ON LEVEL ONE CUSPIDAL BIANCHI MODULAR FORMS

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Abstract. In this paper, we present significant numerical data strongly suggesting the rareness of level one cuspidal Bianchi modular forms which are not lifts of classical holomorphic modular forms.

Let $K = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field. Bianchi modular forms over K are modular forms associated to the $\mathbb{Q}$-algebraic group $\mathrm{Res}_{K/\mathbb{Q}}(\mathrm{SL}_2)$. Up to date, no one has been able to produce a formula for the dimension of the (finite dimensional) space of Bianchi modular forms over K for given level and weight, although we should note that there is recent progress made by Calegari-Emerton [CE09] and Marshall [Mar10] in producing asymptotic upper bounds.

Let $S_k(1)$ denote the space of level one weight $k+2$ cuspidal Bianchi modular forms over K. In a recent paper, Finis, Grunewald and Tirao computed the dimension of the subspace $L_k(1)$ of $S_k(1)$ which consists of lifts of classical holomorphic modular forms. To investigate the natural question "how much more is there beyond the lifts?", they carried out machine computations to compute the actual size of $S_k(1)$ for ten fields K and numerous weights k. In Table 1, we summarize the range of their computations.

d	1	2	3	7	11	19	5	6	10	14
$k \leqslant$	104	141	116	132	153	60	60	60	60	60

Table 1. Finis-Grunewald-Tirao test range

A remarkable outcome of the data they collected is that except in two of the 946 spaces they computed, the subspace $L_k(1)$ exhausts all of $S_k(1)$. The exceptional cases are $(d, k) = (7, 10)$ and $(d, k) = (11, 12)$. In both cases, there is a two-dimensional complement to $L_k(1)$ inside $S_k(1)$.

In this paper, we further investigate this phenomenon. Using a completely different and more efficient approach, we computed the dimension of $S_k(1)$ for many more fields K and weights k. The range of our computations is given in Table 2. Out of the 1132 new spaces we computed, there were only two cases where $L_k(1)$ did not exhaust $S_k(1)$; for $(d, k) = (91, 6)$ and $(d, k) = (643, 0)$, where we found a one-dimensional and a two dimensional (respectively) complement to $L_k(1)$.

The starting point of our approach is the so called "Eichler-Shimura-Harder" isomorphism which allows us to replace $S_k(1)$ with the cohomology

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d	1	2	3	5	6	10	13	15	19	22	35	37	43
$k \leqslant$	200	200	196	100	100	73	52	100	94	45	86	31	77
d	51	58	67	91	123	163	187	191	235	259	267	427	643
$k \leqslant$	68	21	58	50	35	33	25	15	21	17	21	13	8

TABLE 2. the scope of our computations

of the relevant Bianchi group with special non-trivial coefficients. Then to compute this cohomology space, we use the program *Bianchi.gp* [Rah], which analyzes the structure of the Bianchi group via its action on hyperbolic 3-space (which is isomorphic to the associated symmetric space $\mathrm{SL}_2(\mathbb{C})/\mathrm{SU}_2$). We then feed this group-geometric information into an equivariant spectral sequence that gives us an explicit description of the second cohomology of the Bianchi group, with the relevant coefficients.

There is a widely believed conjectural connection between Bianchi newforms of weight 2 over K and abelian varieties of GL_2-type defined over K (see [EGM82],[Cre92],[Tay95]). In particular, an abelian variety of GL_2-type over K, that is not definable over $\mathbb{Q}$ nor of CM-type, with everywhere good reduction is expected give rise to newforms in $S_0(1)$ that are not in $L_0(1)$. We know by Krämer [Kra84] that there is such an elliptic curve over $\mathbb{Q}(\sqrt{-643})$ which accounts for the non-lift classes we encountered in the case $(d, k) = (643, 0)$, see Scheutzow [Sch92].

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1. Background

Let K be an imaginary quadratic field with ring of integers $\mathcal{O}$. Let Γ be the Bianchi group $\mathrm{SL}_2(\mathcal{O})$. It is a discrete subgroup of the real Lie group $\mathrm{SL}_2(\mathbb{C})$ and thus acts discontinuously on hyperbolic 3-space. Let Y_Γ be the quotient hyperbolic 3-fold. Denote by X_Γ the Borel/Serre compactification [Ser70, appendix] of Y_Γ. Then X_Γ is a compact 3-fold with boundary ∂X_Γ, and its interior is homeomorphic to Y_Γ. It is well known that when the discriminant of K is smaller than -4, this boundary consists of h_K disjoint 2-tori where h_K is the class number of K. For an ordered pair of nonnegative integers (n, m), denote by $\mathcal{E}_n$ the locally constant sheaf on Y_Γ induced by the irreducible finite dimensional complex representation E_n of $\mathrm{SL}_2(\mathbb{C})$ of highest weight (n, n). Consider the long exact sequence

$$\ldots \to H_c^{i-1}(X_\Gamma, \bar{\mathcal{E}}_n) \to H^i(X_\Gamma, \bar{\mathcal{E}}_n) \to H^i(\partial X_\Gamma, \bar{\mathcal{E}}_n) \to \ldots$$

where H_c^i denotes the compactly supported cohomology and $\bar{\mathcal{E}}_n$ is a certain natural extension of $\mathcal{E}_n$.

The *cuspidal cohomology* H^i_{cusp} is defined as the image of the compactly supported cohomology. The *Eisenstein cohomology* H^i_{Eis} is the complement of the cuspidal cohomology inside H^i and it is isomorphic to the image of the restriction map inside the cohomology of the boundary. The decomposition $H^i = H^i_{cusp} \oplus H^i_{Eis}$ respects the Hecke action which is defined, as usual, via correspondences on X_Γ.

Let $S_n(1)$ denote the space of level one cuspidal Bianchi modular forms (over K) of weight (n,n). It is well known that

$$S_n(1) \simeq H^1_{cusp}(X_\Gamma, \bar{\mathcal{E}}_n) \simeq H^2_{cusp}(X_\Gamma, \bar{\mathcal{E}}_n)$$

as Hecke modules. Here the first isomorphism was established by Harder and the second follows from duality, see [AS86]. Consider the subspace of $S_n(1)$ which is formed by the Bianchi modular forms which arise by base change from classical holomorphic modular forms. We denote the corresponding subspace of $H^i_{cusp}(X_\Gamma, \bar{\mathcal{E}}_n)$ by $H^i_{bc}(X_\Gamma, \bar{\mathcal{E}}_n)$. In [FGT10], a formula for the dimension of this space has been given for all weights n.

By construction, the embedding $Y_\Gamma \hookrightarrow X_\Gamma$ is a homotopy invariance. Together with the fact that Y_Γ is a $K(\Gamma,1)$-space, we get the isomorphisms

$$H^i(X_\Gamma, \bar{\mathcal{E}}_n) \simeq H^i(Y_\Gamma, \mathcal{E}_n) \simeq H^i(\Gamma, E_n).$$

Via these isomorphisms, we define the cuspidal and Eisenstein parts of $H^i(\Gamma, E_n)$.

The following Proposition will allow us to deduce the size of the cuspidal cohomology once we have computed the size of the whole cohomology.

Proposition 1. *Let K be an imaginary quadratic field of discriminant smaller than -4; and let $\mathcal{O}$ be its ring of integers. Let Γ be the associated Bianchi group $\mathrm{SL}_2(\mathcal{O})$. Then*

$$dim\ H^2_{Eis}(X_\Gamma, \bar{\mathcal{E}}_n) = h_K - \delta(n,0)$$

where h_K is the class number of K and δ is the Kronecker delta function.

Proof. Studying the long exact sequence above, taking into account that the virtual cohomological dimension of $\mathrm{SL}_2(\mathcal{O})$ is two, we obtain for $n > 0$ that

$$H^2_{Eis}(X_\Gamma, \bar{\mathcal{E}}_n) \simeq H^2(\partial X_\Gamma, \bar{\mathcal{E}}_n)$$

and for $n = 0$ the left hand term is of complex codimension 1 in the right hand term. Hence to prove the statement, we need to prove that the dimension of the right hand term is h_K. Recall that the boundary ∂X_Γ is a disjoint union of 2-tori, indexed by the class group of K, each closing a cusp of Y_Γ. Hence to understand $H^2(\partial X_\Gamma, \bar{\mathcal{E}}_n)$, it is enough to understand $H^2(T_c, \bar{\mathcal{E}}_n)$ for the 2-torus T_c associated to a fixed cusp c.

