

GLOBAL WELL POSEDNESS FOR THE NAVIER-STOKES EQUATIONS

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ABSTRACT. We investigate general topology properties to show that the set of solutions of the Navier-Stokes is homeomorphic to the set of solutions of regularized Navier-Stokes equations by adding a high-order viscosity term. This result means that the set of solutions is reduced to one solution for each dimension $d \leq 4$. We also prove a high regularity for solution to the Navier-Stokes equations.

1. INTRODUCTION

Two of the profound open problems in the theory of three dimensional viscous flow are the unique solvability theorem for all time and the regularity of solutions.

For the 3D Navier–Stokes system weak solutions of problem are known to exist by a basic result by J. Leray from 1934 [3], only the uniqueness of weak solutions remains as an open problem.

We consider in this paper a “regularized” Navier-Stokes system was proposed by J. L. Lions [4], who added the artificial hyper-viscosity $\varepsilon(-\Delta)^l$, $l > 1$, to the Navier–Stokes equation. Here $\varepsilon > 0$ is the artificial dissipation parameter.

For such a modified problem considered in a bounded domain, J. L. Lions was able to prove (cf. [4, Chap.1, Remarque 6.11]) the existence of a unique regular solution provided ($l \geq (d+2)/2$ for the d -dimensional problem). We have shown in our earlier study [9], the strong convergence of the solution of this problem to the solution of the conventional system as the regularization parameter ε goes to zero for $l \geq \sup(\frac{d}{2}, \frac{d+2}{4})$ in each $d \leq 4$.

A natural question then is to investigate the possibility of establishing such results for the conventional Navier-Stokes equations by a limit process.

We shall denote by $\mathfrak{S}_\varepsilon^l = S(f, u_{\varepsilon_0}, \varepsilon)$ the set of solutions u_ε of the regularized problem. In order to show the uniqueness of weak solutions of the Navier-Stokes equations in each dimension $d \leq 4$, we investigate general topology properties to identify $\mathfrak{S}_\varepsilon^l$ to the set of solutions u of the conventional problem. We show that the set of solutions of the Navier Stokes equations is homeomorphic to $\mathfrak{S}_\varepsilon^l$ for such l . We finish by establishing a high regularity result for the solutions of Navier-Stokes equations.

In this paper we give an answer to the uniqueness question for the Navier-Stokes equations for $d \leq 4$. It appears that this method is an interesting way to establish the uniqueness by a limit process in other branches of partial differential equations.

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2. PRELIMINARY

We denote by $H^m(\Omega)$, the Sobolev space of L -periodic functions endowed with the inner product

$$(u, v) = \sum_{|\beta| \leq m} (D^\beta u, D^\beta v)_{L^2(\Omega)} \text{ and the norm } \|u\|_m = \left(\sum_{|\beta| \leq m} \|D^\beta u\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}} \quad (2.1)$$

and by $H^{-m}(\Omega)$ the dual space of $H^m(\Omega)$. We define the spaces V_s as completions of smooth, divergence-free, periodic, zero-average functions with respect to the H^s norms. V'_s denote the dual space of V_s and V denote the space V_0 .

We denote by A the Stokes operator $Au = -\Delta u$ for $u \in D(A) = V_2 \cap V_0$. The spectral theory of A allows us to define the powers A^l of A for $l \geq 1$, A^l is an unbounded self-adjoint operator in V_0 . We set here

$$A^l u = (-\Delta)^l u \text{ for } u \in D(A^l) = V_{2l} \cap V_0. \quad (2.2)$$

Now define the trilinear form $b(\cdot, \cdot, \cdot)$ associated with the inertial terms

$$b(u, v, w) = \sum_{i,j=1}^3 \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j dx. \quad (2.3)$$

The continuity property of the trilinear form enables us to define (using Riesz representation theorem) a bilinear continuous operator $B(u, v); V_2 \times V_2 \rightarrow V'_2$ will be defined by

$$\langle B(u, v), w \rangle = b(u, v, w), \quad \forall w \in V_2. \quad (2.4)$$

Recall that for u satisfying $\nabla \cdot u = 0$ we have

$$b(u, u, u) = 0 \text{ and } b(u, v, w) = -b(u, w, v). \quad (2.5)$$

Hereafter, $c_i \in \mathbb{N}$, will denote a dimensionless scale invariant positive constant which might depend on the shape of the domain. We recall some inequalities that we will be using in what follows.

Young's inequality

$$ab \leq \frac{\sigma}{p} a^p + \frac{1}{q\sigma^{\frac{q}{p}}} b^q, \quad a, b, \sigma > 0, p > 1, q = \frac{p}{p-1}. \quad (2.6)$$

Poincaré's inequality

$$\lambda_1 \|u\|^2 \leq \|u\|_1^2 \quad (2.7)$$

for all $u \in V$, where λ_1 is the smallest eigenvalue of the Stokes operator A .

