

PERTURBATIONS OF DIAGONAL MATRICES BY BAND RANDOM MATRICES

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ABSTRACT. We exhibit an explicit formula for the spectral density of a (large) random matrix which is a diagonal matrix whose spectral density converges, perturbed by the addition of a symmetric matrix with Gaussian entries and a given (small) limiting variance profile.

1. PERTURBATION OF THE SPECTRAL DENSITY OF A LARGE DIAGONAL MATRIX

In this paper, we consider the spectral measure of a random matrix D_n^ε defined by $D_n^\varepsilon = D_n + \sqrt{\frac{\varepsilon}{n}}X_n$, for D_n a deterministic diagonal matrix whose spectral measure converges and X_n an Hermitian or real symmetric matrix whose entries are Gaussian independent variables, with a limiting variance profile (such matrices are called *band matrices*). We give a first order Taylor expansion, as $\varepsilon \rightarrow 0$, of the limit spectral density, as $n \rightarrow \infty$, of D_n^ε .

The proof is elementary and based on a formula given in [12] for the Cauchy transform of the limit spectral distribution of D_n^ε as $n \rightarrow \infty$.

For each n , we consider an Hermitian or real symmetric random matrix $X_n = [x_{i,j}^n]_{i,j=1}^n$ and a real diagonal matrix $D_n = \text{diag}(a_n(1), \dots, a_n(n))$. We suppose that:

- (a) the entries $x_{i,j}^n$ of X_n are independent (up to symmetry), centered, Gaussian with variance denoted by $\sigma_n^2(i, j)$,
- (b) for a certain bounded function σ defined on $[0, 1] \times [0, 1]$ and a certain bounded real function f defined on $[0, 1]$, we have, in the L^∞ topology,

$$\sigma_n^2(\lfloor nx \rfloor, \lfloor ny \rfloor) \xrightarrow{n \rightarrow \infty} \sigma^2(x, y) \quad \text{and} \quad a_n(\lfloor nx \rfloor) \xrightarrow{n \rightarrow \infty} f(x),$$
- (c) the set of discontinuities of the function σ is closed and intersects a finite number of times any vertical line of the square $[0, 1]^2$.

For $\varepsilon \geq 0$, let us define, for all n ,

$$D_n^\varepsilon = D_n + \sqrt{\frac{\varepsilon}{n}}X_n.$$

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It is known, from Shlyakhtenko in [12, Th. 4.3] (see also [2], which also provides a fluctuation result), that as n tends to infinity, the spectral distribution of D_n^ε tends to a limit μ_ε with Cauchy transform

$$C_\varepsilon(z) = \int_{x=0}^1 C_\varepsilon(x, z) dx,$$

where $C_\varepsilon(x, \cdot)$ is defined by the fact that it is analytic, maps the upper half-plane \mathbb{C}^+ into the lower one \mathbb{C}^- , and satisfies the relation

$$(1) \quad C_\varepsilon(x, z) = \frac{1}{z - f(x) - \varepsilon \int_{y=0}^1 \sigma^2(x, y) C_\varepsilon(y, z) dy}.$$

Our goal here is to understand $\mu_\varepsilon - \mu$ for small values of ε . Let us introduce the set \mathcal{T} of test functions we shall use here. We define

$$\mathcal{T} = \left\{ t \mapsto \frac{1}{z - t}; z \in \mathbb{C}^+ \right\}.$$

Let us now define the *Hilbert transform*, denoted by $H[u]$, of a function u :

$$H[u](s) := \text{p. v.} \int_{t \in \mathbb{R}} \frac{u(t)}{s - t} dt = \int_{y \in \mathbb{R}} \frac{u(s - y) - u(s)}{y} dy.$$

Before stating our main result, let us make some assumptions on the functions σ and f :

- (d) the push-forward μ of the uniform measure on $[0, 1]$ by the function f has a density ρ with respect to the Lebesgue measure on \mathbb{R} ,
- (e) there exists a symmetric function $\tau(\cdot, \cdot)$ such that for all x, y , $\sigma^2(x, y) = \tau(f(x), f(y))$,
- (f) there exist $\eta_0 > 0, \alpha > 0$ and $C < \infty$ such that for almost all $s \in \mathbb{R}$, for all $t \in [s - \eta_0, s + \eta_0]$, $|\tau(s, t)\rho(t) - \tau(s, s)\rho(s)| \leq C|t - s|^\alpha$.

