A general moment bound for a product of Gaussian vector's functionals and a central limit theorem for subordinated Gaussian triangular arrays

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Abstract

A general moment bound for a product of Gaussian vector's functions extending the moment bound in Taqqu (1977, Lemma 4.5) is established. A very general central limit theorem for triangular arrays of nonlinear functionals of multidimensional Gaussian sequences generalizing the results of Arcones (1994) is deduced. The stationarity of the Gaussian vector sequences is not required.

Keywords: Central limit theorem for triangular arrays; Moment bound for Gaussian vector's functions; Hermitian decomposition; Diagram formula; Long memory processes.

1 Introduction and the main results

This paper is devoted to the statement of two new results concerning functions of Gaussian vectors. The first one (see Lemma 1, Section 2) consists in a moment bound for products of such functions of Gaussian vectors in a general frame. It is an extension of an important lemma by Taqqu (1977, Lemma 4.5). Its proof uses the Hermitian decompositions of functions and the diagram formula. Note that in Soulier (2001) a similar moment bound is also proved using an elegant technique which does not make use of diagram formula. However its assumptions are more interesting when the components of the vectors are not independent than when the family of vectors are not independent. Such result is very interesting for proving limit theorem, such as strong law of large numbers (see for instance Bardet and Surgailis, 2011).

Another application of Lemma 1 is studied here with the statement of a central limit theorem (CLT) for triangular arrays of Gaussian vector's functions, call Theorem 1 in the sequel. Roughly speaking, this new CLT is a generalization of the CLT of Arcones (1994, Theorem 2) to non-stationary triangular arrays of subordinated Gaussian vectors. Such a result is motivated by the numerous statistical applications of triangular arrays. For instance, let us note statistics for time series and essentially based on triangular arrays like estimators of regression parameters, kernel density estimators or statistics applied to a discretization of a continuous process (for example $(X_{\delta_n}, X_{2\delta_n}, \cdots, X_{n\delta_n})$). Two examples of applications (a central limit theorem for the Increment Ratio statistic of a Gaussian process admitting a tangent process and a central limit theorem for functions of locally stationary Gaussian process) are studied in Section 4.

From the famous Lindeberg Theorem for independent random variables, there exists numerous paper devoted to CLT for triangular arrays. Proofs of such results are not only straightforward applications of CLTs for sequences of random variables. Following the different assumptions on the dependence, we may cite as reference the book of Jacod and Shiryaev (1987) for martingale differences and the paper of Rio (1995) for strongly mixing sequences. From our knowledge, the more recent and important papers devoted to this topic are those of Coulon-Prieur and Doukhan (2000) with a weak dependence condition and Dedecker and Merlevede (2002) with a necessary and sufficient condition on stable convergence of the normalized partial sums. The case of linear triangular arrays (*i.e.* sequence $(a_{in}\xi_i)_{1\leq i\leq n,\ n\in\mathbb{N}}$ was treated in details in Peligrad and Utev (1997) for several form of dependence conditions on $(\xi_i)_i$. Note that all these results are obtained under a stationarity condition.

Here, we will consider non linear functional of Gaussian vectors. Numerous articles were already devoted to establish CLT or non-CLT in such a frame: we may cite the seminal papers of Taqqu (1975), Dobrushin and Major (1979), Taqqu (1979), Breuer and major (1983) and Giraitis and Surgailis (1985). Extensions were also obtained by Chambers and Slud (1989), Sanchez de Naranjo (1993) and recently in Nourdin *et al.* (2010). However here even if the case of long memory sequences is studied, we will not consider the case of non-CLT and therefore we can cite Arcones (1994) and Soulier (2001) as more recent and closest references (the paper of Arcones is written for unidimensional sequences but can be easily extended to multidimensional sequences).
Differences between our results and Arcones (1994) and Soulier (2001) articles are the following. Firstly and contrarily to Arcones' paper, we will not assume the stationarity property which is however a particular case. In Soulier the nonstationarity is essential since under a stationarity assumption its Theorem 3.1. requires that the sequence $(X_{k,n})_k$ of random vectors are independent when $n\to\infty$. Secondly, instead of their proofs based on the convergence of moments, we will prove the CLT (see Theorem 1 below) from the asymptotic behavior of cumulants. Such a proof already used in Giraitis and Surgailis (1985) is quite shorter since it only requires to show that the cumulants of order greater or equal to 3 converge to 0. Finally we will consider triangular arrays of the form $\big(f_{k,n}(\boldsymbol{Y}_n(k))\big)_{1\leq k\leq n,\ n\in\mathbb{N}}$ where $(\boldsymbol{Y}_n(k))_{1\leq k\leq n, n\in\mathbb{N}}$ be a triangular array of standardized Gaussian vectors, which notably extends the results of Arcones (devoted to $(f(X_j))_{j\in\mathbb{N}}$) and Soulier (devoted to $\big(\beta_{n,k}f(\boldsymbol{Y}_n(k))\big)_{1\leq k\leq n,\ n\in\mathbb{N}}$ with $\sum_{k=1}^n\beta_{n,k}^2=1$). Using the moment bound obtained in Lemma 1, we will show that the assumptions of Theorem 1 will essentially concern the Hermite rank of the functions $(f_{k,n})$ and the asymptotic behavior of the covariances of the triangular array of Gaussian vectors (as in Arcones, 1994).

The CLT for triangular arrays (Theorem 1) and its proof are provided in Section 3 and both the applications are given in Section 4. The following section Section 2 is devoted to the essential moment bound inequality.

