Partial functional quantization and generalized bridges

Sylvain Corlay* †

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Abstract

In this article, we develop a new approach to functional quantization, which consists in discretizing only the first Karhunen-Loève coordinates of a continuous Gaussian semimartingale X. Using filtration enlargement techniques, we prove that the conditional distribution of X knowing its first Karhunen-Loève coordinates is a Gaussian semimartingale with respect to its natural filtration.

This allows to define the partial quantization of a solution of a stochastic differential equation with respect to X by simply plugging the partial functional quantization of X in the SDE.

Then, we provide an upper bound of the L^p-partial quantization error for the solution of SDE involving the $L^{p+\varepsilon}$-partial quantization error for X, for $\varepsilon > 0$. The *a.s.* convergence is also investigated.

Incidentally, we show that the conditional distribution of a Gaussian semimartingale X knowing that it stands in some given Voronoi cell of its functional quantization is a (non-Gaussian) semimartingale. As a consequence, the functional stratification method developed in [6], amounted in the case of solutions of SDE to use the Euler scheme of these SDE in each Voronoi cell.

Keywords: Gaussian semimartingale, functional quantization, vector quantization, Karhunen-Loève, Gaussian process, Brownian motion, Brownian bridge, Ornstein-Uhlenbeck, filtration enlargement, stratification, Cameron-Martin space, Wiener integral.

*Natixis, Equity Derivatives and Arbitrage. E-mail: sylvain.corlay@natixis.com. The author would like to thank the members of Natixis equity derivatives quantitative R&D team for fruitful discussions.

†Laboratoire de Probabilités et Modèles Aléatoires, UMR 7599, Université Paris 6, case 188, 4, pl. Jussieu, F-75252 Paris Cedex 5, France.

Introduction

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and E a reflexive separable Banach space. The norm on E is denoted $|\cdot|$. The quantization of a E-valued random variable X consists in its approximation by a random variable Y taking finitely many values. The resulting error of this discretization is measured by the L^p norm of $|X - Y|$. If we settle to a fixed maximum cardinal for $Y(\Omega)$, the minimization of the error comes to the following minimization problem:

$$\min \left\{ \left\| |X - Y| \right\|_p, Y : \Omega \to E \text{ measurable }, \mathrm{card}(Y(\Omega)) \leq N \right\}. \tag{1}$$

A solution of (1) is an optimal quantizer of X. This problem, initially investigated as a signal discretization method [9] has then been introduced in numerical probability, to devise cubature methods [21] or solving multi-dimensional stochastic control problems [3]. Since the early 2000's, the infinite dimensional setting has been extensively investigated from both constructive numerical and theoretical viewpoints with a special attention paid to functional quantization, especially in the quadratic case [16] but also in some other Banach spaces [27]. Stochastic processes are viewed as random variables taking values in their path spaces such as $L^2_T := L^2([0,T], dt)$.

We now assume that X is a bi-measurable stochastic process on $[0,T]$ verifying $\int_0^T \mathbb{E}\left[|X_t|^2\right] dt < +\infty$, so that this can be viewed as a random variable valued in the separable Hilbert space $L^2([0,T])$. We assume that its covariance function Γ^X is continuous. In the seminal article on Gaussian functional quantization [16], it is shown that in the centered Gaussian case, linear subspaces U of $L^2([0,T])$ spanned by L^2-optimal quantizers correspond to principal components of X, in other words, are spanned by the first eigenvectors of the covariance operator of X. Thus, the quadratic optimal quantization of Gaussian processes consists in exploiting its Karhunen-Loève decomposition $(e_n^X, \lambda_n^X)_{n\geq 1}$.

If Y is a quadratic N-optimal quantizer of the Gaussian process X and $d^X(N)$ is the dimension of the subspace of $L^2([0,T])$ spanned by $Y(\Omega)$, the quadratic quantization error $\mathcal{E}_N(X)$ verifies

$$\mathcal{E}_N^2(X) = \sum_{j\geq m+1} \lambda_j^X + \mathcal{E}_N^2\left(\bigotimes_{j=1}^m \mathcal{N}\left(0, \lambda_j^X\right)\right) \text{ for } m \geq d^X(N). \tag{2}$$

$$\mathcal{E}_N^2(X) < \sum_{j\geq m+1} \lambda_j^X + \mathcal{E}_N^2\left(\bigotimes_{j=1}^m \mathcal{N}\left(0, \lambda_j^X\right)\right) \text{ for } 1 \leq m < d^X(N). \tag{3}$$

To perform optimal quantization, the decomposition is first truncated at a fixed order m and then the $\mathbb{R}^m$-valued Gaussian vector constituted of the m first coordinates of the process on its Karhunen-Loève decomposition is quantized. To reach optimality, we have to determine the optimal rank of truncation $d^X(N)$ (the quantization dimension) and the optimal $d^X(N)$-dimensional quantizer corresponding to the first coordinates $\bigotimes_{j=1}^{d^X(N)} \mathcal{N}\left(0, \lambda_j^X\right)$. Usual examples of such processes are the standard Brownian motion on $[0,T]$, the Brownian bridge on $[0,T]$, Ornstein-Uhlenbeck processes and the fractional Brownian motion. In Figure 1, we display the quadratic optimal $N-$quantizer of the fractional Brownian motion on $[0,1]$ with Hurst exponent $h = 0.25$ and $N = 20$.

Another possibility is to use a product quantization of the distribution $\bigotimes_{j=1}^{d^X(N)} \mathcal{N}(0, \lambda_j^X)$. The product quantization is the Cartesian product of the optimal quadratic quantizers of the standard one-dimensional Gaussian distributions $\mathcal{N}(0, \lambda_i^X)_{1\leq i\leq d^X(N)}$. In the case of independent marginals, this yields a stationary quantizer, i.e. a quantizer Y of X which satisfies $\mathbb{E}[X|Y] = Y$. This property, shared with optimal quantizers, makes us reach an order in the convergence rate of cubature rules based on quantization. One advantage of this setting is that the one-dimensional Gaussian quantization is a fast procedure. In [22], deterministic optimization methods (as Newton-Raphson) are shown to converge rapidly to the unique optimal quantizer of the one-dimensional Gaussian distribution. Moreover, a sharply optimized database of quantizers of standard univariate and multivariate Gaussian distributions is available on the web site `www.quantize.maths-fi.com` [24] for download. Still, we have to determine the quantization size on each direction to obtain optimal product quantization. In this case, the minimization of the

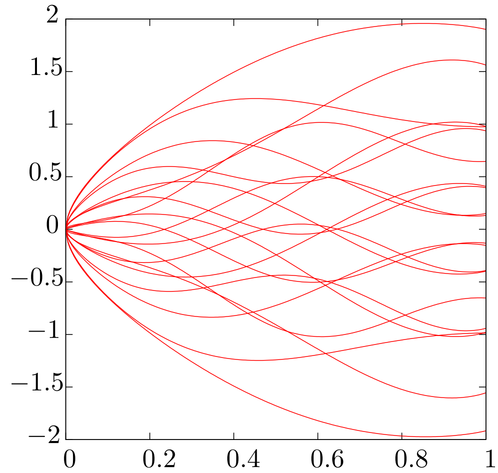

Figure 1: Quadratic N-optimal quantizer of the fractional Brownian motion on $[0,1]$ with Hurst's parameter $h=0.25$ and $N=20$. The quantization dimension is 3.

distortion (2) comes to:

$$\min\left\{\sum_{j=1}^{d}\mathcal{E}_{N_j}^2\left(\mathcal{N}(0,\lambda_j^X)\right)+\sum_{j\geq d+1}\lambda_j^X, N_1\times\cdots\times N_d\leq N, d\geq 1\right\}. \quad (4)$$

In [16], the rate of convergence to zero of the quantization error is investigated. A complete solution is provided for the case of Gaussian processes under rather general conditions on the eigenvalues of the covariance operator. Rates of convergence are available for the above cited examples of Gaussian processes. The asymptotics of the quantization dimension $d^X(N)$ is investigated in [17, 19].

From a constructive viewpoint, the numerical computation of the optimal quantization or the optimal product quantization requires a numerical evaluation of the Karhunen-Loève eigenfunctions and eigenvalues, at least the very first terms. (As seen in [16, 17, 19], under rather general conditions on its eigenvalues, the quantization dimension of a Gaussian process increases asymptotically as the logarithm of the size of the quantizer. Hence it is most likely that it is small. For instance, the quantization dimension of the Brownian motion with $N=10000$ is 9.) The Karhunen-Loève decomposition of several usual Gaussian processes have a closed-form expression. It is the case of the standard Brownian motion, the Brownian bridge and Ornstein-Uhlenbeck processes. (The case of Ornstein-Uhlenbeck processes is derived in [6], in the general setting of an arbitrary initial variance σ_0. A pseudo-algorithm for the computation of ω_λ is also provided.) Another example of explicit Karhunen-Loève expansion is derived in [7] by Deheuvels and Martynov.

In the general case, no closed form expression of the Karhunen-Loève expansion is available. For instance, the Karhunen-Loève expansion of the fractional Brownian motion is not known. To fulfill the requirement of a numerical evaluation of those functions, it is possible to use numerical methods related to integral equations to solve the eigenvalue problem that defines the Karhunen-Loève expansion. A review of these methods is available in [2]. In [5], the so-called "Nyström method" is used to compute the first terms of the Karhunen-Loève decomposition of the fractional Brownian motion for its optimal functional quantization.

An application of the quantization of a Gaussian process X, is to perform a quantization of the solution of a SDE with respect X, when a stochastic integration with respect to X can be defined. In the following, we will assume that X is a continuous Gaussian semimartingale on $[0,T]$. The Brownian motion, the Brownian bridge and Ornstein-Uhlenbeck processes are semimartingales, but the fractional Brownian motion with Hurst exponent $h\neq\frac{1}{2}$ is not. We can obtain a stationary quantizer of the diffusion by inserting the quantizer of the Gaussian process in the diffusion equation written in the Stratonovich sense. In [25], Pagès and Sellami proved the a.s. convergence of this quantization when the quantizer size goes to infinity. The rate of convergence is also investigated. This work is mostly specific to the Brownian motion but main results remain valid for continuous semimartingales which satisfy the Kolmogorov criterion as the Brownian bridge and Ornstein-Uhlenbeck processes.

1 Quantization based cubature and related inequalities

The idea of quantization-based cubature method is to approach the probability distribution of the random variable X by the distribution of a quantizer Y of X. As Y is a discrete random variable, we can write $\mathbb{P}_Y = \sum_{i=1}^{N} p_i \delta_{y_i}$. If $F : E \to \mathbb{R}$ is a Borel functional,

$$\mathbb{E}[F(Y)] = \sum_{i=1}^{N} p_i F(y_i). \tag{5}$$

Hence, if we have access to the weighed discrete distribution $(y_i, p_i)_{1 \le i \le N}$ of Y, we are able to compute the sum (5). Now, we review some error bounds that can be derived when approaching $\mathbb{E}[F(X))]$ by the quantity (5). See [23] for more details on error bounds.

1. *If $X \in L^2$, Y a quantizer of X of size N and F is Lipschitz-continuous, then*

$$|\mathbb{E}[F(X)] - \mathbb{E}[F(Y)]| \le [F]_{Lip} \|X - Y\|_2. \tag{6}$$

In particular, if $(Y_N)_{N \ge 1}$ is a sequence of quantizers such that $\lim_{N\to\infty} \|X - Y_N\|_2 = 0$, then the distribution $\sum_{i=1}^{N} p_i^N \delta_{x_i^N}$ of Y_N converges weakly to the distribution $\mathbb{P}_X$ of X as $N \to \infty$.

This first error bound is a straightforward consequence of $|F(X) - F(Y)| \le [F]_{\mathrm{Lip}} |X - Y|$.

2. *If Y is a stationary quantizer of X, i.e. $Y = \mathbb{E}[X|Y]$, and F is differentiable with and α-Hölder differential DF ($\alpha \in (0, 1]$), then*

$$|\mathbb{E}[F(X)] - \mathbb{E}[F(Y)]| \le [DF]_\alpha \|X - Y\|_2^{1+\alpha}. \tag{7}$$

In the particular case where F has a Lipschitz-continuous derivative ($\alpha = 1$), we have. $[DF_1] = DF_{Lip}$. For example, if F is twice differentiable and D^2F is bounded, then $DF_{Lip} = \frac{1}{2}\|D^2F\|_\infty$.

This particular inequality comes from the Taylor expansion of F around X and the stationarity of Y.

3. *If F is a convex functional and Y is a stationary quantizer of X,*

$$\mathbb{E}[F(Y)] \le \mathbb{E}[F(X)]. \tag{8}$$

This inequality is a straightforward consequence of the stationarity property and Jensen's inequality.

$$\mathbb{E}[F(Y)] = \mathbb{E}[F(\mathbb{E}[X|Y])] \le \mathbb{E}[\mathbb{E}[F(X)|Y]] = \mathbb{E}[F(X)].$$

2 Functional quantization and generalized bridges

2.1 Generalized bridges

Let $(X_t)_{t\in[0,T]}$ be a continuous centered Gaussian semimartingale starting from 0 on $(\Omega, \mathcal{A}, \mathbb{P})$ with respect to a filtration $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}$. Fernique's theorem ensures that $\int_0^T \mathbb{E}\left[X_t^2\right] dt < +\infty$ (see Janson [12]).

We aim here to compute the conditioning with respect to a finite family $\overline{Z}_T := (Z_T^i)_{i\in I}$ of Gaussian random variables, which are measurable with respect to $\sigma(X_t, t \in [0, T])$. ($I \subset \mathbb{N}$ is a finite subset of $\mathbb{N}^*$.) As Alili in [1] we settle to the case where $(Z_T^i)_{i\in I}$ are the terminal values of processes of the form $Z_t^i = \int_0^t f_i(s) dX_s$, $i \in I$, for some given finite set $\overline{f} = (f_i)_{i\in I}$ of $L^2_{loc}([0, T])$ functions. The *generalized* *bridge* for $(X_t)_{t\in[0,T]}$ corresponding to $\overline{f}$ with end-point $\overline{z} = (z_i)_{i\in I}$ is the process $\left(X^{\overline{f},\overline{z}}\right)_{t\in[0,T]}$ which has the distribution

$$X^{\overline{f},\overline{z}} \overset{\mathcal{L}}{\sim} \mathcal{L}\left(X \middle| Z_T^i = z_i, i \in I\right). \tag{9}$$

For example, in the case where X is a standard Brownian motion with $|I| = 1$, $\overline{f} = \{f\}$ and $f \equiv 1$, this is the Brownian bridge on $[0, T]$. If X is an Ornstein-Uhlenbeck process this is an Ornstein-Uhlenbeck bridge.

Let H be the Gaussian Hilbert space spanned by $(X_s)_{s\in[0,T]}$ and $H_{\overline{Z}_T}$ the closed subspace of H spanned by $(Z_T^i)_{i\in I}$. We denote $H_{\overline{Z}_T}^{\perp}$ its orthogonal complement in H. Any Gaussian random variable G of H can be orthogonally decomposed in $G = \text{Proj}_{\overline{Z}_T}(G) \overset{\perp\!\!\!\perp}{+} \text{Proj}_{\overline{Z}_T}^{\perp}(G)$, where $\text{Proj}_{\overline{Z}_T}$ and $\text{Proj}_{\overline{Z}_T}^{\perp}$ are the orthogonal projections on $H_{\overline{Z}_T}$ and $H_{\overline{Z}_T}^{\perp}$. ($\text{Proj}_{\overline{Z}_T}^{\perp} = Id_H - \text{Proj}_{\overline{Z}_T}$). Within these notations, $\mathbb{E}\left[G \middle| (Z_T^i)_{i\in I}\right] = \text{Proj}_{\overline{Z}_T}(G)$.

Other definitions of generalized bridges exist in the literature, see e.g. [20].