Note that by hypothesis (f) and by the boundedness of the function f , the function

$$s \mapsto \rho(s)H[\tau(s, \cdot)\rho(\cdot)](s)$$

is well defined and compactly supported.

Theorem 1. *Under the hypotheses (a) to (f), as $\varepsilon \rightarrow 0$, for all $g \in \mathcal{T}$,*

$$\int g(s) d\mu_\varepsilon(s) = \int g(s) d\mu(s) - \varepsilon \int g'(s) F(s) ds + o(\varepsilon),$$

with $F(s) := -\rho(s)H[\tau(s, \cdot)\rho(\cdot)](s)$.

As a consequence, if the function $F(\cdot)$ has bounded variations, then

$$\mu_\varepsilon = \mu + \varepsilon dF + o(\varepsilon).$$

Remark 1. *Roughly speaking, this theorem states that*

$$\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{\text{spectral law}(D_n^\varepsilon) - \text{spectral law}(D_n)}{\varepsilon} = dF.$$

It would be interesting to let ε and n tend to 0 and ∞ together, and to find out the adequate rate of convergence to get a deterministic limit or non degenerated fluctuations. We are working on this question.

Remark 2. *This result provides an analogue, for our random matrix model, of the following formula about real random variables (valid when Y is centered and independent of X):*

$$\text{density}_{X+\sqrt{\varepsilon}Y}(s) = \text{density}_X(s) + \varepsilon \frac{\mathbb{E}[Y^2]}{2} \text{density}_X''(s) + o(\varepsilon).$$

Remark 3. *In the case where X_n is a GUE or GOE matrix, the limiting spectral distribution of D_n^ε as $n \rightarrow \infty$ is the free convolution of the limiting spectral distribution of D_n with a semi-circle distribution. Several papers are devoted to the study of qualitative properties (like regularity) of the free convolution (see [8, 7, 4, 3, 6]). Besides, it has recently been proved that type-B free probability theory allows to give Taylor expansions, for small values of t , of the moments of $\mu_t \boxplus \nu_t$ for two time-dependent probability measures μ_t and ν_t (see [5, 10, 9]). Our work differs from the ones mentioned above by the fact that we allow to perturb D_n by any band matrix, but also by the fact that it is focused on the density and not on the moments, giving an explicit formula rather than qualitative properties.*

Proof. For all $z \in \mathbb{C}^+$, we have

$$(2) \quad |C_\varepsilon(x, z)| \leq \frac{1}{\Im z}.$$

Indeed, for all y, z such that $z \in \mathbb{C}^+$, $C_\varepsilon(y, z) \in \mathbb{C}^-$. As a consequence, the imaginary part of the denominator of the right hand term of (1) is larger than $\Im(z)$.

Hence by (1) and (2), as $\varepsilon \rightarrow 0$, $C_\varepsilon(x, z) \rightarrow \frac{1}{z-f(x)}$ uniformly in x .

From what precedes,

$$\begin{aligned} C_\varepsilon(x, z) - \frac{1}{z-f(x)} &= \frac{\varepsilon \int_{y=0}^1 \sigma^2(x, y) C_\varepsilon(y, z) dy}{(z-f(x) - \varepsilon \int_{y=0}^1 \sigma^2(x, y) C_\varepsilon(y, z) dy)(z-f(x))} \\ &= \varepsilon \frac{1}{(z-f(x))^2} \int_{y=0}^1 \sigma^2(x, y) C_\varepsilon(y, z) dy + o(\varepsilon) \\ &= \varepsilon \frac{1}{(z-f(x))^2} \int_{y=0}^1 \frac{\sigma^2(x, y)}{z-f(y)} dy + o(\varepsilon) \end{aligned}$$

where each $o(\varepsilon)$ is uniform in $x \in [0, 1]$.