3. THE REGULARIZED NAVIER-STOKES SYSTEM

In this paper, we study the uniqueness of weak solutions of the modified Navier-Stokes equations in each dimension $d \leq 4$. We regularized the Navier-Stokes system by adding a high order artificial viscosity term to the conventional system

$$\begin{aligned} \frac{du_\varepsilon}{dt} + \varepsilon(-\Delta)^l u_\varepsilon - \nu \Delta u_\varepsilon + (u_\varepsilon \cdot \nabla) u_\varepsilon + \nabla p &= f(x), \text{ in } \Omega \times (0, \infty) \\ \operatorname{div} u_\varepsilon &= 0, \text{ in } \Omega \times (0, \infty), \\ p(x + Le_i, t) &= p(x, t), \quad u(x + Le_i, t) = u(x, t) \quad i = 1, \dots, d \quad t \in (0, \infty) \\ u_\varepsilon(x, 0) &= u_{\varepsilon 0}(x), \text{ in } \Omega, \end{aligned} \quad (3.1)$$

where $\Omega = (0, L)^d$ with periodic boundary conditions and (e_1, \dots, e_d) is the natural basis of \mathbb{R}^d . Here $\varepsilon > 0$ is the artificial dissipation parameter, u_ε is the velocity vector field, p is the pressure, $\nu > 0$ is the kinematic viscosity of the fluid and f is

a given force field. For $\varepsilon = 0$, the model is reduced to Navier-Stokes system. For further discussion of theoretical results concerning (3.1), see [1, 4, 9].

Using the operators defined in the previous section, we can write the modified system (3.1) in the evolution form

$$\begin{aligned} \partial_t u_\varepsilon + \varepsilon A^l u_\varepsilon + \nu A u_\varepsilon + B(u_\varepsilon, u_\varepsilon) &= f(x), \text{ in } \Omega \times (0, \infty) \\ u_\varepsilon(x, 0) &= u_{\varepsilon 0}, \text{ in } \Omega. \end{aligned} \quad (3.2)$$

The existence and uniqueness results for initial value problem (3.1) can be found in Lions [4].

The following theorem collects the main result in this work.

Theorem 3.1. *For $l \geq \frac{d+2}{4}$, d is the space dimension, for $\varepsilon > 0$ fixed, $f \in L^2(0, T; V_0')$ and $u_{\varepsilon 0} \in V_0$ be given. There exists a unique weak solution of (3.1) which satisfies*

$$u_\varepsilon \in L^2(0, T; V_l) \cap L^\infty(0, T; V_0), \forall T > 0.$$

Notice that the conventional Navier-Stokes system can be written in the evolution form

$$\begin{aligned} \frac{du}{dt} + \nu A u + B(u, u) &= f(x), \text{ in } \Omega \times (0, \infty) \\ u(0) &= u_0, \text{ in } \Omega. \end{aligned} \quad (3.3)$$

Theorem 3.2. *For $d \leq 4$, for $f \in L^2(0, T; V_0)$ and $u_0 \in V_0$ be given. There exists a weak solution of (3.3) which satisfies $u \in L^\infty(0, T; V_0) \cap L^2(0, T; V_1)$, for $T > 0$. For $d = 2$, u is unique (J. Lions [4]).*

In the work [9], we demonstrated the following theorem

Theorem 3.3. *For $l \geq \sup(\frac{d}{2}, \frac{d+2}{4})$ and for $d \leq 4$, the weak solution u_ε of the modified Navier-Stokes equations (3.1) given by Theorem 3.1 converges strongly in $L^2(0, T; V_0)$ as $\varepsilon \rightarrow 0$ to u a weak solution of the system (3.3).*

At the first we prove a priori estimates for the solution $u_\varepsilon(t)$, we show that the system has a global attractor.

A compact set $\mathfrak{A} \in E$ is said to be a global attractor of a semigroup $\{S(t), t > 0\}$ acting in a Banach or Hilbert space E if \mathfrak{A} is strictly invariant with respect to $\{S(t)\} : S(t)\mathfrak{A} = \mathfrak{A} \forall t \geq 0$ and \mathfrak{A} attracts any bounded set $B \subset E$:

$$\text{dist}(S(t)B, \mathfrak{A}) \rightarrow 0 \text{ when } t \rightarrow \infty, \quad (3.4)$$

see for instance Temam [8] or Chepyzhov and Vishik [5].

For $\varepsilon > 0$ and $f \in V_0$ a time independent function, let $S^\varepsilon(t)$ denote the semiflow generated by the weak solutions of the regularized Navier-Stokes equations (3.1). Thus

$$S^\varepsilon(t) = S^\varepsilon(f, t) u_{\varepsilon 0}, \quad (3.5)$$

where $u_\varepsilon(t) = S^\varepsilon(f, t) u_{\varepsilon 0}$ is the weak solution of (3.1) that satisfies $u_\varepsilon(0) = u_{\varepsilon 0}$.

