

AN EFFICACY ALGORITHM FOR THE TWO DIMENSIONAL STEADY NAVIER-STOKES EQUATIONS BY MULTI-GRID SCHEME

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ABSTRACT. In this paper, we propose a multi-grid algorithm for solving the Navier-Stokes equations and we analyze its convergence. The proposed method consists of multi-level grids, where in the first one (the very coarse mesh $h_0 = H$) we solve one small nonlinear system of discrete Navier-Stokes equations, then, in every intermediate mesh ($h_i = h_{i-1}^2 = H^{2^i}$) we solve two linear problems and in the last one (the very fine mesh $h_n = H^{2^n}$) we solve one linear system. Moreover, the algorithm produces a numerical solution with an optimal asymptotic error estimate with respect to $h = h_n$. Finally, we give some numerical illustrations showing the efficiency of the new multi-grid algorithm.

1. INTRODUCTION.

The purpose of the present work is to solve the non-stationary incompressible Navier-Stokes problem using a multi-grid scheme on different levels of grids and to show that this algorithm global error is similar to the one of the direct resolution of the non-linear problem on the finest grid. This strategy is a general method for solving a non-linear Partial Differential Equation (PDE) with solution u . This technique proceeds as follows : In a first step (step of order $i = 0$), we discretize the fully non-linear PDE on a very coarse grid of mesh-size $h_0 = H$ and we compute an approximate solution u_H . Then recursively, for a step of order i ($i = 1, \dots, n - 1$), we solve two linear problems. The first one consists of linearizing the Navier-Stokes around $u_{h_{i-1}}$ to obtain the solution $u_{h_i}^{1/2}$ and the second one around $u_{h_i}^{1/2}$ to obtain u_{h_i} . Finally, at the step of order n , we do only the first linearized problem of the previous step. We prove that, under suitable assumptions and if the mesh sizes h_i ($i = 0, \dots, n$) are well-chosen, the order of the global error of the multi-grid algorithm $\|u - u_{h_n}\|$ is similar to that of the non-linear problem discretized directly on the finest grid $h = h_n = H^{2^n}$.

Multi-grid discretizations (in particular two-grid ones) have been widely applied to linear and non-linear elliptic boundary value problems: J. Xu has pioneered their development in [22], [23], [24]. These methods have been extended to the steady Navier-Stokes equations, cf. for instance the work of W. Layton [13], W. Layton & W. Lenferink [14] and V. Girault & J.-L. Lions [7]. Also, these methods have been applied to the time-dependent Navier-Stokes problem, cf. V. Girault & J.-L. Lions [8] for an analysis of a semi-discrete algorithm, H. Abboud & T. Sayah [2] and H. Abboud, V. Girault & T. Sayah [3] for an analysis of a fully-discrete in time and space algorithm. In this work, we generalize the result obtained by J.L. Lions and V. Girault in [7].

Let Ω be a convex bounded domain of \mathbb{R}^2 with a polygonal boundary $\partial\Omega$. Consider the following Navier-Stokes problem for an incompressible fluid

$$-\nu\Delta u(x) + u(x) \cdot \nabla u(x) + \nabla p(x) = f(x) \text{ in } \Omega, \quad (1.1)$$

with the incompressibility condition

$$\operatorname{div} u(x) = 0 \text{ in } \Omega \quad (1.2)$$

and the homogeneous Dirichlet boundary condition

$$u(x) = 0 \text{ on } \partial\Omega, \quad (1.3)$$

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where u and p represent respectively the velocity and the pressure of the fluid. All the quantities are taken at the point $x = (x_1, x_2) \in \mathbb{R}^2$. We suppose that the fluid density is constant ($\rho = 1$); f denotes the external forces applied to the fluid and ν denotes the viscosity. The notations $u \cdot \nabla u$, Δu and $\operatorname{div} u$ mean :

$$u \cdot \nabla u = \sum_{i=1}^2 u_i \frac{\partial u}{\partial x_i}, \quad \Delta u = \sum_{i=1}^2 \frac{\partial^2 u}{\partial^2 x_i} \quad \text{and} \quad \operatorname{div} u = \sum_{i=1}^2 \frac{\partial u_i}{\partial x_i}.$$

The term $u \cdot \nabla u$ is the convection term and $\nu \Delta u$ is the diffusion one.

Setting $L_0^2(\Omega) = \{q \in L^2(\Omega); \int_{\Omega} q \, dx = 0\}$ and assuming that f belongs to $H^{-1}(\Omega)^2$, it is well known that (1.1)–(1.3) has the following variational formulation: find $u \in H_0^1(\Omega)^2$, such that

$$\forall v \in H_0^1(\Omega)^2, \quad \nu(\nabla u, \nabla v) + (u \cdot \nabla u, v) - (p, \operatorname{div} v) = \langle f, v \rangle, \quad (1.4)$$

$$\forall q \in L_0^2(\Omega), \quad (q, \operatorname{div} u) = 0. \quad (1.5)$$

This problem has one and only one solution (up) and we have the following regularity result:

Theorem 1.1. *If Ω is convex and $f \in L^2(\Omega)^2$, then*

$$u \in H^2(\Omega)^2 \text{ and } p \in H^1(\Omega). \quad (1.6)$$

For discretizing (1.4)–(1.5), let $\eta > 0$ be a discretization parameter in space and for each η , let \mathcal{T}_η be a corresponding regular (or non-degenerate) family of triangulations of $\bar{\Omega}$, consisting of triangles such that any two triangles are either disjoint or share a vertex or an entire side. For an arbitrary triangle κ , we denote by η_κ the diameter of κ and by ρ_κ the diameter of the circle inscribed in κ . Then η denotes the maximum of η_κ and we assume that \mathcal{T}_η is regular in the sense of Ciarlet [6] : there exists a constant σ independent of η such that

$$\sup_{\kappa \in \mathcal{T}_\eta} \frac{\eta_\kappa}{\rho_\kappa} = \sigma_\kappa \leq \sigma. \quad (1.7)$$