2 A moment bound

Let $\boldsymbol{X}=(X^{(1)},\ldots,X^{(\nu)})\in\mathbb{R}^\nu$ be a standardized Gaussian vector, with zero mean $\mathrm{E}X^{(u)}=0$ and covariances $\mathrm{E}X^{(u)}X^{(v)}=\delta_{uv}$ $(u,v=1,\ldots,\nu)$. Let $\mathbb{L}^2(\boldsymbol{X})$ denote the class of all measurable functions $G=G(\boldsymbol{x}),\boldsymbol{x}=(x^{(1)},\ldots,x^{(\nu)})\in\mathbb{R}^\nu$ such that $\|G\|^2:=\mathrm{E}G^2(\boldsymbol{X})<\infty$. For any multiindex $\boldsymbol{k}=(k^{(1)},\ldots,k^{(\nu)})\in\mathbb{Z}_+^\nu:=\{(j^{(1)},\ldots,j^{(\nu)})\in\mathbb{Z}^\nu,j^{(u)}\geq 0\ (1\leq u\leq\nu)\}$, let $H_{\boldsymbol{k}}(\boldsymbol{x})=H_{k^{(1)}}(x^{(1)})\cdots H_{k^{(\nu)}}(x^{(\nu)})$ be the (product) Hermite polynomial; $H_k(x):=(-1)^k\mathrm{e}^{x^2/2}(\mathrm{e}^{-x^2/2})^{(k)}$, $k=0,1,\ldots$ are standard Hermite polynomials. Write $|\boldsymbol{k}|:=k^{(1)}+\ldots+k^{(\nu)}$, $\boldsymbol{k}!:=k^{(1)}!\cdots k^{(\nu)}!$ $(\boldsymbol{k}=(k^{(1)},\ldots,k^{(\nu)})\in\mathbb{Z}_+^\nu)$. A function $G\in\mathbb{L}^2(\boldsymbol{X})$ is said to have a Hermite rank $m\geq 0$ if $J_G(\boldsymbol{k}):=\mathrm{E}G(\boldsymbol{X})H_{\boldsymbol{k}}(\boldsymbol{X})=0$ for any $\boldsymbol{k}\in\mathbb{Z}_+^\nu,|\boldsymbol{k}|<m$, and $J_G(\boldsymbol{k})\neq 0$ for some $\boldsymbol{k},|\boldsymbol{k}|=m$. It is well-known that any $G\in\mathbb{L}^2(\boldsymbol{X})$ having a Hermite rank $m\geq 0$

admits the Hermite expansion

$$G(\boldsymbol{x}) = \sum_{|\boldsymbol{k}|\ge m} \frac{J_G(\boldsymbol{k})}{\boldsymbol{k}!} H_{\boldsymbol{k}}(\boldsymbol{x}),$$

which converges in $\mathbb{L}^2(\boldsymbol{X})$. Let $(\boldsymbol{X}_1, \dots, \boldsymbol{X}_N)$ be a collection of standardized Gaussian vectors $\boldsymbol{X}_t = (X_t^{(1)}, \dots, X_t^{(\nu)}) \in \mathbb{R}^\nu$ having a joint Gaussian distribution in $\mathbb{R}^{\nu N}$. Let $\varepsilon \in [0,1]$ be a fixed number. Following Taqqu (1977), we call $(\boldsymbol{X}_1, \dots, \boldsymbol{X}_N)$ $\varepsilon-$*standard* if $|\mathrm{E} X_t^{(u)} X_s^{(v)}| \le \varepsilon$ for any $t \ne s, 1 \le t, s \le N$ and any $1 \le u, v \le \nu$.

Lemma 1 below is an extension of an important lemma by Taqqu (1977, Lemma 4.5), to the case of a vector-valued Gaussian family $(\boldsymbol{X}_1, \dots, \boldsymbol{X}_N)$, taking values in $\mathbb{R}^\nu (\nu \ge 1)$. The lemma concerns a bound for $\sum' \mathrm{E}|G_{1,t_1,N}(\boldsymbol{X}_{t_1}) \dots G_{p,t_p,N}(\boldsymbol{X}_{t_p})|$, where $G_{1,t,N}, \dots, G_{p,t,N}$ are square integrable functions among which the first $0 \le \alpha \le p$ functions $G_{1,t,N}, \dots, G_{\alpha,t,N}$ for any $1 \le t \le N$ have a Hermite rank at least equal to $m \ge 1$ and where $\sum'$ is the sum over all different indices $1 \le t_i \le N$ $(1 \le i \le p), t_i \ne t_j (i \ne j)$. In the case when $G_{j,t,N} = G_j$ does not depend on t, N, the bound of Lemma 1 coincides with that of Taqqu (1977, Lemma 4.5) provided $m\alpha$ is even, but is worse than Taqqu's bound in the more delicate case when $m\alpha$ is odd. An advantage of our proof is its relative simplicity (we do not use the graph-theoretical argument as in Taqqu, 1977, but rather a simple Hölder inequality). A different approach towards moment inequalities for functions in vector-valued Gaussian variables is discussed in Soulier (2001) but is especially interesting when the component of vectors are not independent.

Lemma 1 *Let* $(\boldsymbol{X}_1, \dots, \boldsymbol{X}_N)$ *be* $\varepsilon-$*standard Gaussian vector,* $\boldsymbol{X}_t = (X_t^{(1)}, \dots, X_t^{(\nu)}) \in \mathbb{R}^\nu$ $(\nu \ge 1)$, *and let* $G_{j,t,N} \in \mathbb{L}^2(\boldsymbol{X}), 1 \le j \le p$ $(p \ge 2), 1 \le t \le N$ *be some functions. For given integers* $m \ge 1, 0 \le \alpha \le p, N \ge 1$, *define*

$$Q_N \quad := \quad \max_{1\le t\le N} \sum_{1\le s\le N, s\ne t} \max_{1\le u,v\le \nu} |\mathrm{E} X_t^{(u)} X_s^{(v)}|^m. \tag{2.1}$$

Assume that the functions $G_{1,t,N}, \dots, G_{\alpha,t,N}$ *have a Hermite rank at least equal to* m *for any* $N \ge 1, 1 \le t \le N$, *and that*

$$\varepsilon < \frac{1}{\nu p - 1}. \tag{2.2}$$

Then

$$\sum\nolimits' \mathrm{E}|G_{1,t_1,N}(\boldsymbol{X}_{t_1}) \cdots G_{p,t_p,N}(\boldsymbol{X}_{t_p})| \quad \le \quad C(\varepsilon, p, m, \alpha, \nu) K N^{p-\frac{\alpha}{2}} Q_N^{\frac{\alpha}{2}},$$

where the constant $C(\varepsilon, p, m, \alpha, \nu)$ *depends on* $\varepsilon, p, m, \alpha, \nu$ *only, and*

$$K = \prod_{j=1}^p \max_{1\le t\le N} \|G_{j,t,N}\|. \tag{2.3}$$

Proof. Fix a collection $(t_1, \dots, t_p)$ of disjoint indices $t_i \ne t_j (i \ne j)$, and write $G_j = G_{j,t_j,N},\ 1 \le j \le p$ for brevity. Let $J_j(\boldsymbol{k}) := J_{G_j}(\boldsymbol{k}) = \mathrm{E} G_j(\boldsymbol{X}) H_{\boldsymbol{k}}(\boldsymbol{X})$ be the coefficients of the Hermite expansion of G_j. Then,

$$\begin{aligned} |J_j(\boldsymbol{k})| \quad &\le \quad \|G_j\| \prod_{i=1}^\nu \mathrm{E}^{1/2} H^2_{k^{(i)}}(X) \\ &\le \quad \|G_j\| \prod_{i=1}^\nu (k^{(i)}!)^{1/2} = \|G_j\| (\boldsymbol{k}!)^{1/2}. \end{aligned}$$

