view of $N\tau \leq s_0$, we have

$$\begin{aligned} \tilde{A}_{1.\lambda_1}(\tau) &\lesssim \|\hat{g}\|_{L^\infty} \iint_{\mathbb{R}^6} \frac{|\hat{f}(\xi_*)|}{\langle \xi_* \rangle^{3+\gamma-s_0}} \frac{|\hat{h}(\xi)|}{\langle \xi \rangle^{3/2+\varepsilon}} d\xi d\xi_* \\ &\lesssim \|\hat{f}\|_{L^{3/(2s')}} \|g\|_{L^1} \|h\|_{L^2} \lesssim \|f\|_{L^{3/(3-2s')}} \|g\|_{L^1} \|h\|_{L^2} \end{aligned}$$

because of the Hölder inequality and the fact that $(3 + \gamma - s_0)\{3/(3 - 2s')\} > 3$. By means of Lemma 3.8, we have for some $\ell_0 > 0$

$$\begin{aligned} \tilde{A}_{1,\lambda_1} &\lesssim \Big(\|\langle v \rangle^\gamma f\|_{L^{3/(3-2s)}} + \|f\|_{L^1_{\ell_0}} \Big) \|g\|_{L^1} \|h\|_{L^2} \\ &\lesssim \Big(\|\langle v \rangle^{\gamma/2} \sqrt{f}\|^2_{H^s} + \|f\|_{L^1_{\ell_0}} \Big) \|g\|_{L^1} \|h\|_{L^2}\,. \end{aligned}$$

Putting $f = g = f(\tau, v)$ and $h = M^\delta_{\lambda_1(\tau)} f(\tau, v)$, we have a term coming from $\tilde{A}_{1,\lambda_1}$ in estimating

$$\int_0^t \Big(M^\delta_{\lambda_1} Q(f(\tau), f(\tau)) - Q(f(\tau), M^\delta_{\lambda_1} f(\tau)), M^\delta_{\lambda_1} f(\tau) \Big) d\tau$$

as follows:

$$
\begin{aligned}
&\Big(\sup_{\tau\in[0,t]}\|f(\tau)\|_{L^1}\|M^\delta_{\lambda_1(\tau)}f(\tau)\|_{L^2}\Big)\int_0^t\|\langle v\rangle^{\gamma/2}\sqrt{f(\tau)}\|^2_{H^s}d\tau\\
&+\Big(\sup_{\tau\in[0,t]}\|f(\tau)\|^2_{L^1_{\ell_0}}\Big)\sqrt{t}\Big(\int_0^t\|M^\delta_{\lambda_1(\tau)}f(\tau)\|^2_{L^2}d\tau\Big)^{1/2}\\
&\le\frac{1}{10}\sup_{\tau\in[0,t]}\|M^\delta_{\lambda_1(\tau)}f(\tau)\|^2_{L^2}+C_f\Big\{\Big(\int_0^t D(f(\tau),f(\tau))d\tau\Big)^2\\
&\qquad\qquad+t\sup_{\tau\in[0,t]}\|f(\tau)\|^4_{L^1_{\ell_0}}+\int_0^t\|M^\delta_{\lambda_1(\tau)}f(\tau)\|^2_{L^2}d\tau\Big\},
\end{aligned}
$$

where we have used Corollary 2.4 in the last inequality. Instead of (4.6), we obtain

$$
\begin{aligned}
&\|M^\delta_{\lambda_1(t)}f(t)\|^2_{L^2}-\frac{1}{10}\sup_{\tau\in[0,t]}\|M^\delta_{\lambda_1(\tau)}f(\tau)\|^2_{L^2}+\int_0^t\|M^\delta_{\lambda_1(\tau)}f(\tau)\|^2_{H^{s'}}d\tau\\
&\lesssim\|M^\delta_{\lambda_1(0)}f(0)\|^2_{L^2}+\Big(\int_0^t D(f(\tau),f(\tau))d\tau\Big)^2\\
&\qquad\qquad+t\sup_{\tau\in[0,t]}\|f(\tau)\|^4_{L^1_{\ell_0}}+\int_0^t\|f(\tau)\|^2_{H^a_\ell}d\tau\,.
\end{aligned}
$$

If we consider $\tau\in[0,t]$ instead of t then the first term on the right hand side can be replaced by $\sup_{\tau\in[0,t]}\|M^\delta_{\lambda_1(\tau)}f(\tau)\|^2_{L^2}$, which absorbs the second term on the right hand side. Letting $\delta\to 0$ we obtain, in view of $Nt=s_0$,

$$\|\langle D\rangle^{s_0-3/2-\varepsilon}f(t)\|_{L^2}<\infty, \tag{5.6}$$

and

$$\int_0^t\|\langle D\rangle^{N\tau-3/2-\varepsilon}f(\tau)\|^2_{H^{s'}}d\tau<\infty. \tag{5.7}$$