Let X_η and M_η be a "stable" pair of finite-element spaces for discretizing the velocity u and the pressure p , stable in the sense that it satisfies a uniform discrete inf-sup condition: there exists a constant $\beta^* \geq 0$, independent of η , such that

$$\forall q_\eta \in M_\eta, \quad \sup_{v_\eta \in X_\eta} \frac{1}{|v_\eta|_{H^1(\Omega)}} \int_{\Omega} q_\eta \operatorname{div} v_\eta \, dx \geq \beta^* \|q_\eta\|_{L^2(\Omega)}. \quad (1.8)$$

Let \mathbb{P}_κ denote the space of polynomials with total degree less than or equal to κ . As the multi-grid scheme is better adapted to finite-elements of low degree, we may choose for instance the "mini-element" (see D. Arnold, F. Brezzi and M. Fortin in [5]), where in each triangle κ , the pressure p is a polynomial of \mathbb{P}_1 and each component of the velocity is the sum of a polynomial of \mathbb{P}_1 and a "bubble" function b_κ .

Denoting the vertices of κ by $a_i, 1 \leq i \leq 3$, and its corresponding barycentric coordinates by λ_i , the basic bubble function b_κ is the polynomial of degree three given by

$$b_\kappa(x) = \lambda_1(x)\lambda_2(x)\lambda_3(x).$$

We observe that $b_\kappa(x) = 0$ on $\partial\kappa$ and that $b_\kappa(x) > 0$ on κ . The graph of b_κ looks like a bulb attached to the boundary of κ , whence its name.

Therefore, the finite-element spaces are :

$$X_\eta = \{v_\eta \in C^0(\bar{\Omega})^2; \forall \kappa \in \mathcal{T}_\eta, v_{\eta|\kappa} \in \mathcal{P}(\kappa), v_{\eta|\partial\Omega} = 0\}, \quad (1.9)$$

$$M_\eta = \left\{ q_\eta \in C^0(\bar{\Omega}); \forall \kappa \in \mathcal{T}_\eta, q_{\eta|\kappa} \in \mathbb{P}_1, \int_{\Omega} q_\eta \, dx = 0 \right\}, \quad (1.10)$$

where

$$\mathcal{P}(\kappa) = [\mathbb{P}_1 \oplus \operatorname{Vect}(b_\kappa)]^2. \quad (1.11)$$

There exists an approximation operator $P_\eta \in \mathcal{L}(H_0^1(\Omega)^2; X_\eta)$ such that (see [9]) :

$$\forall v \in H_0^1(\Omega)^2, \quad \forall q_\eta \in M_\eta, \quad \int_\Omega q_\eta \operatorname{div}(P_\eta(v) - v) dx = 0, \quad (1.12)$$

for $k = 0$ or 1 ,

$$\forall v \in [H^{1+k}(\Omega) \cap H_0^1(\Omega)]^2, \quad \|P_\eta(v) - v\|_{L^2(\Omega)} \leq C\eta^{1+k}|v|_{H^{1+k}(\Omega)} \quad (1.13)$$

and for all $r \geq 2, k = 0$ or 1 ,

$$\forall v \in [W^{1+k,r}(\Omega) \cap H_0^1(\Omega)]^2, \quad |P_\eta(v) - v|_{W^{1,r}(\Omega)} \leq C_r \eta^k |v|_{W^{1+k,r}(\Omega)}. \quad (1.14)$$

In addition, as M_η contains all polynomials of degree one, there exists an operator $r_\eta \in \mathcal{L}(L_0^2(\Omega); M_\eta)$, such that for any real number $s \in [0, 2]$, we have

$$\forall q \in H^s(\Omega) \cap L_0^2(\Omega), \quad \|r_\eta(q) - q\|_{L^2(\Omega)} \leq C\eta^s |q|_{H^s(\Omega)}. \quad (1.15)$$

With these spaces, we propose the following multi-grid scheme for discretizing (1.4)–(1.5). Let H be the mesh step of the very coarse mesh and let h_i be the sequence defined by $h_i = h_{i-1}^2 = H^{2^i}$. We use multi-regular nested triangulations \mathcal{T}_{h_i} ($i = 0, \dots, n$) of $\bar{\Omega}$ such that \mathcal{T}_{h_i} is a refinement of $\mathcal{T}_{h_{i-1}}$. In that case, the interpolation/projection procedure is easy.

On each of these, we define the same stable pair of finite-element spaces (X_{h_i}, M_{h_i}) such that $X_{h_{i-1}} \subset X_{h_i}$ and $M_{h_{i-1}} \subset M_{h_i}$. The multi-grid algorithm reads :