Now, we show that the semigroup $S^\varepsilon(t)$ has an absorbing ball in V_0 and an absorbing ball in V_1 . Then we show that $S^\varepsilon(t)$ admits a compact attractor in V_0 for each $\varepsilon > 0$.

Theorem 3.4. *For $l \geq \frac{d+2}{4}$ in each $d \leq 4$ and $u_0 \in V_0$ be given the dynamical system associated with the regularized Navier-Stokes equations (3.1) possesses a compact attractor \mathfrak{A}_ε for all $\varepsilon > 0$ fixed, which attracts bounded sets of V_0 .*

Proof. We take the inner product of (3.1) with u_ε , we obtain

$$\frac{d}{dt} \|u_\varepsilon\|^2 + 2\varepsilon (A^l u_\varepsilon, u_\varepsilon) + 2\nu \|\nabla u_\varepsilon\|^2 = 2(f, u_\varepsilon). \quad (3.6)$$

Here we have used the fact that $b(u_\varepsilon, u_\varepsilon, u_\varepsilon) = 0$.

By applying Young's inequality (2.6) and the Poincaré's inequality (2.7), we get

$$\frac{d}{dt} \|u_\varepsilon\|^2 + 2\varepsilon \|A^{\frac{l}{2}} u_\varepsilon\|^2 + \nu \|\nabla u_\varepsilon\|^2 \leq \frac{\|f\|^2}{\nu\lambda_1}, \quad (3.7)$$

we drop the term $2\varepsilon \|A^{\frac{l}{2}} u_\varepsilon\|^2$, we obtain

$$\frac{d}{dt} \|u_\varepsilon\|^2 + \nu\lambda_1 \|u_\varepsilon\|^2 \leq \frac{\|f\|^2}{\nu\lambda_1}, \quad (3.8)$$

by integrating the above inequality from 0 to t , we get

$$\|u_\varepsilon(t)\|^2 \leq \|u_{\varepsilon 0}\|^2 e^{-\nu\lambda_1 t} + \rho_0^2 (1 - e^{-\nu\lambda_1 t}), \quad t > 0, \quad (3.9)$$

where $\rho_0 = \frac{1}{\nu\lambda_1} \|f\|$. Hence for any ball $B_{R_0} = \{u_{\varepsilon 0} \in V_0; \|u_{\varepsilon 0}\| \leq R_0\}$ there is a ball $B(0, \delta_0)$ in V_0 centered at origin with radius $\delta_0 > \rho_0$ ($R_0 > \delta_0$) such that

$$S^\varepsilon(t)B_{R_0} \subset B_{r_0} \text{ for } t \geq t_0(B_{R_0}) = \frac{1}{\nu\lambda_1} \log \frac{R_0^2 - \rho_0^2}{\delta_0^2 - \rho_0^2}. \quad (3.10)$$

The ball B_{δ_0} is said to be absorbing and invariant under the action of $S^\varepsilon(t)$. Taking the limit in (3.9) we get,

$$\limsup_{t \rightarrow \infty} \|u_\varepsilon(t)\| \leq \rho_0. \quad (3.11)$$

We integrate (3.8) from t to $t+r$, we obtain for $u_{\varepsilon 0} \in B_{R_0}$

$$\int_t^{t+r} \|u_\varepsilon\|_1^2 ds \leq \frac{1}{\nu} \left(\frac{r \|f\|^2}{\nu\lambda_1} + \|u_\varepsilon(t)\|^2 \right), \quad \forall r > 0, \quad \forall t \geq t_0(B_{R_0}). \quad (3.12)$$

With the use of (3.11) we conclude that

$$\limsup_{t \rightarrow \infty} \int_t^{t+r} \|u_\varepsilon\|_1^2 ds \leq \frac{r}{\nu^2 \lambda_1} \|f\|^2 + \frac{\|f\|^2}{\nu^3 \lambda_1^2}. \quad (3.13)$$

To show that the semigroup $S^\varepsilon(t)$ has an absorbing set in V_1 , we consider the strong solutions and take the inner product of (3.1) with Au_ε , we obtain

$$\frac{1}{2} \frac{d}{dt} \|A^{\frac{l}{2}} u_\varepsilon\|^2 + \varepsilon (A^l u_\varepsilon, Au_\varepsilon) + \nu \|Au_\varepsilon\|^2 = -b(u_\varepsilon, u_\varepsilon, Au_\varepsilon) + (f, Au_\varepsilon). \quad (3.14)$$

By applying Young's inequality (2.6), we get

$$\begin{aligned} (f, Au_\varepsilon) &\leq \|f\| \|Au_\varepsilon\| \\ &\leq \frac{\nu}{4} \|Au_\varepsilon\|^2 + \frac{1}{\nu} \|f\|^2. \end{aligned} \quad (3.15)$$