But for all $a \neq b$, $\frac{1}{(z-a)^2(z-b)} = \frac{1}{(a-b)^2} \left(\frac{1}{z-b} - \frac{1}{z-a} - \frac{b-a}{(z-a)^2} \right)$, hence since the Lebesgue measure of the set $\{y \in [0, 1]; f(y) = f(x)\}$ is null, we have

$$\frac{1}{(z-f(x))^2} \int_{y=0}^1 \frac{\sigma^2(x, y)}{z-f(y)} dy = \int_{y=0}^1 \frac{\sigma^2(x, y)}{(f(x)-f(y))^2} \left(\frac{1}{z-f(y)} - \frac{1}{z-f(x)} - \frac{f(y)-f(x)}{(z-f(x))^2} \right) dy.$$

As a consequence, it follows by an integration in $x \in [0, 1]$ that

$$C_\varepsilon(z) - C(z) = \varepsilon \int_{x=0}^1 \int_{y=0}^1 \frac{\sigma^2(x, y)}{(f(x)-f(y))^2} \left(\frac{1}{z-f(y)} - \frac{1}{z-f(x)} - \frac{f(y)-f(x)}{(z-f(x))^2} \right) dy dx + o(\varepsilon),$$

where $C(\cdot)$ is the Cauchy transform of μ .

Let us now recall that the push-forward of the uniform law on $[0, 1]$ by f is the measure $\rho(x)dx$ and that $\sigma^2(x, y)$ can be rewritten $\sigma^2(x, y) = \tau(f(x), f(y))$. Hence

$$C_\varepsilon(z) - C(z) = \varepsilon \int_{s \in \mathbb{R}} \int_{t \in \mathbb{R}} \left\{ \frac{1}{z-t} - \frac{1}{z-s} - \frac{1}{(z-s)^2} (t-s) \right\} \frac{\tau(s, t)}{(s-t)^2} \rho(s)\rho(t) dt ds + o(\varepsilon).$$

This allows us to write that for any test function $g \in \mathcal{T}$,

$$\lim_{\varepsilon \rightarrow 0} \frac{\mu_\varepsilon(g) - \mu(g)}{\varepsilon} = \Lambda(g),$$

where

$$\Lambda(g) = \int_{(s,t) \in \mathbb{R}^2} \{g(t) - g(s) - g'(s)(t-s)\} \frac{\tau(s,t)}{(t-s)^2} \rho(s)\rho(t) dt ds.$$

Note that by the Taylor-Lagrange formula, for all s, t ,

$$\left| \{g(t) - g(s) - g'(s)(t-s)\} \frac{\tau(s,t)}{(t-s)^2} \rho(s)\rho(t) \right| \leq \frac{\rho(s)\rho(t) \times \|\tau(\cdot, \cdot)\|_{L^\infty} \|g''\|_{L^\infty}}{2},$$

so that, since ρ is a density, by dominated convergence,

$$\Lambda(g) = \lim_{\eta \rightarrow 0} \int_{\substack{(s,t) \in \mathbb{R}^2 \\ |s-t| > \eta}} \{g(t) - g(s) - g'(s)(t-s)\} \frac{\tau(s,t)}{(t-s)^2} \rho(s)\rho(t) ds dt.$$

But by symmetry, for all $\eta > 0$,

$$\int_{\substack{(s,t) \in \mathbb{R}^2 \\ |s-t| > \eta}} \{g(t) - g(s)\} \frac{\tau(s,t)}{(t-s)^2} \rho(s)\rho(t) ds dt = 0.$$

As a consequence, $\Lambda(g) = \lim_{\eta \rightarrow 0} \Lambda_\eta(g)$, with

$$\Lambda_\eta(g) := \int_{\substack{(s,t) \in \mathbb{R}^2 \\ |s-t| > \eta}} g'(s) \frac{\tau(s,t)}{s-t} \rho(s)\rho(t) ds dt.$$

Let us prove that almost all $s \in \mathbb{R}$, $\lim_{\eta \rightarrow 0} \int_{\substack{t \in \mathbb{R} \\ |s-t| > \eta}} \frac{\tau(s,t)\rho(s)\rho(t)}{s-t} dt$ exists and that

$$\Lambda(g) = \int_{s \in \mathbb{R}} g'(s) \left(\lim_{\eta \rightarrow 0} \int_{\substack{t \in \mathbb{R} \\ |s-t| > \eta}} \frac{\tau(s,t)\rho(s)\rho(t)}{s-t} dt \right) ds.$$