• **Step** $H = h_0$ (non-linear problem on coarse grid): find (u_H, p_H) with values in $X_H \times M_H$, solution of

$$\forall v_H \in X_H, \quad \nu(\nabla u_H, \nabla v_H) + (u_H \cdot \nabla u_H, v_H) - (p_H, \operatorname{div} v_H) = \langle f, v_H \rangle, \quad (1.16)$$

$$\forall q_H \in M_H, \quad (q_H, \operatorname{div} u_H) = 0. \quad (1.17)$$

• **Step** h_i ($i = 1, \dots, n-1$) (linearized problem on fine grid h_i): Having $(u_{h_{i-1}}, p_{h_{i-1}})$, find (u_{h_i}, p_{h_i}) with values in $X_{h_i} \times M_{h_i}$ solution of

$$\forall v_{h_i} \in X_{h_i}, \quad \nu(\nabla u_{h_i}^{1/2}, \nabla v_{h_i}) + (u_{h_{i-1}} \cdot \nabla u_{h_i}^{1/2}, v_{h_i}) - (p_{h_i}^{1/2}, \operatorname{div} v_{h_i}) = \langle f, v_{h_i} \rangle, \quad (1.18)$$

$$\forall q_{h_i} \in M_{h_i}, \quad (q_{h_i}, \operatorname{div} u_{h_i}^{1/2}) = 0, \quad (1.19)$$

then

$$\forall v_{h_i} \in X_{h_i}, \quad \nu(\nabla u_{h_i}, \nabla v_{h_i}) + (u_{h_i}^{1/2} \cdot \nabla u_{h_i}, v_{h_i}) - (p_{h_i}, \operatorname{div} v_{h_i}) = \langle f, v_{h_i} \rangle \quad (1.20)$$

$$\forall q_{h_i} \in M_{h_i}, \quad (q_{h_i}, \operatorname{div} u_{h_i}) = 0. \quad (1.21)$$

• **Step** h_n (linearized problem on fine grid h_n): Having $(u_{h_{n-1}}, p_{h_{n-1}})$, find (u_{h_n}, p_{h_n}) with values in $X_{h_n} \times M_{h_n}$ solution of

$$\forall v_{h_n} \in X_{h_n}, \quad \nu(\nabla u_{h_n}, \nabla v_{h_n}) + (u_{h_{n-1}} \cdot \nabla u_{h_n}, v_{h_n}) - (p_{h_n}, \operatorname{div} v_{h_n}) = \langle f, v_{h_n} \rangle, \quad (1.22)$$

$$\forall q_{h_n} \in M_{h_n}, \quad (q_{h_n}, \operatorname{div} u_{h_n}) = 0. \quad (1.23)$$

The purpose of this two-grid algorithm is to reduce the time of computation for both velocity and pressure without losing precision. We will obtain the a priori error estimate

$$|u_{h_n} - u|_{1,\Omega} \leq Ch$$

where $h = h_n$ and C a constant independent of h . This last error is similar to that of the non-linear Navier-Stokes problem solved directly on the mesh of step h .

In what follows, all constants are positive and independent of h_i ($i = 0, \dots, n$), and we refer them by a generic one denoted C .

Remark 1.2. In theory, the discrete nonlinear term of the first step (1.16)-(1.17) is written in an antisymmetric form (cf. for instance [19]):

$$(u_H \cdot \nabla u_H, v_H) + \frac{1}{2}(\operatorname{div} u_H, u_H \cdot v_H),$$

so that it vanishes when $v_H = u_H$. In fact, this is not necessary; for the reader's convenience, we refer to [7] for the existence of the solutions for h small enough and their strong convergence in $H_0^1(\Omega)^2 \times L_0^2(\Omega)$, without restrictions on the data f and ν .

The remainder of this paper is organized as follows: in section 2, we recall some conventions and notations that will be used throughout the article. In section 3, we establish an error estimation for the algorithm and, finally, in section 4, we confirm these results numerically.

2. PRELIMINARIES.

Let (k_1, k_2) denote a pair of non-negative integers, set $|k| = k_1 + k_2$ and define the partial derivative operator ∂^k by $\partial^k v = \frac{\partial^{|k|} v}{\partial x_1^{k_1} \partial x_2^{k_2}}$. For any non-negative integer m and real number $r \geq 1$, we introduce the space

$$W^{m,r}(\Omega) = \{v \in L^r(\Omega); \partial^k v \in L^r(\Omega), \forall |k| \leq m\}$$

equipped with the seminorm

$$|v|_{W^{m,r}(\Omega)} = \left[\sum_{|k|=m} \int_{\Omega} |\partial^k v|^r dx \right]^{1/r},$$

and the norm

$$\|v\|_{W^{m,r}(\Omega)} = \left[\sum_{0 \leq |k| \leq m} |v|_{W^{k,r}(\Omega)}^r \right]^{1/r},$$

with the usual extension when $r = \infty$. In the case where $r = 2$, this space is the Hilbert space $H^m(\Omega)$. In particular, the scalar product of $L^2(\Omega)$ is denoted by (\cdot, \cdot) .

For functions that vanish on the boundary, we recall Poincaré's inequality: there exists a constant \mathcal{P} such that

$$\forall v \in H_0^1(\Omega), \|v\|_{L^2(\Omega)} \leq \mathcal{P} |v|_{H^1(\Omega)}. \quad (2.1)$$

More generally, recall the inequalities of Sobolev imbedding in two dimensions: for each $r \in [2, \infty[$, there exists a constant S_r such that

$$\forall v \in H_0^1(\Omega), \|v\|_{L^r(\Omega)} \leq S_r |v|_{H^1(\Omega)}, \quad (2.2)$$

where

$$|v|_{H^1(\Omega)} = \|\nabla v\|_{L^2(\Omega)}. \quad (2.3)$$

When $r = 2$, (2.2) reduces to Poincaré's inequality and S_2 is Poincaré's constant. The case $r = \infty$ is excluded and is replaced by: for any $r > 2$, there exists a constant M_r such that

$$\forall v \in W_0^{1,r}(\Omega), \|v\|_{L^\infty(\Omega)} \leq M_r |v|_{W^{1,r}(\Omega)}. \quad (2.4)$$

Owing to (2.1), we use the seminorm $|\cdot|_{H^1(\Omega)}$ as a norm on $H_0^1(\Omega)$ and we use it to define the norm of the dual space $H^{-1}(\Omega)$ of $H_0^1(\Omega)$:

$$\|f\|_{H^{-1}(\Omega)} = \sup_{v \in H_0^1(\Omega)} \frac{\langle f, v \rangle}{|v|_{H^1(\Omega)}},$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

3. ERROR ESTIMATIONS FOR THE SOLUTION OF THE ALGORITHM

In this section, we establish the error estimates of the algorithm.