Following Taqqu (1977, p. 213, bottom, p. 214, top), we obtain

$$\begin{aligned}|\mathrm{E}G_1(\boldsymbol{X}_{t_1})\cdots G_p(\boldsymbol{X}_{t_p})| &= \left|\sum_{q=0}^{\infty}\sum_{|\boldsymbol{k}_1|+\ldots+|\boldsymbol{k}_p|=2q}\left\{\prod_{j=1}^{p}\frac{J_j(\boldsymbol{k}_j)}{\boldsymbol{k}_j!}\right\}\mathrm{E}H_{\boldsymbol{k}_1}(\boldsymbol{X}_{t_1})\cdots H_{\boldsymbol{k}_p}(\boldsymbol{X}_{t_p})\right| \\ &\le K_1\sum_{q=0}^{\infty}\sum_{|\boldsymbol{k}_1|+\ldots+|\boldsymbol{k}_p|=2q}\frac{|\mathrm{E}H_{\boldsymbol{k}_1}(\boldsymbol{X}_{t_1})\cdots H_{\boldsymbol{k}_p}(\boldsymbol{X}_{t_p})|}{(\boldsymbol{k}_1!\cdots\boldsymbol{k}_p!)^{1/2}} \\ &\le K_1\sum_{q=0}^{\infty}\sum_{|\boldsymbol{k}_1|+\ldots+|\boldsymbol{k}_p|=2q}\frac{\varepsilon^{(|\boldsymbol{k}_1|+\ldots+|\boldsymbol{k}_p|)/2}\mathrm{E}\prod_{1\le u\le\nu}\prod_{1\le j\le p}H_{k_j^{(u)}}(X)}{(\boldsymbol{k}_1!\cdots\boldsymbol{k}_p!)^{1/2}} \\ &\le K_1\sum_{q=0}^{\infty}\sum_{|\boldsymbol{k}_1|+\ldots+|\boldsymbol{k}_p|=2q}(\varepsilon(\nu p-1))^{(|\boldsymbol{k}_1|+\ldots+|\boldsymbol{k}_p|)/2}<\infty,\end{aligned}$$

where $X\sim\mathcal{N}(0,1)$ and

$$K_1 := \|G_{1,t_1,N}\|\cdots\|G_{p,t_p,N}\| \ \le\ K,$$

where K is defined in (2.3) and K is independent of $t_1,\ldots,t_p$, and where we used the assumption (2.2) to get the convergence of the last series. Therefore,

$$\sum{}'\,\mathrm{E}|G_{1,t_1,N}(\boldsymbol{X}_{t_1})\cdots G_{p,t_p,N}(\boldsymbol{X}_{t_p})| \ \le\ K\sum_{q=0}^{\infty}\sum_{\substack{|\boldsymbol{k}_1|+\ldots+|\boldsymbol{k}_p|=2q\\ |\boldsymbol{k}_1|\ge m,\ldots,|\boldsymbol{k}_\alpha|\ge m}}\sum{}'\,\frac{|\mathrm{E}H_{\boldsymbol{k}_1}(\boldsymbol{X}_{t_1})\cdots H_{\boldsymbol{k}_p}(\boldsymbol{X}_{t_p})|}{(\boldsymbol{k}_1!\cdots\boldsymbol{k}_p!)^{1/2}}.$$

Now, the following bound remains to be proved: for any integers $m\ge 1, 0\le\alpha\le p, N\ge 1$ and any multiindices $\boldsymbol{k}_1,\ldots\boldsymbol{k}_p\in\mathbb{Z}_+^\nu$ satisfying $|\boldsymbol{k}_1|+\ldots+|\boldsymbol{k}_p|=2q,\ |\boldsymbol{k}_1|\ge m,\ldots,|\boldsymbol{k}_\alpha|\ge m$,

$$\sum{}'\,|\mathrm{E}H_{\boldsymbol{k}_1}(\boldsymbol{X}_{t_1})\cdots H_{\boldsymbol{k}_p}(\boldsymbol{X}_{t_p})| \ \le\ C_1(\varepsilon(\nu p-1))^{(|\boldsymbol{k}_1|+\ldots+|\boldsymbol{k}_p|)/2}(\boldsymbol{k}_1!\cdots\boldsymbol{k}_p!)^{1/2}N^{p-\frac{\alpha}{2}}Q_N^{\frac{\alpha}{2}}, \tag{2.4}$$

where C_1 is some constant depending only on p,ν,α,ε, and independent of $\boldsymbol{k}_1,\ldots,\boldsymbol{k}_p,N$.

First, we write the expectation on the left hand side of (2.4) as a sum of contributions of diagrams. Let

$$T := \begin{pmatrix}(1,1) & (1,2) & \ldots & (1,k_1)\\ (2,1) & (2,2) & \ldots & (1,k_2)\\ \ldots & & & \\ (p,1) & (p,2) & \ldots & (p,k_p)\end{pmatrix} \tag{2.5}$$

be a table having p rows $\tau_1,\ldots,\tau_p$ of respective lengths $|\tau_u|=k_u=|\boldsymbol{k}_u|=k_u^{(1)}+\ldots+k_u^{(\nu)}$ (we write $T=\bigcup_{u=1}^p\tau_u$). A *subtable* of T is a table $T'=\bigcup_{u\in U}\tau_u,\ U\subset\{1,\ldots,p\}$ consisting of some rows of T written from top to bottom in the same order as rows in T; clearly any subtable T' of T can be identified with a (nonempty) subset $U\subset\{1,\ldots,p\}$. A *diagram* is a partition γ of the table T by pairs (called *edges* of the diagram) such that no pair belongs to the same row. A diagram γ is called *connected* if the table T cannot be written as a union $T=T'\cup T''$ of two disjoint subtables T',T'' so that T' and T'' are partitioned by γ separately. Write $\Gamma(T),\Gamma_c(T)$ for the class of all diagrams and the class of all connected diagrams over the table T, respectively. Let

$$\rho(t,s) := \max_{1\le u,v\le\nu}|\mathrm{E}X_t^{(u)}X_s^{(v)}| \qquad (t\ne s).$$