For $\eta > 0$ and $s \in \mathbb{R}$, set

$$\theta_\eta(s) := \int_{\substack{t \in \mathbb{R} \\ |s-t| > \eta}} \frac{\tau(s,t)\rho(s)\rho(t)}{s-t} dt.$$

Set also $M := \|f\|_{L^\infty}$. Then the support of the function ρ is contained in $[-M, M]$, and so does the support of the function θ_η , for any $\eta > 0$. For almost all $s \in [-M, M]$, $\lim_{\eta \rightarrow 0} \theta_\eta(s)$ exists by the formula

$$\theta_\eta(s) = \int_{t \in [s-2M, s-\eta] \cup [s+\eta, s+2M]} \frac{\tau(s,t)\rho(s)\rho(t) - \tau(s,s)\rho(s)\rho(s)}{s-t} dt$$

and by Hypothesis (f). Moreover, for η_0 as in Hypothesis (f),

$$\begin{aligned} |\theta_\eta(s)| &\leq 2C\rho(s) \int_{t=s+\eta}^{s+\eta_0} (s-t)^{\alpha-1} dt + \int_{t \in [s-2M, s-\eta_0] \cup [s+\eta_0, s+2M]} \frac{\tau(s,t)\rho(s)\rho(t)}{s-t} dt \\ &\leq \frac{2C\rho(s)}{\alpha} (\eta_0)^\alpha + \frac{1}{\eta_0} \int_{t \in \mathbb{R}} \tau(s,t)\rho(s)\rho(t) ds dt \\ &\leq \frac{2C\rho(s)}{\alpha} (\eta_0)^\alpha + \frac{\|\tau(\cdot, \cdot)\|_{L^\infty}}{\eta_0} \rho(s). \end{aligned}$$

Hence by dominated convergence, $\int_{s \in \mathbb{R}} g'(s) \lim_{\eta \rightarrow 0} \theta_\eta(s) ds = \lim_{\eta \rightarrow 0} \int_{s \in \mathbb{R}} g'(s) \theta_\eta(s) ds$, i.e.

$$\Lambda(g) = \int_{s \in \mathbb{R}} g'(s) \left(\lim_{\eta \rightarrow 0} \int_{\substack{t \in \mathbb{R} \\ |s-t| > \eta}} \frac{\tau(s,t)\rho(s)\rho(t)}{s-t} dt \right) ds.$$

□

2. EXAMPLES

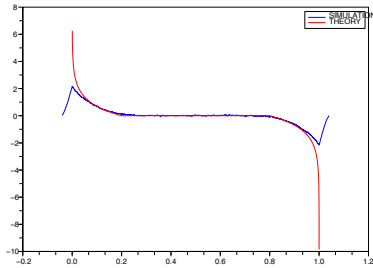
2.1. Perturbation of a uniform distribution by a standard band matrix. Let us consider the case where $f(x) = x$ (so that μ is the uniform distribution on $[0, 1]$) and $\sigma^2(x, y) = \mathbb{1}_{|y-x| \leq \ell}$, where ℓ is a fixed parameter in $[0, 1]$ (the width of the band). In this case, $\tau(\cdot, \cdot) = \sigma^2(\cdot, \cdot)$ and

$$F(s) = \mathbb{1}_{(0,1)}(s) \log \left(\frac{\ell \wedge (1-s)}{\ell \wedge s} \right).$$

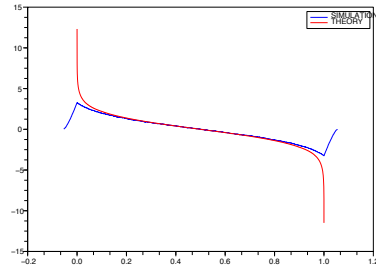
For small values of ε and large values of n , the density ρ_ε of the eigenvalue distribution μ_ε of D_n^ε is approximately

$$\rho_\varepsilon(s) = \rho(s) + \varepsilon \frac{\partial}{\partial s} F(s) + o(\varepsilon) = \mathbb{1}_{(0,1)}(s) - \varepsilon \left(\frac{\mathbb{1}_{(0,\ell)}(s)}{s} + \frac{\mathbb{1}_{(1-\ell,1)}(s)}{1-s} \right) + o(\varepsilon),$$

which means that the additive perturbation $\sqrt{\frac{\varepsilon}{n}} X_n$ alters the spectrum of D_n essentially by decreasing the amount of extreme eigenvalues. This phenomenon is illustrated by Figure 1 (where we plotted the cumulative distribution functions rather than the densities for visual reasons).