We begin with the first step with the size mesh $h_0 = H$. The error analysis of (1.16)-(1.17) cannot be done without additional assumptions. This can be easily seen by taking the difference between (1.4) and (1.16). Either we impose on the size of the data the same restriction that guarantees uniqueness (cf. [15], [19], or [9]), or we allow for multiple solutions and we assume that the particular solution we want to discretize is nonsingular, as in [9]. We shall adopt here the second option, because it is less restrictive. Thus, from now on, we assume that u is a nonsingular solution of (1.16)-(1.17). Then, according to [9], there exists $\eta_0 > 0$ such that for all $H \leq \eta_0$, (1.16) has a nonsingular solution u_H that is unique in a neighborhood \mathcal{O} of u , and the radius of \mathcal{O} can be bounded by a constant independent of h .

If the solution u is sufficiently smooth, we refer to [7] for the following result: there exists a constant η_1 with $0 < \eta_1 \leq \eta_0$, such that for all $H \leq \eta_1$ we have

$$\begin{aligned} |u_H|_{1,\Omega} &\leq C_1 \|f\|_{0,\Omega}, \\ |u - u_H|_{1,\Omega} &\leq C_2 H, \\ \|u - u_H\|_{0,\Omega} &\leq C_3 H^2, \end{aligned} \quad (3.1)$$

where C_1 , C_2 and C_3 are constants independent of H .

At every step h_i ($i = 1, \dots, n-1$), we solve two linear problems (1.18)-(1.19) and (1.20)-(1.21). We suppose that $H \leq \eta_1$ and we proceed recursively beginning with $i = 1$.

Theorem 3.1. *If there exists a positive constant C_4 such that $u_{h_{i-1}}$ satisfies*

$$|u - u_{h_{i-1}}|_{1,\Omega} \leq C_4 h_{i-1}, \quad (3.2)$$

and if h_{i-1} is sufficiently small, then, the problem (1.18)-(1.19) has one and only one solution such that

$$|u_{h_i}^{1/2}|_{1,\Omega} \leq C \|f\|_{0,\Omega}. \quad (3.3)$$

Proof. We denote by $a(u_{h_i}^{1/2}, v_{h_i}) = \nu(\nabla u_{h_i}^{1/2}, \nabla v_{h_i}) + (u_{h_{i-1}} \cdot \nabla u_{h_i}^{1/2}, v_{h_i})$ the bilinear form of the problem (1.18)-(1.19). Using the incompressibility condition (1.2), we have :

$$\begin{aligned} a(v_{h_i}, v_{h_i}) &= \nu(\nabla v_{h_i}, \nabla v_{h_i}) + (u_{h_{i-1}} \cdot \nabla v_{h_i}, v_{h_i}) \\ &= \nu |\nabla v_{h_i}|_{1,\Omega}^2 + ((u_{h_{i-1}} - u) \cdot \nabla v_{h_i}, v_{h_i}) \\ &\geq |v_{h_i}|_{1,\Omega}^2 (\nu - S_4^2 |u_{h_{i-1}} - u|_{1,\Omega}) \\ &\geq |v_{h_i}|_{1,\Omega}^2 (\nu - C_4 S_4^2 h_{i-1}). \end{aligned}$$

For $h_{i-1} < \frac{\nu}{C_4 S_4^2}$, the bilinear form $a(\cdot, \cdot)$ is coercive which guarantees the existence and the uniqueness of the solution of (1.18)-(1.19). The bound (3.3) is obtained by taking $v_{h_i} = u_{h_i}^{1/2}$ in equation (1.18). \square

Theorem 3.2. *Under the assumptions of theorem 3.1, if there exists a positive constant C_5 such that*

$$\|u - u_{h_{i-1}}\|_{0,\Omega} \leq C_5 h_{i-1}^2, \quad (3.4)$$

and if the solution (u, p) of (1.4)-(1.5) satisfies $u \in H^2(\Omega)^2$, $\nabla u \in L^\infty(\Omega)^2$ and $p \in H^1(\Omega)$, then, the numerical solution of (1.18)-(1.19) verifies

$$|u - u_{h_i}^{1/2}|_{1,\Omega} \leq C h_i. \quad (3.5)$$

Proof. We choose in equation (1.4) $v = v_{h_i}$, take the difference with equation (1.18), insert $\pm P_{h_i} u$ and then take $v_{h_i} = P_{h_i} u - u_{h_i}^{1/2}$ to obtain:

$$\nu |P_{h_i} u - u_{h_i}^{1/2}|_{1,\Omega} = \nu (P_{h_i} u - u, v_{h_i}) + (p - p_{h_i}^{1/2}, \operatorname{div} v_{h_i}) - (u \nabla u - u_{h_{i-1}} \nabla u_{h_i}^{1/2}, v_{h_i}). \quad (3.6)$$