2.2 The case of the Karhunen-Loève basis

As X is a continuous Gaussian process, it has a continuous covariance function (See [12, VIII.3]). We denote $(e_i^X, \lambda_i^X)_{i\geq 1}$ its Karhunen-Loève eigensystem. Thus, if we define function f_i^X as the primitive of $-e_i^X$ which vanishes at $t = T$, i.e. $f_i^X(t) = \int_t^T e_i^X(s)ds$, an integration by parts yields

$$\int_0^T X_s e_i^X(s)ds = \int_0^T f_i^X(s)dX_s. \tag{10}$$

In other words, with the notations of Section 2.1, we have $Y_i := \int_0^T X_s e_i^X(s)ds = Z_T^i$.
For some finite subset $I \subset \mathbb{N}^*$, we denote $X^{I,\overline{y}}$ and call *K-L generalized bridge* the generalized bridge associated with functions $(f_i^X)_{i\in I}$ and with end-point $\overline{y} = (y_i)_{i\in I}$. This process has the distribution $\mathcal{L}(X|Y_i = y_i, i \in I)$.
In this case, the Karhunen-Loève expansion gives the decomposition

$$X = \underbrace{\sum_{i\in I} Y_i e_i^X}_{=\text{Proj}_{\overline{Z}_T}(X)} \overset{\perp\!\!\!\perp}{+} \underbrace{\sum_{i\in\mathbb{N}^*\setminus I} \sqrt{\lambda_i^X}\xi_i e_i^X}_{=\text{Proj}_{\overline{Z}_T}^{\perp}(X)}, \tag{11}$$

where $(\xi_i)_{i\in\mathbb{N}^*\setminus I}$ are independent standard Gaussian random variables. This gives us the projections $\text{Proj}_{\overline{Z}_T}$ and $\text{Proj}_{\overline{Z}_T}^{\perp}$ defined in Section 2.1. It follows from (11) that a K-L generalized bridge is centered on $\mathbb{E}[X|Y_i = y_i, i \in I]$ and has the covariance function

$$\Gamma^{X|Y}(s,t) = \text{cov}(X_s, X_t) - \sum_{i\in I} \lambda_i^X e_i^X(s) e_i^X(t). \tag{12}$$

We have $\int_0^T \Gamma^{X|Y}(t,t)dt = \sum\limits_{i\in\mathbb{N}^*\setminus I} \lambda_i^X$.

Moreover, thanks to decomposition (11), if $X^{I,\overline{y}}$ is a K-L generalized bridge associated with X with terminal values $\overline{y} = (y_i)_{i\in I}$, it has the same probability distribution as the process

$$\sum_{i\in I} y_i e_i^X(t) + X_t - \sum_{i\in I} \left(\int_0^T X_s e_i^X(s)ds\right) e_i^X(t).$$

This process is then the sum of a semimartingale and a non-adapted finite-variation process.
Let us stress out the fact that the second term in the left-hand side of (11) is the corresponding K-L generalized bridge with end-point 0, i.e. $\text{Proj}_{\overline{Z}_T}^{\perp} = X^{I,\overline{0}}$.
In [6], an algorithm is proposed to exactly simulate marginals of a K-L generalized bridge with a linear additional cost to a prior simulation of $(X_{t_0}, \cdots, X_{t_n})$, for some subdivision $0 = t_0 \leq t_1 \leq \cdots t_n = T$ of $[0, T]$. This was used for variance reduction issues. Note that the algorithm is easily extended to the case of (non-K-L) generalized bridges.

2.3 Generalized bridges as semimartingales

For a random variable L, we denote $\mathbb{P}\left[\cdot|L\right]$ the conditional probability knowing L. We keep the notations and assumptions of previous sections. (X is a continuous Gaussian semimartingale starting from 0.) We consider a finite set $I \subset \{1, 2, \cdots\}$ and $(f_i)_{i\in I}$ a set of bounded measurable functions. Let $X^{\overline{f},\overline{y}}$ be the generalized bridge associated with X with end-point $\overline{y} = (y_i)_{i\in I}$. For $i \in I$, $Z^i_t = \int_0^t f_i(s)dX_s$ and $\overline{Z}_t = (Z^i_t)_{i\in I}$.

Jirina's theorem ensures the existence of a transition kernel

$$\nu_{\overline{Z}_T|\left((X_t)_{t\in[0,s]}\right)} : \mathcal{B}(\mathbb{R}^I) \times C^0\left([0, s], \mathbb{R}\right) \to \mathbb{R}_+,$$

corresponding to the conditional distribution $\mathcal{L}\left(\overline{Z}_t\middle|\left((X_t)_{t\in[0,s]}\right)\right)$.
We now make the additional assumption $(\mathcal{H})$ that, for every $s \in [0, T)$ and for every $(x_u)_{u\in[0,s]} \in C^0\left([0, s], \mathbb{R}\right)$, the probability measure $\nu_{\overline{Z}_T|\left((X_t)_{t\in[0,s]}\right)}\left(d\overline{y}, (x_u)_{u\in[0,s]}\right)$ is absolutely continuous with respect to the Lebesgue measure. We denote $\Pi_{(x_u)_{u\in[0,s]},T}$ its density. The covariance matrix of this Gaussian distribution on $\mathbb{R}^I$ writes

$$Q(s, T) := \mathbb{E}\left[\left(\overline{Z}_T - \mathbb{E}\left[\overline{Z}_T\middle|(X_u)_{u\in[0,s]}\right]\right)\left(\overline{Z}_T - \mathbb{E}\left[\overline{Z}_T\middle|(X_u)_{u\in[0,s]}\right]\right)^*\middle|(X_u)_{u\in[0,s]}\right].$$

If X is a martingale, we have $Q(s, T) = \left(\left(\int_s^T f_i(u)f_j(u)d\langle X\rangle_u\right)\right)_{(i,j)\in I^2}$. We recall in mind that a continuous centered semimartingale X is Gaussian if and only if $\langle X\rangle$ is deterministic (see e.g. [26]). Hence, this additional hypothesis is equivalent to assume that

$$Q(s, T) \quad \text{is invertible for every } s \in [0, T). \qquad (\mathcal{H})$$

The following theorem follows from the same approach as the homologue result in the article of Alili [1] for the Brownian case. It is generalized to the case of a continuous centered Gaussian semimartingale starting from 0.

Theorem 2.1. *Under the $(\mathcal{H})$ hypothesis, for any $s \in [0, T)$, and for $\mathbb{P}_{\overline{Z}_T}-$almost surely $\overline{y} \in \mathbb{R}^I$, $\mathbb{P}\left[\cdot\middle|\overline{Z}_T = \overline{y}\right]$ is equivalent to $\mathbb{P}$ on $\mathcal{F}^X_s$ and its Radon-Nikodym density is given by*

$$\frac{d\mathbb{P}\left[\cdot\middle|\overline{Z}_T = \overline{y}\right]}{d\mathbb{P}}_{|\mathcal{F}^X_s} = \frac{\Pi_{(X_u)_{u\in[0,s]},T}(\overline{y})}{\Pi_{0,T}(\overline{y})}.$$

Proof: Consider F a real bounded $\mathcal{F}^X_s$-measurable random variable and $\phi : \mathbb{R}^I \to \mathbb{R}$ a bounded Borel function.

- On one hand, preconditioning by $\overline{Z}_T$ yields

$$\mathbb{E}[F\phi(\overline{Z}_T)] = \mathbb{E}\left[\mathbb{E}\left[F\middle|\overline{Z}_T\right]\phi(\overline{Z}_T)\right] = \int_{\mathbb{R}^I} \phi(\overline{y})\mathbb{E}\left[F\middle|\overline{Z}_T = \overline{y}\right]\Pi_{0,T}(\overline{y})d\overline{y}. \qquad (13)$$

- On the other hand, as F is measurable with respect to $\mathcal{F}^X_s$, preconditioning with respect to $\mathcal{F}^X_s$ yields

$$\mathbb{E}[F\phi(\overline{Z}_T)] = \mathbb{E}\left[F\mathbb{E}\left[\phi(\overline{Z}_T)\middle|\mathcal{F}^X_s\right]\right] = \mathbb{E}\left[F\int_{\mathbb{R}^I}\phi(\overline{y})\Pi_{(X_t)_{t\in[0,s]},T}(\overline{y})d\overline{y}\right].$$

  Now, thanks to Fubini's theorem

$$\mathbb{E}[F\phi(\overline{Z}_T)] = \int_{\mathbb{R}^I}\phi(\overline{y})\mathbb{E}\left[F\Pi_{(X_t)_{t\in[0,s]},T}(\overline{y})\right]d\overline{y}. \qquad (14)$$

Identifying equations (13) and (14), we see that for $\mathbb{P}_{\overline{Z}_T}$−almost surely $\overline{y} \in \mathbb{R}^I$ and for every real bounded $\mathcal{F}^X_s$-measurable random variable F,

$$\mathbb{E}\left[F\middle|\overline{Z}_T = \overline{y}\right] = \mathbb{E}\left[F \frac{\Pi_{(X_t)_{t\in[0,s]},T}(\overline{y})}{\Pi_{0,T}(\overline{y})}\right]. \tag{15}$$

Equation (15) characterizes the Radon-Nikodym derivative of the probability $\mathbb{P}[\cdot|\overline{Z}_T = \overline{y}]$ on $\mathcal{F}^X_s$. □

We now can use classical filtration enlargement techniques [11, 13, 28].

Proposition 2.2 (Generalized bridges as semimartingales)**.** *Let us define the filtration $\mathcal{G}^X$ by $\mathcal{G}^X_t = \sigma\left(\overline{Z}_T, \mathcal{F}^X_t\right)$, the enlargement of the filtration $\mathcal{F}^X$ corresponding to the above conditioning. We consider the stochastic process $D^{\overline{y}}_s := \frac{d\mathbb{P}[\cdot|\overline{Z}_T]}{d\mathbb{P}}_{|\mathcal{F}^X_s}(\overline{y}) = \frac{\Pi_{(X_t)_{t\in[0,s]},T}(\overline{y})}{\Pi_{0,T}(\overline{y})}$ for $s \in [0,T)$.*

Under the ($\mathcal{H}$) hypothesis, and the assumption that $D^{\overline{y}}$ is continuous, $X^{\overline{f},\overline{y}}$ is a continuous $\mathcal{G}^X$-semimartingale on $[0,T)$.

Proof: $D^{\overline{y}}$ is a strictly positive martingale on $[0,T)$ which is uniformly integrable on every interval $[0,t] \subset [0,T)$. Hence, as we assumed that it is continuous, we can write $D^{\overline{y}}$ as an exponential martingale $D^{\overline{y}}_s = \exp\left(L^{\overline{y}}_s - \frac{1}{2}\langle L^{\overline{y}}\rangle_s\right)$ with $L^{\overline{y}}_t = \int_0^t \left(D^{\overline{y}}_s\right)^{-1} dD^{\overline{y}}_s$ (as $D^{\overline{y}}_0 = 1$).
Now, as X is a continuous $(\mathcal{F}^X, \mathbb{P})$-semimartingale, we write $X = V + M$ its canonical decomposition (under the filtration $\mathcal{F}^X$).

- Thanks to Girsanov theorem, $\widetilde{M^{\overline{y}}} := M - \left\langle M, L^{\overline{y}}\right\rangle$ is a $\left(\mathcal{F}^X, \mathbb{P}[\cdot|\overline{Z}_T = \overline{y}]\right)$-martingale. As a consequence, it is a $\left(\mathcal{G}^X, \mathbb{P}[\cdot|\overline{Z}_T = \overline{y}]\right)$-martingale and thus a $\left(\mathcal{G}^X, \mathbb{P}\right)$-martingale.
- Moreover, conditionally to $\overline{Z}_T = \overline{y}$, V is still a finite variation process V, and is adapted to $\mathcal{G}^X$. □

Remark (Continuous modification)**.** *In Proposition 2.2, if one only assumes that $D^{\overline{y}}$ has a continuous modification $\mathcal{D}^{\overline{y}}$, and for each one of its continuous modifications is associated a continuous $\mathcal{G}^X$-semimartingale on $[0,T)$, $X^{\overline{f},\mathcal{D}^{\overline{y}}}$ and all these semimartingales are modifications of each other.*

Proposition 2.3 (Continuity of $D^{\overline{y}}$)**.** *If $\mathcal{F}^X$ is a standard Brownian filtration, then $D^{\overline{y}}$ has a continuous modification.*

Proof: Consider $s \in [0,T)$. Under the ($\mathcal{H}$) hypothesis, the density $\Pi_{(X_u)_{u\in[0,s]},T}$ writes

$$\Pi_{(X_u)_{u\in[0,s]},T}(\overline{y}) = (2\pi \det Q(s,T))^{-\frac{|I|}{2}} \exp\left(\left(\overline{y} - \mathbb{E}\left[\overline{Z}_T\middle|(X_u)_{u\in[0,s]}\right]\right) Q(s,T)^{-1} \left(\overline{y} - \mathbb{E}\left[\overline{Z}_T\middle|(X_u)_{u\in[0,s]}\right]\right)^*\right). \tag{16}$$

The continuity of $s \to \det Q(s,T)$ and $s \to Q(s,T)^{-1}$ follows from the definition of $Q(s,T)$. The point now is to establish the continuity of the stochastic process $\overline{H}$ defined by $\overline{H}_s := \mathbb{E}\left[\overline{Z}_T\middle|(X_u)_{u\in[0,s]}\right]$.

The so-defined process H is a $\mathcal{F}^X$ local martingale. Thus, thanks the Brownian representation theorem, H has a Brownian representation and has a continuous modification and so does $D^{\overline{y}}$. □

Remark.
- *The measurability assumption with respect to a Brownian filtration is verified in the cases of the Brownian bridge and Ornstein-Uhlenbeck processes.*
- *This hypothesis is not necessary so long as the continuity of the martingale $\overline{H}_s = \mathbb{E}\left[\overline{Z}_T\middle|(X_u)_{u\in[0,s]}\right]$ can be proved by any means.*

2.3.1 On the canonical decomposition

Observing that $\langle M, L^{\overline{y}}\rangle = \langle X, L^{\overline{y}}\rangle$ we can compute the canonical decomposition of $X^{\overline{f},\overline{y}}$. We have

$$L^{\overline{y}}_t = \int_0^t \frac{d\Pi_{(X_u)_{u\in[0,s]},T}(\overline{y})}{\Pi_{(X_u)_{u\in[0,s]},T}(\overline{y})},$$

and

$$\ln\left(\Pi_{(X_u)_{u\in[0,s]},T}(\overline{y})\right) = -\frac{|I|}{2}\ln\left(2\pi\det Q(s,T)\right) - \frac{1}{2}\left(\overline{y}-\mathbb{E}\left[\overline{Z}_T\big|(X_u)_{u\in[0,s]}\right]\right)Q(s,T)^{-1}\left(\overline{y}-\mathbb{E}\left[\overline{Z}_T\big|(X_u)_{u\in[0,s]}\right]\right)^*.$$

Using that for a semimartingale S, $d\ln S = \frac{dS}{S} - \frac{1}{2}d\left\langle \frac{1}{S}\cdot S\right\rangle$, we obtain

$$\begin{aligned}\frac{d\Pi_{(X_u)_{u\in[0,s]},T}(\overline{y})}{\Pi_{(X_u)_{u\in[0,s]},T}(\overline{y})} &= d\ln\left(\Pi_{(X_u)_{u\in[0,s]},T}(\overline{y})\right) + \left(\begin{array}{c}\text{finite variation}\\ \text{process}\end{array}\right)\\ &= -\tfrac{1}{2}d\left(\left(\overline{y}-\mathbb{E}\left[\overline{Z}_T\big|(X_u)_{u\in[0,s]}\right]\right)Q^{-1}(s,T)\left(\overline{y}-\mathbb{E}\left[\overline{Z}_T\big|(X_u)_{u\in[0,s]}\right]\right)^*\right) + (\text{f. v. p.})\\ &= \left(d\mathbb{E}\left[\overline{Z}_T\big|(X_u)_{u\in[0,s]}\right]\right)Q^{-1}(s,T)\left(\overline{y}-\mathbb{E}\left[\overline{Z}_T\big|(X_u)_{u\in[0,s]}\right]\right)^* + (\text{f. v. p.})\,.\end{aligned}$$

Hence,

$$d\left\langle X, L^{\overline{y}}\right\rangle_s = d\left\langle X, \mathbb{E}\left[\overline{Z}_T\big|(X_u)_{u\in[0,\cdot]}\right]\right\rangle_s Q^{-1}(s,T)\left(\overline{y}-\mathbb{E}\left[\overline{Z}_T\big|(X_u)_{u\in[0,s]}\right]\right)^*.$$

There exists a continuous linear map $\Lambda_s^T : C^0([0,s],\mathbb{R}) \to \mathbb{R}^I$ such that for every $(x_u)_{u\in[0,s]} \in C^0([0,s],\mathbb{R})$, $\mathbb{E}\left[\overline{Z}_T\big|(X_u)_{u\in[0,s]} = (x_u)_{u\in[0,s]}\right] = \Lambda_s^T\left((x_u)_{u\in[0,s]}\right)$. Riesz-Markov theorem ensures that there exists a set of (signed) σ-finite Borel measures $\overline{\mu}_s = (\mu_s^i)_{i\in I}$ such that for every $i\in I$, $\Lambda_s^T(f)_i = \int_0^s f d\mu_s^i$. Hence, in this setting, $d\langle X, \mathbb{E}\left[\overline{Z}_T\big|(X_u)_{u\in[0,\cdot]}\right]\rangle = \overline{\mu}_s(\{s\})d\langle X\rangle$. Denoting $\overline{a}(s) = \overline{\mu}_s(\{s\})$, we get

$$\begin{aligned}d\langle X, L^{\overline{y}}\rangle_s &= \left(\overline{a}(s)Q^{-1}(s,T)\left(\overline{y}-\mathbb{E}\left[Z_T^j\Big|(X_u)_{u\in[0,s]}\right]\right)^*\right)d\langle X\rangle_s\\ &= \sum_{i\in I}a_i(s)\sum_{j\in I}\left(Q(s,T)^{-1}\right)_{ij}(y_j-\mathbb{E}\left[Z_T^j\Big|(X_u)_{u\in[0,s]}\right])d\langle X\rangle_s.\end{aligned} \tag{17}$$

As a consequence, $M - \sum_{i\in I}a_i(s)\sum_{j\in I}\left(Q(s,T)^{-1}\right)_{ij}(y_j-\mathbb{E}\left[Z_T^j\Big|(X_u)_{u\in[0,s]}\right])d\langle X\rangle_s$ is a $\left(\mathcal{G}^X, \mathbb{P}\left[\cdot\big|\overline{Z}_t=\overline{y}\right]\right)$-martingale.