(a) Case where $n = 4.10^3$, $\varepsilon = 10^{-2}$, with width $\ell = 0.2$



(b) Case where $n = 4.10^3$, $\varepsilon = 10^{-2}$, with width $\ell = 0.9$

FIGURE 1. Perturbation of a uniform distribution by a standard band matrix: plot of the functions $F(\cdot)$ and $\frac{F_{D_n^\varepsilon}(\cdot) - F_{D_n}(\cdot)}{\varepsilon}$ (with $F_{D_n^\varepsilon}(\cdot)$ and $F_{D_n}(\cdot)$ the cumulative eigenvalue distribution functions of D_n^ε and D_n) for different values of ℓ .

2.2. Perturbation of the triangular pulse distribution by a GOE matrix. Let us consider the case where $\rho(x) = (1 - |x|)\mathbb{1}_{[-1,1]}(x)$ and $\sigma^2 \equiv 1$ (what follows can be adapted to the case $\sigma^2(x, y) = \mathbb{1}_{|y-x| \leq \ell}$, but the formulas are a bit heavy). In this case, thanks to the formula (9.6) of $H[\rho(\cdot)]$ given p. 509 of [11], we get

$$F(s) = (1 - |s|)\mathbb{1}_{[-1,1]}(s) \{ (1 - s) \log(1 - s) - (1 + s) \log(1 + s) + 2s \log |s| \}.$$

For small values of ε and large values of n , the density ρ_ε of the eigenvalue distribution μ_ε of D_n^ε is approximately

$$\rho_\varepsilon(s) = \rho(s) + \varepsilon \frac{\partial}{\partial s} F(s) + o(\varepsilon),$$

which implies that the additive perturbation $\sqrt{\frac{\varepsilon}{n}} X_n$ alters the spectrum of D_n by increasing the amount of eigenvalues in $[-1, -0.5] \cup [0.5, 1]$ and decreasing the amount of eigenvalues around zero. This phenomenon is illustrated by Figure 2.

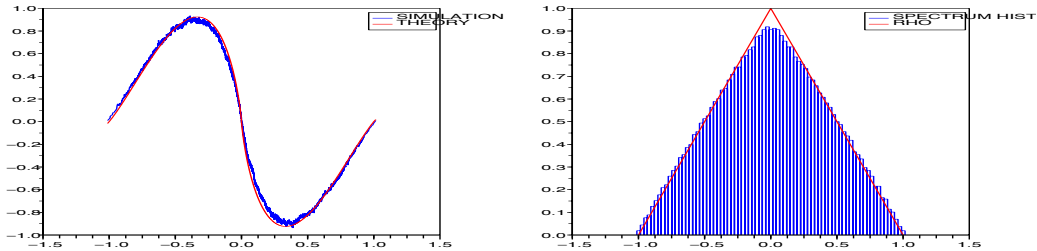


FIGURE 2. **Perturbation of the triangular pulse distribution by a GOE matrix:** *Left:* plot of the functions $F(\cdot)$ and $\frac{F_{D_n^\varepsilon}(\cdot) - F_{D_n}(\cdot)}{\varepsilon}$ (with $F_{D_n^\varepsilon}(\cdot)$ and $F_{D_n}(\cdot)$ the cumulative eigenvalue distribution functions of D_n^ε and D_n). *Right:* plot of the eigenvalues histogram of D_n^ε and of the spectral density ρ of D_n . On the right figure, the (infinitesimal) increase of eigenvalues with respect to ρ on $[-1, -0.5] \cup [0.5, 1]$ and the (infinitesimal) decrease around zero can be observed, in agreement with the fact that, as the left figure shows, $F' \gg 0$ on (approximately) $[-1, -0.5] \cup [0.5, 1]$ and $F' \ll 0$ around zero. Both figures were made with the same simulation ($n = 6.10^3$ and $\varepsilon = 10^{-2}$).