For the second member, we have by Holder's inequality that

$$|b(u_\varepsilon, u_\varepsilon, Au_\varepsilon)| \leq \|u_\varepsilon\|_\infty \|u_\varepsilon\|_1 \|Au_\varepsilon\|, \quad (3.16)$$

using (3.11) we get

$$|b(u_\varepsilon, u_\varepsilon, Au_\varepsilon)| \leq \rho_0 \|u_\varepsilon\|_1 \|Au_\varepsilon\|. \quad (3.17)$$

By applying Young's inequality (2.6), we get

$$|b(u_\varepsilon, u_\varepsilon, Au_\varepsilon)| \leq \frac{\rho_0^2}{\nu} \|u_\varepsilon\|_1^2 + \frac{\nu}{4} \|Au_\varepsilon\|^2. \quad (3.18)$$

Substituting the above result into (3.14), we obtain

$$\frac{d}{dt} \|u_\varepsilon\|_1^2 + 2\varepsilon \|A^{\frac{l+1}{2}} u_\varepsilon\|^2 + \nu \|Au_\varepsilon\|^2 \leq \frac{2}{\nu} \|f\|^2 + \frac{\rho_0^2}{\nu} \|u_\varepsilon\|_1^2. \quad (3.19)$$

If we drop the positive term associated with ε , we obtain

$$\frac{d}{dt} \|u_\varepsilon\|_1^2 \leq \frac{2}{\nu} \|f\|^2 + \frac{\rho_0^2}{\nu} \|u_\varepsilon\|_1^2. \quad (3.20)$$

We apply the uniform Gronwall Lemma to (3.20). Thanks to (3.9)-(3.13) we estimate the quantities a_1, a_2, a_3 by

$$a_1 = \frac{\rho_0^2}{\nu}, \quad a_2 = \frac{2}{\nu} \|f\|^2, \quad a_3 = \frac{r \|f\|^2}{\nu^2 \lambda_1} + \frac{\|f\|^2}{\nu^3 \lambda_1^2}. \quad (3.21)$$

Then, we obtain

$$\|u_\varepsilon(t)\|_1^2 \leq \left(\frac{a_3}{r} + a_2\right) \exp(a_1) = R_1^2 \text{ for } t \geq t_0, \quad t_0 \text{ as in (3.10)}. \quad (3.22)$$

Hence, for any ball B_{R_1} , there exists a ball B_{δ_1} , in V_1 centered at origin with radius $R_1 > \delta_1 > \rho_1$ such that

$$S^\varepsilon(t)B_{R_1} \subset B_{\delta_1} \text{ for } t \geq t_1(B_{R_0}) = t_0(B_{R_0}) + 1 + \frac{1}{\nu \lambda_1} \log \frac{R_1^2 - \rho_1^2}{\delta_1^2 - \rho_1^2}. \quad (3.23)$$

The ball B_{δ_1} is said to be absorbing and invariant for the semigroup $S^\varepsilon(t)$.

Furthermore, if B is any bounded set of V_0 , then $S^\varepsilon(t)B \subset B_{\delta_1}$ for $t \geq t_1(B, R_0)$, this shows the existence of an absorbing set in V_1 . Since the embedding of V_1 in V_0 is compact, we deduce that $S^\varepsilon(t)$ maps a bounded set in V_0 into a compact set in V_0 . In addition, the operators $S^\varepsilon(t)$ are uniformly compact for $t \geq t_1(B, R_0)$. That is,

$$\bigcup_{t \geq t_1} S^\varepsilon(t, 0, B_{R_0}) \quad (3.24)$$

is relatively compact in V_0 .

Due to a standard procedure (cf., for example, Temam [8, Theorem I.1.1] for details), there is a global attractor \mathfrak{A}_ε for the operators $S^\varepsilon(t)$ for $\varepsilon > 0$.

Note that the global attractor \mathfrak{A}_ε must be contained in the absorbing balls V_0 and V_1

$$\mathfrak{A}_\varepsilon = \bigcap_{t_1 \geq 0} \overline{\bigcup_{t \geq t_1} B_{\delta_1}(t)} \subset B_{\delta_0} \cap B_{\delta_1}. \quad (3.25)$$

□

It is well known (cf. J. L. Lions [3]) that the problem reduces to a functional equation involving only u_ε . We recall that $\mathfrak{S}_\varepsilon^l = S(\varepsilon, f, u_{\varepsilon_0})$ denotes the set of solutions u_ε of the problem (3.1) for all $\varepsilon > 0$ fixed.

We shall first list some properties of the set $\mathfrak{S}_\varepsilon^l$.