Let $c \in K \cup \{\infty\}$ be a cusp and let Γ_c be its stabiliser in Γ (which is a parabolic subgroup). Then Γ_c is the fundamental group of T_c. In fact, T_c is a $K(\Gamma_c, 1)$–Eilenberg/MacLane space. Hence we can turn our attention to computing $H^2(\Gamma_c, E_n)$. Composition of the cup product and the well-known perfect pairing $(\cdot,\cdot) : E_n \otimes_{\mathbb{C}} E_n \to \mathbb{C}$ (see, for example, Section 2.4. [Ber08]) gives us a pairing

$$\begin{array}{ccc} H^0(\Gamma_c, E_n) \times H^2(\Gamma_c, E_n) & \xrightarrow{\cup} & H^2(\Gamma_c, E_n \otimes_{\mathbb{C}} E_n) \\ & & \big\downarrow{\scriptstyle (\cdot,\cdot)} \\ & & H^2(\Gamma_c, \mathbb{C}) \simeq \mathbb{C}. \end{array}$$

Here the last isomorphism follows from the fact that T_c is a compact 2-fold (see also proof of Prop.3.5. of [Sen10] for a direct algebraic argument). Thus the dimension we are looking for is equal to that of $H^0(\Gamma_c, E_n)$. Clearly, if $n = 0$, the latter dimension is 1 and thus the dimension of $H^2(\partial X_\Gamma, \bar{\mathcal{E}}_n)$ is h_K as desired.

Let us now assume that $n \neq 0$. By translating c to the vertex at infinity, we obtain an isomorphism $\Gamma_c \simeq \Gamma_\infty = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \subset \mathrm{SL}_2(\mathcal{O})$. Consider the normal subgroup $\Gamma_\infty^+ := \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ of Γ_∞. Then Γ_∞^+ is a free abelian group on two generators. We are now going to determine the submodule $E_n^{\Gamma_\infty^+}$ of E_n invariant under its action. Let $\mathbb{C}[x,y]_n$ denote the space of homogeneous complex coefficient polynomials of degree n with variables x, y. $\mathrm{SL}_2(\mathcal{O})$ acts on this space in an obvious way. Then the $\mathrm{SL}_2(\mathcal{O})$-module $\mathbb{C}[x,y]_n \otimes_{\mathbb{C}} \overline{\mathbb{C}[x,y]_n}$ is isomorphic to E_n. Here, the overlined notation of the second factor is to indicate that the action on this factor is twisted with complex conjugation. As the generators are of the form $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$, it is clear that the vector $x^n \otimes y^n$ is fixed by Γ_∞^+. One shows, proceeding as in Lemma 2.4. of [Wie07], that there are no other fixed vectors. Hence

$$H^0(\Gamma_\infty^+, E_n) = E_n^{\Gamma_\infty^+} = \langle x^n \otimes y^n \rangle$$

is of complex dimension 1. Let $\mu := \Gamma_\infty / \Gamma_\infty^+ = \left\{ \left(\begin{smallmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{smallmatrix}\right) \right\}$. As we are considering modules over $\mathbb{C}$, it follows that

$$H^0(\Gamma_\infty, E_n) \simeq H^0(\Gamma_\infty^+, E_n)^\mu$$

is the invariant submodule under μ. We easily check that the action of μ on E_n is trivial, and so

$$H^0(\Gamma_c, E_n) \simeq H^0(\Gamma_\infty^+, E_n)$$

is again of complex dimension 1. In the quotient space, we find h_K cusps, so the claim follows. $\square$

2. The Bianchi fundamental polyhedron

Let m be a squarefree positive integer and $K = \mathbb{Q}(\sqrt{-m})$ be an imaginary quadratic number field with ring of integers $\mathcal{O}_{-m}$, which we also just denote by $\mathcal{O}$. Consider the familiar action (we give an explicit formula for it in lemma 3) of the group

$\Gamma := \mathrm{SL}_2(\mathcal{O}) \subset \mathrm{GL}_2(\mathbb{C})$ on hyperbolic three-space, for which we will use the upper-half space model $\mathcal{H}$. As a set,

$$\mathcal{H} = \{(z, \zeta) \in \mathbb{C} \times \mathbb{R} \mid \zeta > 0\}.$$

We will call the coordinate ζ the *height*.

The Bianchi/Humbert theory [Bia92]/[Hum15] gives a fundamental domain for this action. We will start by giving a geometric description of it, and the arguments why it is a fundamental domain.

Definition 1. *A pair of elements* $(\mu, \lambda) \in \mathcal{O}^2$ *is called* unimodular *if the ideal sum* $\mu\mathcal{O} + \lambda\mathcal{O}$ *equals* $\mathcal{O}$.

The boundary of $\mathcal{H}$ is the Riemann sphere $\partial\mathcal{H} = \mathbb{C} \cup \{\infty\}$ (as a set), which contains the complex plane $\mathbb{C}$. The totally geodesic surfaces in $\mathcal{H}$ are the Euclidean vertical planes (we define *vertical* as orthogonal to the complex plane) and the Euclidean hemispheres centred on the complex plane.

Notation 1. *Given a unimodular pair* $(\mu, \lambda) \in \mathcal{O}^2$ *with* $\mu \neq 0$, *let* $S_{\mu,\lambda} \subset \mathcal{H}$ *denote the hemisphere given by the equation* $|\mu z - \lambda|^2 + |\mu|^2\zeta^2 = 1$.

This hemisphere has centre λ/μ *on the complex plane* $\mathbb{C}$, *and radius* $1/|\mu|$. *Let*

$$B := \big\{(z, \zeta) \in \mathcal{H}\colon \textit{The inequality } |\mu z - \lambda|^2 + |\mu|^2\zeta^2 \geqslant 1 \textit{ is fulfilled for all unimodular pairs } (\mu, \lambda) \in \mathcal{O}^2 \textit{ with } \mu \neq 0 \big\}.$$

Then B *is the set of points in* $\mathcal{H}$ *which lie above or on all hemispheres* $S_{\mu,\lambda}$.

Lemma 1 ([Swa71])**.** *The set* B *contains representatives for all the orbits of points under the action of* $\mathrm{SL}_2(\mathcal{O})$ *on* $\mathcal{H}$.

Proof. Consider hyperbolic three-space as the set of positive definite Hermitian forms f in two complex variables, modulo homotheties. The action of $\mathrm{GL}_2(\mathbb{C})$ on the variables by linear automorphisms of $\mathbb{C}^2$ induces an action on this set by the formula $\gamma \cdot f(z) := f(\gamma^{-1}z)$ for $\gamma \in \mathrm{GL}_2(\mathbb{C})$, $z \in \mathbb{C}^2$. The latter action corresponds to the familiar action on $\mathcal{H}$, which Swan even defines this way. Now the set B corresponds to the forms which take their "proper minimum" at the argument $(1, 0)$. From Humbert [Hum15], it follows that for any binary Hermitian form f, there exists an element $\gamma \in \mathrm{SL}_2(\mathcal{O})$ such that $\gamma \cdot f$ takes its proper minimum at $(1, 0)$. □

The action extends continuously to the boundary $\partial\mathcal{H}$, which is a Riemann sphere.

In $\Gamma := \mathrm{SL}_2(\mathcal{O}_{-m})$, consider the stabiliser subgroup Γ_∞ of the point $\infty \in \partial\mathcal{H}$. In the cases $m = 1$ and $m = 3$, the latter group contains some rotation matrices like $\begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}$, which we want to exclude. These two cases have been treated in [Men79], [SV83] and others, and we assume $m \neq 1$, $m \neq 3$ throughout this chapter. Then,

$$\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \mid \lambda \in \mathcal{O} \right\},$$

which performs translations by the elements of $\mathcal{O}$ with respect to the Euclidean geometry of the upper-half space $\mathcal{H}$.

Notation 2. *A fundamental domain for Γ_∞ in the complex plane (as a subset of $\partial\mathcal{H}$) is given by the rectangle*

$$D_0 := \begin{cases} \{x + y\sqrt{-m} \in \mathbb{C} \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1\}, & m \equiv 1 \text{ or } 2 \mod 4, \\ \{x + y\sqrt{-m} \in \mathbb{C} \mid \frac{-1}{2} \leqslant x \leqslant \frac{1}{2},\ 0 \leqslant y \leqslant \frac{1}{2}\}, & m \equiv 3 \mod 4. \end{cases}$$

And a fundamental domain for Γ_∞ in $\mathcal{H}$ is given by

$$D_\infty := \{(z, \zeta) \in \mathcal{H} \mid z \in D_0\}.$$

Definition 2. *We define the* Bianchi fundamental polyhedron *as*

$$D := D_\infty \cap B.$$

It is a polyhedron in hyperbolic space up to the missing vertex ∞, and up to missing vertices at the singular points if $\mathcal{O}$ is not a principal ideal domain (see subsection 5.1). As Lemma 1 states $\Gamma \cdot B = \mathcal{H}$, and as $\Gamma_\infty \cdot D_\infty = \mathcal{H}$ yields $\Gamma_\infty \cdot D = B$, we have $\Gamma \cdot D = \mathcal{H}$. We observe the following notion of strictness of the fundamental domain: the interior of the Bianchi fundamental polyhedron contains no two points which are identified by Γ.
Swan proves the following theorem, which implies that the boundary of the Bianchi fundamental polyhedron consists of finitely many cells.