In the case where X is a standard Brownian motion, a simple computation shows that $\forall i\in I, a_i(s) = f_i(s)$ and $\mathbb{E}\left[Z_T^j\Big|(X_u)_{u\in[0,s]}\right] = \int_0^s f_j(u)dX_u$. We recover Alili's result on the generalized Brownian bridge [1].

2.3.2 Generalized bridges and functional stratification

Within the same set of notations, we set $Y = \overline{Z}_T$ and $\widehat{Y}^\Gamma = \mathrm{Proj}_\Gamma(Y) = \sum_{i=1}^N \gamma_i \mathbf{1}_{C_i}(Y)$ a stationary quantizer of Y (where $\Gamma = \{\gamma_1,\cdots,\gamma_N\}$ and $C = \{C_1,\cdots,C_N\}$ are respectively the associated codebook and Voronoi partition).

Proposition 2.4 (Stratification). *Under the ($\mathcal{H}$) hypothesis, for any $s\in[0,T)$, for any $k\in\{1,\cdots,N\}$, $\mathbb{P}\left[\widehat{Y}^\Gamma=\gamma_k\right]>0$ and the conditional probability $\mathbb{P}\left[\cdot\Big|\widehat{Y}^\Gamma=\gamma_k\right]$ is equivalent to $\mathbb{P}$ on $\mathcal{F}_s^X$.*

Proof: Obviously, if $A\in\mathcal{F}_s^X$ is such that $\mathbb{P}[A]=0$, we have $\mathbb{P}\left[A\Big|\widehat{Y}^\Gamma=\gamma_k\right]=0$. Conversely, $B\in\mathcal{F}_s^X$ satisfies $\mathbb{P}\left[B\Big|\widehat{Y}^\Gamma=\gamma_k\right]=0$, then pre-conditioning by Y, we get $\mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_B|Y\right]\Big|\widehat{Y}^\Gamma=\gamma_k\right]=0$. Thus, $\int_{\overline{y}\in C_k}\mathbb{P}\left[B|Y=\overline{y}\right]d\mathbb{P}_Y(\overline{y})=0$. Hence $\mathbb{P}\left[B|Y=\overline{y}\right]=0$ for $\mathbb{P}_Y-$almost every $\overline{y}\in C_k$.
Since $\mathbb{P}_Y(C_k)>0$, there exists at least an $\overline{y}\in C_k$ such that $\mathbb{P}\left[B|Y=\overline{y}\right]=0$. Now thanks to Theorem 2.1, $\mathbb{P}[B]=0$. □

Proposition 2.5 (Stratification). *Let us define the filtration $\mathcal{G}^X$ by $\mathcal{G}_t^X = \sigma\left(\mathcal{F}_t^X, \widehat{Y}^\Gamma\right)$, the enlargement of $\mathcal{F}^X$ corresponding to the conditioning with respect to $\widehat{Y}^\Gamma$. For $k\in\{1,\cdots,N\}$, we consider the stochastic process $D_s^k := \frac{d\mathbb{P}\left[\cdot\big|\widehat{Y}^\Gamma=\gamma_k\right]}{d\mathbb{P}}_{|\mathcal{F}_s^X}(\overline{y})$ for $s\in[0,T)$.*

Under the ($\mathcal{H}$) hypothesis, and the assumption that D^k is continuous, the conditional distribution $\mathcal{L}\left(X\big|\overline{Z}_T=\gamma_k\right)$ of X knowing that $\overline{Z}_T$ falls in some Voronoi cell C_k is the probability distribution of a $\mathcal{G}^X$-semimartingale on $[0,T)$.

Proof: Using that $\mathbb{P}\left[\cdot\middle|\widehat{Y}^\Gamma = \gamma_k\right]$ is equivalent to $\mathbb{P}$ on $\mathcal{F}_s^X$, thanks to Proposition 2.4, we can mutatis mutandis use the same arguments as for Proposition 2.2, $\mathbb{P}\left[\cdot|\overline{Z}_T = \overline{y}\right]$ being replaced by $\mathbb{P}\left[\cdot\middle|\widehat{Y}^\Gamma = \gamma_k\right]$.

D^k is a strictly positive martingale on $[0,T)$ uniformly integrable on every $[0,t] \subset [0,T)$. Hence, as D^k is continuous by hypothesis, it is an exponential martingale $D^k = \exp\left(L_s^k - \frac{1}{2}\langle L^k\rangle_s\right)$, with $L_t^k = \int_0^t \left(D_s^k\right)^{-1} dD_s^k$ (as $D_0^k = 1$). Now, as X is a continuous $(\mathcal{F}^X, \mathbb{P})$-semimartingale, we write $X = V + M$ its canonical decomposition (under the filtration $\mathcal{F}^X$).

- Thanks to Girsanov theorem, $\widetilde{M}^k := M - \langle M, L^k\rangle$ is a $\left(\mathcal{F}^X, \mathbb{P}[\cdot|\widehat{Y}^\Gamma = \gamma_k]\right)$-martingale. As a consequence, it is a $\left(\mathcal{G}^X, \mathbb{P}[\cdot|\widehat{Y}^\Gamma = \gamma_k]\right)$-martingale and thus a $\left(\mathcal{G}^X, \mathbb{P}\right)$-martingale.
- Moreover, conditionally to $\widehat{Y}^\Gamma = \gamma_k$, V is still a finite variation process V, and is adapted to $\mathcal{G}^X$. □

Proposition 2.6 (Continuity of D^k)**.** *If $\mathcal{F}^X$ is a Brownian filtration, then D^k has a continuous modification.*

Proof: By definition, D^k is a $\mathcal{F}^X$ local martingale on $[0,T]$. The conclusion is a straightforward consequence of the Brownian representation theorem. □

Considering the partition of $L^2([0,T])$ corresponding to the Voronoi cells of a functional quantizer of X, the last two propositions show that the conditional distribution of the X in each Voronoi cell (strata) is a Gaussian semimartingale with respect to its own filtration. This allows to define the corresponding functional stratification of the solutions of stochastic differential equations driven by X.

In [6], an algorithm is proposed to simulate the conditional distribution of the marginals $(X_{t_0}, \cdots, X_{t_n})$ of X for a given subdivision $0 = t_0 < t_1 < \cdots < t_n = T$ of $[0,T]$ conditionally to a given Voronoi cell (strata) of a functional quantization of X. The simulation complexity has an additional linear complexity to an unconditioned simulation of $(X_{t_0}, \cdots, X_{t_n})$. We refer to [6] for more details.

To deal with solutions of SDE, it was proposed in [6] to simply insert these marginals in the Euler scheme of the SDE. Proposition 2.5 now shows that this amounts to simulate the Euler scheme of the SDE driven by the corresponding (non-Gaussian) semimartingale.

2.4 About the $(\mathcal{H})$ hypothesis

2.4.1 The martingale case

In the case where X is a continuous Gaussian martingale, the matrix $Q(s,t)$ defined in Section 2.3 writes $Q(s,t) = \left(\left(\int_s^t f_i(u) f_j(u) d\langle X\rangle_u\right)\right)_{(i,j)\in I^2}$.

For $1 \le s < t \le T$, the map $(\cdot|\cdot) : (f,g) \to \int_s^t f(u)g(u)d\langle X\rangle_u$ defines a scalar product on $L^2([s,t], d\langle X\rangle)$. Hence $Q(s,t)$ is the Gram matrix of the vectors of $L^2([s,t], d\langle X\rangle)$ defined by the restrictions to $[s,t]$ of the functions $(f_i)_{i\in I}$. Thus, it is invertible if and only if these restrictions form a linearly independent family of $L^2([s,t], d\langle X\rangle)$. (Another consequence, is that if $Q(s,t)$ is invertible for some $0 \le s < t \le T$, then for every (u,v) such that $[s,t] \subset [u,v]$, $Q(u,v)$ is invertible).

For instance, if X is a standard Brownian motion on $[0,T]$, the functions $(f_i^X)_{i\in I}$ (associated with the Karhunen-Loève decomposition) are trigonometric functions with strictly different frequencies. Hence, they form a linearly independent family of continuous functions on every no-empty interval $[s,T) \subset [0,T)$. Moreover, the measure $d\langle X\rangle$ is proportional to the Lebesgue measure on $[0,T]$ and thus $Q(s,T)$ is invertible for any $s \in [0,T)$. *Hence, the $(\mathcal{H})$ hypothesis is fulfilled in the case of K-L generalized bridges of the standard Brownian motion.*

2.4.2 The standard Brownian bridge and Ornstein-Uhlenbeck processes

The Brownian bridge and the Ornstein-Uhlenbeck process are not martingales. Hence, this criteria is not sufficient and the invertibility of matrix $Q(s,T)$ has to be proved by other means.

Following from the definitions of $Q(s,T)$ and $\overline{Z}_T$, in the case of the K-L generalized bridge

$$\begin{aligned} Q(s,T)_{ij} &= \mathbb{E}\Big[\left(\int_s^T f_i^X(u)dX_u - \mathbb{E}\left[\int_s^T f_i^X(u)dX_u \Big| (X_u)_{u\in[0,s]} \right] \right) \\ &\quad \times \left(\int_s^T f_j^X(u)dX_u - \mathbb{E}\left[\int_s^T f_j^X(u)dX_u \Big| (X_u)_{u\in[0,s]} \right] \right)^* \Big| (X_u)_{u\in[0,s]} \Big] \\ &= \operatorname{cov}\left(\int_s^T f_i^X(u)dX_u^{(s)}, \int_s^T f_j^X(u)dX_u^{(s)} \right), \end{aligned}$$

where $\left(X_u^{(s)}\right)_{u\in[s,T]}$ has the conditional distribution of X knowing $(X_u)_{u\in[0,s]}$.

- When X is a standard Brownian bridge on $[0,T]$, $X_u^{(s)}$ is a Brownian bridge on $[s,T]$, starting from X_s and arriving at 0.

  It is the sum of an affine function and a standard centered Brownian bridge on $[s,T]$.

- When X is a centered Ornstein-Uhlenbeck process, $X_u^{(s)}$ is an Ornstein-Uhlenbeck process on $[0,T]$ starting from X_s, with the same mean reversion parameter as X.

  It is also the sum of a deterministic function and an Ornstein-Uhlenbeck process starting from 0.

As a consequence, in these two cases, the quantity $\operatorname{cov}\left(\int_s^T f_i^X(u)dX_u^{(s)}, \int_s^T f_j^X(u)dX_u^{(s)} \right)$ can be computed by plugging either a centered Brownian bridge on $[s,T]$ or an Ornstein-Uhlenbeck starting from 0 instead of $X^{(s)}$. This means that $Q(s,T)$ is the Gram matrix of the random variables $\left(\int_s^T f_i^X(u)dG_u \right)_{i\in I}$, where the centered Gaussian process $(G_u)_{u\in[s,T]}$ is either a standard Brownian bridge on $[s,T]$ or an Ornstein-Uhlenbeck process starting from 0 at s. Thus it is singular if and only if there exists $(\alpha_i)_{i\in I} \neq 0$ in $\mathbb{R}^I$ such that

$$\int_s^T \underbrace{\left(\sum_{i\in I} \alpha_i f_i^X(u) \right)}_{:=g(u)} dG_u = 0 \quad a.s.. \tag{18}$$

The case of the Brownian bridge
In the case where X is the standard Brownian bridge on $[0,T]$, functions $(f_i^X)_{i\in I}$ are C^∞ functions and G is a standard Brownian bridge on $[s,T]$. An integration by parts gives $\int_s^T G_s g'(s)ds = 0 \quad a.s.$ and thus $g' \equiv 0$ on (s,t) and thus g is constant on $[0,T]$. The functions $(f_i^X)_{i\in I}$ form an linearly independent set of functions and, as they are trigonometric functions with different frequencies, they clearly not span constant functions, so that Equation (18) yields $\alpha_1 = \cdots = \alpha_n = 0$. *Hence the ($\mathcal{H}$) hypothesis is fulfilled in the case of K-L generalized bridges of the standard Brownian bridge.*

The case of the Ornstein-Uhlenbeck process
In the case where X is an Ornstein-Uhlenbeck process on $[0,T]$, G is an Ornstein-Uhlenbeck process on $[s,T]$ starting from 0. The injectivity property of the Wiener integral related to the Ornstein-Uhlenbeck process stated in Proposition 2.7 below, applied on $[s,T]$, shows that Equation (18) amounts to $g \overset{L^2([s,T],dt)}{=} 0$ and thus

$$\sum_{i\in I} \alpha_i f_i^X \overset{L^2([s,T],dt)}{=} 0. \tag{19}$$

Again, as $(f_i^X)_{i\in I}$ are linearly independent, we have $\alpha_1 = \cdots = \alpha_n = 0$. *Hence the ($\mathcal{H}$) hypothesis is fulfilled in the case of K-L generalized bridges of the Ornstein-Uhlenbeck processes.*

Proposition 2.7 (Injectivity of the Wiener integral related to centered Ornstein-Uhlenbeck processes starting from 0)**.** *Let G be an Ornstein-Uhlenbeck process starting from 0 on $[0,T]$ defined by the SDE*

$$dG_t = -\theta G_t dt + \sigma dW_t \quad \text{with } \sigma > 0 \text{ and } \theta > 0.$$

If $g \in L^2([0,T])$, then we have

$$\int_0^T g(s)dG_s = 0 \quad \Leftrightarrow \quad g \overset{L^2([0,T])}{=} 0.$$

Proof: If $g \in L^2([0,T])$ and $\int_0^T g(s)dG_s = 0$, then $\theta \int_0^T g(s)G_s ds = \sigma \int_0^T g(s)dW_s$, and thus

$$\theta^2 \int_0^T \int_0^T g(s)g(t)\Gamma^X(s,t)dsdt = \sigma^2 \int_0^T g(s)^2 ds. \tag{20}$$

Applying Schwarz's inequality twice, we get

$$\int_0^T \int_0^T g(s)g(t)\Gamma^X(s,t)dsdt \leq \int_0^T g(t)^2 dt \sqrt{\int_0^T \int_0^T \Gamma^X(s,t)dsdt}.$$

Moreover, provided that

$$\sqrt{\int_0^T \int_0^T \Gamma^X(s,t)dsdt} < \frac{\sigma^2}{\theta^2}, \tag{21}$$

equality (20) implies $\int_0^T g(s)^2 ds = 0$.