Theorem 3.5. *Assume that $d \leq 4$ and $\varepsilon > 0$ fixed, let $S(\varepsilon, f, u_{\varepsilon_0})$ the set of solutions of the modified Navier-Stokes equations (3.1) given by Theorem 3.1. Then*

- i). $S(\varepsilon, f, u_{\varepsilon_0})$ is not empty (existence of solutions),*
- ii). $S(\varepsilon, f, u_{\varepsilon_0})$ is closed and bounded in V_0 , compact in V_0 , for $l \geq \frac{d+2}{4}$.*
- iii). $S(\varepsilon, f, u_{\varepsilon_0})$ is reduced to one point (uniqueness of solutions) if $l \geq \frac{d+2}{4}$.*

Proof. The existence of solutions (point *i*) is established in [4, 6.163]. The closedness of the set $S(\varepsilon, f, u_{\varepsilon_0})$ and its compactness in V_0 , are consequences of the Theorem 3.4 . The proof of (*iii*) is also established in (cf. [4, 6.164]). \square

The set of solutions u of problem (3.3) is denoted by $\mathfrak{S}_0^1 = S(0, f, u_0)$. The set \mathfrak{S}_0^1 contains all solutions of the system (3.3).

Lemma 3.6. *Let v_n and u_n be two convergent sequences such that u_n converges to u and v_n converges to v in V_0 , then for all $\epsilon > 0$ there exists a positive number $N \in \mathbb{N}$ such that for all $n \geq N$ we have*

1).

$$\|u_n - v_n\| \leq \|u - v\| + \epsilon \quad (3.26)$$

2).

$$\|u - v\| \leq \|u_n - v_n\| + \epsilon.$$

Proof. We have

$$\|u_n - v_n\| \leq \|u_n - u\| + \|u - v\| + \|v - v_n\|$$

thus, for all $\epsilon > 0$ there exists N such that for all $n \geq N$

$$\|u_n - v_n\| \leq \epsilon/2 + \|u - v\| + \epsilon/2$$

Whence the result.

The proof of part 2) is similar to that of part 1).

$$\begin{aligned} \|u - v\| &\leq \|u_n - u\| + \|u_n - v_n\| + \|v - v_n\| \\ &\leq \|u_n - v_n\| + \epsilon, \end{aligned}$$

for all $\epsilon > 0$ and for all $n \geq N$. \square

An interesting question is then whether we can construct a continuous operator from $\mathfrak{S}_\varepsilon^l$ into \mathfrak{S}_0^1 .

Theorem 3.7. *Assume that $l \geq \frac{d}{2}$ in each $d \leq 4$, then there exists $\varepsilon' > 0$ such that for all $\varepsilon \leq \varepsilon'$, there exists a sequence of continuous functions Φ_α , from $\mathfrak{S}_\varepsilon^l$ to \mathfrak{S}_0^1 , for all $\alpha > 0$.*

Proof. Let $v_\varepsilon(t)$ and $u_\varepsilon(t)$ two weak solutions of the problem (3.1), we obtain for $w_\varepsilon(t) = v_\varepsilon(t) - u_\varepsilon(t)$

$$\frac{dw_\varepsilon}{dt} + \varepsilon A^l w_\varepsilon + \nu A w_\varepsilon + B(w_\varepsilon, v_\varepsilon) + B(v_\varepsilon, w_\varepsilon) - B(w_\varepsilon, w_\varepsilon) = 0 \quad (3.27)$$

and $w_\varepsilon(0) = v_{\varepsilon 0} - u_{\varepsilon 0} = 0$.

Taking the inner product of the last equation with w_ε , we obtain

$$\frac{d}{dt} \|w_\varepsilon\|^2 + 2\varepsilon \|A^{\frac{l}{2}} w_\varepsilon\|^2 + 2\nu \|w_\varepsilon\|_1^2 \leq |2b(w_\varepsilon, v_\varepsilon, w_\varepsilon)|. \quad (3.28)$$

The second member satisfies [4, 6.168]

$$\frac{d}{dt} \|w_\varepsilon\|^2 \leq 2c_1 \|w_\varepsilon\|^2 \|v_\varepsilon\|_1^{\frac{1}{2}}. \quad (3.29)$$

with

$$\theta + \frac{d}{4l} \leq 1. \quad (3.30)$$

Using (3.22), we obtain the following differential inequality

$$\frac{d}{dt} \|w_\varepsilon\|^2 \leq c_1 R_1^{\frac{1}{\theta}} \|w_\varepsilon\|^2. \quad (3.31)$$

Integrating from 0 to t to obtain

$$\|w_\varepsilon\|^2 \leq c_1 R_1^{\frac{1}{\theta}} \int_0^t \|w_\varepsilon\|^2 ds. \quad (3.32)$$

Integrating again from 0 to T to obtain

$$\|w_\varepsilon\|_{L^2(0,T;V_0)}^2 \leq c_1 T R_1^{\frac{1}{\theta}} \|w_\varepsilon\|_{L^2(0,T;V_0)}^2. \quad (3.33)$$