Theorem 1 ([Swa71])**.** *There is only a finite number of unimodular pairs (λ, μ) such that the intersection of $S_{\mu,\lambda}$ with the Bianchi fundamental polyhedron is non-empty.*

He also proves a corollary, from which it can be deduced that the action of Γ on $\mathcal{H}$ is properly discontinuous.

Corollary 1 ([Swa71])**.** *There are only finitely many matrices $\gamma \in \mathrm{SL}_2(\mathcal{O})$ such that $D \cap \gamma \cdot D \neq \emptyset$.*

3. The reduced cellular complex

In order to obtain a cell complex with compact quotient space, we proceed in the following way due to Flöge [Flö83]. The boundary of $\mathcal{H}$ is the Riemann sphere $\partial\mathcal{H}$, which, as a topological space, is made up of the complex plane $\mathbb{C}$ compactified with the cusp ∞. The totally geodesic surfaces in $\mathcal{H}$ are the Euclidean vertical planes (we define *vertical* as orthogonal to the complex plane) and the Euclidean hemispheres centred on the complex plane. The action of the Bianchi groups extends continuously to the boundary $\partial\mathcal{H}$. The cellular closure of the refined cell complex in $\mathcal{H} \cup \partial\mathcal{H}$ consists of $\mathcal{H}$ and $\left(\mathbb{Q}(\sqrt{-m}) \cup \{\infty\}\right) \subset (\mathbb{C} \cup \{\infty\}) \cong \partial\mathcal{H}$. The $\mathrm{SL}_2(\mathcal{O}_{-m})$–orbit of a cusp $\frac{\lambda}{\mu}$ in $\left(\mathbb{Q}(\sqrt{-m}) \cup \{\infty\}\right)$ corresponds to the ideal class $[(\lambda, \mu)]$ of $\mathcal{O}_{-m}$. It is well-known that this does not depend on the choice of the representative $\frac{\lambda}{\mu}$. We extend the refined cell complex to a cell complex $\widetilde{X}$ by joining to it, in the case that $\mathcal{O}_{-m}$ is not a principal ideal domain, the $\mathrm{SL}_2(\mathcal{O}_{-m})$–orbits of the cusps $\frac{\lambda}{\mu}$ for which the ideal (λ, μ) is not principal. At these cusps, we equip $\widetilde{X}$ with the "horoball topology" described in [Flö83]. This simply means that the set of cusps, which is discrete in $\partial\mathcal{H}$, is located at the hyperbolic extremities of $\widetilde{X}$: No neighbourhood of a cusp, except the whole $\widetilde{X}$, contains any other cusp.

We retract $\widetilde{X}$ in the following, $\mathrm{SL}_2(\mathcal{O}_{-m})$–equivariant, way. On the Bianchi fundamental polyhedron, the retraction is given by the vertical projection (away from the cusp ∞) onto its facets which are closed in $\mathcal{H} \cup \partial\mathcal{H}$. The latter are the facets which do not touch the cusp ∞, and are the bottom facets with respect to our vertical direction. The retraction is continued on $\mathcal{H}$ by the group action. It is proven in [Flö80] that this retraction is continuous. We call the retract of $\widetilde{X}$ the *Flöge cellular complex* and denote it by X. So in the principal ideal domain cases, X is a retract of the refined cell complex, obtained by contracting the Bianchi fundamental polyhedron onto its cells which do not touch the boundary of $\mathcal{H}$. In [RF11], it is checked that the Flöge cellular complex is contractible.

4. The spectral sequence

Let X be our Flöge complex constructed as above. Next we will consider the spectral sequence associated to the double complex $\mathrm{Hom}_{\mathbb{Z}\Gamma}(\Theta_*, C^*_{\mathbb{Z}}(X, M))$, where Θ_* is the standard resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$ and $C^*(X, M)$ is the cellular cochain complex of X with $\mathbb{Z}\Gamma$-module coefficients M. We can (see [Bro82], p. 164) derive the first-quadrant spectral sequence

$$E_1^{p,q}(M) = \bigoplus_{\sigma \in \Sigma_p} H^q(\Gamma_\sigma, M) \Longrightarrow H^{p+q}(\Gamma, M)$$

where Σ_p denotes the Γ-conjugacy classes of p-cells of X. Observe that Γ_σ will be a finite group whose order is divisible only by 2 and/or 3 unless σ is the class of a singular cusp, in which case Γ_σ is a free abelian group on two unipotent generators.

Assume that M admits an additional module structure over a ring where 6 is invertible (in fact we are interested in the case where M is a complex vector space). Then the higher cohomology groups of the Γ_σ which are finite vanish. Thus, when there are no singular cusps (equivalently, when the class number of $\mathcal{O}$ is one), the spectral sequence concentrates on the row $q = 0$ and stabilizes on the E^2-page. Otherwise, the spectral sequence concentrates on the rows $q = 0, 1, 2$ and stabilizes at the E^3-page.

In the cases where $\mathcal{O}$ is a Euclidean ring (which excludes singular cusps), we have

$$H^2(\Gamma, M) \simeq E_2^{2,0} \simeq E_1^{2,0}/\mathrm{Im}(d_1^{1,0}),$$

where the differential $d_1^{1,0}$ is given as

$$E_1^{1,0} \simeq \bigoplus_{\sigma \in \Sigma_1} M^{\Gamma_\sigma} \longrightarrow M \simeq E_1^{2,0}.$$

In the more complicated general case, we have

$$H^2(\Gamma, M) \simeq E_3^{2,0} \oplus E_3^{0,2}.$$

Here $E_3^{0,2} \simeq \bigoplus_s H^2(\Gamma_s, M)$ where the summation is over Γ-classes of singular cusps s. Moreover $E_3^{2,0} = E_2^{2,0}/\mathrm{Im}(d_2^{0,1})$ where the differential $d_2^{0,1}$ is given as

$$\bigoplus_{s\ singular} H^1(\Gamma_s, M) \longrightarrow E_2^{2,0}(\simeq E_1^{2,0}/\mathrm{Im}(d_2^{1,0})).$$

Hence, more precisely, we have

$$H^2(\Gamma, M) \simeq \left(\bigoplus_s H^2(\Gamma_s, M) \right) \oplus \left(E_2^{2,0}/\mathrm{Im}(d_2^{0,1}) \right)$$

Note that the proof of Proposition 1 shows that

$$\mathrm{dim} H^0(\Gamma_s, M) = \mathrm{dim} H^2(\Gamma_s, M) = 1.$$

As Γ_s is the fundamental group of a torus, we have

$$\mathrm{dim} H^1(\Gamma_s, M) = 2 \cdot \mathrm{dim} H^2(\Gamma_s, M) = 2.$$

The above discussion shows that in the general case, one needs to understand the size of the image of the differential $d_2^{0,1}$ in order to compute the size of $H^2(\Gamma, M)$. Thanks to the existence of lower bounds (due to Finis-Grunewald-Tirao, as discussed in the introduction), we are able to compute the size of $H^2(\Gamma, M)$ as long as we know that the differential $d_2^{0,1}$ does not vanish. We have considerable computational evidence for the following.

Conjecture 1. *Let s be a singular cusp. The restriction of the differential $d_2^{0,1}$ on*

$$H^1(\Gamma_s, M) \longrightarrow E_2^{2,0}$$

is nonzero.

5. Swan's concept to determine the Bianchi fundamental polyhedron

This section recalls Richard G. Swan's work [Swa71], which gives a concept — from the theoretical viewpoint — for an algorithm to compute the Bianchi fundamental polyhedron. Such algorithm has been implemented by Cremona [Cre84] for the five cases where $\mathcal{O}$ is Euclidean, and by his students Whitley [Whi90] for the non-Euclidean principal ideal domain cases, Bygott [Byg98] for a case of class number 2 and Lingham [Lin05] for some cases of class number 3. In subsection 6, we give an algorithm worked out from Swan's concept indepently from the mentioned implementations. This algorithm has been implemented for all Bianchi groups [Rah], and we make use of it in our computations.