Now, we come to the proof of Inequality (21). The covariance function of the Ornstein-Uhlenbeck process starting from 0 writes

$$\Gamma^{OU}(s,t) = \frac{\sigma^2}{2\theta} e^{-\theta(s+t)} \left(e^{2\theta \min(s,t)} - 1\right).$$

If $t \in [0,T]$, we have

$$\int_0^T \Gamma^{OU}(s,t)ds \quad = \int_0^t \Gamma^{OU}(s,t)ds + \int_t^T \Gamma^{OU}(s,t)ds = \tfrac{\sigma^2}{2\theta^2}\left(2 - 2e^{-\theta t} + e^{-\theta(t+T)} - e^{\theta(t-T)}\right),$$

and thus

$$\int_0^T \int_0^T \Gamma^{OU}(s,t)dsdt = \frac{\sigma^2}{2\theta^3}\left(2T\theta + 4e^{-\theta T} - 3 - e^{-2\theta T}\right).$$

Consequently, the function ϕ defined by $\phi(\theta) := \int_0^T \int_0^T \Gamma^{OU}(s,t)dsdt - \frac{\sigma^2}{\theta^2}$ writes

$$\phi(\theta) = \frac{\sigma^2}{2\theta^3}\left(2T\theta + 4e^{-\theta T} - 3 - e^{-2\theta T} - 2\theta\right).$$

We see that $\phi(0) = 0$ and $\phi'(\theta) = \frac{\sigma^2}{2\theta^2}\left(-4Te^{-\theta T} + 2T + 2Te^{-2\theta T} - 2\right) = \frac{\sigma^2}{2\theta^2}\left(2T\left(1 - e^{-\theta T}\right)^2 - 2\right)$, so that $\phi'(\theta) < 0$ if $\theta > 0$. Finally, if $\theta > 0$, Inequality (21) holds.
The inverse implication is obvious. □

In fact, the injectivity property stated in Proposition 2.7 also holds when dealing with an Ornstein-Uhlenbeck process with a positive initial variance. This case, of a (strictly) positive initial variance is proved thanks to a simpler argument.

Proposition 2.8 (Injectivity of the Wiener integral related to centered Ornstein-Uhlenbeck processes with a (strictly) positive initial variance)**.** *Let G be an Ornstein-Uhlenbeck process $[0,T]$ defined by the SDE*

$$dG_t = -\theta G_t dt + \sigma dW_t \quad \textit{with } \sigma > 0 \textit{ and } \theta > 0,$$

where $G_0 \overset{\mathcal{L}}{\sim} \mathcal{N}(0,\sigma_0^2)$ is independent from the standard Brownian motion W and $\sigma_0^2 > 0$.
If $g \in L^2([0,T])$, then we have

$$\int_0^T g(s)dG_s = 0 \quad \Leftrightarrow \quad g \overset{L^2([0,T])}{=} 0.$$

Proof: We recall in mind that the solution of the SDE that defines the Ornstein-Uhlenbeck process is

$$G_t = G_0 e^{-\theta t} + \underbrace{\int_0^t \sigma e^{\theta(s-t)} dW_s}_{:=G_t^0}.$$

If $g \in L^2([0,T])$ and $\int_0^T g(s)dG_s = 0$, then

$$\theta \int_0^T g(s)G_s ds = \sigma \int_0^T g(s)dW_s. \tag{22}$$

Moreover

$$\theta \int_0^T g(s)G_s ds = \underbrace{\theta G_0 \int_0^T e^{-\theta t} g(s)ds}_{\text{independent of } W} + \theta \int_0^T g(s)G_s^0 ds.$$

As a consequence, Equation (22) yields $\theta G_0 \int_0^T e^{-\theta t} g(s)ds = 0$ $a.s.$ and thus $g \stackrel{L^2([0,T])}{=} 0$. (This argument only holds if $\sigma_0^2 > 0$.) The inverse implication is obvious. □

The case of a more general Gaussian semimartingale
In Appendix A, we investigate the problem for more general Gaussian processes.

3 K-L generalized bridges and partial functional quantization

We keep the notations and assumptions of Section 2.2. As we have seen, Equation (11) decomposes the process X as the sum of a linear combination of $Y := (Y_i)_{i\in I}$ and an independent remainder term. We now consider $\widehat{Y}^\Gamma$ a stationary Voronoi N-quantization of Y. $\widehat{Y}^\Gamma$ can be written as a nearest neighbor projection of Y on a finite codebook $\Gamma = (\gamma_1, \cdots, \gamma_N)$.

$$\widehat{Y}^\Gamma = \text{Proj}_\Gamma(Y), \qquad \text{where } \text{Proj}_\Gamma \text{ is a nearest neighbor projection on } \Gamma.$$

For example, $\widehat{Y}^\Gamma$ can be a stationary product quantization or an optimal quadratic quantization of Y. We now define the stochastic process $\widetilde{X}^{I,\Gamma}$ by replacing Y by $\widehat{Y}^\Gamma$ in the decomposition (11). We denote $\widetilde{X}^{I,\Gamma} = \text{Proj}_{I,\Gamma}(X)$.

$$\widetilde{X}^{I,\Gamma} = \sum_{i\in I} \widehat{Y}_i^\Gamma e_i^X \overset{\perp\!\!\!\perp}{+} \sum_{i\in\mathbb{N}^*\setminus I} \sqrt{\lambda_i^X}\xi_i e_i^X.$$

The conditional distribution of $\widetilde{X}^{I,\Gamma}$ given that Y falls in the Voronoi cell of γ_k is the probability distribution of the K-L generalized bridge with end-point γ_k. In other words, we have quantized the Karhunen-Loève coordinates of X corresponding to $i \in I$, and not the other ones.
The so-defined process $\widetilde{X}^{I,\Gamma}$ is called a *partial functional quantization of* X.

3.1 Partial functional quantization of stochastic differential equations

Let X be a continuous centered Gaussian semimartingale on $[0,T]$ with $X_0 = 0$. We consider the SDE

$$dS_t = b(t,S_t)dt + \sigma(t,S_t)dX_t, \qquad S_0 = x \in \mathbb{R}, \qquad \text{and } t \in [0,T], \tag{23}$$

where $b(t,x)$ and $\sigma(t,x)$ are Borel functions, Lipschitz-continuous with respect to x uniformly in t, σ and $|b(\cdot,0)|$ are bounded. This SDE admits a unique strong solution S.

The conditional distribution given that $Y_i = y_i$ for $i \in I$ of S is the strong solution of the stochastic differential equation $dS_t = b(t,S_t)dt + \sigma(t,S_t)dX_t^{I,\overline{y}}$, with $S_0 = x \in \mathbb{R}$, and for $t \in [0,T]$, where $X_t^{I,\overline{y}}$ is the corresponding K-L generalized bridge.

Under the $(\mathcal{H})$, this suggests to define the partial quantization of S from a partial quantization $\widetilde{X}^{I,\Gamma}$ of X by replacing X by $\widetilde{X}^{I,\Gamma}$ in the SDE (23). We define the *partial quantization* $\widetilde{S}^{I,\Gamma}$ as the process whose conditional distribution given that Y falls in the Voronoi cell of γ_k is the strong solution of the same SDE where X is replaced by the K-L generalized bridge with end-point γ_k. We write

$$d\widetilde{S}_t^{I,\Gamma} = b\left(t, \widetilde{S}_t^{I,\Gamma}\right)dt + \sigma\left(t, \widetilde{S}_t^{I,\Gamma}\right)d\widetilde{X}_t^{I,\Gamma}. \tag{24}$$

3.2 Convergence of partially quantized SDE

We start by stating some useful inequalities for the sequel. Then, we recall the so-called Zador's theorem which will be used in the proof of the *a.s.* convergence of partially quantized SDE.

Lemma 3.1 (Gronwall inequality for locally finite measures). *Consider $\mathcal{I}$ an interval of the form $[a,b)$ or $[a,b]$ with $a<b$ or $[a,\infty)$. Let μ be a locally finite measure on the Borel σ-algebra of $\mathcal{I}$. We consider u a measurable function defined on $\mathcal{I}$ such that for all $t\in\mathcal{I}$, $\int_a^t |u(s)|\mu(ds)<+\infty$. We assume that there exists a Borel function ψ on $\mathcal{I}$ such that*

$$u(t)\leq\psi(t)+\int_{[a,t)}u(s)\mu(ds),\quad \forall t\in\mathcal{I}.$$

If | either ψ is non-negative,
| or $t\to\mu([a,t))$ is continuous on $\mathcal{I}$ and for all $t\in\mathcal{I}$, $\int_a^t|\psi(s)|\mu(ds)<\infty$,

then u satisfies the Gronwall inequality.

$$u(t)\leq\psi(t)+\int_{[a,t)}\psi(s)\exp(\mu([s,t)))\mu(ds).$$

A proof of this result is available in [8] (Appendix 5.1).

Lemma 3.2 (A Gronwall-like inequality in the non-decreasing case). *Consider $\mathcal{I}$ an interval of the form $[a,b)$ or $[a,b]$ with $a<b$ or $[a,\infty)$. Let μ be a locally finite measure on the Borel σ-algebra of $\mathcal{I}$. We consider u a measurable* non-decreasing *function defined on $\mathcal{I}$ such that for all $t\in\mathcal{I}$, $\int_a^t|u(s)|\mu(ds)<+\infty$. We assume that there exists a Borel function ψ on $\mathcal{I}$, and two non-negative constants $(A,B)\in\mathbb{R}_+^2$ such that*

$$u(t)\leq\psi(t)+A\int_{[a,t)}u(s)\mu(ds)+B\sqrt{\int_{[a,t)}u(s)^2\mu(ds)},\quad\forall t\in\mathcal{I}. \tag{25}$$

If | either ψ is non-negative,
| or $t\to\mu([a,t))$ is continuous on $\mathcal{I}$ and for all $t\in\mathcal{I}$, $\int_a^t|\psi(s)|\mu(ds)<\infty$,

then u satisfies the following Gronwall inequality.

$$u(t)\leq 2\psi(t)+2\left(2A+B^2\right)\int_{[a,t)}\psi(s)\exp\left(\left(2A+B^2\right)\mu([s,t))\right)\mu(ds).$$

Proof: Using that for $(x,y)\in\mathbb{R}_+^2$, $\sqrt{xy}\leq\frac{1}{2}\left(\frac{x}{B}+By\right)$, we have

$$\left(\int_{[a,t)}u(s)^2\mu(ds)\right)^{\frac{1}{2}}\leq\left(u(t)\int_{[a,t)}u(s)\mu(ds)\right)^{\frac{1}{2}}\leq\frac{u(t)}{2B}+\frac{B}{2}\int_{[a,t)}u(s)\mu(ds).$$

Plugging this in the original inequality (25) yields

$$u(t)\leq 2\psi(t)+\left(2A+B^2\right)\int_{[a,t)}u(s)\mu(ds).$$

Applying the regular Gronwall's inequality (Lemma 3.1) yields the announced result. □

Theorem 3.3 (Zador, Bucklew, Wise, Graf, Luschgy, Pagès).

1. (Sharp rate) *Consider $r>0$, and X be a $\mathbb{R}^d$-valued random variable such that $X\in L^{r+\eta}$ for some $\eta>0$. Let $\mathbb{P}_X(d\xi)=\phi(\xi)d\xi+\nu(d\xi)$ be the Radon-Nikodym decomposition of the probability distribution of X. (ν and the Lebesgue's measure are singular). Then, if $\phi\not\equiv 0$,*

$$\mathcal{E}_{N,r}(X)\underset{N\to\infty}{\sim}\widetilde{J}_{r,d}\times\left(\int_{\mathbb{R}^d}\phi^{\frac{d}{d+r}}(u)du\right)^{\frac{1}{d}+\frac{1}{r}}\times N^{-\frac{1}{d}},$$

where $\widetilde{J}_{r,d}\in(0,\infty)$.

2. (Non asymptotic upper bound) *There exists* $C_{d,r,\eta} \in (0, \infty)$ *such that, for every* $\mathbb{R}^d$*-valued random vector* X*,*

$$\forall N \geq 1, \quad \mathcal{E}_{N,r}(X) \leq C_{d,r,\eta} \|X\|_{r+\eta} N^{-\frac{1}{d}}.$$

The first statement of the theorem was first established for probability distributions with compact support by Zador [29], and extended by Bucklew and Wise to general probability distributions on $\mathbb{R}^d$ [4]. The first mathematically rigorous proof can be found in [10]. The proof of the second statement is available in [18].

The real constant $\widetilde{J}_{r,d}$ corresponds to the case of the uniform probability distribution over the unit hypercube $[0,1]^d$. We have $\widetilde{J}_{r,1} = \frac{1}{2}(r+1)^{-\frac{1}{r}}$ and $\widetilde{J}_{2,2} = \sqrt{\frac{5}{18\sqrt{2}}}$. (See [10].)

Proposition 3.4 (Some inequalities related to the Gaussian distribution)**.** *Let* G *be a standard Gaussian random variable valued in* $\mathbb{R}$*. We have*

$$\mathbb{E}\left[G^2 \mathbf{1}_{|G|>M}\right] = \frac{2M \exp\left(-\frac{M^2}{2}\right)}{\sqrt{2\pi}} + 2\mathcal{N}(-M). \tag{26}$$

Moreover

$$\mathcal{N}(-M) = \mathbb{P}(G > M) \leq \frac{1}{2} \exp\left(-\frac{M^2}{2}\right). \tag{27}$$

Thus

$$\mathbb{E}\left[G^2 \mathbf{1}_{|G|>M}\right] \leq \left(\frac{2M}{\sqrt{2\pi}} + 1\right) \exp\left(-\frac{M^2}{2}\right)$$

Under the additional assumption that $M > 1$*, we obtain*

$$\mathbb{E}\left[G^2 \mathbf{1}_{|G|>M}\right] \leq M\left(\frac{2}{\sqrt{2\pi}} + 1\right) \exp\left(-\frac{M^2}{2}\right) \quad \textit{if} \quad M > 1.$$

Proof: The proof of equality (26) is left to the reader. Inequality (27) comes from

$$\begin{aligned} \mathcal{N}(-M) = \mathbb{P}(G > M) &= \tfrac{1}{\sqrt{2\pi}} \int_M^\infty e^{-\frac{x^2}{2}} dx = \tfrac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{(x+M)^2}{2}} dx \\ &= \tfrac{1}{\sqrt{2\pi}} e^{-\frac{M^2}{2}} \int_0^\infty e^{-tx} e^{-\frac{x^2}{2}} dx \leq \tfrac{1}{\sqrt{2\pi}} e^{-\frac{M^2}{2}} \int_0^\infty e^{-\frac{x^2}{2}} dx = \tfrac{1}{2} e^{-\frac{M^2}{2}}. \end{aligned}$$

And the proof of the last claim is straightforward. □

Proposition 3.5 (The non standard case and reverse inequality)**.** *If* $H := \sigma G$ *has a variance of* σ^2*, we obtain*

$$\mathbb{E}\left[H^2 \mathbf{1}_{|H|>M}\right] = \sigma^2 \left[G^2 \mathbf{1}_{|G|>\frac{M}{\sigma}}\right] = \frac{2\sigma M}{\sqrt{2\pi}} \exp\left(-\frac{M^2}{2\sigma^2}\right) + 2\sigma^2 \mathcal{N}\left(-\frac{M}{\sigma}\right) \leq \left(\frac{2\sigma M}{\sqrt{2\pi}} + \sigma^2\right) \exp\left(-\frac{M^2}{2\sigma^2}\right).$$

And if $M > 1$*, we get* $\mathbb{E}\left[H^2 \mathbf{1}_{|H|>M}\right] \leq \underbrace{\left(\frac{2\sigma}{\sqrt{2\pi}} + \sigma^2\right) M \exp\left(-\frac{M^2}{2\sigma^2}\right)}_{:=\eta_M}$*. Conversely, for some settled* $\eta > 0$*, and if* $M > 1$*, we have*

$$M \geq \underbrace{\sqrt{-\operatorname{LambertW}\left(-\frac{\eta^2}{\left(\frac{2\sigma}{\sqrt{2\pi}} + \sigma^2\right)^2}\right)}}_{:=M_\eta} \quad \Rightarrow \quad \eta_M \leq \eta$$

where LambertW *is the Lambert* W *function.*

3.2.1 Quadratic convergence of partially quantized SDE

In the following theorem, we restrict to the case where X is a Gaussian martingale starting from 0. Still, we can easily extend the result to the case of a Gaussian *semimartingale* (see remark 3.2.3).

Theorem 3.6 (L^2 quantization error of partially quantized SDE)**.** *Let X be a continuous centered Gaussian martingale on $[0,T]$ with $X_0 = 0$. Let S be the strong solution of the SDE*

$$dS_t = b(t, S_t)dt + \sigma(t, S_t)dX_t, \quad S_0 = x,$$

where $b(t,x)$ and $\sigma(t,x)$ are Borel functions, Lipschitz-continuous with respect to x uniformly in t, σ and $|b(\cdot, 0)|$ are bounded.
We consider $\widetilde{X}^{I,\Gamma}$ a stationary partial functional quantization of X and $\widetilde{S}^{I,\Gamma}$ the corresponding partial functional quantization of S, i.e. the strong solutions of

$$d\widetilde{S}^{I,\Gamma}_t = b\left(t, \widetilde{S}^{I,\Gamma}_t\right)dt + \sigma\left(t, \widetilde{S}^{I,\Gamma}_t\right)d\widetilde{X}^{I,\Gamma}_t, \quad \widetilde{S}^{I,\Gamma}_0 = x.$$

Then, for every $\varepsilon > 0$ and $t \in [0,T)$, we have

$$\mathbb{E}\left[\sup_{u\in[0,t]}\left|S_u - \widetilde{S}^{I,\Gamma}_u\right|^2\right] = O\left(\mathbb{E}\left[\left|Y - \widehat{Y}^{\Gamma}\right|^{2+\varepsilon}\right]^{\frac{2}{2+\varepsilon}}\right), \tag{28}$$

where Y is defined from X by Equation (11) and $\widehat{Y}^{\Gamma}$ is the nearest neighbor projection of Y on Γ.