2.3. Free convolution with a semi-circular distribution and complex Burger's equation. Let us consider the case where $\sigma^2 \equiv 1$, which happens for example if the matrix X_n is taken in the Gaussian Orthogonal Ensemble. In this case, by the theory of free probability developed by Dan Voiculescu (see e.g. [13] or [1, Cor 5.4.11 (ii)]), for all $t \geq 0$,

$$\mu_t = \mu \boxplus \lambda_t,$$

where λ_t is the *semi-circular distribution with variance t* , i.e. the distribution with support $[-2\sqrt{t}, 2\sqrt{t}]$ and density $\frac{1}{2\pi t} \sqrt{4t - x^2}$. In this case, we know by the work of Biane [8, Cor. 2] that for all $t > 0$, μ_t admits a density ρ_t . By the implicit function theorem, and the formula given in [8, Cor. 2], one easily sees that the function $(s, t) \mapsto \rho_t(s)$ is regular. Then, by Theorem 1 and the fact that the linear span of \mathcal{T} is dense in the set of continuous functions on the real line with null limit at infinity, one easily recovers the following PDE, which is a kind of projection on the real axis of the imaginary part of complex Burger's equation given in [8, Intro.]

$$(3) \quad \begin{cases} \frac{\partial}{\partial t} \rho_t(s) + \frac{\partial}{\partial s} \{ \rho_t(s) H[\rho_t(\cdot)](s) \} = 0, \\ \rho_0(s) = \rho(s). \end{cases}$$

For example, if $\mu = \lambda_c$ for a certain $c > 0$, then by the semi-group property of the semi-circle distribution [1, Ex. 5.3.26], for all $t \geq 0$, $\mu_t = \lambda_{c+t}$ and $\rho_t(s) = \frac{1}{2\pi(c+t)} \sqrt{4(c+t) - s^2}$. One can

then verify (3), using the formula (9.21) of $H[\rho_t(\cdot)]$ given p. 511 of [11].

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REFERENCES

- [1] G. Anderson, A. Guionnet, O. Zeitouni. *An Introduction to Random Matrices*. Cambridge studies in advanced mathematics, 118 (2009).
- [2] G. Anderson, O. Zeitouni. *A CLT for a band matrix model*. Probab. Theory Related Fields 134 (2005), 283–338.
- [3] S.T. Belinschi. *A note on regularity for free convolutions*, Ann. Inst. H. Poincaré Probab. Statist. 42 (2006), no. 5, 635–648.
- [4] S.T. Belinschi. *The Lebesgue decomposition of the free additive convolution of two probability distributions*, Probab. Theory Related Fields 142 (2008), no. 1–2, 125–150.
- [5] S.T. Belinschi, D. Shlyakhtenko. *Free probability of type B: analytic aspects and applications*, to appear in American J. Math.
- [6] S.T. Belinschi, F. Benaych-Georges, A. Guionnet. *Regularization by Free Additive Convolution, Square and Rectangular Cases*. Complex Analysis and Operator Theory. Vol. 3, no. 3 (2009) 611–660.
- [7] H. Bercovici and D. Voiculescu. *Regularity questions for free convolution*, Nonselfadjoint operator algebras, operator theory, and related topics, 37–47, Oper. Theory Adv. Appl. 104, Birkhäuser, Basel, 1998.
- [8] P. Biane. *On the Free convolution by a semi-circular distribution*. Indiana University Mathematics Journal, Vol. 46, (1997), 705–718.
- [9] M. Février. *Higher order infinitesimal freeness*, arXiv.
- [10] M. Février, A. Nica. *Infinitesimal non-crossing cumulants and free probability of type B*, J. Funct. Anal. 258, 2983–2023 (2010).
- [11] F. W. King. *Hilbert transforms. Vol. 2*. Encyclopedia of Mathematics and its Applications, 125. Cambridge University Press, Cambridge, 2009
- [12] D. Shlyakhtenko. *Random Gaussian band matrices and freeness with amalgamation*. Internat. Math. Res. Notices 1996, no. 20, 1013–1025.
- [13] D. Voiculescu. *Limit laws for random matrices and free products*. Inventiones Mathematicae, 104 (1991) 201–220.

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