The set B which determines the Bianchi fundamental polyhedron has been defined using infinitely many hemispheres. But we will see that only a finite number of them are significant for this purpose and need to be computed. We will state a criterion for what is an appropriate choice that gives us precisely the set B. This criterion is easy to verify in practice.

Suppose we have made a finite selection of n hemispheres. The index i running from 1 through n, we denote the i-th hemisphere by $S(\alpha_i)$, where α_i is its centre and given by a fraction $\alpha_i = \frac{\lambda_i}{\mu_i}$ in the number field K. Here, we require the ideal (λ_i, μ_i) to be the whole ring of integers $\mathcal{O}$. This requirement is just the one already made for all the hemispheres in the definition of B. Now, we can do an approximation of notation 1, using, modulo the translation group Γ_∞, a finite number of hemispheres.

Notation 3. *Let* $B(\alpha_1, \ldots, \alpha_n) := \big\{(z, \zeta) \in \mathcal{H}$*: The inequality* $|\mu z - \lambda|^2 + |\mu|^2\zeta^2 \geqslant 1$ *is fulfilled for all unimodular pairs* $(\mu,\, \lambda) \in \mathcal{O}^2$ *with* $\frac{\lambda}{\mu} = \alpha_i + \gamma$, *for some* $i \in \{1, \ldots, n\}$ *and some* $\gamma \in \mathcal{O}\ \big\}$.

Then $B(\alpha_1, \ldots, \alpha_n)$ *is the set of all points in* $\mathcal{H}$ *lying above or on all hemispheres* $S(\alpha_i + \gamma)$, $i = 1, \ldots, n$*; for any* $\gamma \in \mathcal{O}$.

The intersection $B(\alpha_1, \ldots, \alpha_n) \cap D_\infty$ with the fundamental domain D_∞ for the translation group Γ_∞, is our candidate to equal the Bianchi fundamental polyhedron.

Convergence of the approximation. We will give a method to decide when $B(\alpha_1, \ldots, \alpha_n) = B$. This gives us an effective way to find B by adding more and more elements to the set $\{\alpha_1, \ldots, \alpha_n\}$ until we find $B(\alpha_1, \ldots, \alpha_n) = B$.

We consider the boundary $\partial B(\alpha_1, \ldots, \alpha_n)$ of $B(\alpha_1, \ldots, \alpha_n)$ in $\mathcal{H} \cup \mathbb{C}$. It consists of the points $(z, \zeta) \in \mathcal{H} \cup \mathbb{C}$ satisfying all the non-strict inequalities $|\mu z - \lambda|^2 + |\mu|^2\zeta^2 \geqslant 1$ that we have used to define $B(\alpha_1, \ldots, \alpha_n)$, and satisfy the additional condition that at least one of these non-strict inequalities is an equality.

We will see below that $\partial B(\alpha_1, \ldots, \alpha_n)$ carries a natural cell structure. This, together with the following definitions, makes it possible to state the criterion which tells us when we have found all the hemispheres relevant for the Bianchi fundamental polyhedron.

Definition 3. *We shall say that the hemisphere* $S_{\mu,\lambda}$ *is* strictly below *the hemisphere* $S_{\beta,\alpha}$ *at a point* $z \in \mathbb{C}$ *if the following inequality is satisfied:*

$$\left|z - \frac{\alpha}{\beta}\right|^2 - \frac{1}{|\beta|^2} < \left|z - \frac{\lambda}{\mu}\right|^2 - \frac{1}{|\mu|^2}.$$

This is, of course, an abuse of language because there may not be any points on $S_{\beta,\alpha}$ or $S_{\mu,\lambda}$ with coordinate z. However, if there is a point (z, ζ) on $S_{\mu,\lambda}$, the right hand side of the inequality is just $-\zeta^2$. Thus the left hand side is negative and so of the form $-(\zeta')^2$. Clearly, $(z, \zeta') \in S_{\beta,\alpha}$ and $\zeta' > \zeta$.

We will further say that a point $(z, \zeta) \in \mathcal{H} \cup \mathbb{C}$ is *strictly below* a hemisphere $S_{\mu,\lambda}$, if there is a point $(z, \zeta') \in S_{\mu,\lambda}$ with $\zeta' > \zeta$.

5.1. **Singular points.** We call *cusps* the elements of the number field considered as points in the boundary of hyperbolic space, via the inclusion $K \subset \mathbb{C} \cup \{\infty\} \cong \partial\mathcal{H}$. We write $\infty = \frac{1}{0}$, which we also consider as a cusp. It is well-known that the set of cusps is closed under the action of $\mathrm{SL}_2(\mathcal{O})$ on $\partial\mathcal{H}$; and that we have the following bijective correspondence between the $\mathrm{SL}_2(\mathcal{O})$-orbits of cusps and the ideal classes in $\mathcal{O}$. A cusp $\frac{\lambda}{\mu}$ is in the $\mathrm{SL}_2(\mathcal{O})$-orbit of the cusp $\frac{\lambda'}{\mu'}$, if and only if the ideals (λ', μ') and (λ, μ) are in the same ideal class. It immediately follows that the orbit of the cusp $\infty = \frac{1}{0}$ corresponds to the principal ideals. Let us call *singular* the cusps $\frac{\lambda}{\mu}$ such that (λ, μ) is not principal. And let us call *singular points* the singular cusps which lie in ∂B. It follows from the characterisation of the singular points by Bianchi that they are precisely the points in $\mathbb{C} \subset \partial\mathcal{H}$ which cannot be strictly below any hemisphere. In the cases where $\mathcal{O}$ is a principal ideal domain, $K \cup \{\infty\}$ consists of only one $\mathrm{SL}_2(\mathcal{O})$-orbit, so there are no singular points. We use the following formulae derived by Swan, to compute representatives modulo the translations by Γ_∞, of the singular points.

Lemma 2 ([Swa71]). *The singular points of K, $\mod \mathcal{O}$, are given by $\frac{p(r+\sqrt{-m})}{s}$, where $p, r, s \in \mathbb{Z}$, $s > 0$, $\frac{-s}{2} < r \leqslant \frac{s}{2}$, $s^2 \leqslant r^2 + m$, and*

- *if $m \equiv 1$ or $2 \mod 4$,*
  $s \neq 1$, $s \mid r^2 + m$, the numbers p and s are coprime, and p is taken $\mod s$;
- *if $m \equiv 3 \mod 4$,*
  s is even, $s \neq 2$, $2s \mid r^2 + m$, the numbers p and $\frac{s}{2}$ are coprime; p is taken $\mod \frac{s}{2}$.

The singular points need not be considered in Swan's termination criterion, because they cannot be strictly below any hemisphere $S_{\mu,\lambda}$.

5.2. **Swan's termination criterion.** We observe that the set of $z \in \mathbb{C}$ over which some hemisphere is strictly below another is $\mathbb{C}$ or an open half-plane. In the latter case, the boundary of this is a line.

Notation 4. *Denote by $L(\frac{\alpha}{\beta}, \frac{\lambda}{\mu})$ the set of $z \in \mathbb{C}$ over which neither $S_{\beta,\alpha}$ is strictly below $S_{\mu,\lambda}$ nor vice versa.*

This line is computed by turning the inequality in definition 3 into an equation. Swan calls it the line over which the two hemispheres *agree*, and we will see later that the most important edges of the Bianchi fundamental polyhedron lie on the preimages of such lines.

We now restrict our attention to a set of hemispheres which is finite modulo the translations in Γ_∞.

Consider a set of hemispheres $S(\alpha_i + \gamma)$, where the index i runs from 1 through n, and γ runs through $\mathcal{O}$. We call this set of hemispheres a *collection*, if every non-singular point $z \in \mathbb{C} \subset \partial\mathcal{H}$ is strictly below some hemisphere in our set.

Now consider a set $B(\alpha_1, \dots, \alpha_n)$ which is determined by such a collection of hemispheres.

Theorem 2 (Swan's termination criterion [Swa71])**.** *We have* $B(\alpha_1, \dots, \alpha_n) = B$ *if and only if no vertex of* $\partial B(\alpha_1, \dots, \alpha_n)$ *can be strictly below any hemisphere* $S_{\mu,\lambda}$*.*

In other words, no vertex v of $\partial B(\alpha_1, \dots, \alpha_n)$ can lie strictly below any hemisphere $S_{\mu,\lambda}$.

With this criterion, it suffices to compute the cell structure of $\partial B(\alpha_1, \dots, \alpha_n)$ to see if our choice of hemispheres gives us the Bianchi fundamental polyhedron. This has only to be done modulo the translations of Γ_∞, which preserve the height and hence the situations of being strictly below. Thus our computations only need to be carried out on a finite set of hemispheres.