Note $0\le\rho(t,s)\le\varepsilon$ and $Q_N=\max_{1\le t\le N}\sum_{1\le s\le N,s\ne t}\rho^m(t,s)$. By the diagram formula for moments of Hermite (Wick) polynomials (see e.g. Surgailis, 2000),

$$\begin{aligned}|\mathrm{E}H_{\boldsymbol{k}_1}(\boldsymbol{X}_{t_1})\cdots H_{\boldsymbol{k}_p}(\boldsymbol{X}_{t_p})| &\le \sum_{\gamma\in\Gamma(T)}\prod_{1\le u<v\le p}(\rho(t_u,t_v))^{\ell_{uv}} &(2.6)\\ &= \sum_{(U_1,\ldots,U_h)}\prod_{r=1}^{h}\sum_{\gamma\in\Gamma_c(U_r)}\prod_{u,v\in U_r,u<v}(\rho(t_u,t_v))^{\ell_{uv}}, &(2.7)\end{aligned}$$

where ℓ_{uv} is the number of edges between rows τ_u and τ_v in the diagram γ over table T, and the sum $\sum_{(U_1,\dots,U_h)}$ is taken over all partitions $(U_1,\dots,U_h), h = 1,2,\dots,[p/2]$ of $\{1,\dots,p\}$ by nonempty subsets U_r of cardinality $|U_r| \ge 2$. (Thus, (2.7) follows from (2.6) by decomposing $\gamma \in \Gamma(T)$ into connected components $\gamma_r \in \Gamma_c(U_r), r = 1,\dots,h; h = 1,\dots,[p/2]$; the restriction $|U_r| \ge 2$ stems from the fact that any edge must necessarily connect different rows.) From (2.7) we obtain

$$\sum\nolimits' \mathrm{E}|G_1(\boldsymbol{X}_{t_1})\cdots G_p(\boldsymbol{X}_{t_p})| \;\le\; \sum_{(U_1,\dots,U_h)} \prod_{r=1}^{h} \sum_{\gamma\in\Gamma_c(U_r)} I_{N,U_r}(\gamma), \tag{2.8}$$

where, for any subtable $U \subset T$ having at least two rows and for any connected diagram $\gamma \in \Gamma_c(U)$, the quantity $I_{N,U}(\gamma)$ is defined by

$$I_{N,U}(\gamma) := \sum\nolimits' \prod_{u,v\in U, u<v} (\rho(t_u,t_v))^{\ell_{uv}}$$

where (recall) the product is taken over all ordered pairs of rows $(\tau_u,\tau_v), u < v$ of the table U, and ℓ_{uv} is the number of edges in γ between the uth and the vth rows. Below we prove the bound

$$I_{N,U}(\gamma) \le K_3 \epsilon^{|\boldsymbol{k}_U|/2} N^{|U|-\frac{\alpha(U)}{2}} (NQ_N)^{\frac{\alpha(U)}{2}}, \tag{2.9}$$

where $|\boldsymbol{k}_U| := \sum_{u\in U} k_u$ is the number of points of table U and $\alpha(U) := |\{1,\dots,\alpha\} \bigcap U| = \#\{u \in U : |\boldsymbol{k}_u| \ge m\}$ is the number of rows in U having at least m points. Clearly, it suffices to show (2.9) for $U = T$.

Next, let for $1 \le u,v \le p, u \ne v$, denote

$$R_{uv} \quad := \quad \Big(\sum\nolimits_{1\le t\le N} \Big(\sum\nolimits_{1\le s\le N, s\ne t} \rho^{k_u}(s,t)\Big)^{k_v/k_u}\Big)^{\ell_{uv}/k_v}. \tag{2.10}$$

Let $A := \{1,\dots,\alpha\}, A' := \{1,\dots,p\}\backslash A = \{\alpha+1,\dots,p\}$. It follows immediately by definition of R_{uv} and $\rho(s,t)$ that

$$R_{uv} \quad \le \quad \begin{cases} N^{\frac{\ell_{uv}}{k_v}} Q_N^{\frac{\ell_{uv}}{k_u}} \varepsilon^{(1-\frac{m}{k_u})\ell_{uv}}, & \text{if } u \in A, \\ N^{\frac{\ell_{uv}}{k_u}+\frac{\ell_{uv}}{k_v}} \varepsilon^{\ell_{uv}}, & \text{if } u \in A^c. \end{cases} \tag{2.11}$$

By the Hölder inequality (see Giraitis and Surgailis, 1985, p.202, for details),

$$I_{N,T}(\gamma) \quad \le \quad \min\left(\prod_{1\le u<v\le p} R_{uv}, \prod_{1\le u<v\le p} R_{vu}\right). \tag{2.12}$$

For any subset $U \subset \{1,\dots,p\}$, let

$$L(U) := \sum_{u\in U} \sum_{u<v\le p} \frac{\ell_{uv}}{k_u}, \qquad L^*(U) := \sum_{u\in U} \sum_{1\le v<u} \frac{\ell_{uv}}{k_u}, \tag{2.13}$$