Proof: As we have seen in Section 2.2, Equation (11) decomposes the process X into

$$X_t = \sum_{i\in I} Y_i e^X_i(t) \overset{\perp\!\!\!\perp}{+} X^{I,\overline{0}}_t,$$

and the partial quantization $\widetilde{X}^{\Gamma,I}_t$ of X is decomposed into

$$\widetilde{X}^{\Gamma,I}_t = \sum_{i\in I} \widehat{Y}^{\Gamma}_i e^X_i(t) \overset{\perp\!\!\!\perp}{+} X^{I,\overline{0}}_t,$$

where $X^{I,\overline{0}}_t$ is the associated K-L generalized bridge with end-point $\overline{0}$ and $\widehat{Y}^{\Gamma}$ is the nearest neighbor projection of Y on Γ. Thanks to the Theorem (2.2), it is a Gaussian semimartingale with respect to the corresponding enlarged filtration.
Pre-conditioning with respect to $\widehat{Y}^{\Gamma}$ yields

$$\mathbb{E}\left[\sup_{v\in[0,t]}\left|S_v - \widetilde{S}^{\Gamma,I}_v\right|^2\right] = \mathbb{E}\left[\mathbb{E}\left[\sup_{v\in[0,t]}\left|S_v - \widetilde{S}^{\Gamma,I}_v\right|^2\middle|\widehat{Y}^{\Gamma}\right]\right] = \sum_{k=1}^{N} p_k \mathbb{E}\left[\sup_{v\in[0,t]}\left|S_v - \widetilde{S}^{\Gamma,I}_v\right|^2\middle|\widehat{Y}^{\Gamma} = \gamma_k\right]. \tag{29}$$

For some $k \in \{1, \cdots, N\}$, conditionally to $\widehat{Y}^{\Gamma} = \gamma_k$, we have

$$\begin{aligned} S_t - \widetilde{S}^{I,\Gamma}_t = &\int_0^t \left(b(u, S_u) - b\left(u, \widetilde{S}^{I,\Gamma}_u\right)\right)du + \sum_{i\in I}\int_0^t \left(\sigma(u, S_u)Y_i - \sigma\left(u, \widetilde{S}^{I,\Gamma}_u\right)\widehat{Y}^{\Gamma}_i\right)de^X_i(u) \\ &+ \int_0^t \left(\sigma(u, S_u) - \sigma\left(u, \widetilde{S}^{I,\Gamma}_u\right)\right)dX^{I,\overline{0}}_u. \end{aligned}$$

Hence, (conditionally to $\widehat{Y}^{\Gamma} = \gamma_k$)

$$\begin{aligned} \left|S_t - \widetilde{S}^{I,\Gamma}_t\right|^2 \le\ & 3\left|\int_0^t \left(b(u, S_u) - b\left(u, \widetilde{S}^{I,\Gamma}_u\right)\right)du\right|^2 + 3\left|\sum_{i\in I}\int_0^t \left(\sigma(u, S_u)Y_i - \sigma\left(u, \widetilde{S}^{I,\Gamma}_u\right)\widehat{Y}^{\Gamma}_i\right)de^X_i(u)\right|^2 \\ &+ 3\left|\int_0^t \left(\sigma(u, S_u) - \sigma\left(u, \widetilde{S}^{I,\Gamma}_u\right)\right)dX^{I,\overline{0}}_u\right|^2. \end{aligned}$$

From the canonical decomposition of $X^{I,\overline{0}} = \underbrace{\left\langle X, L^{\overline{0}}\right\rangle}_{:=\widetilde{V}} + \underbrace{X - \left\langle X, L^{\overline{0}}\right\rangle}_{:=\widetilde{M}}$, we get, conditionally to $\widehat{Y}^{\Gamma} = \gamma_k$,

$$\left|S_t - \widetilde{S}_t^{I,\Gamma}\right|^2 \leq 3\left|\int_0^t \left(b(u,S_u) - b\left(u,\widetilde{S}_u^{I,\Gamma}\right)\right)du\right|^2 + 3\left|\sum_{i\in I}\int_0^t \left(\sigma(u,S_u)Y_i - \sigma\left(u,\widetilde{S}_u^{I,\Gamma}\right)\widehat{Y}_i^{\Gamma}\right)de_i^X(u)\right|^2$$
$$+ 6\left|\int_0^t \left(\sigma(u,S_u) - \sigma\left(u,\widetilde{S}_u^{I,\Gamma}\right)\right)d\widetilde{V}_u\right|^2 + 6\left|\int_0^t \left(\sigma(u,S_u) - \sigma\left(u,\widetilde{S}_u^{I,\Gamma}\right)\right)d\widetilde{M}_u\right|^2. \quad (30)$$

We decompose the second term of the right hand side of Equation (30) into

$$3\left|\sum_{i\in I}\int_0^t \left(\sigma(u,S_u)Y_i - \sigma\left(u,\widetilde{S}_u^{I,\Gamma}\right)\widehat{Y}_i^{\Gamma}\right)de_i^X(u)\right|^2 \leq 9\left|\sum_{i\in I}\left(Y_i - \widehat{Y}_i^{\Gamma}\right)\int_0^t \sigma(u,S_u)de_i^X(u)\right|^2$$
$$+ 9\left|\sum_{i\in I}\widehat{Y}_i^{\Gamma}\int_0^t \left(\sigma(u,S_u) - \sigma\left(u,\widetilde{S}_u^{I,\Gamma}\right)\right)de_i^X(u)\right|^2.$$

Moreover, Equation (17) yields

$$d\widetilde{V}_s = d\left\langle X, L^{\overline{0}}\right\rangle_s = \underbrace{-\sum_{i\in I}a_i(s)\sum_{j\in I}(Q(s,T)^{-1})_{ij}\mathbb{E}\left[Z_T^j\middle|\left(X_u^{I,\overline{0}}\right)_{u\in[0,s]}\right]}_{:=G_s}d\langle X\rangle_s = G_s d\langle X\rangle_s,$$

where the so-defined process $(G_s)_{s\in[0,T]}$ is Gaussian. Hence, thanks to Schwarz's inequality

$$\begin{aligned}\left|\int_0^t \left(\sigma(u,S_u) - \sigma\left(u,\widetilde{S}_u^{I,\Gamma}\right)\right)d\widetilde{V}_u\right|^2 &= \left|\int_0^t \left(\sigma(u,S_u) - \sigma\left(u,\widetilde{S}_u^{I,\Gamma}\right)\right)G_u d\langle X\rangle_u\right|^2 \\ &\leq \left(\int_0^t |G_u|^2 d\langle X\rangle_u\right)\left(\int_0^t \left|\sigma(u,S_u) - \sigma\left(u,\widetilde{S}_u^{I,\Gamma}\right)\right|^2 d\langle X\rangle_u\right).\end{aligned}$$

Hence, with Doob's inequality and using that $\langle X\rangle = \left\langle\widetilde{M}\right\rangle = \left\langle X^{I,\overline{0}}\right\rangle$,

$$\mathbb{E}\left[\sup_{v\in[0,t]}\left|S_v - \widetilde{S}_v^{I,\Gamma}\right|^2\middle|\widehat{Y}^{\Gamma} = \gamma_k\right] \leq 3t\mathbb{E}\left[\sup_{v\in[0,t]}\int_0^v \left|b(u,S_u) - b\left(u,\widetilde{S}_u^{I,\Gamma}\right)\right|^2 du\middle|\widehat{Y}^{\Gamma} = \gamma_k\right]$$
$$+ 3\mathbb{E}\left[\sup_{v\in[0,t]}\left|\sum_{i\in I}\int_0^v \left(\sigma(u,S_u)Y_i - \sigma\left(u,\widetilde{S}_u^{I,\Gamma}\right)\widehat{Y}_i^{\Gamma}\right)de_i^X(u)\right|^2\middle|\widehat{Y}^{\Gamma} = \gamma_k\right]$$
$$+ 6\mathbb{E}\left[\sup_{v\in[0,t]}\left|\int_0^v \left(\sigma(u,S_u) - \sigma\left(u,\widetilde{S}_u^{I,\Gamma}\right)\right)d\widetilde{V}_u\right|^2\middle|\widehat{Y}^{\Gamma} = \gamma_k\right]$$
$$+ 24\int_0^t \mathbb{E}\left[\left|\sigma(u,S_u) - \sigma\left(u,\widetilde{S}_u^{I,\Gamma}\right)\right|^2\middle|\widehat{Y}^{\Gamma} = \gamma_k\right]d\langle X\rangle_u.$$

Now, using that b and σ are Lipschitz-continuous, and thanks to $\left(\sum_{i\in I} a_i\right)^2 \leq |I| \sum_{i\in I} a_i^2$,

$$\mathbb{E}\left[\sup_{v\in[0,t]}\left|S_v - \widetilde{S}_v^{I,\Gamma}\right|^2 \middle| \widehat{Y}^\Gamma = \gamma_k\right] \leq \underbrace{9[\sigma]_{\max}^2 T \max_{\substack{i\in I\\ u\in[0,T]}}\left\{\left|\left(e_i^X\right)'(u)\right|^2\right\}|I|}_{:=A_I^X}\mathbb{E}\left[\left|Y-\widehat{Y}^\Gamma\right|^2\middle|\widehat{Y}^\Gamma = \gamma_k\right]$$
$$+ 3T[b]_{\text{Lip}}^2 \int_0^t \mathbb{E}\left[\sup_{v\in[0,u]}\left|S_v - \widetilde{S}_v^{I,\Gamma}\right|^2 \middle| \widehat{Y}^\Gamma = \gamma_k\right] du$$
$$+ 9[\sigma]_{\text{Lip}}^2 |I| T \left(\max_{\substack{i\in I\\ u\in[0,T]}} \left(e_i^X\right)'(u)\right)^2 \left(\max_{i\in I}\left(\widehat{Y}_i^\Gamma\right)^2\right) \int_0^t \mathbb{E}\left[\sup_{v\in[0,u]}\left|S_v - \widetilde{S}_v^{I,\Gamma}\right|^2 \middle| \widehat{Y}^\Gamma = \gamma_k\right] du$$
$$+ 24[\sigma]_{\text{Lip}}^2 \int_0^t \mathbb{E}\left[\sup_{v\in[0,u]}\left|S_v - \widetilde{S}_v^{I,\Gamma}\right|^2 \middle| \widehat{Y}^\Gamma = \gamma_k\right] d\langle X\rangle_u$$
$$+ 6\mathbb{E}\left[\left(\int_0^t \left(\sigma(u,S_u) - \sigma\left(u,\widetilde{S}_u\right)\right) G_u d\langle X\rangle_u\right)^2 \middle| \widehat{Y}^\Gamma = \gamma_k\right]. \quad (31)$$

Now, for some $M > 1$, we decompose the expectation in the last term of Equation (31) on $\{|G_u| > M\}$ and $\{|G_u| \leq M\}$.

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^t \left(\sigma(u,S_u) - \sigma\left(u,\widetilde{S}_u\right)\right) G_u d\langle X\rangle_u\right)^2 \middle| \widehat{Y}^\Gamma = \gamma_k\right] \\
&\quad\leq \langle X\rangle_t \int_0^t \mathbb{E}\left[G_u^2\left(\sigma(u,S_u) - \sigma\left(u,\widetilde{S}_u\right)\right)^2 \middle| \widehat{Y}^\Gamma = \gamma_k\right] d\langle X\rangle_u \\
&\quad= \langle X\rangle_t \int_0^t \mathbb{E}\left[\mathbf{1}_{|G_u|\geq M} G_u^2\left(\sigma(u,S_u) - \sigma\left(u,\widetilde{S}_u\right)\right)^2 \middle| \widehat{Y}^\Gamma = \gamma_k\right] d\langle X\rangle_u \\
&\qquad\qquad + \langle X\rangle_t \int_0^t \mathbb{E}\left[\mathbf{1}_{|G_u|\leq M} G_u^2\left(\sigma(u,S_u) - \sigma\left(u,\widetilde{S}_u\right)\right)^2 \middle| \widehat{Y}^\Gamma = \gamma_k\right] d\langle X\rangle_u \\
&\quad\leq 4[\sigma]_{\max}^2 \langle X\rangle_t \int_0^t \mathbb{E}\left[\mathbf{1}_{|G_u|\leq M} G_u^2\right] d\langle X\rangle_u + \langle X\rangle_t M^2[\sigma]_{\text{Lip}}^2 \int_0^t \mathbb{E}\left[\left|S_u - \widetilde{S}_u^{I,\Gamma}\right|^2 \middle| \widehat{Y}^\Gamma = \gamma_k\right] d\langle X\rangle_u.
\end{aligned}$$

We obtain, thanks to Proposition 3.4

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^t \left(\sigma(u,S_u) - \sigma\left(u,\widetilde{S}_u\right)\right) G_u d\langle X\rangle_u\right)^2 \middle| \widehat{Y}^\Gamma = \gamma_k\right] \\
&\quad\leq \underbrace{4[\sigma]_{\max}^2 \langle X\rangle_t^2 \left(\frac{2v_t}{\sqrt{2\pi}} + v_t^2\right) M \exp\left(-\frac{M^2}{2v_t^2}\right)}_{:=\eta_M} + \langle X\rangle_t M^2 [\sigma]_{\text{Lip}}^2 \int_0^t \mathbb{E}\left[\left|S_u - \widetilde{S}_u^{I,\Gamma}\right|^2 \middle| \widehat{Y}^\Gamma = \gamma_k\right] d\langle X\rangle_u,
\end{aligned}$$

where $v_t^2 = \max_{u\in[0,t]} \left(\text{Var}\left(G_u\right)\right)$. Plugging this last inequality into Equation (31), we get

$$\begin{aligned}
\mathbb{E}\left[\sup_{v\in[0,t]}\left|S_v - \widetilde{S}_v^{I,\Gamma}\right|^2 \middle| \widehat{Y}^\Gamma = \gamma_k\right] &\leq A_I^X \mathbb{E}\left[\left|Y - \widehat{Y}^\Gamma\right|^2 \middle| \widehat{Y}^\Gamma = \gamma_k\right] + 6\eta_M \\
&+ B_I^{X,\gamma_k} \int_0^t \mathbb{E}\left[\sup_{v\in[0,u]}\left|S_v - \widetilde{S}_v^{I,\Gamma}\right|^2 \middle| \widehat{Y}^\Gamma = \gamma_k\right] du \\
&+ C^{X,M} \int_0^t \mathbb{E}\left[\sup_{v\in[0,u]}\left|S_v - \widetilde{S}_v^{I,\Gamma}\right|^2 \middle| \widehat{Y}^\Gamma = \gamma_k\right] d\langle X\rangle_u,
\end{aligned}$$

where B_I^{X,γ_k} is an affine function of $\max_{i\in I} |\gamma_{k i}|$ and $C^{X,M} = 24[\sigma]_{\text{Lip}}^2 + 6[\sigma]_{\text{Lip}}^2 M^2$. We can then apply the Gronwall lemma 3.1 for locally finite measures to the function

$$\mathbb{E}\left[\sup_{v\in[0,t]}\left|S_v - \widetilde{S}_v^{I,\Gamma}\right|^2 \middle| \widehat{Y}^\Gamma = \gamma_k\right]$$

with the locally finite measure μ defined by $\mu(du) = du + d\langle X\rangle_u$, and we obtain

$$\mathbb{E}\left[\sup_{v\in[0,t]}\left|S_v - \widetilde{S}_v^{I,\Gamma}\right|^2 \middle| \widehat{Y}^\Gamma = \gamma_k\right] \leq \left(A_I^X \mathbb{E}\left[\left|Y - \widehat{Y}^\Gamma\right|^2 \middle| \widehat{Y}^\Gamma = \gamma_k\right] + 6\eta_M\right) \exp\left(\left(E_I^{X,\gamma_k} + C^{X,M}\right)\mu([0,t))\right)$$
$$\left(A_I^X \mathbb{E}\left[\left|Y - \widehat{Y}^\Gamma\right|^2 \middle| \widehat{Y}^\Gamma = \gamma_k\right] + 6\eta_M\right) \underbrace{\exp\left(E_I^{X,\gamma_k}\mu([0,t))\right)}_{:=\phi(\gamma_k)} \exp\left(C^{X,M}\mu([0,t))\right),$$

where E_I^{X,Γ_k} is an affine function of $\max_{i\in I}|\gamma_k|_i$. Plugging it in Equation (29) yields

$$\mathbb{E}\left[\sup_{v\in[0,t]}\left|S_v - \widetilde{S}_v^{I,\Gamma}\right|^2\right] \leq \exp\left(C^{X,M}\mu([0,t))\right)\mathbb{E}\left[\phi\left(\widehat{Y}^\Gamma\right)\left(A_I^X \mathbb{E}\left[\left|Y - \widehat{Y}^\Gamma\right|^2 \middle| \widehat{Y}^\Gamma\right] + 6\eta_M\right)\right]. \quad (32)$$