5.3. **Computing the cell structure in the complex plane.** We will in a first step compute the image of the cell structure under the homeomorphism from $\partial B(\alpha_1, \dots, \alpha_n)$ to $\mathbb{C}$ given by the vertical projection. For each 2-cell of this structure, there is an associated hemisphere $S_{\mu,\lambda}$. The interior of this 2-cell consists of the points $z \in \mathbb{C}$ where all other hemispheres in our collection are strictly below $S_{\mu,\lambda}$. Swan shows that this is the interior of a convex polygon.

The edges of these polygons lie on real lines in $\mathbb{C}$ specified in notation 4.

A vertex is an intersection point z of any two of these lines involving the same hemisphere $S_{\mu,\lambda}$, if all other hemispheres in our collection are strictly below, or agree with, $S_{\mu,\lambda}$ at z.

5.3.1. *Lifting the cell structure back to hyperbolic space.* Now we can lift the cell structure back to $\partial B(\alpha_1, \dots, \alpha_n)$, using the projection homeomorphism onto $\mathbb{C}$. The preimages of the convex polygons of the cell structure on $\mathbb{C}$, are totally geodesic hyperbolic polygons each lying on one of the hemispheres in our collection. These are the 2-cells of $\partial B(\alpha_1, \dots, \alpha_n)$.

The edges of these hyperbolic polygons lie on the intersection arcs of pairs of hemispheres in our collection. As two Euclidean 2-spheres intersect, if they do so non-trivially, in a circle centred on the straight line which connects the two 2-sphere centres, such an intersection arc lies on a semicircle centred in the complex plane. The plane which contains this semicircle must be orthogonal to the connecting line, hence a vertical plane in $\mathcal{H}$. We can alternatively conclude the latter facts observing that an edge which two totally geodesic polygons have in common must be a geodesic segment.

Lifting the vertices becomes now obvious from their definition. This enables us to check Swan's termination criterion.

We will now sketch Swan's proof of this criterion. Let P be one of the convex polygons of the cell structure on $\mathbb{C}$. The preimage of P lies on one hemisphere $S(\alpha_i)$ of our collection. Now the condition stated in theorem 2 says that at the vertices of P, the hemisphere $S(\alpha_i)$ cannot be strictly below any other hemisphere. The points where $S(\alpha_i)$ can be strictly below some hemisphere constitute an open half-plane in $\mathbb{C}$, and hence cannot lie in the convex hull of the vertices of P, which is P. Theorem 2 now follows because $\mathbb{C}$ is tessellated by these convex polygons.

6. Realization of Swan's algorithm

From now on, we will work on putting Swan's concept into practice.

We can reduce the set of hemispheres on which we carry out our computations, with the help of the following notion.

Definition 4. *A hemisphere $S_{\mu,\lambda}$ is said to be* everywhere below *a hemisphere $S_{\beta,\alpha}$ when:*

$$\left|\frac{\lambda}{\mu} - \frac{\alpha}{\beta}\right| \leqslant \frac{1}{|\beta|} - \frac{1}{|\mu|}.$$

Note that this is also the case when $S_{\mu,\lambda} = S_{\beta,\alpha}$. Any hemisphere which is everywhere below another one, does not contribute to the Bianchi fundamental polyhedron, in the following sense.

Proposition 2. *Let $S(\alpha_n)$ be a hemisphere everywhere below some other hemisphere $S(\alpha_i)$, where $i \in \{1, \ldots, n-1\}$.*
Then $B(\alpha_1, \ldots, \alpha_n) = B(\alpha_1, \ldots, \alpha_{n-1})$.

Proof. Write $\alpha_n = \frac{\lambda}{\mu}$ and $\alpha_i = \frac{\theta}{\tau}$ with $\lambda, \mu, \theta, \tau \in \mathcal{O}$. We take any point (z, ζ) strictly below $S_{\mu,\lambda}$ and show that it is also strictly below $S_{\tau,\theta}$. In terms of notation 3, this problem looks as follows: we assume that the inequality $|\mu z - \lambda|^2 + |\mu|^2\zeta^2 < 1$ is satisfied, and show that this implies the inequality $|\tau z - \theta|^2 + |\tau|^2\zeta^2 < 1$. The first inequality can be transformed into $\left|z - \frac{\lambda}{\mu}\right|^2 + \zeta^2 < \frac{1}{|\mu|^2}$. Hence, $\sqrt{\left|z - \frac{\lambda}{\mu}\right|^2 + \zeta^2} < \frac{1}{|\mu|}$. We will insert this into the triangle inequality for the Euclidean distance in $\mathbb{C} \times \mathbb{R}$ applied to the three points (z, ζ), $(\frac{\lambda}{\mu}, 0)$ and $(\frac{\theta}{\tau}, 0)$, which is

$$\sqrt{\left|z - \frac{\theta}{\tau}\right|^2 + \zeta^2} \; < \left|\frac{\lambda}{\mu} - \frac{\theta}{\tau}\right| + \sqrt{\left|z - \frac{\lambda}{\mu}\right|^2 + \zeta^2}.$$

So we obtain $\sqrt{\left|z - \frac{\theta}{\tau}\right|^2 + \zeta^2} \; < \left|\frac{\lambda}{\mu} - \frac{\theta}{\tau}\right| + \frac{1}{|\mu|}$. By definition 4, the expression on the right hand side is smaller than or equal to $\frac{1}{|\tau|}$. Therefore, we take the square and obtain $\left|z - \frac{\theta}{\tau}\right|^2 + \zeta^2 < \frac{1}{|\tau|^2}$, which is equivalent to the claimed inequality. □

Another notion that will be useful for our algorithm, is the following.

Definition 5. *Let $z \in \mathbb{C}$ be a point lying within the vertical projection of $S_{\mu,\lambda}$. Define the* lift on the hemisphere $S_{\mu,\lambda}$ *of z as the point on $S_{\mu,\lambda}$ the vertical projection of which is z.*

Notation 5. *Denote by the* hemisphere list *a list into which we will record a finite number of hemispheres $S(\alpha_1),\ldots,S(\alpha_n)$. Its purpose is to determine a set $B(\alpha_1,\ldots,\alpha_n)$ in order to approximate, and finally obtain, the Bianchi fundamental polyhedron.*

6.0.2. *The algorithm computing the Bianchi fundamental polyhedron.* We now describe the algorithm that we have realized using Swan's description; it is decomposed into algorithms 1 through 3 below.

Initial step. We begin with the smallest value which the norm of a non-zero element $\mu \in \mathcal{O}$ can take, namely 1. Then μ is a unit in $\mathcal{O}$, and for any $\lambda \in \mathcal{O}$, the pair (μ, λ) is unimodular. And we can rewrite the fraction $\frac{\lambda}{\mu}$ such that $\mu = 1$. We obtain the unit hemispheres (of radius 1), centred at the imaginary quadratic integers $\lambda \in \mathcal{O}$. We record into the hemisphere list the ones which touch the Bianchi fundamental polyhedron, i.e. the ones the centre of which lies in the fundamental rectangle D_0 (of notation 2) for the action of Γ_∞ on the complex plane.

Step A. Increase $|\mu|$ to the next higher value which the norm takes on elements of $\mathcal{O}$. Run through all the finitely many μ which have this norm. For each of these μ, run through all the finitely many λ with $\frac{\lambda}{\mu}$ in the fundamental rectangle D_0. Check that $(\mu, \lambda) = \mathcal{O}$ and that the hemisphere $S_{\mu,\lambda}$ is not everywhere below a hemisphere $S_{\beta,\alpha}$ in the hemisphere list. If these two checks are passed, record (μ, λ) into the hemisphere list.

We repeat step **A** until $|\mu|$ has reached an expected value. Then we check if we have found all the hemispheres which touch the Bianchi fundamental polyhedron, as follows.

Step B. We compute the lines $L(\frac{\alpha}{\beta}, \frac{\lambda}{\mu})$ of definition 4, over which two hemispheres agree, for all pairs $S_{\beta,\alpha}$, $S_{\mu,\lambda}$ in the hemisphere list which touch one another.
Then, for each hemisphere $S_{\beta,\alpha}$, we compute the intersection points of each two lines $L(\frac{\alpha}{\beta}, \frac{\lambda}{\mu})$ and $L(\frac{\alpha}{\beta}, \frac{\theta}{\tau})$ referring to $\frac{\alpha}{\beta}$.
We drop the intersection points at which $S_{\beta,\alpha}$ is strictly below some hemisphere in the list.
We erase the hemispheres from our list, for which less than three intersection points remain. We can do this because a hemisphere which touches the Bianchi fundamental polyhedron only in two vertices shares only an edge with it and no 2-cell.
Now, the vertices of $B(\alpha_1,\ldots,\alpha_n) \cap D_\infty$ are the lifts of the remaining intersection points. Thus we can check Swan's termination criterion (theorem 2), which we do as follows. We pick the lowest value $\zeta > 0$ for which $(z, \zeta) \in \mathcal{H}$ is the lift inside Hyperbolic Space of a remaining intersection point z.