The property (1.13) leads to the following bound for the first term of the right hand side

$$|\nu(P_{h_i}u - u, v_{h_i})| \leq Ch_i. \quad (3.7)$$

Using (1.12), the second term of the right hand side writes:

$$(p - p_{h_i}^{1/2}, \operatorname{div}(P_{h_i}u - u_{h_i}^{1/2})) = (p - r_{h_i}p, \operatorname{div}(P_{h_i}u - u_{h_i}^{1/2})) \leq Ch|P_{h_i}u - u_{h_i}^{1/2}|_{1,\Omega}.$$

From the the incompressibility condition (1.2), the third term of the right hand side can be written as

$$\begin{aligned} (u\nabla u - u_{h_{i-1}}\nabla u_{h_i}^{1/2}, v_{h_i}) &= (u\nabla(u - u_{h_i}^{1/2}), v_{h_i}) + ((u - u_{h_{i-1}})\nabla(u_{h_i}^{1/2} - u), v_{h_i}) \\ &\quad + ((u - u_{h_{i-1}})\nabla u, v_{h_i}) \\ &= (u\nabla(u - P_{h_i}u), v_{h_i}) + ((u - u_{h_{i-1}})\nabla(u_{h_i}^{1/2} - P_{h_i}u), v_{h_i}) \\ &\quad + ((u - u_{h_{i-1}})\nabla(P_{h_i}u - u), v_{h_i}) + ((u - u_{h_{i-1}})\nabla u, v_{h_i}). \end{aligned} \quad (3.8)$$

Using (2.2) and theorem 3.2, we get the bound

$$\begin{aligned} (u\nabla u - u_{h_{i-1}}\nabla u_{h_i}^{1/2}, v_{h_i}) &\leq C_6 \left(|v_{h_i}|_{1,\Omega} |u|_{1,\Omega} |u - P_{h_i}u|_{1,\Omega} + |u - u_{h_{i-1}}|_{1,\Omega} |v_{h_i}|_{1,\Omega}^2 \right. \\ &\quad \left. + |u - u_{h_{i-1}}|_{1,\Omega} |P_{h_i}u - u|_{1,\Omega} |v_{h_i}|_{1,\Omega} \right. \\ &\quad \left. + \|u - u_{h_{i-1}}\|_{0,\Omega} \|\nabla u\|_{L^\infty(\Omega)^2} \|v_{h_i}\|_{0,\Omega} \right) \\ &\leq C_7(h_{i-1}^2 + h_i) |v_{h_i}|_{1,\Omega} + C_6 h_{i-1} |v_{h_i}|_{1,\Omega}^2. \end{aligned} \quad (3.9)$$

Then, (3.6) gives

$$\nu |P_{h_i}u - u_{h_i}^{1/2}|_{1,\Omega} \leq C_8(h_{i-1}^2 + h_i) + C_6 h_{i-1} |v_{h_i}|_{1,\Omega}.$$

For $h_{i-1} < \frac{\nu}{C_6}$, we get

$$|P_{h_i}u - u_{h_i}^{1/2}|_{1,\Omega} \leq C_9(h_{i-1}^2 + h_i),$$

which leads, with the triangular inequality $|u - u_{h_i}^{1/2}|_{1,\Omega} \leq |u - P_{h_i}u|_{1,\Omega} + |P_{h_i}u - u_{h_i}^{1/2}|_{1,\Omega}$, to the result. \square

Theorem 3.3. *Under the assumptions of theorem 3.2 and if h_i is sufficiently small, the problem (1.20)-(1.21) has a unique solution which satisfies*

$$|u_{h_i}|_{1,\Omega} \leq C \|f\|_{0,\Omega}. \quad (3.10)$$

Proof. The proof is analogue to that of theorem 3.1. \square

Theorem 3.4. *Under the assumptions of theorem 3.3, the solution of (1.20)-(1.21) satisfies*

$$|u - u_{h_i}|_{1,\Omega} \leq Ch_i \quad (3.11)$$

and

$$\|u - u_{h_i}\|_{0,\Omega} \leq Ch_i^2. \quad (3.12)$$

Proof. To establish the bound (3.11), we choose in equation (1.4) $v = v_{h_i}$, take the difference with equation (1.20), insert $\pm P_{h_i}u$ and take $v_{h_i} = P_{h_i}u - u_{h_i}$ to obtain:

$$\nu |P_{h_i}u - u_{h_i}|_{1,\Omega}^2 = \nu(P_{h_i}u - u, v_{h_i}) + (p - p_{h_i}, \operatorname{div} v_{h_i}) - (u\nabla u - u_{h_i}^{1/2}\nabla u_{h_i}, v_{h_i}). \quad (3.13)$$

The property (1.13) leads to the following bound for the first term of the right hand side

$$|\nu(P_{h_i}u - u, v_{h_i})| \leq Ch_i. \quad (3.14)$$

Using (1.12), the second term of the right hand side writes:

$$(p - p_{h_i}, \operatorname{div}(P_{h_i}u - u_{h_i})) = (p - r_{h_i}p, \operatorname{div}(P_{h_i}u - u_{h_i})) \leq Ch|P_{h_i}u - u_{h_i}|_{1,\Omega}.$$

From the incompressibility condition (1.2), the third term of the right hand side can be written as

$$\begin{aligned}
 (u \nabla u - u_{h_i}^{1/2} \nabla u_{h_i}, v_{h_i}) &= (u \nabla(u - u_{h_i}), v_{h_i}) + ((u - u_{h_i}^{1/2}) \nabla(u_{h_i} - u), v_{h_i}) \\
 &\quad + ((u - u_{h_i}^{1/2}) \nabla u, v_{h_i}) \\
 &= (u \nabla(u - P_{h_i} u), v_{h_i}) + ((u - u_{h_i}^{1/2}) \nabla(u_{h_i} - P_{h_i} u), v_{h_i}) \\
 &\quad + ((u - u_{h_i}^{1/2}) \nabla(P_{h_i} u - u), v_{h_i}) + ((u - u_{h_i}^{1/2}) \nabla u, v_{h_i}).
 \end{aligned} \tag{3.15}$$