Now, for $\varepsilon > 0$ and $\tilde{p} = 1 + \frac{\varepsilon}{2}$ and $\tilde{q} = \frac{\tilde{p}}{\tilde{p}-1} = 1 + \frac{2}{\varepsilon}$ the conjugate exponent of $\tilde{p}$, we have, thanks to Hölder's inequality

$$\begin{aligned}\mathbb{E}\left[\phi\left(\widehat{Y}^\Gamma\right)\left(A_I^X \mathbb{E}\left[\left|Y - \widehat{Y}^\Gamma\right|^2 \middle| \widehat{Y}^\Gamma\right] + 6\eta_M\right)\right] &\leq \left\|\phi\left(\widehat{Y}^\Gamma\right)\right\|_{\tilde{q}} \left\|A_I^X \mathbb{E}\left[\left|Y - \widehat{Y}^\Gamma\right|^2 \middle| \widehat{Y}^\Gamma\right] + 6\eta_M\right\|_{\tilde{p}} \\ &\leq \left\|\phi\left(\widehat{Y}^\Gamma\right)\right\|_{\tilde{q}} \left(A_I^X \mathbb{E}\left[\left|Y - \widehat{Y}^\Gamma\right|^{2\tilde{p}}\right]^{1/\tilde{p}} + 6\eta_M\right).\end{aligned}$$

Now, as the so-defined function ϕ is convex and as $\widehat{Y}^\Gamma$ is a stationary quantizer of Y, we have thanks to Equation (8), $\left\|\phi\left(\widehat{Y}^\Gamma\right)\right\|_{\tilde{q}} \leq \|\phi(Y)\|_{\tilde{q}}$. Hence

$$\mathbb{E}\left[\sup_{v\in[0,t]}\left|S_v - \widetilde{S}_v^{I,\Gamma}\right|^2\right] \leq \exp\left(C^{X,M}\mu([0,t))\right)\|\phi(Y)\|_{\tilde{q}}\left(A_I^X \mathbb{E}\left[\left|Y - \widehat{Y}^\Gamma\right|^{2+\varepsilon}\right]^{\frac{2}{2+\varepsilon}} + 6\eta_M\right). \quad (33)$$

Now, thanks to Proposition 3.5, we can ensure that $\eta_M \leq \eta := \mathbb{E}\left[\left|Y - \widehat{Y}^\Gamma\right|^{2+\varepsilon}\right]^{\frac{2}{2+\varepsilon}}$ by taking $M = \sqrt{-\operatorname{LambertW}\left(-\frac{\eta^2}{C_t^2}\right)}$, with $C_t = 4\langle X\rangle_t^2[\sigma]_{\max}^2\left(\frac{2v_t}{\sqrt{2\pi}} + v_t^2\right)$. We finaly have the following error bound

$$\begin{aligned}\mathbb{E}\left[\sup_{v\in[0,t]}\left|S_v - \widetilde{S}_v^{I,\Gamma}\right|^2\right] \leq \|\phi(Y)\|_{\tilde{q}} \exp\left(24[\sigma]_{\text{Lip}}^2 - 4[\sigma_{\text{Lip}}]^2 \operatorname{LambertW}\left(-\frac{\left(\mathbb{E}\left[\left|Y - \widehat{Y}^\Gamma\right|^{2+\varepsilon}\right]^{\frac{2}{2+\varepsilon}}\right)^2}{C_t^2}\right)\right) \\ \times\left(A_I^X + 6\right)\left(\mathbb{E}\left[\left|Y - \widehat{Y}^\Gamma\right|^{2+\varepsilon}\right]^{\frac{2}{2+\varepsilon}}\right). \quad (34)\end{aligned}$$

Finally, we can conclude by observing that $\operatorname{LambertW}(u) \underset{u\to 0}{\to} 0$. □

Remark (On the time-dependence)**.** *Considering Equation (17), we can see that* $\operatorname{Var}(G_u) \underset{u\to T}{\to} \infty$*, and so does* $v_t^2 = \max_{u\in[0,t]} \operatorname{Var}(G_u)$*. As a consequence, the constant* C_t *involved in Inequality (34) also goes to infinity. This means that Theorem 3.6 cannot be extended to* $t = T$*.*

Corollary 3.7 (Quadratic convergence)**.** *Within the same notations and hypothesis as in Theorem 3.6, consider* $\left(\widetilde{X}^{I,\Gamma_n}\right)_{n\in\mathbb{N}}$ *a sequence of stationary partial functional quantizers of* X *and* $\left(\widetilde{S}^{I,\Gamma_n}\right)_{n\in\mathbb{N}}$ *the corresponding sequence of partial quantizers of* S*.*
If we make the additional assumption that the associated sequence of quantizers $\left(\widehat{Y}^{\Gamma_n}\right)_{n\in\mathbb{N}}$ *is rate optimal for the* $L^{2+\varepsilon}$ *convergence for some* $\varepsilon > 0$*, then for every* $t \in [0,T)$ *we have*

$$\mathbb{E}\left[\sup_{u\in[0,t]}\left|S_u - \widetilde{S}_u^{I,\Gamma_n}\right|^2\right] = O\left(n^{-\frac{2+\varepsilon}{|I|}}\right).$$

Proof: This is a straightforward consequence of Theorem 3.6 and Zador's theorem 3.3, which defines the optimal convergence rate of a sequence of quantizers. □

3.2.2 L^p convergence of partially quantized SDE

We now focus on the general L^p framework, following the same steps except that the Doob's inequality for continuous (local) martingale is replaced by the Burkholder-Davis-Gundy inequality which holds for every $p > 0$.

Lemma 3.8 (Generalized Minkowski inequality for locally finite measures)**.** *Consider $\mathcal{I}$ an interval of the form $[a, b)$ or $[a, b]$ with $a < b$ or $[a, \infty)$. Let μ be a locally finite measure on the Borel σ-algebra of $\mathcal{I}$. Then for any non-negative bi-measurable process $X = (X_t)_{t\in\mathcal{I}}$ and every $p \in [1, \infty)$,*

$$\left\| \int_{\mathcal{I}} X_t \mu(dt) \right\|_p \leq \int_{\mathcal{I}} \|X_t\|_p \mu(dt).$$

Proposition 3.9 (Burkholder-Davis-Gundy inequality)**.** *For every $p \in (0, \infty)$, there exists two positive real constants c_p^{BDG} and C_p^{BDG} such that for every continuous local martingale $(X_t)_{t\in[0,T]}$ null at 0,*

$$c_p^{BDG} \left\| \sqrt{\langle X \rangle_T} \right\|_p \leq \left\| \sup_{s\in[0,T]} |X_s| \right\|_p \leq C_p^{BDG} \left\| \sqrt{\langle X \rangle_T} \right\|_p.$$

We refer to [26] for a detailed proof.

Proposition 3.10 (L^p inequality)**.** *Let G be a standard Gaussian random variable valued in $\mathbb{R}$. There exists a constant $C_p > 0$ such that for every $M > 1$*

$$\mathbb{E}\left[G^p \mathbf{1}_{|G|>M} \right] \leq C_p^p M^{p-1} \exp\left(-\frac{M^2}{2} \right).$$

Consequently

$$\left\| G \mathbf{1}_{|G|>M} \right\|_p \leq C_p M^{\frac{1}{q}} \exp\left(-\frac{M^2}{2p} \right),$$

where q is the conjugate exponent of p.

Proposition 3.11 (The non standard case and L^p reverse inequality)**.** *If $H := \sigma G$ has a variance of σ^2, we obtain*

$$\begin{aligned} \left\| H \mathbf{1}_{|H|>M} \right\|_p &\leq \sigma \left\| G \mathbf{1}_{|G|>\frac{M}{\sigma}} \right\|_p = \sigma C_p \left(\tfrac{M}{\sigma} \right)^{\frac{1}{q}} \exp\left(-\tfrac{M^2}{2p\sigma^2} \right), \\ &= \underbrace{\sigma^{\frac{1}{p}} C_p M^{\frac{1}{q}} \exp\left(-\frac{M^2}{2p\sigma^2} \right)}_{:=\eta_M}. \end{aligned} \tag{35}$$

Conversely, for some settled $\eta > 0$, and if $M > 1$, we have

$$M \geq \underbrace{\sqrt{-\sigma^2(p-1)\,\mathrm{LambertW}\left(-\frac{q\eta^{2q}}{p\sigma^2 \left(C_p^{2q} \sigma^{2q/p} \right)} \right)}}_{:=M_\eta} \Rightarrow \eta_M \leq \eta \tag{36}$$

where LambertW *is the Lambert W function.*

Theorem 3.12 (L^p quantization of partially quantized SDE)**.** *Let X be a continuous centered Gaussian martingale on $[0, T]$ with $X_0 = 0$. Let S be the strong solution of the SDE*

$$dS_t = b(t, S_t)dt + \sigma(t, S_t)dX_t, \quad S_0 = x,$$

where $b(t, x)$ and $\sigma(t, x)$ are Borel functions, Lipschitz-continuous with respect to x uniformly in t, σ and $|b(\cdot, 0)|$ are bounded.
We consider $\widetilde{X}^{I,\Gamma}$ a stationary partial functional quantization of X and $\widetilde{S}^{I,\Gamma}$ the corresponding partial functional quantization of S, i.e. the strong solutions of

$$d\widetilde{S}_t^{I,\Gamma} = b\left(t, \widetilde{S}_t^{I,\Gamma}\right) dt + \sigma\left(t, \widetilde{S}_t^{I,\Gamma}\right) d\widetilde{X}_t^{I,\Gamma}, \quad \widetilde{S}_0^{I,\Gamma} = x.$$

Then, for every $p \in (0,\infty)$, $\varepsilon > 0$ and $t \in [0,T)$, we have

$$\left\| \sup_{v\in[0,t]} \left| S_v - \widetilde{S}_v^{I,\Gamma} \right| \right\|_p = O\left(\left\| Y - \widehat{Y}^\Gamma \right\|_{p+\varepsilon} \right), \tag{37}$$

where Y is defined from X by Equation (11) and $\widehat{Y}^\Gamma$ is the nearest neighbor projection on Γ.

Proof: As in the proof of Theorem 3.6, we decompose the process X into $X_t = \sum_{i\in I} Y_i e_i^X(t) + X_t^{I,\overline{0}}$ and $\widetilde{X}^{I,\Gamma}$ into $\widetilde{X}_t^{I,\Gamma} = \sum_{i\in I} \widehat{Y}_i^\Gamma e_i^X(t) + X_t^{I,\overline{0}}$, where $\widehat{Y}^\Gamma$ is the nearest neighbor projection of Y on Γ.

For some $k \in \{1,\cdots,N\}$, conditionally to $\widehat{Y}^\Gamma = \gamma_k$, we have

$$\begin{aligned} S_t - \widetilde{S}_t^{I,\Gamma} = & \int_0^t \left(b(u,S_u) - b\left(u,\widetilde{S}_t^{I,\Gamma}\right)\right) du + \sum_{i\in I} \int_0^t \left(\sigma(u,S_u) - \sigma\left(u,\widetilde{S}_u^{I,\Gamma}\right)\right) \widehat{Y}_i^\Gamma de_i^X(u) \\ & + \sum_{i\in I} \int_0^t \left(Y_i - \widehat{Y}_i^\Gamma\right) \sigma(u,S_u) de_i^X(u) + \int_0^t \left(\sigma(u,S_u) - \sigma\left(u,\widetilde{S}_u^{I,\Gamma}\right)\right) G_u d\langle X\rangle_u \\ & + \int_0^t \left(\sigma(u,S_u) - \sigma\left(u,\widetilde{S}_u^{I,\Gamma}\right)\right) d\widetilde{M}_u. \end{aligned}$$

This gives (conditionally to $\widehat{Y}^\Gamma = \gamma_k$)

$$\begin{aligned} \left| S_t - \widetilde{S}_t^{I,\Gamma} \right| \le & [b]_{\mathrm{Lip}} \int_0^t \left| S_u - \widetilde{S}_u^{I,\Gamma}\right| du + [\sigma]_{\mathrm{Lip}} |I| \max_{\substack{i\in I \\ u\in[0,T]}} \left| \left(e_i^X\right)'(u)\right| \int_0^t \left| S_u - \widetilde{S}_u^{I,\Gamma}\right| du \\ & + [\sigma]_{\max} |I| \max_{\substack{i\in I \\ u\in[0,T]}} \left| \left(e_i^X\right)'(u)\right| T \sum_{i\in I} \left| Y_i - \widehat{Y}_i^\Gamma\right| + \left| \int_0^t \left(\sigma(u,S_u) - \sigma\left(u,\widetilde{S}_u^{I,\Gamma}\right)\right) G_u d\langle X\rangle_u \right| \\ & + \left| \int_0^t \left(\sigma(u,S_u) - \sigma\left(u,\widetilde{S}_u^{I,\Gamma}\right)\right) d\widetilde{M}_u \right|. \end{aligned}$$

As a consequence, conditionally to $\widehat{Y}^\Gamma = \gamma_k$,

$$\begin{aligned} \max_{v\in[0,t]} \left| S_v - \widetilde{S}_v^{I,\Gamma} \right| \le & [b]_{\mathrm{Lip}} \int_0^t \max_{v\in[0,u]} \left| S_v - \widetilde{S}_v^{I,\Gamma}\right| du + [\sigma]_{\mathrm{Lip}} |I| \max_{\substack{i\in I \\ u\in[0,T]}} \left| \left(e_i^X\right)'(u)\right| \int_0^t \max_{v\in[0,u]} \left| S_v - \widetilde{S}_v^{I,\Gamma}\right| du \\ & + [\sigma]_{\max} |I| \max_{\substack{i\in I \\ u\in[0,T]}} \left| \left(e_i^X\right)'(u)\right| T \sum_{i\in I} \left| Y_i - \widehat{Y}_i^\Gamma\right| + \max_{v\in[0,t]} \left| \int_0^v \left(\sigma(u,S_u) - \sigma\left(u,\widetilde{S}_u^{I,\Gamma}\right)\right) G_u d\langle X\rangle_u \right| \\ & + \max_{v\in[0,t]} \left| \int_0^v \left(\sigma(u,S_u) - \sigma\left(u,\widetilde{S}_u^{I,\Gamma}\right)\right) d\widetilde{M}_u \right|. \end{aligned}$$

To shorten the notations, we denote, for a random variable V and a non-negligible event A, $\|V\|_{p,A} := \mathbb{E}\left[V^p | A\right]^{1/p}$. Hence, using the Minkowski inequality and the generalized Minkowski inequality for locally finite measures (Lemma 3.8), we get

$$
\begin{aligned}
\left\|\max_{v\in[0,t]}\left|S_v-\widetilde{S}_v^{I,\Gamma}\right|\right\|_{p,\{\widehat{Y}^\Gamma=\gamma_k\}} &\leq [b]_{\mathrm{Lip}}\int_0^t\left\|\max_{v\in[0,u]}\left|S_v-\widetilde{S}_v^{I,\Gamma}\right|\right\|_{p,\{\widehat{Y}^\Gamma=\gamma_k\}}du\\
&+[\sigma]_{\mathrm{Lip}}|I|\max_{\substack{i\in I\\ u\in[0,T]}}\left|\left(e_i^X\right)'(u)\right|\left(\max_{i\in I}\left|\widehat{Y}_i^\Gamma\right|\right)\int_0^t\left\|\max_{v\in[0,u]}\left|S_v-\widetilde{S}_v^{I,\Gamma}\right|\right\|_{p,\{\widehat{Y}^\Gamma=\gamma_k\}}du\\
&+[\sigma]_{\mathrm{Lip}}|I|\max_{\substack{i\in I\\ u\in[0,T]}}\left|\left(e_i^X\right)'(u)\right|T\left\|\sum_{i\in I}\left|Y_i-\widehat{Y}_i^\Gamma\right|\right\|_{p,\{\widehat{Y}^\Gamma=\gamma_k\}}\\
&+\left\|\max_{v\in[0,t]}\left|\int_0^v\left(\sigma(u,S_u)-\sigma\left(u,\widetilde{S}_u^{I,\Gamma}\right)\right)G_u d\langle X\rangle_u\right|\right\|_{p,\{\widehat{Y}^\Gamma=\gamma_k\}}\\
&+\left\|\max_{v\in[0,t]}\left|\int_0^v\left(\sigma(u,S_u)-\sigma\left(u,\widetilde{S}_u^{I,\Gamma}\right)\right)d\widetilde{M}_u\right|\right\|_{p,\{\widehat{Y}^\Gamma=\gamma_k\}}.
\end{aligned}
$$