If $\zeta \geqslant \frac{1}{|\mu|}$, then all (infinitely many) remaining hemispheres have radius equal or smaller than ζ, so (z, ζ) cannot be strictly below them. So Swan's termination criterion is fulfilled, we have found the Bianchi fundamental polyhedron, and can proceed by determining its cell structure.
Else, ζ becomes the new expected value for $\frac{1}{|\mu|}$. We repeat step **A** until $|\mu|$ reaches $\frac{1}{\zeta}$ and then proceed again with step **B**.

Algorithm 1 Computation of the Bianchi fundamental polyhedron

Input: A square-free positive integer m.
Output: The hemisphere list, containing entries $S(\alpha_1)$,...,$S(\alpha_n)$ such that $B(\alpha_1, \ldots, \alpha_n) = B$.

Let $\mathcal{O}$ be the ring of integers in $\mathbb{Q}(\sqrt{-m})$.
Let $h_{\mathcal{O}}$ be the class number of $\mathcal{O}$. Compute $h_{\mathcal{O}}$.
Estimate the highest value for $|\mu|$ which will occur in notation 3 by the formula $E := \begin{cases} \frac{5m}{2}h_{\mathcal{O}} - 2m + \frac{1}{2}, & m \equiv 3 \mod 4, \\ 21mh_{\mathcal{O}} - 19m, & \text{else.} \end{cases}$
$\mathcal{N} := 1$.
Swan's_cancel_criterion_fulfilled := false.

while Swan's_cancel_criterion_fulfilled = false, **do**
  while $\mathcal{N} \leqslant E$ **do**
    Execute algorithm 2 with argument $\mathcal{N}$.
    Increase $\mathcal{N}$ to the next greater value in the set $\{\sqrt{n^2m + j^2} \mid n, j \in \mathbb{N}\}$ of values of the norm on $\mathcal{O}$.
  end while
  Compute ζ with algorithm 3.
  if $\zeta \geqslant \frac{1}{\mathcal{N}}$, **then**
    `All (infinitely many) remaining hemispheres have radius smaller than` ζ`,`
    `so` (z, ζ) `cannot be strictly below any of them.`
    Swan's_cancel_criterion_fulfilled := true.
  else
    ζ `becomes the new expected lowest value for` $\frac{1}{\mathcal{N}}$`:`
    $E := \frac{1}{\zeta}$.
  end if
end while

Proposition 3. *The hemisphere list, as computed by algorithm* 1*, determines the Bianchi fundamental polyhedron. This algorithm terminates within finite time.*

Proof.

- The value ζ is the minimal height of the non-singular vertices of the cell complex $\partial B(\alpha_1, \ldots, \alpha_n)$ determined by the hemisphere list $\{S(\alpha_1), \ldots, S(\alpha_n)\}$. All the hemispheres which are not in the list, have radius smaller than $\frac{1}{\mathcal{N}}$. By remark 1, the inequality $\zeta \geqslant \frac{1}{\mathcal{N}}$ will become satisfied; and then no non-singular vertex of $\partial B(\alpha_1, \ldots, \alpha_n)$ can be strictly below any of them.
Hence by theorem 2, $B(\alpha_1, \ldots, \alpha_n) = B$; and we obtain the Bianchi fundamental polyhedron as $B(\alpha_1, \ldots, \alpha_n) \cap D_\infty$.
- We now consider the run-time. By theorem 1, the set of hemispheres

$$\{S_{\mu,\lambda} \mid S_{\mu,\lambda} \text{ touches the Bianchi Fundamental Polyhedron}\}$$

is finite. So, there exists an $S_{\mu,\lambda}$ for which the norm of μ takes its maximum on this finite set. The variable $\mathcal{N}$ reaches this maximum for $|\mu|$ after a finite number of steps; and then Swan's termination criterion is fulfilled. The latter steps require a finite run-time because of propositions 4 and 5.

□

Swan explains furthermore how to obtain an a priori bound for the norm of the $\mu \in \mathcal{O}$ occurring for such hemispheres $S_{\mu,\lambda}$. But he states that this upper bound for $|\mu|$ is much too large. So instead of the theory behind theorem 1, we use Swan's termination criterion (theorem 2 above) to limit the number of steps in our computations. We then get the following.

Observation 1. *We can give bounds for $|\mu|$ in the cases where K is of class number 1 or 2 (there are nine cases of class number 1 and eighteen cases of class number 2, and we have done the computation for all of them). They are the following:*

$$\begin{cases} K \textit{ of class number 1:} & |\mu| \leqslant \frac{|\Delta|+1}{2}, \\ K \textit{ of class number 2:} & \begin{cases} |\mu| \leqslant 3|\Delta|, & m \equiv 3 \mod 4, \\ |\mu| \leqslant (5 + \frac{61}{116})|\Delta|, & \text{else}, \end{cases} \end{cases}$$

where Δ is the discriminant of $K = \mathbb{Q}(\sqrt{-m})$,
i.e., $|\Delta| = \begin{cases} m, & m \equiv 3 \mod 4, \\ 4m, & \text{else}. \end{cases}$

Remark 1. *In algorithm 1, we have chosen the value E by an extrapolation formula for observation 1. If this is greater than the exact bound for $|\mu|$, the algorithm computes additional hemispheres which do not contribute to the Bianchi fundamental polyhedron. On the other hand, if E is smaller than the exact bound for $|\mu|$, it will be increased in the outer while loop of the algorithm, until it is sufficiently large. But then, the algorithm performs some preliminary computations of the intersection lines and vertices, which cost additional run-time. Thus our extrapolation formula is aimed at choosing E slightly greater than the exact bound for $|\mu|$ we expect.*

Algorithm 2 Recording the hemispheres of radius $\frac{1}{\mathcal{N}}$

Input: The value $\mathcal{N}$, and the hemisphere list (empty by default).
Output: The hemisphere list with some hemispheres of radius $\frac{1}{\mathcal{N}}$ added.

for a running from 0 through $\mathcal{N}$ within $\mathbb{Z}$, **do**
  for b in $\mathbb{Z}$ such that $|a + b\omega| = \mathcal{N}$, **do**
    Let $\mu := a + b\omega$.
    for all the $\lambda \in \mathcal{O}$ with $\frac{\lambda}{\mu}$ in the fundamental rectangle D_0, **do**
      if the pair (μ, λ) is unimodular, **then**
        Let $\mathcal{L}$ be the length of the hemisphere list.
        everywhere_below := false, $j := 1$.
        while everywhere_below = false and $j \leqslant \mathcal{L}$, **do**
          Let $S_{\beta,\alpha}$ be the j'th entry in the hemisphere list;
          if $S_{\mu,\lambda}$ is everywhere below $S_{\beta,\alpha}$, **then**
            everywhere_below := true.
          end if
          Increase j by 1.
        end while
        if everywhere_below = false, **then**
          Record $S_{\mu,\lambda}$ into the hemisphere list.
        end if
      end if
    end for
  end for
end for

`We recall that the notion ‘‘everywhere below’’ has been made precise in definition 4; and that the fundamental rectangle` D_0 `has been specified in notation 2.`

Proposition 4. *Algorithm 2 finds all the hemispheres of radius $\frac{1}{N}$, on which a 2-cell of the Bianchi fundamental polyhedron can lie. This algorithm terminates within finite time.*

Proof.

- Directly from the definition of the hemispheres $S_{\mu,\lambda}$, it follows that the radius is given by $\frac{1}{|\mu|}$. So our algorithm runs through all μ in question. By construction of the Bianchi fundamental polyhedron D, the hemispheres on which a 2-cell of D lies must have their centre in the fundamental rectangle D_0. By proposition 2, such hemispheres cannot be everywhere below some other hemisphere in the list.
- Now we consider the run-time of the algorithm. There are finitely many $\mu \in \mathcal{O}$ the norm of which takes a given value. And for a given μ, there are finitely many $\lambda \in \mathcal{O}$ such that $\frac{\lambda}{\mu}$ is in the fundamental rectangle D_0. Therefore, this algorithm consists of finite loops and terminates within finite time.