$L := L(T), L^* := L^*(T)$. Clearly,

$$L(U) + L^*(U) \;=\; \sum_{u\in U} \frac{1}{k_u} \sum_{v=1,\dots,p, v\ne u} \ell_{uv} \;=\; |U| \tag{2.14}$$

is the number of points in U. From (2.11) - (2.12),

$$I_{N,T}(\gamma) \;\le\; \min\Big(N^{L^*+L(A^c)} Q_N^{L(A)} \varepsilon^{|T|/2-mL(A)}, N^{L+L^*(A^c)} Q_N^{L^*(A)} \varepsilon^{|T|/2-mL^*(A)}\Big),$$

where $|T| = \sum_{u=1}^p k_i$. As $0 \le L(A), L^*(A) \le p$, see (2.14), we obtain

$$\begin{aligned} I_{N,T}(\gamma) \quad &\le \quad \varepsilon^{|T|/2-mp} \min\Big(N^{L^*(A)+L^*(A^c)+L(A^c)} Q_N^{L(A)}, N^{L(A)+L(A^c)+L^*(A^c)} Q_N^{L^*(A)}\Big) \\ &= \quad \varepsilon^{|T|/2-mp} N^{p-\alpha} \min\Big(N^{L^*(A)} Q_N^{L(A)}, N^{L(A)} Q_N^{L^*(A)}\Big) \\ &= \quad \varepsilon^{|T|/2-mp} N^{p-\alpha} (NQ_N)^{\frac{\alpha}{2}} \min\Big((N/Q_N)^{\frac{\alpha}{2}-L(A)}, (N/Q_N)^{L(A)-\frac{\alpha}{2}}\Big) \\ &\le \quad \varepsilon^{|T|/2-mp} N^{p-\alpha} (NQ_N)^{\frac{\alpha}{2}}, \end{aligned}$$

proving (2.9).

With (2.9)-(2.8) in mind,

$$\begin{aligned}
{\sum}' |\mathrm{E} H_{\boldsymbol{k}_1}(\boldsymbol{X}_{t_1}) \cdots H_{\boldsymbol{k}_p}(\boldsymbol{X}_{t_p})| &\le C_3 \varepsilon^{|T|/2} \sum_{(U_1,\ldots,U_h)} \prod_{r=1}^{h} \sum_{\gamma \in \Gamma_c(U_r)} N^{|U_r| - \frac{\alpha(U_r)}{2}} Q_N^{\frac{\alpha(U_r)}{2}} \\
&= C_3 \varepsilon^{|T|/2} N^{p-\frac{\alpha}{2}} Q_N^{\frac{\alpha}{2}} \sum_{(U_1,\ldots,U_h)} \prod_{r=1}^{h} \sum_{\gamma \in \Gamma_c(U_r)} 1 \\
&= C_3 \varepsilon^{|T|/2} N^{p-\frac{\alpha}{2}} Q_N^{\frac{\alpha}{2}} \sum_{\gamma \in \Gamma(T)} 1,
\end{aligned}$$

where the last sum (= the number of all diagrams over the table T) does not exceed

$$|\mathrm{E} H_{k_1^{(1)}}(X) \cdots H_{k_1^{(\nu)}}(X) \cdots H_{k_p^{(1)}}(X) \cdots H_{k_p^{(\nu)}}(X)| \le (p\nu - 1)^{(|\boldsymbol{k}_1| + \ldots + |\boldsymbol{k}_p|)/2} (\boldsymbol{k}_1! \cdots \boldsymbol{k}_p!)^{1/2},$$

see Taqqu (1977, Lemma 3.1). This proves the bound (2.4) and the lemma, too. □

3 A central limit for triangular arrays of functions Gaussian's vectors and its proof

Let $(\boldsymbol{Y}_n(k))_{1 \le k \le n, n \in \mathbb{N}}$ be a triangular array of standardized Gaussian vectors with values in $\mathbb{R}^\nu$, $\boldsymbol{Y}_n(k) = (Y_n^{(1)}(k), \ldots, Y_n^{(\nu)}(k))$, $\mathrm{E} Y_n^{(p)}(k) = 0$, $\mathrm{E} Y_n^{(p)}(k) Y_n^{(q)}(k) = \delta_{pq}$. Now define,

$$r_n^{(p,q)}(j,k) := \mathrm{E} Y_n^{(p)}(j) Y_n^{(q)}(k) \qquad (1 \le j, k \le n).$$

For a given integer $m \ge 1$, introduce the following assumptions: for any $1 \le p, q \le \nu$,

$$\sup_{n \ge 1} \max_{1 \le k \le n} \sum_{1 \le j \le n} \left| r_n^{(p,q)}(j,k) \right|^m < \infty, \tag{3.1}$$

$$\sup_{n \ge 1} \frac{1}{n} \sum_{\substack{1 \le j, k \le n \\ |j-k| > K}} \left| r_n^{(p,q)}(j,k) \right|^m \underset{K \to \infty}{\longrightarrow} 0, \tag{3.2}$$

$$\forall (j,k) \in \{1, \ldots, n\}^2, \qquad \left| r_n^{(p,q)}(j,k) \right| \le |\rho(j-k)| \quad \text{with} \quad \sum_{j \in \mathbb{Z}} |\rho(j)|^m < \infty. \tag{3.3}$$

Note (3.3) ⇒ (3.1) and (3.3) ⇒ (3.2).

Theorem 1 *Let $(\boldsymbol{Y}_n(k))_{1 \le k \le n, n \in \mathbb{N}}$ be a triangular array of standardized Gaussian vectors.*

(i) Assume (3.1). Let $f_k \in \mathbb{L}_0^2(\boldsymbol{X})$ $(1 \le k \le n)$ have Hermite rank at least $m \in \mathbb{N}^*$ with $\mathbb{L}_0^2(\boldsymbol{X}) = \{f \in \mathbb{L}^2(\boldsymbol{X}) : \mathrm{E} f(\boldsymbol{X}) = 0\}$. Then there exists a constant C independent of n and $f_k, 1 \le k \le n$ such that

$$\mathrm{E}\Big(n^{-1/2} \sum_{k=1}^{n} f_k\left(\boldsymbol{Y}_n(k)\right)\Big)^2 \le C \max_{1 \le k \le n} \|f_k\|^2. \tag{3.4}$$

(ii) Assume (3.1) and (3.2). Let $f_{k,n} \in \mathbb{L}_0^2(\boldsymbol{X})$ $(n \ge 1, 1 \le k \le n)$ be a triangular array of functions all having Hermite rank at least $m \in \mathbb{N}^*$. Assume that there exists a $\mathbb{L}_0^2(\boldsymbol{X})-$valued continuous function $\phi_\tau, \tau \in [0,1]$, such that

$$\sup_{\tau \in [0,1]} \|f_{[\tau n],n} - \phi_\tau\|^2 = \sup_{\tau \in [0,1]} \mathrm{E}(f_{[\tau n],n}(\boldsymbol{X}) - \phi_\tau(\boldsymbol{X}))^2 \underset{n \to \infty}{\longrightarrow} 0. \tag{3.5}$$