Now, from the Burkholder-Davis-Gundy inequality,

$$
\begin{aligned}
\left\|\max_{v\in[0,t]}\left|S_v-\widetilde{S}_v^{I,\Gamma}\right|\right\|_{p,\{\widehat{Y}^\Gamma=\gamma_k\}} &\leq [b]_{\mathrm{Lip}}\int_0^t\left\|\max_{v\in[0,u]}\left|S_v-\widetilde{S}_v^{I,\Gamma}\right|\right\|_{p,\{\widehat{Y}^\Gamma=\gamma_k\}}du\\
&+[\sigma]_{\mathrm{Lip}}|I|\max_{\substack{i\in I\\ u\in[0,T]}}\left|\left(e_i^X\right)'(u)\right|\left(\max_{i\in I}\left|\widehat{Y}_i^\Gamma\right|\right)\int_0^t\left\|\max_{v\in[0,u]}\left|S_v-\widetilde{S}_v^{I,\Gamma}\right|\right\|_{p,\{\widehat{Y}^\Gamma=\gamma_k\}}du\\
&+[\sigma]_{\mathrm{Lip}}|I|\max_{\substack{i\in I\\ u\in[0,T]}}\left|\left(e_i^X\right)'(u)\right|T\left\|\sum_{i\in I}\left|Y_i-\widehat{Y}_i^\Gamma\right|\right\|_{p,\{\widehat{Y}^\Gamma=\gamma_k\}}\\
&+\left\|\int_0^t\left|\sigma(u,S_u)-\sigma\left(u,\widetilde{S}_u^{I,\Gamma}\right)\right||G_u|d\langle X\rangle_u\right\|_{p,\{\widehat{Y}^\Gamma=\gamma_k\}}\\
&+C_p^{BDG}\left\|\sqrt{\int_0^t\left(\sigma(u,S_u)-\sigma\left(u,\widetilde{S}_u^{I,\Gamma}\right)\right)^2d\langle X\rangle_u}\right\|_{p,\{\widehat{Y}^\Gamma=\gamma_k\}}. \qquad (38)
\end{aligned}
$$

Now, from Schwarz's inequality

$$
\left\|\sum_{i\in I}\left|Y_i-\widehat{Y}_i^\Gamma\right|\right\|_{p,\{\widehat{Y}^\Gamma=\gamma_k\}}\leq\left\|\sqrt{|I|}\sqrt{\sum_{i\in I}\left|Y_i-\widehat{Y}_i^\Gamma\right|^2}\right\|_{p,\{\widehat{Y}^\Gamma=\gamma_k\}}=\sqrt{|I|}\left\|Y-\widehat{Y}^\Gamma\right\|_{p,\{\widehat{Y}^\Gamma=\gamma_k\}}.
$$

From the generalized Minkowsky inequality

$$
\begin{aligned}
&\left\|\int_0^t\left|\sigma(u,S_u)-\sigma\left(u,\widetilde{S}_u^{I,\Gamma}\right)\right||G_u|d\langle X\rangle_u\right\|_{p,\{\widehat{Y}^\Gamma=\gamma_k\}}\leq\int_0^t\left\|\left(\sigma(u,S_u)-\sigma\left(u,\widetilde{S}_u^{I,\Gamma}\right)\right)G_u\right\|_{p,\{\widehat{Y}^\Gamma=\gamma_k\}}d\langle X\rangle_u\\
&=\int_0^t\left\|\left(\sigma(u,S_u)-\sigma\left(u,\widetilde{S}_u^{I,\Gamma}\right)\right)G_u\mathbf{1}_{|G_u|\geq M}+\left(\sigma(u,S_u)-\sigma\left(u,\widetilde{S}_u^{I,\Gamma}\right)\right)G_u\mathbf{1}_{|G_u|\leq M}\right\|_{p,\{\widehat{Y}^\Gamma=\gamma_k\}}d\langle X\rangle_u\\
&\leq\int_0^t\left\|\left(\sigma(u,S_u)-\sigma\left(u,\widetilde{S}_u^{I,\Gamma}\right)\right)G_u\mathbf{1}_{|G_u|\geq M}\right\|_{p,\{\widehat{Y}^\Gamma=\gamma_k\}}d\langle X\rangle_u\\
&\qquad+\int_0^t\left\|\left(\sigma(u,S_u)-\sigma\left(u,\widetilde{S}_u^{I,\Gamma}\right)\right)G_u\mathbf{1}_{|G_u|\leq M}\right\|_{p,\{\widehat{Y}^\Gamma=\gamma_k\}}d\langle X\rangle_u\\
&\leq 2[\sigma]_{\max}\int_0^t\left\|G_u\mathbf{1}_{|G_u|\geq M}\right\|_{p,\{\widehat{Y}^\Gamma=\gamma_k\}}d\langle X\rangle_u+M[\sigma]_{\mathrm{Lip}}\int_0^t\left\|S_u-\widetilde{S}_u^{I,\Gamma}\right\|_{p,\{\widehat{Y}^\Gamma=\gamma_k\}}d\langle X\rangle_u.
\end{aligned}
$$

We obtain, thanks to Proposition 3.11

$$
\begin{aligned}
&\left\|\int_0^t\left|\sigma(u,S_u)-\sigma\left(u,\widetilde{S}_u^{I,\Gamma}\right)\right||G_u|d\langle X\rangle_u\right\|_{p,\{\widehat{Y}^\Gamma=\gamma_k\}}\\
&\leq\underbrace{2[\sigma]_{\max}\langle X\rangle_t C_p v_t^{\frac{1}{p}}M^{\frac{1}{q}}\exp\left(-\frac{M^2}{2pv_t^2}\right)}_{:=\eta_M}+M[\sigma]_{\mathrm{Lip}}\int_0^t\left\|S_u-\widetilde{S}_u^{I,\Gamma}\right\|_{p,\{\widehat{Y}^\Gamma=\gamma_k\}}d\langle X\rangle_u,
\end{aligned}
$$

where $v_t^2 = \max_{u\in[0,t]} (\mathrm{Var}(G_u))$. Moreover

$$\left\| \sqrt{\int_0^t \left(\sigma(u,S_u) - \sigma\left(u, \widetilde{S}_u^{I,\Gamma}\right)\right)^2 d\langle X\rangle_u} \right\|_{p,\{\widehat{Y}^\Gamma=\gamma_k\}} \le \sqrt{\int_0^t \left\| \max_{\substack{i\in I\\ v\in[0,u]}} \left| S_v - \widetilde{S}_v^{I,\Gamma}\right| \right\|^2_{p,\{\widehat{Y}^\Gamma=\gamma_k\}} d\langle X\rangle_u}.$$

Hence, Equation (38) becomes

$$\begin{aligned}
\left\| \max_{v\in[0,t]} \left| S_v - \widetilde{S}_v^{I,\Gamma}\right| \right\|_{p,\{\widehat{Y}^\Gamma=\gamma_k\}} \le & \underbrace{[\sigma]_{\mathrm{Lip}}|I| \max_{\substack{i\in I\\ u\in[0,T]}} \left|\left(e_i^X\right)'(u)\right| \sqrt{|I|}}_{:=A_i^X} \left\| Y - \widehat{Y}^\Gamma\right\|_{p,\{\widehat{Y}^\Gamma=\gamma_k\}} + \eta_M \\
&+ [b]_{\mathrm{Lip}} \int_0^t \left\| \max_{v\in[0,u]} \left| S_v - \widetilde{S}_v^{I,\Gamma}\right| \right\|_{p,\{\widehat{Y}^\Gamma=\gamma_k\}} du \\
&+ [\sigma]_{\mathrm{Lip}}|I| \max_{\substack{i\in I\\ u\in[0,T]}} \left|\left(e_i^X\right)'(u)\right| \left(\max_{i\in I}\left|\widehat{Y}_i^\Gamma\right|\right) \int_0^t \left\| \max_{v\in[0,u]} \left| S_v - \widetilde{S}_v^{I,\Gamma}\right| \right\|_{p,\{\widehat{Y}^\Gamma=\gamma_k\}} du \\
&+ C_p^{BDG} \left(\int_0^t 2 \left\| \max_{\substack{i\in I\\ v\in[0,u]}} \left| S_v - \widetilde{S}_v^{I,\Gamma}\right| \right\|^2_{p,\{\widehat{Y}^\Gamma=\gamma_k\}} d\langle X\rangle_u \right)^{1/2} \\
&+ \underbrace{M[\sigma]_{\mathrm{Lip}}}_{:=C^{X,M}} \int_0^t \left\| \max_{v\in[0,u]} \left| S_v - \widetilde{S}_v^{I,\Gamma}\right| \right\|_{p,\{\widehat{Y}^\Gamma=\gamma_k\}} d\langle X\rangle_u. \qquad (39)
\end{aligned}$$

We can then apply the "Gronwall-like" lemma 3.2 for locally finite measures to the non-decreasing function

$$\left\| \sup_{v\in[0,t]} \left| S_v - \widetilde{S}_v^{I,\Gamma}\right| \right\|_{p,\{\widehat{Y}^\Gamma=\gamma_k\}} = \mathbb{E}\left[\sup_{v\in[0,t]} \left| S_v - \widetilde{S}_v^{I,\Gamma}\right|^p \middle| \widehat{Y}^\Gamma = \gamma_k \right]^{1/p}$$

and with the locally finite measure μ defined by $\mu(du) = du + d\langle X\rangle_u$, and we obtain

$$\begin{aligned}
\left\| \sup_{v\in[0,t]} \left| S_v - \widetilde{S}_v^{I,\Gamma}\right| \right\|_{p,\{\widehat{Y}^\Gamma=\gamma_k\}} &\le \left(A_I^X \mathbb{E}\left[\left| Y - \widehat{Y}^\Gamma\right|^p \middle| \widehat{Y}^\Gamma = \gamma_k\right]^{1/p} + \eta_M \right) \exp\left(\left(E_I^{X,\gamma_k} + C^{X,M}\right) \mu([0,t))\right) \\
&\le \left(A_I^X \mathbb{E}\left[\left| Y - \widehat{Y}^\Gamma\right|^p \middle| \widehat{Y}^\Gamma = \gamma_k\right]^{1/p} + \eta_M \right) \underbrace{\exp\left(E_I^{X,\gamma_k}\mu([0,t))\right)}_{:=\phi(\gamma_k)} \exp\left(C^{X,M}\mu([0,t))\right),
\end{aligned}$$

where E_I^{X,γ_k} is an affine function of $\max_{i\in I} |\gamma_k|_i$. This yields

$$\left\| \sup_{v\in[0,t]} \left| S_v - \widetilde{S}_v^{I,\Gamma}\right| \right\|_p \le \left(A_I^X \left\| \mathbb{E}\left[\left| Y - \widehat{Y}^\Gamma\right|^p \middle| \widehat{Y}^\Gamma\right]^{1/p} \phi\left(\widehat{Y}^\Gamma\right) \right\|_p + \eta_M \left\| \phi\left(\widehat{Y}^\Gamma\right)\right\|_p \right) \exp\left(C^{X,M}\mu([0,t))\right).$$

Now, for $\varepsilon > 0$ and $\tilde{p} = 1 + \frac{\varepsilon}{p}$ and $\tilde{q} = \frac{\tilde{p}}{\tilde{p}-1} = 1 + \frac{p}{\varepsilon}$ the conjugate exponent of $\tilde{p}$, we have, thanks to Hölder's inequality

$$\begin{aligned}
\mathbb{E}\left[\phi\left(\widehat{Y}^\Gamma\right)^p \mathbb{E}\left[\left| Y - \widehat{Y}^\Gamma\right|^p \middle| \widehat{Y}^\Gamma\right]\right] &\le \left\| \phi\left(\widehat{Y}^\Gamma\right)^p \right\|_{\tilde{q}} \left\| \mathbb{E}\left[\left| Y - \widehat{Y}^\Gamma\right|^p \middle| \widehat{Y}^\Gamma\right]\right\|_{\tilde{p}} \\
&\le \left\| \phi\left(\widehat{Y}^\Gamma\right)^p \right\|_{\tilde{q}} \mathbb{E}\left[\left| Y - \widehat{Y}^\Gamma\right|^{p+\varepsilon}\right]^{\frac{p}{p+\varepsilon}}.
\end{aligned}$$

Hence,

$$\left\| \mathbb{E}\left[\left| Y - \widehat{Y}^\Gamma\right|^p \middle| \widehat{Y}^\Gamma\right]^{1/p} \phi\left(\widehat{Y}^\Gamma\right) \right\|_p \le \left\| \phi\left(\widehat{Y}^\Gamma\right)^p \right\|_{\tilde{q}}^{1/p} \mathbb{E}\left[\left| Y - \widehat{Y}^\Gamma\right|^{p+\varepsilon}\right]^{\frac{1}{p+\varepsilon}}.$$

Now, as the so-defined function ϕ is convex and as $\widehat{Y}^\Gamma$ is a stationary quantizer of Y, we have thanks to Equation (8), $\left\|\phi\left(\widehat{Y}^\Gamma\right)^p\right\|_{\tilde{q}} \leq \|\phi\left(Y\right)^p\|_{\tilde{q}}$ and $\left\|\phi\left(\widehat{Y}^\Gamma\right)\right\|_p \leq \|\phi\left(Y\right)\|_p$.

Now, thanks to Proposition 3.11, we can ensure that $\eta_M \leq \eta := \left\|Y - \widehat{Y}^\Gamma\right\|_{p+\varepsilon}$ by taking $M = \sqrt{-v_t(p-1)\,\mathrm{LambertW}\left(-\frac{q\left\|Y-\widehat{Y}^\Gamma\right\|_{p+\varepsilon}^{2q}}{pv_t^2 C_p^{2q} v_t^{2q/p}}\right)}$ where q is the conjugate exponent of p. We finaly have the following error bound

$$\left\|\sup_{v\in[0,t]}\left|S_v - \widetilde{S}_v^{I,\Gamma}\right|\right\|_p \leq C_{X,\varepsilon,I}\exp\left([\sigma]_{\mathrm{Lip}}\sqrt{-v_t(p-1)\,\mathrm{LambertW}\left(-\frac{q\left\|Y-\widehat{Y}^\Gamma\right\|_{p+\varepsilon}^{2q}}{pv_t^2 C_p^{2q} v_t^{2q/p}}\right)}\right)\left\|Y-\widehat{Y}^\Gamma\right\|_{p+\varepsilon}.$$

Finally, we can conclude by observing that $\mathrm{LambertW}(u) \underset{u\to 0}{\to} 0$. □

Remark (Without the stationarity property)**.** *The last step of the demonstration of Theorem 3.12 (the use of the Jensen's inequality) relies on the stationarity of the quantizer $\widehat{Y}$.*
Now, without this stationarity hypothesis and under the additional assumption

$$\Gamma \cap B(0,1) \neq \emptyset, \tag{$\mathcal{A}$}$$

we have for every $i \in I$

$$\left|\widehat{Y}_i\right| \leq \left|Y_i - \widehat{Y}_i\right| + |Y_i| \leq |Y_i| + \left|Y_i - \gamma_i^{k_0}\right| \leq 2|Y_i| + \left|\gamma_i^{k_0}\right| \leq 2|Y_i| + 1, \quad \textit{where } \gamma^{k_0} \in \Gamma \cap B(0,1).$$

Hence

$$\max_{i\in I}\left|\widehat{Y}_i\right| \leq 2\max_{i\in I}|Y_i| + 1.$$

Now, we can notice that the function $\phi(x)$ defined in the demonstration of Theorem 3.12 writes $\phi(x) = \psi(\max_{i\in I} x_i)$ for some non decreasing function ψ. This implies

$$\phi\left(\widehat{Y}\right) = \psi\left(\max_{i\in I}\widehat{Y}_i\right) \leq \psi\left(\max_{i\in I}\left(2|Y_i|+1\right)\right) = \phi(2|Y|+1).$$

Hence, we can obtain the same conclusion as in Theorem 3.12.