$\square$

Proposition 5. *Algorithm 3 finds the minimal height occurring amongst the non-singular vertices of $\partial B(\alpha_1, \dots, \alpha_n)$. This algorithm erases only such hemispheres from the list, which do not change $\partial B(\alpha_1, \dots, \alpha_n)$. It terminates within finite time.*

Proof.

- The heights of the points in $\mathcal{H}$ are preserved by the action of the translation group Γ_∞, so we only need to consider representatives in the fundamental domain D_∞ for this action. Our algorithm computes the entire cell structure of $\partial B(\alpha_1, \dots, \alpha_n) \cap D_\infty$, as described in subsection 5.3. The number of lines to intersect is smaller than the square of the length of the hemisphere list, and thus finite. As a consequence, the minimum of the height has to be taken only on a finite set of intersection points, whence the first claim.
- If a cell of $\partial B(\alpha_1, \dots, \alpha_n)$ lies on a hemisphere, then its vertices are lifts of intersection points. So we can erase the hemispheres which are strictly below some other hemispheres at all the intersection points, without changing $\partial B(\alpha_1, \dots, \alpha_n)$.
- Now we consider the run-time. This algorithm consists of loops running through the hemisphere list, which has finite length. Within one of these loops, there is a loop running through the set of pairs of lines $L(\frac{\alpha}{\beta}, \frac{\lambda}{\mu})$. A (far too large) bound for the cardinality of this set is given by the fourth power of the length of the hemisphere list.
The steps performed within these loops are very delimited and easily seen to be of finite run-time.

$\square$

Algorithm 3 Computing the minimal proper vertex height

Input: The hemisphere list $\{S(\alpha_1), \ldots, S(\alpha_n)\}$.
Output: The lowest height ζ of a non-singular vertex of $\partial B(\alpha_1, \ldots, \alpha_n)$. And the hemisphere list with some hemispheres removed which do not touch the Bianchi fundamental polyhedron.

for all pairs $S_{\beta,\alpha}$, $S_{\mu,\lambda}$ in the hemisphere list which touch one another, **do**
  compute the line $L(\frac{\alpha}{\beta}, \frac{\lambda}{\mu})$ of notation 4.
end for

for each hemisphere $S_{\beta,\alpha}$ in the hemisphere list, **do**
  for each two lines $L(\frac{\alpha}{\beta}, \frac{\lambda}{\mu})$ and $L(\frac{\alpha}{\beta}, \frac{\theta}{\tau})$ referring to $\frac{\alpha}{\beta}$, **do**
    Compute the intersection point of $L(\frac{\alpha}{\beta}, \frac{\lambda}{\mu})$ and $L(\frac{\alpha}{\beta}, \frac{\theta}{\tau})$, if it exists.
  end for
end for
Drop the intersection points at which $S_{\beta,\alpha}$ is strictly below some hemisphere in the list.
Erase the hemispheres from our list, for which no intersection points remain.
`Now the vertices of` $B(\alpha_1, \ldots, \alpha_n) \cap D_\infty$ `are the lifts (specified in definition 5) on the appropriate hemispheres of the remaining intersection points.`
Pick the lowest value $\zeta > 0$ for which $(z, \zeta) \in \mathcal{H}$ is the lift on some hemisphere of a remaining intersection point z.
Return ζ.

7. The cell complex and its orbit space

With the method described in subsection 5.3, we obtain a cell structure on the boundary of the Bianchi fundamental polyhedron. The cells in this structure which touch the cusp ∞ are easily determined: they are four 2-cells each lying on one of the Euclidean vertical planes bounding the fundamental domain D_∞ for Γ_∞ specified in notation 2; and four 1-cells each lying on one of the intersection lines of these planes. The other 2-cells in this structure lie each on one of the hemispheres determined with our realization of Swan's algorithm.

As the Bianchi fundamental polyhedron is a hyperbolic polyhedron up to some missing cusps, its boundary cells can be oriented as its facets. Once the cell structure is subdivided until the cells are fixed pointwise by their stabilisers, this cell structure with orientation is transported onto the whole hyperbolic space by the action of Γ.

7.1. **Computing the vertex stabilisers and identifications.** Let us state explicitly the Γ-action on the upper-half space model $\mathcal{H}$, in the form in which we will use it rather than in its historical form.

Lemma 3 (Poincaré). *If* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$*, the action of* γ *on* $\mathcal{H}$ *is given by* $\gamma \cdot (z, \zeta) = (z', \zeta')$*, where*

$$\zeta' = \frac{|\det \gamma| \zeta}{|cz - d|^2 + \zeta^2 |c|^2},$$

$$z' = \frac{\left(\overline{d - cz}\right)(az - b) - \zeta^2 \bar{c} a}{|cz - d|^2 + \zeta^2 |c|^2}.$$

From this operation formula, we establish equations and inequalities on the entries of a matrix sending a given point (z, ζ) to another given point (z', ζ') in $\mathcal{H}$. We will use them in algorithm 4 to compute such matrices. For the computation of the vertex stabilisers, we have $(z, \zeta) = (z', \zeta')$ which simplifies the below equations and inequalities as well as the pertinent algorithm.

First, we fix a basis for $\mathcal{O}$ as the elements 1 and

$$\omega := \begin{cases} \sqrt{-m}, & m \equiv 1 \text{ or } 2 \mod 4, \\ -\frac{1}{2} + \frac{1}{2}\sqrt{-m}, & m \equiv 3 \mod 4. \end{cases}$$

As we have put $m \neq 1$ and $m \neq 3$, the only units in the ring $\mathcal{O}$ are ± 1. We will use the notations $\lceil x \rceil := \min\{n \in \mathbb{Z} \mid n \geqslant x\}$
and $\lfloor x \rfloor := \max\{n \in \mathbb{Z} \mid n \leqslant x\}$ for $x \in \mathbb{R}$.

Lemma 4. *Let* $m \equiv 3 \mod 4$*. Let* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O})$ *be a matrix sending* (z, r) *to* $(\zeta, \rho) \in \mathcal{H}$*. Write* c *in the basis as* $j + k\omega$ *, where* $j, k \in \mathbb{Z}$*. Then* $|c|^2 \leqslant \frac{1}{r\rho}$*,* $|j| \leqslant \sqrt{\frac{1+\frac{1}{m}}{r\rho}}$ *and*

$$\frac{2j}{m+1} - 2\frac{\sqrt{\frac{m+1}{r\rho} - j^2 m}}{m+1} \leqslant k \leqslant \frac{2j}{m+1} + 2\frac{\sqrt{\frac{m+1}{r\rho} - j^2 m}}{m+1}.$$

Algorithm 4 Computation of the matrices identifying two points in $\mathcal{H}$.

Input: The points (z, r), (ζ, ρ) in the interior of $\mathcal{H}$, where z, $\zeta \in K$ and r^2, $\rho^2 \in \mathbb{Q}$.
Output: The set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_{-m})$, $m \equiv 3 \mod 4$, with nonzero entry c, sending the first of the input points to the second one.

c `will run through` $\mathcal{O}$ `with` $0 < |c|^2 \leqslant \frac{1}{r\rho}$.
Write c in the basis as $j + k\omega$, where $j, k \in \mathbb{Z}$.

for j running from $-\left\lceil \sqrt{\frac{1+\frac{1}{m}}{r\rho}} \right\rceil$ through $\left\lceil \sqrt{\frac{1+\frac{1}{m}}{r\rho}} \right\rceil$ **do**
  $k^{\pm}_{\text{limit}} := 2\frac{j}{m+1} \pm 2\frac{\sqrt{\frac{m+1}{r\rho} - j^2 m}}{m+1}$.
  for k running from $\lfloor k^{-}_{\text{limit}} \rfloor$ through $\lceil k^{+}_{\text{limit}} \rceil$ **do**
    $c := j + k\omega$;
    if $|c|^2 \leqslant \frac{1}{r\rho}$ and c nonzero, **then**
      Write cz in the basis as $R(cz) + W(cz)\omega$ with $R(cz), W(cz) \in \mathbb{Q}$.
      d `will run through` $\mathcal{O}$ `with` $|cz - d|^2 + r^2|c|^2 = \frac{r}{\rho}$.
      Write d in the basis as $q + s\omega$, where $q, s \in \mathbb{Z}$.
      $s^{\pm}_{\text{limit}} := W(cz) \pm 2\sqrt{\frac{\frac{r}{\rho} - r^2|c|^2}{m}}$.
      for s running from $\lfloor s^{-}_{\text{limit}} \rfloor$ through $\lceil s^{+}_{\text{limit}} \rceil$ **do**
        $\Delta := \frac{r}{\rho} - r^2|c|^2 - m\left(\frac{W(cz)}{2} - \frac{s}{2}\right)^2$;
        if Δ is a rational square, **then**
          $q_{\pm} := R(cz) - \frac{W(cz)}{2} + \frac{s}{2} \pm \sqrt{\Delta}$.
          Do the following for both $q_{\pm} = q_+$ and $q_{\pm} = q_-$ if $\Delta \neq 0$.
          if $q_{\pm} \in \mathbb{Z}$, **then**
            $d := q_{\pm} + s\omega$;
            if $|cz - d|^2 + r^2|c|^2 = \frac{r}{\rho}$ and (c, d) unimodular, **then**
              $a := \frac{\rho}{r}\overline{d} - \frac{\rho}{r}\overline{cz} - c\zeta$.
              if a is in the ring of integers, **then**
                b `is determined by the determinant` 1:
                $b := \frac{ad-1}{c}$.
                if b is in the ring of integers, **then**
                  Check that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z, r) = (\zeta, \rho)$.
                  Return $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
                end if
              end if
            end if
          end if
        end if
      end for
    end if
  end for
end for