Moreover, let

$$\sigma_n^2 := \mathrm{E}\Big(n^{-1/2} \sum_{k=1}^{n} f_{k,n}\left(\boldsymbol{Y}_n(k)\right)\Big)^2 \underset{n \to \infty}{\longrightarrow} \sigma^2, \tag{3.6}$$

where $\sigma^2 > 0$. *Then*

$$n^{-1/2}\sum_{k=1}^{n} f_{k,n}\left(\boldsymbol{Y}_n(k)\right) \ \xrightarrow[n\to\infty]{\mathcal{D}}\ \mathcal{N}(0,\sigma^2). \tag{3.7}$$

(iii) Assume (3.3). Moreover, assume that for any $\tau \in [0,1]$ *and any* $J \in \mathbb{N}^*$,

$$\left(\boldsymbol{Y}_n([n\tau]+j)\right)_{-J\le j\le J} \ \xrightarrow[n\to\infty]{\mathcal{D}}\ \left(\boldsymbol{W}_\tau(j)\right)_{-J\le j\le J}, \tag{3.8}$$

where $(\boldsymbol{W}_\tau(j))_{j\in\mathbb{Z}}$ *is a stationary Gaussian process taking values in* $\mathbb{R}^\nu$ *and depending on parameter* $\tau \in (0,1)$. *Let* $f_{k,n} \in \mathbb{L}^2_0(\boldsymbol{X})$ $(n \ge 1, 1 \le k \le n)$ *satisfy the same conditions as in part (ii), with exception of (3.6). Then (3.6) and (3.7) hold, with*

$$\sigma^2 \ = \ \int_0^1 \mathrm{d}\tau\Big(\sum_{j\in\mathbb{Z}} \mathrm{E}\big[\phi_\tau\left(\boldsymbol{W}_\tau(0)\right)\phi_\tau\left(\boldsymbol{W}_\tau(j)\right)\big]\Big). \tag{3.9}$$

It can be observed that if part (i) and (ii) are natural extensions of Theorem 2 of Arcones (1994) the part (iii) is new. It is particularly interesting when $\boldsymbol{Y}_n(j) = \boldsymbol{X}_{j/n}$ and $(\boldsymbol{X}_t)_t$ is a vector valued continuous time process.

Proof. (i) Using Arcones' inequality (see Arcones, 1994, (2.44) or Soulier, 2001, (2.4)), one obtains

$$\begin{aligned}
\mathrm{E}\Big(n^{-1/2}\sum_{k=1}^{n} f_k\left(\boldsymbol{Y}_n(k)\right)\Big)^2 \ &= \ \frac{1}{n}\sum_{k=1}^{n}\|f_k\|^2 + \frac{1}{n}\sum{}' \,\mathrm{E} f_k(\boldsymbol{Y}_n(k))f_\ell(\boldsymbol{Y}_n(\ell)) \\
&\le \ \max_{1\le k\le n}\|f_k\|^2 + C\big(\max_{1\le k\le n}\|f_k\|\big)^2 \max_{1\le k\le n}\sum_{1\le \ell\le n, \ell\ne k}\max_{1\le p,q\le\nu}\left|r_n^{(p,q)}(k,\ell)\right|^m,
\end{aligned}$$

where C is a positive real number not depending on n or f_k. Now, using assumption (3.1), (i) is proved.

(ii) We use the following well-known fact. Let $(Z_n)_{n\ge1}$ be a sequence of r.v.'s with zero mean and finite variance. Then $Z_n \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0,\sigma^2)$ if and only if for any $\epsilon > 0$ one can find an integer $n_0(\epsilon) \ge 1$ and a sequence $(Z_{n,\epsilon})_{n\ge1}$ satisfying $Z_{n,\epsilon} \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0,\sigma^2_\epsilon)$ and $\forall n > n_0(\epsilon)$, $\mathrm{E}(Z_n - Z_{n,\epsilon})^2 < \epsilon$.

Let $Z_n := n^{-1/2}\sum_{k=1}^n f_{k,n}\left(\boldsymbol{Y}_n(k)\right)$. We shall construct an approximating sequence $Z_{n,\epsilon}$ with the above properties in two steps.

Firstly, by condition (3.5) and continuity of ϕ_τ, for a given $\epsilon > 0$ one can find integers $M, n_0(\epsilon)$ and a partition $0 =: \tau_0 < \tau_1 < \ldots < \tau_M < \tau_{M+1} := 1$ such that $\forall\ n > n_0(\epsilon)$,

$$\max_{0\le i\le M}\ \max_{k/n\in[\tau_i,\tau_{i+1})}\|f_{k,n} - \phi_{\tau_i}\| = \max_{0\le i\le M}\ \max_{k/n\in[\tau_i,\tau_{i+1})}\left(\mathrm{E}(f_{k,n}(\boldsymbol{X}) - \phi_{\tau_i}(\boldsymbol{X}))^2\right)^{1/2} \ < \ \epsilon. \tag{3.10}$$

Put

$$\widetilde{Z}_{n,\epsilon} := n^{-1/2}\sum_{i=0}^{M}\ \sum_{k/n\in[\tau_i,\tau_{i+1})}\phi_{\tau_i}\left(\boldsymbol{Y}_n(k)\right).$$

Note for any $\tau \in [0,1]$, the function ψ_τ has Hermite rank not less than m, being the limit of a sequence of $\mathbb{L}^2_0(\boldsymbol{X})-$valued functions of Hermite rank $\ge m$. Therefore for the difference $Z_n - \widetilde{Z}_{n,\epsilon}$ the inequality (3.4) applies, yielding $\forall\ n > n_0(\epsilon)$

$$\mathrm{E}(Z_n - \widetilde{Z}_{n,\epsilon})^2 \ \le\ C\max_{0\le i\le M}\ \max_{k/n\in[\tau_i,\tau_{i+1})}\|f_{k,n} - \phi_{\tau_i}\|^2 \le C\epsilon^2 \tag{3.11}$$

in view of (3.10), with a constant C independent of n, ϵ.