Since w_ε converges strongly in V to $w = u - v$ a weak solution of the conventional Navier-Stokes [9, Theorem 3.9.] for $l \geq \frac{d}{2}$, using 2) in Lemma 3.6 we can show that there exists a positive constant ε' , such that at the limit we find, there exists a real parameter α , such that for all $\alpha > 0$,

$$\|w\|_{L^2(0,T;V_0)}^2 \leq c_1 T R_1^{\frac{1}{\theta}} \|w_\varepsilon\|_{L^2(0,T;V_0)}^2 + \alpha \text{ for each } \varepsilon \leq \varepsilon'. \quad (3.34)$$

The inequality (3.34) describes a relationship between solutions of system (3.1) and system (3.3). We will use this relationship to define a mapping between $\mathfrak{S}_\varepsilon^l$ and \mathfrak{S}_0^1 . Now, let Φ_α be a function maps $\mathfrak{S}_\varepsilon^l$ into \mathfrak{S}_0^1 . The function Φ_α is defined as follows

$$\Phi_\alpha(u_\varepsilon) = u. \quad (3.35)$$

The inequality (3.34) means that Φ_α also satisfies

$$\|\Phi_\alpha(v_\varepsilon) - \Phi_\alpha(u_\varepsilon)\|_{L^2(0,T;V_0)} \leq k \|v_\varepsilon - u_\varepsilon\|_{L^2(0,T;V_0)} + \alpha, \quad (3.36)$$

where

$$k = c_1 T R_1^{\frac{1}{\theta}}. \quad (3.37)$$

For all $\varepsilon > 0$, there exists a $\eta \geq 0$, such that for all $v_\varepsilon, u_\varepsilon \in \mathfrak{S}_\varepsilon^l$ satisfy

$$\|v_\varepsilon - u_\varepsilon\|_{L^2(0,T;V_0)} \leq \eta \quad (3.38)$$

we get

$$\|\Phi_\alpha(v_\varepsilon) - \Phi_\alpha(u_\varepsilon)\|_{L^2(0,T;V_0)} \leq \varepsilon. \quad (3.39)$$

We have taken $\alpha = \varepsilon/2$ and $k\eta = \varepsilon/2$. This shows that Φ_α is uniformly continuous. \square

We need the following result

Lemma 3.8. *The solutions u_ε and u are bounded in $L^2(0, T; V_2)$ uniformly, in each dimension $d \leq 4$.*

Proof. If we drop the term $2\varepsilon \|A^{\frac{l+1}{2}} u_\varepsilon\|^2$ in (3.19) and integrate from 0 to T ,

$$\nu \int_0^T \|Au_\varepsilon\|^2 ds \leq \frac{2}{\nu} \int_0^T \|f\|^2 ds + \frac{\rho_0^2}{\nu} \int_0^T \|u_\varepsilon\|_1^2 ds + \|u_{\varepsilon 0}\|_1^2. \quad (3.40)$$

Using (3.22) we get,

$$\int_0^T \|Au_\varepsilon\|^2 ds \leq \frac{2}{\nu^2} \|f\|_{L^2(0,T;V)}^2 + \frac{\rho_0^2}{\nu^2} T R_1^2 + \frac{R_1^2}{\nu} = R_2. \quad (3.41)$$

This means that the strong solution u_ε belongs to $L^2(0, T; V_2)$ uniformly in ε . From this we can deduce that $u \in L^2(0, T; V_2)$ for all $u_0, u_{\varepsilon 0} \in V_0$. \square

We need the following result

Proposition 3.9. *For $l = 2$ in each $d \leq 4$ and $\varepsilon \leq \varepsilon'$, the set $\mathfrak{S}_\varepsilon^2$ is dense in \mathfrak{S}_0^1 .*

Proof. By subtracting (3.3) from (3.2), we obtain for $\xi = u_\varepsilon(t) - u(t)$

$$\partial_t \xi + \varepsilon A^l u_\varepsilon + A\xi + B(u_\varepsilon, u_\varepsilon) - B(u, u) = 0. \quad (3.42)$$

By taking inner product with ξ for the above equation we get

$$\frac{1}{2} \frac{d}{dt} \|\xi\|^2 + \varepsilon(A^l u_\varepsilon, \xi) + \nu \|A^{\frac{1}{2}} \xi\|^2 = b(\xi, \xi, u_\varepsilon). \quad (3.43)$$

The second term of left hand side of equation (3.43) can be written as

$$\begin{aligned} \varepsilon(A^l u_\varepsilon, \xi) &= \varepsilon(A^l u_\varepsilon, u_\varepsilon) - \varepsilon(A^l u_\varepsilon, u) \\ &= \varepsilon \|A^{\frac{1}{2}} u_\varepsilon\|^2 - \varepsilon(A^l u_\varepsilon, u) \\ &= \varepsilon \|A^{\frac{1}{2}} u_\varepsilon\|^2 - \varepsilon(A^{l-1} u_\varepsilon, Au). \end{aligned} \quad (3.44)$$