Proof. From the operation equation $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z, r) = (\zeta, \rho)$, we deduce $|cz - d|^2 + r^2|c|^2 = \frac{r}{\rho}$ and conclude $r^2|c|^2 \leqslant \frac{r}{\rho}$, whence the first inequality. We insert $|c|^2 = \left(j - \frac{k}{2}\right)^2 + m\left(\frac{k}{2}\right)^2 = j^2 + \frac{m+1}{4}k^2 - jk$ into it, and obtain

$$0 \geqslant k^2 - \frac{4j}{m+1}k + \frac{4}{m+1}\left(j^2 - \frac{1}{r\rho}\right) =: f(k).$$

We observe that $f(k)$ is a quadratic function in $k \in \mathbb{Z} \subset \mathbb{R}$, taking its values exclusively in $\mathbb{R}$. Hence its graph has the shape of a parabola, and the negative values of $f(k)$ appear exactly on the interval where k is between its two zeroes,

$$k_\pm = \tfrac{2j}{m+1} \pm 2\tfrac{\sqrt{\Delta}}{m+1}, \qquad \text{where } \Delta = \tfrac{m+1}{r\rho} - j^2 m.$$

This implies the third and fourth claimed inequalities. As k is a real number, Δ must be non-negative in order that $f(k)$ be non-positive. Hence $j^2 \leqslant \frac{1+\frac{1}{m}}{r\rho}$, which gives the second claimed inequality. □

Lemma 5. *Under the assumptions of lemma 4, write d in the basis as $q + s\omega$, where $q, s \in \mathbb{Z}$. Write cz in the basis as $R(cz) + W(cz)\omega$, where $R(cz), W(cz) \in \mathbb{Q}$. Then $W(cz) - 2\sqrt{\frac{\frac{r}{\rho} - r^2|c|^2}{m}} \leqslant s \leqslant W(cz) + 2\sqrt{\frac{\frac{r}{\rho} - r^2|c|^2}{m}}$, and*

$$q = R(cz) - \frac{W(cz)}{2} + \frac{s}{2} \pm \sqrt{\frac{r}{\rho} - r^2|c|^2 - m\left(\frac{W(cz)}{2} - \frac{s}{2}\right)^2}.$$

Proof. Recall that $\omega = -\frac{1}{2} + \frac{1}{2}\sqrt{-m}$, so $\overline{q + s\omega} = q - \frac{s}{2} - \frac{s}{2}\sqrt{-m}$. The operation equation yields $|cz - d|^2 + r^2|c|^2 = \frac{r}{\rho}$. From this, we derive

$$\begin{aligned} \tfrac{r}{\rho} - r^2|c|^2 &= (cz - (q + s\omega))\left(\overline{cz} - (q - \tfrac{s}{2} - \tfrac{s}{2}\sqrt{-m})\right) \\ &= \left(\mathrm{Re}(cz) - q + \tfrac{s}{2}\right)^2 + \left(\mathrm{Im}(cz) - \tfrac{s}{2}\sqrt{m}\right)^2 \\ &= \mathrm{Re}(cz)^2 + q^2 - qs + \tfrac{s^2}{4} - 2\mathrm{Re}(cz)q + \mathrm{Re}(cz)s + \left(\mathrm{Im}(cz) - \tfrac{s}{2}\sqrt{m}\right)^2. \end{aligned}$$

We solve for q,

$$q^2 + (-2\mathrm{Re}(cz) - s)\,q + \left(\mathrm{Re}(cz) + \frac{s}{2}\right)^2 + \left(\mathrm{Im}(cz) - \frac{s}{2}\sqrt{m}\right)^2 - \frac{r}{\rho} + r^2|c|^2 = 0$$

and find

$$q_\pm = \mathrm{Re}(cz) + \tfrac{s}{2} \pm \sqrt{\Delta}, \qquad \text{where } \Delta = \tfrac{r}{\rho} - r^2|c|^2 - \left(\mathrm{Im}(cz) - \tfrac{s}{2}\sqrt{m}\right)^2.$$

We express this as

$$q_\pm = R(cz) - \tfrac{W(cz)}{2} + \tfrac{s}{2} \pm \sqrt{\Delta}, \qquad \text{where } \Delta = \tfrac{r}{\rho} - r^2|c|^2 - m\left(\tfrac{W(cz)}{2} - \tfrac{s}{2}\right)^2,$$

which is the claimed equation. The condition that q must be a rational integer implies $\Delta \geqslant 0$, which can be rewritten in the claimed inequalities. □

We further state a simple inequality in order to prove that algorithm 4 terminates in finite time.

Lemma 6. *Let $K = \mathbb{Q}(\sqrt{-m})$ with $m \neq 3$. Let $c, z \in K$. Write their product cz in the $\mathbb{Q}$-basis $\{1, \omega\}$ for K as $R(cz) + W(cz)\omega$.*
Then $|W(cz)| \leqslant |c| \cdot |z|$.

Proof. Let $x + y\omega \in K$ with $x, y \in \mathbb{Q}$. Our first step is to show that $|y| \leqslant |x + y\omega|$. Consider the case $m \equiv 1$ or $2 \mod 4$. Then

$$|x + y\omega| = \sqrt{x^2 + my^2} \geqslant \sqrt{m}|y| \geqslant |y|,$$

and we have shown our claim. Else consider the case $m \equiv 3 \mod 4$. Then,

$$|x + y\omega| = \sqrt{(x + \omega y)(x + \overline{\omega} y)} = \sqrt{\left(x^2 - 2x\frac{y}{2} + \frac{y^2}{4}\right) + \frac{m}{4}y^2} \geqslant \frac{\sqrt{m}}{2}|y|,$$

and our claim follows for $m > 3$. Now we have shown that $|W(cz)| \leqslant |cz|$; and we use some embedding of K into $\mathbb{C}$ to verify the equation $|cz| = |c| \cdot |z|$. □

Proposition 6. *Let $m \equiv 3 \mod 4$. Then algorithm 4 gives all the matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O})$ with $c \neq 0$, sending (z, r) to $(\zeta, \rho) \in \mathcal{H}$. It terminates in finite time.*

Proof.
- The first claim is easily established using the bounds and formulae stated in lemmata 4 and 5.
- Now we consider the run-time. This algorithm consists of three loops the limits of which are at most linear expressions in $\frac{1}{\sqrt{r\rho}}$. For $s^{\pm}_{\text{limit}}$, we use lemma 6 and $r^2|c|^2 \leqslant \frac{r}{\rho}$ to see this (we get a factor $|z|$ here, which we can neglect).

□

8. Computations

We compute the differentials $d_1^{2,0}$ and $d_2^{0,1}$ of our equivariant spectral sequence, as well as the implied cell stabilisers and identifications with the program *Bianchi.gp* [Rah]. The second author has implemented a MAGMA script that computes from this data the relevant vector spaces, as described in Section 4. As linear algebra over number fields is more expensive compared to working over finite fields, we employ the following shortcut. Recall that by the universal coefficients theorem, the dimension of $H^2(\Gamma, M(\mathbb{F}_p))$ ("the mod p dimension") is greater than or equal to the dimension of $H^2(\Gamma, M(\mathbb{C}))$ ("the complex dimension"). We start computing, under conjecture 1, the mod p-dimensions for primes $p \leqslant 200$. If we find for a particular p for which the mod p dimension is equal to the lower bound of Finis-Grunewald-Tirao then we infer that the complex dimension is equal to the mod p dimension. Note that by Prop. 3.2 (d) of [Sen10], this implies that $H^2(\Gamma, M(\mathcal{O}))$ has no p-torsion. If this is not the case for the primes in our range, then we compute the complex dimension directly by computing $H^2(\Gamma, M(K))$.

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