2nd step: Let $\kappa>0$ be small arbitrarily. Considering $\tau\in[\kappa,t]$ instead of t in (5.6), we may assume

$$\sup_{\tau\in[\kappa,t]}\|\langle D\rangle^{s_0-3/2-\varepsilon}f(\tau)\|_{L^2}<\infty\,.$$

For arbitrary $t>\kappa$ and $N>0$ satisfying $N(t-\kappa)=s_0$ we set

$$\lambda_2(\tau)=s_0+N(\tau-\kappa)-\frac{3}{2}-\varepsilon\ \text{ for }\ \tau\in[\kappa,t]\,.$$

If we substitute $\lambda=\lambda_2(\tau)$ into (5.5) then we have

$$
\begin{aligned}
&\tilde A_{1.\lambda_2}(\tau)\lesssim\int_{\mathbb{R}^3}\frac{|\langle\xi\rangle^{s'}\hat h(\xi)|}{\langle\xi\rangle^{3/2+\varepsilon}}\\
&\Big(\int_{\mathbb{R}^3}\frac{\langle\xi_*\rangle^{s_0-\frac32-\varepsilon}|\hat f(\xi_*)|\langle\xi_*-\xi\rangle^{N(\tau-\kappa)-\frac32-\varepsilon+s'}|\hat g(\xi-\xi_*)|}{\langle\xi_*\rangle^{3+\gamma+2s'-3-2\varepsilon}}\Big(\frac{\langle\xi\rangle}{\langle\xi_*\rangle}\Big)^{1-s'}d\xi_*\Big)d\xi\\
&\lesssim\|f\|_{H^{s_0-\frac32-\varepsilon}}\|\langle D\rangle^{s'+N(\tau-\kappa)-\frac32-\varepsilon}g\|_{L^2}\|h\|_{H^{s'}}\,,
\end{aligned}
$$

if $\gamma + 2s' > 2\varepsilon$. Putting $f = g = f(\tau, v)$ and $h = M^{\delta}_{\lambda_2(\tau)} f(\tau, v)$ we have a term coming from $\tilde{A}_{1,\lambda_2}$ in estimating

$$|\int_\kappa^t \Big(M^\delta_{\lambda_2} Q(f(\tau), f(\tau)) - Q(f(\tau), M^\delta_{\lambda_2} f(\tau)), M^\delta_{\lambda_2} f(\tau) \Big) d\tau|$$

as follows:

$$\Big(\sup_{\tau\in[\kappa,t]} \|f(\tau)\|_{H^{s_0-\frac{3}{2}-\varepsilon}} \Big) \Big\{ \int_\kappa^t \|M^\delta_{\lambda_2(\tau)} f(\tau)\|^2_{H^{s'}} d\tau \Big\}$$

$$+ \int_\kappa^t \|\langle D\rangle^{s'+N(\tau-\kappa)-\frac{3}{2}-\varepsilon} f(\tau)\|^2_{L^2} d\tau \Big\} .$$

In order to avoid the confusion we write $N = N_2 = s_0/(t-\kappa)$ in this second step and $N = N_1 = s_0/t$ in (5.7). Then we have

$$N_2(\tau - \kappa) \le N_1 \tau \quad \text{if} \quad \tau \in [\kappa, t] \,,$$

from which we can use (5.7) to estimate the term coming from $\tilde{A}_{1,\lambda_2}$. In this step we finally obtain, in view of $N(t-\kappa) = s_0$,

$$\|\langle D\rangle^{2s_0-3/2-\varepsilon} f(t)\|_{L^2} < \infty$$

and

$$\int_\kappa^t \|\langle D\rangle^{s_0-3/2-\varepsilon} f(\tau)\|^2_{H^{s'}} d\tau < \infty. \tag{5.8}$$

3rd step: For $k \ge 2$, suppose that

$$\sup_{\tau\in[(k-1)\kappa,t]} \|\langle D\rangle^{(k-1)s_0-3/2-\varepsilon} f(\tau)\|_{L^2} < \infty \,.$$

For arbitrary $t > k\kappa$ and $N > 0$ satisfying $N(t - k\kappa) = s_0$ we set

$$\lambda_k(\tau) = (k-1)s_0 + N(\tau - \kappa) - \frac{3}{2} - \varepsilon \;\; \text{for} \;\; \tau \in [\kappa, t] \,.$$

Consider $M^\delta_{\lambda_k(\tau)}$. Then, using (5.8) instead of (5.7), we can proceed the induction method by the almost same way as in the second step. Since $\kappa > 0$ is arbitrary we obtain the desired conclusion.

Acknowledgements : The research of the first author was supported in part by the Zhiyuan foundation and Shanghai Jiao Tong University. The research of the second author was supported by Grant-in-Aid for Scientific Research No.22540187, Japan Society of the Promotion of Science. The research of the fourth author was supported partially by " the Fundamental Research Funds for the Central Universities". The last author's research was supported by the General Research Fund of Hong Kong, CityU No.103109, and the Lou Jia Shan Scholarship programme of Wuhan University.

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