This leads to

$$\frac{1}{2} \frac{d}{dt} \|\xi\|^2 + \varepsilon \|A^{\frac{1}{2}} u_\varepsilon\|^2 + \nu \|A^{\frac{1}{2}} \xi\|^2 = \varepsilon(A^l u_\varepsilon, u) + b(\xi, \xi, u_\varepsilon). \quad (3.45)$$

By applying Young's inequality (2.6), we find

$$(A^{l-1} u_\varepsilon, Au) \leq \frac{1}{2\nu} \|A^{l-1} u_\varepsilon\|^2 + \frac{\nu}{2} \|Au\|^2. \quad (3.46)$$

The trilinear term can be estimated as in [4, 4.167]

$$|b(\xi, \xi, u_\varepsilon)| \leq \|\xi\|^2 \|u_\varepsilon\|_1^{1/\theta}. \quad (3.47)$$

Combining all these inequalities in (3.45), we obtain

$$\frac{1}{2} \frac{d}{dt} \|\xi\|^2 \leq \|\xi\|^2 \|u_\varepsilon\|_1^{1/\theta} + \frac{\varepsilon}{2\nu} \|A^{l-1} u_\varepsilon\|^2 + \frac{\varepsilon\nu}{2} \|Au\|^2. \quad (3.48)$$

Applying now Gronwall's inequality to (3.48), for $t \geq 0$ we get

$$\begin{aligned} \|\xi(t)\|^2 &\leq \|\xi(0)\|^2 \exp c_4 \int_0^t \|u_\varepsilon\|_1^{1/\theta} ds \\ &\quad + \frac{\varepsilon}{\nu} \int_0^t \left(\frac{1}{2\nu} \|A^{l-1} u_\varepsilon\|^2 + \frac{\nu}{2} \|Au\|^2 \right) \exp c_4 \left(\int_0^t \|u_\varepsilon\|_1^{1/\theta} ds \right) dh, \end{aligned} \quad (3.49)$$

using estimate (3.22) we find

$$\begin{aligned} \|\xi(t)\|^2 &\leq \|\xi(0)\|^2 \exp c_4 T R_1^{\frac{1}{\theta}} \\ &\quad + \frac{\varepsilon \exp c_4 T R_1^{\frac{1}{\theta}}}{\nu} \int_0^t \left(\frac{1}{2\nu} \|A^{l-1} u_\varepsilon\|^2 + \frac{\nu}{2} \|Au\|^2 \right) dh. \end{aligned} \quad (3.50)$$

Since $\xi(0) = u_\varepsilon(0) - u(0) = 0$, this means

$$\|\xi(t)\|^2 \leq \varepsilon \frac{\exp c_4 T R_1^{\frac{1}{\theta}}}{\nu} \int_0^t \left(\frac{1}{2\nu} \|A^{l-1} u_\varepsilon\|^2 + \frac{\nu}{2} \|Au\|^2 \right) dh. \quad (3.51)$$

It follows from Lemma 3.8 (with $l = 2$) that for each $u \in \mathfrak{S}_0^1$ there exists a sequence $u_\varepsilon \in \mathfrak{S}_\varepsilon^2$ such that $\|u_\varepsilon - u\| \rightarrow 0$ as $\varepsilon \rightarrow 0$, from these facts it follows that $\mathfrak{S}_\varepsilon^2$ is dense in \mathfrak{S}_0^1 . \square

Since the uniqueness of weak solutions of the Navier-Stokes equations is not known. It may happen that $\mathfrak{S}_0^1 = S(0, f, u_0)$ the set of solutions of the conventional Navier-Stokes equations (3.3) is not reduced to one point. there can exist “ghost solutions” that solve the original problem, but are not limits of the perturbed problem. For this, we consider two subsets A and B such that $\mathfrak{S}_0^1 = A \cup B$.

We say that $u \in A$ if u is a limit of the perturbed problem and $u \in B$ if u is not limit of the perturbed problem. We clearly have $\overline{A} \cap B = \emptyset$, it follows that $d_V(A, B) > 0$ (d is the Hausdorff distance).

Proposition 3.10. *For $l = 2$ in each $d \leq 4$ and $\varepsilon \leq \varepsilon'$, the two sets $\mathfrak{S}_\varepsilon^2$ and \mathfrak{S}_0^1 are homeomorphic.*

Proof. Since the limits solutions are in A

$$A = \Phi_\alpha(\mathfrak{S}_\varepsilon^2). \quad (3.52)$$

A is a compact set. Thus, $\mathfrak{S}_\varepsilon^2$ is dense in \mathfrak{S}_0^1 , $\mathfrak{S}_\varepsilon^2$ is dense in $A \cup B$, then for each $u' \in B$ and for all $\varepsilon > 0$,

$$d(u', A) = d(u', u) \leq d(u', u_\varepsilon) + d(u_\varepsilon, u) \leq \varepsilon, \quad (3.53)$$

it follows that $u' \in \overline{A}$, but $A = \overline{A}$, we conclude that $A = B$. Consequently,

$$\Phi_\alpha(\mathfrak{S}_\varepsilon^2) = \mathfrak{S}_0^1 = A \text{ for all } \varepsilon \leq \varepsilon' \quad (3.54)$$

and Φ_α is surjective from $\mathfrak{S}_\varepsilon^2$ into \mathfrak{S}_0^1 . The injectivity is a consequence of the uniqueness of the solution of the perturbed system. Since $\mathfrak{S}_\varepsilon^2$ is a compact set, Φ_α is a homeomorphism. \square