Secondly, we expand each ϕ_{τ_i} in Hermite polynomials:

$$\phi_{\tau_i}(\boldsymbol{x}) \ = \ \sum_{m\le|\boldsymbol{k}|}\frac{J_i(\boldsymbol{k})}{\boldsymbol{k}!}H_{\boldsymbol{k}}(\boldsymbol{x}), \qquad (i = 0,1,\ldots,M) \tag{3.12}$$

where

$$J_i(\boldsymbol{k}) := J_{\phi_{\tau_i}}(\boldsymbol{k}) = \mathrm{E}\phi_{\tau_i}(\boldsymbol{X})H_{\boldsymbol{k}}(\boldsymbol{X}), \qquad |J_i(\boldsymbol{k})| \le \|\phi_{\tau_i}\|(\boldsymbol{k}!)^{1/2}.$$

We can choose $t(\epsilon) \in \mathbb{N}$ large enough so that

$$\|\phi_{\tau_i} - \phi_{\tau_i,\epsilon}\| \le \epsilon, \qquad (i = 0, 1, \dots, M), \tag{3.13}$$

where $\phi_{\tau_i,\epsilon}$ is a finite sum of Hermite polynomials:

$$\phi_{\tau_i,\epsilon}(\boldsymbol{x}) \ := \sum_{m \le |\boldsymbol{k}| \le t(\epsilon)} \frac{J_i(\boldsymbol{k})}{\boldsymbol{k}!} H_{\boldsymbol{k}}(\boldsymbol{x}), \qquad (i = 0, 1, \dots, M). \tag{3.14}$$

Note $t(\epsilon)$ does not depend on $i = 0, 1, \dots, M$, and $\epsilon > 0$ is the same as in (3.10). Put

$$Z_{n,\epsilon} := n^{-1/2} \sum_{i=0}^{M} \sum_{k/n \in [\tau_i, \tau_{i+1})} \phi_{\tau_i,\epsilon}\left(\boldsymbol{Y}_n(k)\right). \tag{3.15}$$

Applying (3.4) to the difference $\widetilde{Z}_{n,\epsilon} - Z_{n,\epsilon}$ and using (3.13) and (3.11), we obtain $\forall\ n > n_0(\epsilon)$,

$$\mathrm{E}(Z_n - Z_{n,\epsilon})^2 \ \le \ C\epsilon^2 \tag{3.16}$$

where the constant C is independent of n, ϵ. Let $\sigma^2_{n,\epsilon} := \mathrm{E}Z^2_{n,\epsilon}$. From (3.16) and condition (3.6) it follows that $\forall n > n_0(\epsilon)$,

$$\sigma^2 - C\epsilon \ \le \ \sigma^2_{n,\epsilon} \ \le \ \sigma^2 + C\epsilon, \tag{3.17}$$

with some C independent of n, ϵ. In particular, by choosing $\epsilon > 0$ small enough, it follows that $\liminf_{n\to\infty} \sigma^2_{n,\epsilon} > 0$. We shall prove below that for any fixed $\epsilon > 0$,

$$U_n \ := \ \frac{Z_{n,\epsilon}}{\sigma_{n,\epsilon}} \ = \ \frac{1}{\sigma_{n,\epsilon} n^{1/2}} \sum_{i=1}^{M} \sum_{k/n \in [\tau_i, \tau_{i+1})} \phi_{\tau_i,\epsilon}\left(\boldsymbol{Y}_n(k)\right) \ \underset{n\to\infty}{\overset{\mathcal{D}}{\longrightarrow}} \ \mathcal{N}(0,1). \tag{3.18}$$

As noted in the beginning of the proof of the theorem, the CLT in (3.7) follows from (3.18), (3.16), (3.17). Indeed, write

$$\begin{aligned} \mathrm{E}\mathrm{e}^{\mathrm{i}aZ_n} - \mathrm{e}^{-a^2\sigma^2/2} \ &= \ \left(\mathrm{E}\mathrm{e}^{\mathrm{i}aZ_n} - \mathrm{E}\mathrm{e}^{\mathrm{i}aZ_{n,\epsilon}}\right) + \left(\mathrm{E}\mathrm{e}^{\mathrm{i}a\sigma_{n,\epsilon}U_n} - \mathrm{e}^{-a^2\sigma^2_{n,\epsilon}/2}\right) \\ &+ \ \left(\mathrm{e}^{-a^2\sigma^2_{n,\epsilon}/2} - \mathrm{e}^{-a^2\sigma^2/2}\right) \ := \ \sum_{i=1}^{3} \ell_i(n). \end{aligned}$$

Here, for some constant C independent of n, a, ϵ,

$$\begin{aligned} |\ell_1(n)| \ &\le \ \mathrm{E}^{1/2}\left|\mathrm{e}^{\mathrm{i}a(Z_n - Z_{n,\epsilon})} - 1\right|^2 \ \le \ |a|\mathrm{E}^{1/2}|Z_n - Z_{n,\epsilon}|^2 \ \le \ C|a|\epsilon, \\ |\ell_3(n)| \ &\le \ Ca^2\left|\sigma^2_{n,\epsilon} - \sigma^2\right| \ \le \ Ca^2\epsilon, \end{aligned}$$

and therefore $\ell_i(n), i = 1, 3$ can be made arbitrarily small by choosing $\epsilon > 0$ small enough; see (3.16), (3.17). On the other hand, the convergence in (3.18) implies uniform convergence of characteristic functions on compact intervals and therefore $\sup_{|a| \le A} |\ell_2(n)| \le \sup_{|a| \le 2A} \left|\mathrm{E}\mathrm{e}^{\mathrm{i}aU_n} - \mathrm{e}^{-a^2/2}\right| \underset{n\to\infty}{\longrightarrow} 0$ for any $A > 0$. This proves (3.7).

It remains to prove (3.18). The proof of the corresponding CLTs for sums of Hermite polynomials in Arcones (1994) and Breuer and Major (1983) refer to stationary processes and use Fourier methods. Therefore we present an independent proof of (3.18) based on cumulants and the Hölder inequality in (2.12). Again, our proof appears to be much simpler than computations in the above mentioned papers.