Using (2.2) and theorem 3.2, we get the bound

$$\begin{aligned}
 (u \nabla u - u_{h_i}^{1/2} \nabla u_{h_i}, v_{h_i}) &\leq C \left(|v_{h_i}|_{1,\Omega} |u|_{1,\Omega} |u - P_{h_i} u|_{1,\Omega} + |v_{h_i}|_{1,\Omega}^2 |u - u_{h_i}^{1/2}|_{1,\Omega} \right) \\
 &\quad + |u - u_{h_i}^{1/2}|_{1,\Omega} |P_{h_i} u - u|_{1,\Omega} |v_{h_i}|_{1,\Omega} \\
 &\quad + |u - u_{h_i}^{1/2}|_{1,\Omega} |u|_{1,\Omega} |v_{h_i}|_{1,\Omega} \\
 &\leq C_1(u, p, \Omega) h_i |v_{h_i}|_{1,\Omega} + C_2 h_i |v_{h_i}|_{1,\Omega}^2.
 \end{aligned} \tag{3.16}$$

Then, (3.14) gives

$$\nu |P_{h_i} u - u_{h_i}|_{1,\Omega} \leq C(u, p, \Omega) h_i + C_2 h_i |v_{h_i}|_{1,\Omega}.$$

For $h_i < \frac{\nu}{C_2}$, we get

$$|P_{h_i} u - u_{h_i}|_{1,\Omega} \leq C_3 h_i,$$

which leads, with the triangular inequality $|u - u_{h_i}|_{1,\Omega} \leq |u - P_{h_i} u|_{1,\Omega} + |P_{h_i} u - u_{h_i}|_{1,\Omega}$, to (3.11).

In order to show the bound (3.12), we introduce the following problem of discrete Navier-Stokes equations

$$\begin{aligned}
 &\text{Find } w_{h_i} \in X_{h_i}, \xi_{h_i} \in M_{h_i} \text{ such that} \\
 \forall v_{h_i} \in X_{h_i}, \quad &\nu(\nabla w_{h_i}, \nabla v_{h_i}) - (\xi_{h_i}, \text{div } v_{h_i}) = \langle f, v_{h_i} \rangle - (w_{h_i} \cdot \nabla w_{h_i}, v_{h_i}), \\
 \forall q_{h_i} \in M_{h_i}, \quad &(q_{h_i}, \text{div } w_{h_i}) = 0.
 \end{aligned} \tag{3.17}$$

The same argument used for the first step of the algorithm (step H) gives the following: there exists $\eta_i > 0$ such that, for all $h_i \leq \eta_i$ we have

$$\begin{aligned}
 |w_{h_i}|_{1,\Omega} &\leq C_1 \|f\|_{0,\Omega}, \\
 |u - w_{h_i}|_{1,\Omega} &\leq C_2 h_i, \\
 \|u - w_{h_i}\|_{0,\Omega} &\leq C_3 h_i^2.
 \end{aligned} \tag{3.18}$$

We take the difference between (1.20) and the first equation of (3.17), choose $v_{h_i} = u_{h_i} - w_{h_i}$, use the relations $(\xi_{h_i} - p_{h_i}, \text{div } v_{h_i}) = 0$ and $(v \nabla w, w) + \frac{1}{2}(\text{div } v w, w) = 0$, $w \in H_0^1(\Omega)^2$, and the property (2.4) to obtain:

$$\begin{aligned}
 \nu(\nabla(u_{h_i} - w_{h_i}), \nabla v_{h_i}) &= (w_{h_i} \cdot \nabla w_{h_i} - u_{h_i}^{1/2} \cdot \nabla u_{h_i}, v_{h_i}) \\
 &= -(u_{h_i}^{1/2} \cdot \nabla(u_{h_i} - w_{h_i}), v_{h_i}) - ((u_{h_i}^{1/2} - w_{h_i}) \cdot \nabla(w_{h_i} - u), v_{h_i}) - ((u_{h_i}^{1/2} - w_{h_i}) \cdot \nabla u, v_{h_i}) \\
 &= \frac{1}{2}(\text{div}(u_{h_i}^{1/2} - u)(u_{h_i} - w_{h_i}), v_{h_i}) - ((u_{h_i}^{1/2} - w_{h_i}) \cdot \nabla(w_{h_i} - u), v_{h_i}) - ((u_{h_i}^{1/2} - w_{h_i}) \cdot \nabla u, v_{h_i}) \\
 &\leq C h_i^2 |v_{h_i}|_{1,\Omega} + |((u_{h_i}^{1/2} - u) \cdot \nabla u, v_{h_i})| + |((u - w_{h_i}) \cdot \nabla u, v_{h_i})|
 \end{aligned} \tag{3.19}$$

Using (2.2), we get the following

$$\begin{aligned}
 |((u - w_{h_i}) \cdot \nabla u, v_{h_i})| &\leq \|u - w_{h_i}\|_{0,\Omega} \|\nabla u\|_{L^\infty(\Omega)^2} \|v_{h_i}\|_{0,\Omega} \\
 &\leq C h_i^2 |v_{h_i}|_{1,\Omega}.
 \end{aligned} \tag{3.20}$$