Corollary 3.13 (L^p convergence)**.** *Within the same notations and hypothesis as in Theorem 3.12, consider $\left(\widetilde{X}^{I,\Gamma_n}\right)_{n\in\mathbb{N}}$ a sequence of partial functional quantizers of X and $\left(\widetilde{S}^{I,\Gamma_n}\right)_{n\in\mathbb{N}}$ the corresponding sequence of partial quantizers of S.*
If we make the additional assumption that the associated sequence of quantizers $\left(\widehat{Y}^{\Gamma_n}\right)_{n\in\mathbb{N}}$ is rate optimal for the $L^{p+\varepsilon}$ convergence for some $\varepsilon > 0$, then for every $t \in [0,T)$ we have

$$\mathbb{E}\left[\sup_{u\in[0,t]}\left|S_u - \widetilde{S}_u^{I,\Gamma_n}\right|^p\right]) = O\left(n^{-\frac{p+\varepsilon}{|I|}}\right).$$

Proof: As $\left\|Y - \widehat{Y}^{\Gamma_n}\right\|_p \underset{n\to\infty}{\to} 0$, we have *a.s.* $d\left(\widehat{Y}^{\Gamma_n}, Y\right) \underset{n\to\infty}{\to} 0$. Hence, the exists $N_0 \in \mathbb{N}$ such that for every $n \geq N_0$, Γ_n verifies hypothesis ($\mathcal{A}$). From this observation, the result is straightforward consequence of remark 3.2.2 and Zador's theorem 3.3, which defines the optimal convergence rate of a sequence of quantizers. □

3.2.3 The *a.s.* convergence of partially quantized SDE

Theorem 3.14 (Almost sure convergence of partially quantized SDE)**.** *Let X be a continuous centered Gaussian martingale on $[0,T]$ with $X_0 = 0$. Let S be the strong solution of the SDE*

$$dS_t = b(t,S_t)dt + \sigma(t,S_t)dX_t, \quad S_0 = x,$$

where $b(t,x)$ and $\sigma(t,x)$ are Borel functions, Lipschitz-continuous with respect to x uniformly in t, σ and $|b(\cdot,0)|$ are bounded.
We consider $\left(\widetilde{X}^{I,\Gamma_k}\right)_{k\in\mathbb{N}}$ a sequence of partial functional quantizers of X and $\widetilde{S}^{I,\Gamma_n}$ the corresponding partial functional quantization of S, i.e. the strong solutions of

$$d\widetilde{S}_t^{I,\Gamma_n} = b\left(t, \widetilde{S}_t^{I,\Gamma_n}\right)dt + \sigma\left(t, \widetilde{S}_t^{I,\Gamma_n}\right)d\widetilde{X}_t^{I,\Gamma_n}, \quad \widetilde{S}_0^{I,\Gamma_n} = x.$$

We assume that the sequence of partial quantizers of X is rate optimal for some $p \geq 2|I|$, i.e. that there exists a constant C such that

$$\mathbb{E}\left[\left|Y - \widehat{Y}^{\Gamma_n}\right|^p\right] \leq Cn^{-\frac{p}{|I|}}$$

for every $n \in \mathbb{N}^*$, where Y is defined from X by Equation (11) and $\widehat{Y}^{\Gamma}$ is the nearest neighbor projection on Γ. Then for every $t \in [0,T)$, $\widetilde{S}_t^{I,\Gamma_n}$ converges almost surely to S_t.

Proof: From corollary 3.13, if $t \in [0,T)$, there exists three positive constants $K_{X,\varepsilon,I}$, C_t and K_t and $N_0 \in \mathbb{N}$ such that for $n \geq N_0$,

$$\mathbb{E}\left[\sup_{u\in[0,t]}\left|S_u - \widetilde{S}_u^{I,\Gamma_n}\right|^{p-\varepsilon}\right] = O\left(n^{-\frac{p}{|I|}}\right).$$

Hence, as $\frac{p}{|I|} > 2$, Beppo-Levi's theorem for series with non-negative terms implies

$$\mathbb{E}\left[\sum_{n\geq 1}\sup_{u\in[0,t]}\left|S_u - \widetilde{S}_u^{I,\Gamma_n}\right|^{p-\varepsilon}\right] < +\infty.$$

Thus $\sum_{n\geq 1}\sup_{u\in[0,t]}\left|S_u - \widetilde{S}_u^{I,\Gamma_n}\right|^{p-\varepsilon} < +\infty$ $\mathbb{P}$ *a.s.* so that $\sup_{u\in[0,t]}\left|S_u - \widetilde{S}_u^{I,\Gamma_n}\right| \underset{n\to\infty}{\to} 0$ $\mathbb{P}$ *a.s.*. □

Remark (Extension to semimartingales). *In theorems 3.6, 3.12 and 3.14, we limited ourselves to the case where X is a local martingale. The proofs are easily extended to the case of a semimartingale X as soon as there exists a locally finite measure ν on $[0,T]$ such that for every $\omega \in \Omega$ the finite variation part $dV(\omega)$ in the canonical decomposition of X is absolutely continuous with respect to ν. In particular, it is the case for the Brownian bridge and Ornstein-Uhlenbeck processes whose finite variation parts are absolutely continuous with respect to the Lebesgue measure on $[0,T]$.*

A Injectivity properties of the Wiener integral

In this appendix, we recall some results on the definition of the Wiener integral with respect to a Gaussian process. We focus on the injectivity properties. Here, we pay a special attention to the special case of the Ornstein-Uhlenbeck processes.

The covariance operator and the Cameron-Martin space
Consider X a bi-measurable centered Gaussian process on $[0,T]$ such that $\int_0^T \mathbb{E}[X_t^2]dt < \infty$ and with a continuous covariance function Γ^X on $[0,T]\times[0,T]$. We denote $H := \overline{\operatorname{span}\{X_t, t\in[0,T]\}}^{L^2(\mathbb{P})}$ the Gaussian Hilbert space spanned by $(X_t)_{t\in[0,T]}$. The covariance operator C_X of X is defined by

$$C_X: \begin{array}{lcl} L^2([0,T]) & \to & L^2([0,T]) \\ y & \to & C_X y = \mathbb{E}\left[(y,X)X\right]. \end{array}$$

We have $C_X y = \mathbb{E}\left[(y,X)X\right](t) = \mathbb{E}\left[\int_0^T X_s y(s)ds X_t\right] = \int_0^T \Gamma^X(t,s)y(s)ds$ where $\Gamma^X(t,s) = \mathbb{E}[X_t X_s]$ is the covariance function of X.
The Cameron-Martin space of X, (or reproducing Hilbert space of C_X), which we denote K_X, is the subspace of $L^2([0,T])$ defined by $K_X := \{t \to \mathbb{E}\left[ZX_t\right], Z\in H\}$. K_X is equipped with the scalar product defined by

$$\langle k_1, k_2\rangle_X = \mathbb{E}\left[Z_1 Z_2\right] \quad \text{if} \quad k_i = \mathbb{E}\left[Z_i X_.\right],\ i=1,2,$$

so that $(K_X, \langle\cdot\rangle_X)$ is a Hilbert space, isometric with the Hilbert space $\overline{\{(y,X) : y \in L^2([0,T])\}}^H$. K_X is spanned as a Hilbert space by $\{C_X(y) : y \in L^2([0,T])\}$.

The Wiener integral

Here, we follow the same steps as Lebovits and Lévy-Véhel in [15] and Jost in [14] for the definition of a general Wiener integral. The difference here is that we use the quotient topology in order to define the Wiener integral in a more general setting.
We define the map $U : H \to K_X$ defined by $U(Z)(t) = \mathbb{E}[ZX_t]$. By definition of H and K_X, U is a bijection and for any $s \in [0,T]$, we have $U(X_s) = \Gamma^X(s,\cdot)$. Consequently, K_X is spanned by $\left(\Gamma^X(s,\cdot)\right)_{s\in[0,T]}$ as a Hilbert space. Now, we linearly map the set of the piecewise constant functions $\mathcal{E}([0,T])$ to the Cameron-Martin space K_X by

$$\begin{array}{rlcl} J: & \mathcal{E}([0,T]) & \to & K_X \\ & \mathbf{1}_{|s,t|} & \to & \Gamma^X(t,\cdot) - \Gamma^X(s,\cdot), \end{array}$$

where $|a,b|$ stands either for the interval $[a,b]$, (a,b), $(a,b]$ or $[a,b)$. We equip $\mathcal{E}([0,T])$ with the bilinear form $\langle\cdot,\cdot\rangle_J$ which is defined by

$$\langle f,g\rangle_J := \langle Jf, Jg\rangle_X .$$

It is a bilinear symmetric positive *semidefinite* form.

Remark. *The so-called reproducing property shows that* $\left\langle \mathbf{1}_{|0,t|}, \mathbf{1}_{|0,s|}\right\rangle_J = \Gamma^X(t,s)+\Gamma^X(0,0)-\Gamma^X(0,s)-\Gamma^X(0,t)$. *When* $X_0 = 0$ *a.s., this gives* $\left\langle \mathbf{1}_{|0,t|}, \mathbf{1}_{|0,s|}\right\rangle_J = \Gamma^X(s,t)$.

Now, we define the equivalence relation $\underset{J}{\sim}$ on $\mathcal{E}([0,T])$ by $x \underset{J}{\sim} y$ if $\langle x-y, x-y\rangle_J = 0$. On the quotient space $E([0,T]) := \mathcal{E}([0,T])/\underset{J}{\sim}$, the bilinear form $\langle\cdot,\cdot\rangle_J$ is positive definite and thus it is a scalar product on $E([0,T])$. In this context, J defines an (isometric) linear map from $E([0,T])$ to K_X. Then, considering the completion F of $E([0,T])$ associated with this scalar product, J is extended to F and $U^{-1} \circ J : F \to H$ is an (isometric) injective map that we call Wiener integral associated to X.

$$\int_0^T f(t)dX_t := U^{-1} \circ J(f).$$

Injectivity properties of the Wiener integral

As we have just seen, the Wiener integral is an (isometric) injective map from F to H. Still, for example, when dealing with a standard Brownian bridge on $[0,T]$, $\left\|\mathbf{1}_{[0,T]}\right\|_J = 0$, so that there are functions of $\mathcal{E}([0,T])$ which have a non zero L^2 norm and a zero $\|\cdot\|_J$ norm. Injectivity only holds in the quotient space $E([0,T]) = \mathcal{E}([0,T])/\underset{J}{\sim}$ and its completion F.

It is classical background that in the particular case of a standard Brownian motion, $\|\cdot\|_J$ exactly coincides with the canonical L^2 norm so that $F = L^2([0,T])$.

Study of the case of Ornstein-Uhlenbeck process

From now, we will assume that X is a centered Ornstein-Uhlenbeck process defined by the SDE

$$dX_t = -\theta X_t dt + \sigma dW_t \quad \text{with } \sigma > 0 \text{ and } \theta > 0,$$

where W is a standard Brownian motion and $X_0 \overset{\mathcal{L}}{\sim} \mathcal{N}\left(0,\sigma_0^2\right)$ is independent of W. The covariance function writes

$$\Gamma^X(s,t) = \frac{\sigma^2}{2\theta} e^{-\theta(s+t)}\left(e^{2\min(s,t)} - 1\right) + \sigma_0^2 e^{-\theta(s+t)}.$$

Proposition A.1 (Semi-norm equivalence on $\mathcal{E}([0,T])$)**.** *There exists two positive constants c and C such that for every $f \in \mathcal{E}([0,T])$, $c\|f\|_2 \le \|f\|_J \le C\|f\|_2$.*

Proof: Let us consider $f \in \mathcal{E}([0,T])$. We have

$$\begin{aligned} \|f\|_J^2 &= \operatorname{Var}\left(-\theta \int_0^T f(s)X_s ds + \sigma \int_0^T f(s)dW_s\right) \\ &\le 2\operatorname{Var}\left(\theta \int_0^T f(s)X_s ds\right) + 2\operatorname{Var}\left(\sigma \int_0^T f(s)dW_s\right). \end{aligned}$$

The solution of the Ornstein-Uhlenbeck SDE is

$$X_t = \underbrace{X_0 e^{-\theta t}}_{\text{independent of } W} + \underbrace{\int_0^t \sigma e^{\theta(s-t)} dW_s}_{:=X_t^0}. \tag{40}$$

The so-defined process $\left(X_t^0\right)_{t\in[0,T]}$ is a centered Ornstein-Uhlenbeck process starting from 0. Hence we have

$$\begin{aligned} \|f\|_J^2 &\leq 2\operatorname{Var}\left(X_0\theta \int_0^T f(s)e^{-\theta s}ds\right) + 2\operatorname{Var}\left(\theta\int_0^T f(s)X_s^0 ds\right) + 2\operatorname{Var}\left(\sigma \int_0^T f(s)dW_s\right) \\ &\leq 2\theta^2 T \operatorname{Var}(X_0)\int_0^T f(s)^2 ds + 2\operatorname{Var}\left(\theta\int_0^T f(s)X_s^0 ds\right) + 2\operatorname{Var}\left(\sigma \int_0^T f(s)dW_s\right). \end{aligned}$$

We have seen in the proof of Proposition 2.7 that $\operatorname{Var}\left(\theta\int_0^T f(s)X_s^0 ds\right) \leq \operatorname{Var}\left(\sigma \int_0^T f(s)dW_s\right)$. Hence

$$\|f\|_J^2 \leq \underbrace{\left(2\theta^2 T\sigma_0^2 + 4\sigma^2\right)}_{:=C^2}\int_0^T f(s)^2 ds.$$

This is the desired inequality.

Now we write

$$\int_0^t f(s)dX_s = \underbrace{-\theta\int_0^T f(s)X_0 e^{-\theta s}ds}_{:=G_0^f} + \underbrace{\left(-\theta\int_0^T f(s)X_s^0 ds\right)}_{:=G_1^f} + \underbrace{\sigma\int_0^T f(s)dW_s}_{:=G_2^f},$$

where $\left(G_0^f, G_1^f, G_2^f\right)$ is Gaussian and G_0^f independent of G_1^f and G_2^f. Hence

$$\begin{aligned} \operatorname{Var}\left(\int_0^t f(s)dX_s\right) &\geq \operatorname{Var}\left(G_1^f + G_2^f\right) = \operatorname{Var}\left(G_1^f\right) + \operatorname{Var}\left(G_2^f\right) + 2\operatorname{cov}\left(G_1^f, G_2^f\right) \\ &\geq \operatorname{Var}\left(G_1^f\right) + \operatorname{Var}\left(G_2^f\right) - 2\sqrt{\operatorname{Var}\left(G_1^f\right)\operatorname{Var}\left(G_2^f\right)} = \left(\sqrt{\operatorname{Var}\left(G_2^f\right)} - \sqrt{\operatorname{Var}\left(G_1^f\right)}\right)^2. \end{aligned} \tag{41}$$

It was proved at the begining of the demonstration of Proposition 2.7 that there exists a constant $K < 1$ independent of f such that $\operatorname{Var}(G_1^f) \leq K\operatorname{Var}(G_2^f)$. K was defined by

$$K = \frac{\theta^2}{\sigma^2}\sqrt{\int_0^T\int_0^T \Gamma^{X^0}(s,t)dsdt},$$

where Γ^{X^0} is the covariance function of the Ornstein-Uhlenbeck process starting from 0. Plugging this into Equation (41) yields

$$\begin{aligned} \operatorname{Var}\left(\int_0^t f(s)dX_s\right) \geq &\left(1-\sqrt{K}\right)^2 \operatorname{Var}\left(G_2^f\right) \\ &= \underbrace{\left(1-\sqrt{K}\right)^2\sigma^2}_{:=c^2}\|f\|_2^2. \end{aligned}$$

This is the wanted inequality. □

A straightforward consequence of Proposition A.1 is that $\|f\|_J = 0 \Leftrightarrow \|f\|_2 = 0$ so that equivalent classes in $\mathcal{E}([0,T])$ for the relation $\underset{J}{\sim}$ are *almost surely* equal functions. An other consequence is that the sets of Cauchy sequences and convergent sequences for both norms on $E([0,T])$ coincide, and thus the completions of $E([0,T])$ for both norm are the same. In other words, in the case of Ornstein-Uhlenbeck processes, $F = L^2([0,T])$.

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