Note that all above results are satisfied for $l = 2$. Our main result is then

Theorem 3.11. *For each dimension $d \leq 4$, let $f \in L^2(0, T; V_0)$ and $u_0 \in V_0$ be given. Then there exists a **unique** weak solution u of the Navier-Stokes equations (3.3) which satisfies $u \in L^\infty(0, T; V_0) \cap L^2(0, T; V_1)$, $\forall T > 0$.*

Proof. The existence of a solution to (3.3) follows from Theorem 3.2. We will establish the uniqueness result for solutions of the Navier-Stokes equations (3.3). For $l = 2$ in each $d \leq 4$ and for fixed $\varepsilon \leq \varepsilon'$, the set $\mathfrak{S}_\varepsilon^2$ is reduced to one point and from Proposition 3.10, the map Φ_α is a homeomorphism for all $\alpha > 0$, that is

$$\mathfrak{S}_0^1 = \Phi_\alpha(\mathfrak{S}_\varepsilon^2) = \{u\}. \quad (3.55)$$

Thus, the problem (3.3) has a unique solution. \square

Finally we finish by a regularity theorem

Theorem 3.12. *Assume that $l \geq \frac{d}{2}$ in each $d \leq 4$, then there exists a finite time T^* depending on the data such that for any $t \leq T^*$ there exists a unique solution of the Navier-Stokes equations (3.3) satisfying*

$$u \in L^2(0, T^*; V_1) \cap L^\infty(0, T^*; V_0). \quad (3.56)$$

Proof. Let C be a nonempty subset of a Banach space X and fix a sequence $\{a_n\}$ in $[0, \infty)$ with $a_n \rightarrow 0$. A mapping $\Theta : C \rightarrow C$ will be called nearly Lipschitzian with respect to $\{a_n\}$ if for each $n \in \mathbb{N}$, there exists a constant $k_n \geq 0$ such that

$$\|\Theta^n(u_\varepsilon) - \Theta^n(v_\varepsilon)\| \leq k_n(\|u_\varepsilon - v_\varepsilon\| + a_n) \text{ for all } u_\varepsilon, v_\varepsilon \in C, \quad (3.57)$$

(cf. D. R. Sahu [7]). The infimum of constants k_n for which (3.57) holds will be called nearly Lipschitz constant. A nearly Lipschitzian mapping Φ_{α_n} with sequence $\{(\alpha_n, k_n)\}$ is said to be nearly contraction if

$$\lim_{n \rightarrow \infty} (k_n)^{\frac{1}{n}} < 1 \text{ for all } n \in \mathbb{N}. \quad (3.58)$$

From (3.36) we find that

$$\|\Phi_{\alpha_n}^n(u_\varepsilon) - \Phi_{\alpha_n}^n(v_\varepsilon)\| \leq k^n \|u_\varepsilon - v_\varepsilon\| + a_n \text{ for all } u_\varepsilon, v_\varepsilon \in \mathfrak{S}_\varepsilon^l,$$

with $a_n = \alpha_n(1 + k + \dots + k^n)$. We assume that $\alpha_n = k^{2n}$, then

$$\|\Phi_{\alpha_n}^n(u_\varepsilon) - \Phi_{\alpha_n}^n(v_\varepsilon)\| \leq k^n (\|u_\varepsilon - v_\varepsilon\| + R_n), \quad (3.59)$$

where

$$R_n = k^{n+1} + k^{n+2} + \dots + k^{2n+1}. \quad (3.60)$$

Note that (3.58) implies that there exists a $T^* < c_2 R_1^{\frac{-1}{2\theta}}$, such that for any $t \leq T^*$

$$k < 1. \quad (3.61)$$

Since R_n is the n^{th} remainder of a convergent series, it follows that

$$\lim_{n \rightarrow \infty} R_n = 0. \quad (3.62)$$

Thus Φ_{α_n} is a nearly contracting Lipschitz map on $\mathfrak{S}_\varepsilon^l|_{(0, T^*)}$. As a result, Φ_{α_n} has a unique fixed point such that

$$\Phi_\alpha(u_\varepsilon) = u_\varepsilon = u. \quad (3.63)$$

This fixed point is the limit of the solution of equations (3.1) as ε goes to zero on $(0, T^*)$, which is also unique because of the contraction property. \square

This result gives a new way to the uniqueness and regularity of the Navier-Stokes equations. This method can also be applied to other nonlinear partial differential equations to study the questions of the uniqueness and regularity of solutions.

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