To bound the term $|((u_{h_i}^{1/2} - u) \cdot \nabla u, v_{h_i})|$, we use the integration by part formula. Thus, we obtain, for any any constant C_T on the triangle T

$$\begin{aligned}
((u_{h_i}^{1/2} - u) \cdot \nabla u, v_{h_i}) &= \sum_{T \in \mathcal{T}_{h_i}} \int_T (u_{h_i}^{1/2} - u) \cdot \nabla (u - C_T) v_{h_i} \\
&= - \sum_{T \in \mathcal{T}_{h_i}} \left\{ \int_T \operatorname{div}(u_{h_i}^{1/2} - u) (u - C_T) v_{h_i} + \int_T (u_{h_i}^{1/2} - u) \cdot \nabla v_{h_i} (u - C_T) \right. \\
&\quad \left. - \int_{\partial T} (u_{h_i}^{1/2} - u) \cdot n (u - C_T) v_{h_i} \right\} \\
&\leq C \|\nabla u\|_{L^\infty(\Omega)^2} h_i^2.
\end{aligned} \tag{3.21}$$

Finally, we use the last two bounds in (3.19) and the triangular inequality $\|u - u_{h_i}\|_{0,\Omega} \leq \|u - w_{h_i}\|_{0,\Omega} + \|w_{h_i} - u_{h_i}\|_{0,\Omega}$ to obtain (3.12). \square

Corollary 3.5. *Under the assumptions of theorem 3.4 and for H sufficiently small, we have*

$$|u - u_{h_i}|_{1,\Omega} \leq C h_i, \quad \text{for all } i = 0, \dots, n.$$

Proof. If H is sufficiently small, then $h_i = H^{2^i}$ is also sufficiently small. The result is obtained using theorems 3.1, 3.2, 3.3 and 3.4 recursively for $i = 1, \dots, n$. \square

Remark: The proposed algorithm (multi-grid method) allows us to solve the Navier-Stokes equations much faster than one-grid method without losing precision. The order of the obtained error is similar to the one due to resolution of the Navier-Stokes problem directly on the finest mesh h_n .

4. NUMERICAL RESULTS

In this section, we numerically validate the algorithm describe above. We perform several experiments using the FreeFem++ software (see [11]). On the square domain $]0, 1[\times]0, 1[$, the numerical velocity and the pressure are taken as $(u, p) = (\operatorname{curl} \psi, p)$, where:

$$\psi(x, y) = y^2(y - .5)^2(y - .75)^2(y - 1)^2 \sin(4\pi x)^2 \quad \text{and} \quad p(x, y) = \cos(2\pi x) \sin(2\pi y).$$

For the numerical results, we take the three-grid method where $n = 2$.

In figures 1 and 2, we show color comparison between the exact and the numerical solution for both the velocity and the pressure. We have taken $h_0 = H = 1/3$, $h_1 = 1/9$ and $h_2 = 1/81$.

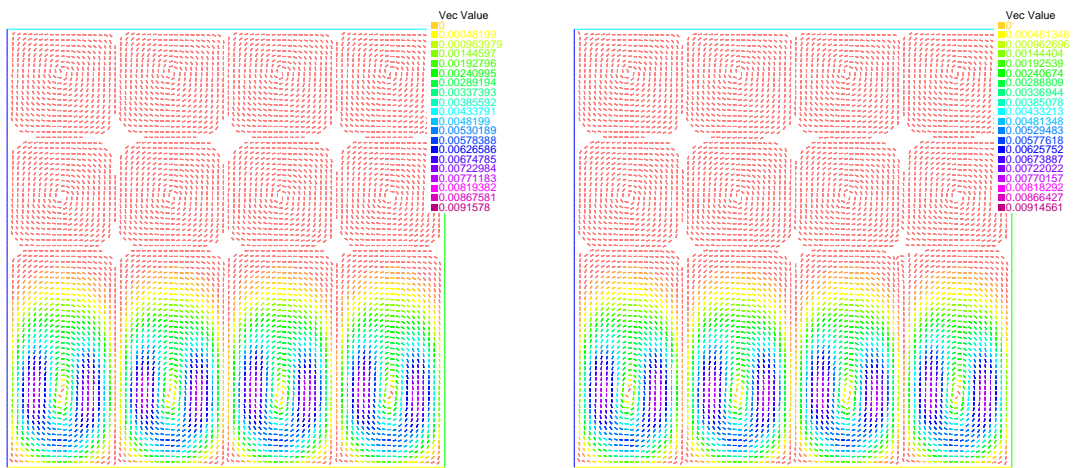


FIGURE 1. Velocity solution : exact (left) and numerical (right).

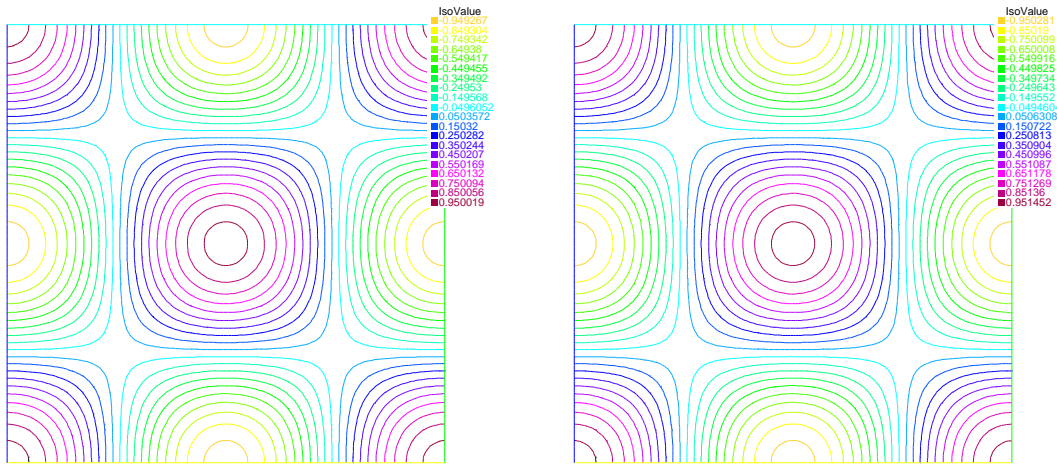


FIGURE 2. Pressure solution : exact (left) and numerical (right).

The values of the error estimations, in the logarithmic scale, are displayed in the following table :

meshes	H^1 velocity error	L^2 pressure error
$H = 1/3, h_1 = 1/4, h_2 = 1/16$	-0.330536	-2.17046
$H = 1/3, h_1 = 1/5, h_2 = 1/25$	-0.718651	-2.55617
$H = 1/3, h_1 = 1/6, h_2 = 1/36$	-1.0353	-2.88805
$H = 1/3, h_1 = 1/7, h_2 = 1/49$	-1.29178	-3.157
$H = 1/3, h_1 = 1/8, h_2 = 1/64$	-1.51285	-3.38946
$H = 1/3, h_1 = 1/9, h_2 = 1/81$	-1.70537	-3.59519
$H = 1/4, h_1 = 1/6, h_2 = 1/36$	-1.03265	-2.88559
$H = 1/4, h_1 = 1/7, h_2 = 1/49$	-1.29237	-3.15699
$H = 1/4, h_1 = 1/8, h_2 = 1/64$	-1.51283	-3.38965
$H = 1/4, h_1 = 1/9, h_2 = 1/81$	-1.70539	-3.5952
$H = 1/4, h_1 = 1/10, h_2 = 1/100$	-1.87084	-3.77885
$H = 1/4, h_1 = 1/11, h_2 = 1/121$	-2.01468	-3.94475
$H = 1/4, h_1 = 1/12, h_2 = 1/144$	-2.13968	-4.09586

For $H = 3$, the $H^1(\Omega)^2$ velocity error slope is of order 1.95 and the $L^2(\Omega)$ pressure error one is of order 2.02. For $H = 4$, the previous slopes are respectively 1.83 and 2.009.

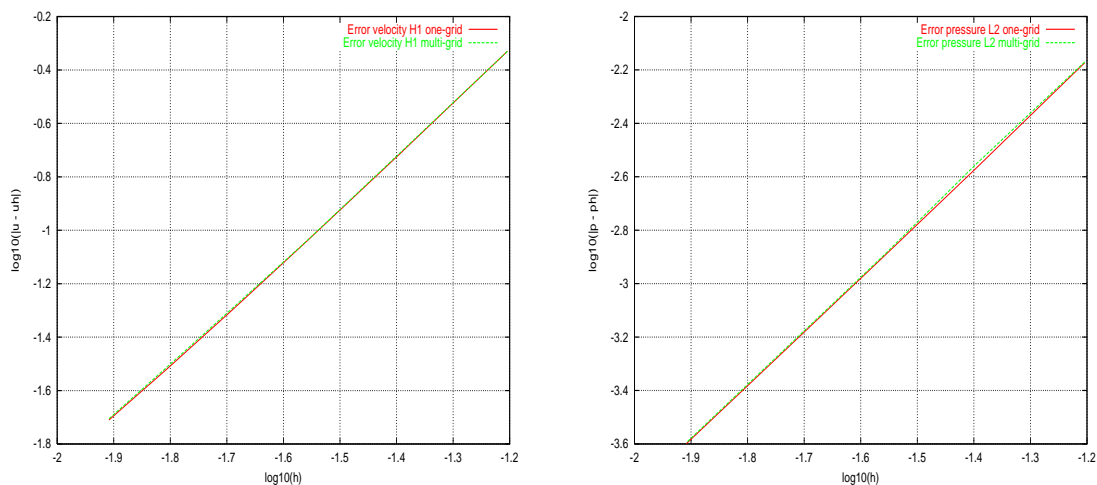


FIGURE 3. Comparison between the errors of the multi-grid algorithm and those of the one-grid method using the finest mesh: $H^1(\Omega)^2$ velocity error (left) and $L^2(\Omega)$ pressure error (right).

Figure 3 shows the curves of the errors obtained by multi-grid method taking $H = 3$, $h_1 = 4, 5, 6, 7, 8, 9$ and $h_2 = h_1^2 = 16, 25, 36, 49, 64, 81$, and the one grid method in the finest mesh. We notice that the slopes of the curves are almost similar and confirm the theoretical results.

CPU comparison and advantage of the multi-grid method.

The goal of the multi-grid strategy is to gain in time of computation of the solution.

We denote by t_{3G} and t_{1G} respectively the computation time (in seconds) of the resolution of the problem by the tree-grid strategy and by the one grid. First we take $h_0 = H = 1/3$, $h_1 = 1/9$ and $h_2 = 1/81$, and second we take $h_0 = H = 1/4$, $h_1 = 1/16$ and $h_2 = 1/144$.

	$h_0 = H = 1/3, h_1 = 1/9, h_2 = 1/81$	$h_0 = H = 1/4, h_1 = 1/16, h_2 = 1/144$
t_{1G} en secondes	208.5	471.5
t_{3G} en secondes	31.03	125
$\frac{ t_{3G} - t_{1G} }{t_{1G}}$ (en %)	85.1	73

TABLE 1. CPU comparison.

As shown in table 1, it is clear that the resolution of the Navier-Stokes problem by the multi-grid strategy is less expensive in time of computation than that obtained by the fine grid method.

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