Accordingly, it suffices to show that cumulants of order $p \ge 3$ of U_n asymptotically vanish. In view of (3.17) and linearity of cumulants, this follows from the fact that for any $p \ge 3$ and any multiindices $\boldsymbol{k}_u = (k_u^{(1)}, \dots, k_u^{(\nu)}) \in \mathbb{Z}_+^\nu,\ u = 1, \dots, p$ with $k_u = |\boldsymbol{k}_u| = k_u^{(1)} + \ldots + k_u^{(\nu)} \ge m$ $(1 \le u \le p)$,

$$\Sigma_n \ := \sum_{t_1,\dots,t_p=1}^{n} |\mathrm{cum}(t_1,\dots,t_p)| = \ o(n^{p/2}), \tag{3.19}$$

where $\mathrm{cum}(t_1,\dots,t_p)$ stands for joint cumulant:

$$\mathrm{cum}(t_1,\dots,t_p) \ := \ \mathrm{cum}\left(H_{\boldsymbol{k}_1}(\boldsymbol{Y}_n(t_1)), \dots, H_{\boldsymbol{k}_p}(\boldsymbol{Y}_n(t_p))\right). \tag{3.20}$$

Split $\Sigma_n = \Sigma_n'(K) + \Sigma_n''(K)$, where

$$\Sigma_n'(K) \ := \sum_{t_1,\dots,t_p=1}^{n} |\mathrm{cum}(t_1,\dots,t_p)| \, \mathbf{1}(|t_i - t_j| \le K \ \forall i \ne j)$$

and where K will be chosen large enough. Then for any fixed K, we have $\Sigma_n'(K) = O(n) = o(n^{p/2})$ as $p \ge 3$. The remaining sum $\Sigma_n''(K)$ does not exceed $\sum_{1 \le i \ne j \le p} \Sigma_{n,i,j}''(K)$, where

$$\Sigma_{n,i,j}''(K) \ := \sum_{t_1,\dots,t_p=1}^{n} |\mathrm{cum}(t_1,\dots,t_p)| \, \mathbf{1}(|t_i - t_j| > K).$$

Therefore, relation (3.19) follows if we show that there exist $\delta(K) \underset{K\to\infty}{\longrightarrow} 0$ and $\tilde{n}_0$ such that for any $1 \le i \ne j \le p$ and any $n > \tilde{n}_0$

$$\limsup_{n\to\infty} \Sigma_{n,i,j}''(K) \ < \ \delta(K) n^{p/2}. \tag{3.21}$$

The proof below is limited to $(i,j) = (1,2)$ as the general case is analogous. It is well-known that the joint cumulant in (3.20), similarly to the joint moment in (2.4), can be expressed as a sum over all *connected* diagrams $\gamma \in \Gamma_c(T)$ over the table T in (2.5). By introducing $\bar{\rho}(s,t) := \max_{1 \le p,q \le \nu} \left| r_n^{(p,q)}(s,t) \right|$, we obtain

$$|\mathrm{cum}(t_1,\dots,t_p)| \ \le \sum_{\gamma \in \Gamma_c(T)} \prod_{1 \le u < v \le p} (\bar{\rho}(t_u, t_v))^{\ell_{uv}}, \tag{3.22}$$

where we use the notation in (2.4). Therefore,

$$\Sigma_{n,1,2}''(K) \ \le \sum_{\gamma \in \Gamma_c(T)} \sum_{t_1,\dots,t_p=1}^{n} \prod_{1 \le u < v \le p} (\bar{\rho}(t_u,t_v))^{\ell_{uv}} \mathbf{1}(|t_1 - t_2| > K) \ := \sum_{\gamma \in \Gamma_c(T)} \bar{I}_{n,T}(\gamma),$$

Next, by applying the Hölder inequality as in (2.12),

$$\bar{I}_{n,T}(\gamma) \quad \le \quad \min\left(\prod_{1 \le u < v \le p} \bar{R}_{uv}, \prod_{1 \le u < v \le p} \bar{R}_{vu} \right). \tag{3.23}$$

where (cf. (2.10))

$$\bar{R}_{uv} \quad := \quad \begin{cases} \left(\sum_{1 \le t \le n} \left(\sum_{1 \le s \le n} \bar{\rho}^{k_u}(s,t) \right)^{k_v/k_u} \right)^{\ell_{uv}/k_v}, & (u,v) \ne (1,2), (2,1), \\ \left(\sum_{1 \le t \le n} \left(\sum_{1 \le s \le n} \bar{\rho}^{k_1}(s,t) \mathbf{1}(|t-s| > K) \right)^{k_2/k_1} \right)^{\ell_{12}/k_2}, & (u,v) = (1,2), \\ \left(\sum_{1 \le t \le n} \left(\sum_{1 \le s \le n} \bar{\rho}^{k_2}(t,s) \mathbf{1}(|t-s| > K) \right)^{k_1/k_2} \right)^{\ell_{12}/k_1}, & (u,v) = (2,1). \end{cases}$$

From assumptions (3.1), (3.2), there exists a constant C and $\delta(K) \underset{K\to\infty}{\longrightarrow} 0$ independent of n such that for any $k \ge m$ and any $n \ge 1$

$$\begin{aligned}\sup_{1\le t\le n} \sum_{s=1}^{n} \bar{\rho}^k(s,t) &\le Cn,\\ \sup_{1\le t\le n} \sum_{s=1}^{n} \bar{\rho}^k(s,t)\mathbf{1}(|t-s|>K) &\le \delta(K)n.\end{aligned}$$

Therefore

$$\bar{R}_{uv} \le \begin{cases} Cn^{\ell_{uv}/k_v}, & (u,v)\ne(1,2),(2,1),\\ \tilde{\delta}(K)n^{\ell_{12}/k_2}, & (u,v)=(1,2),\\ \tilde{\delta}(K)n^{\ell_{12}/k_1}, & (u,v)=(2,1),\end{cases}$$

with some $\tilde{\delta}(K) \underset{K\to\infty}{\longrightarrow} 0$ independent of n. Consequently, the minimum on the right-hand side of (3.23) does not exceed

$$C\tilde{\delta}(K)\min\left(n^{\sum_{1\le u<v\le p}\ell_{uv}/k_v}, n^{\sum_{1\le u<v\le p}\ell_{uv}/k_u}\right) = C\tilde{\delta}(K)n^{\min(L(T),L^*(T))}$$

where the quantities $L(T), L^*(T)$ introduced in (2.13) satisfy $L(T)+L^*(T)=p$, see (2.14), and therefore $\min(L(T),L^*(T)) \le p/2$. This proves (3.21) and the CLT in (3.18), thereby completing the proof of part (ii).

(iii) Let us first prove (3.6) with σ^2 given in (3.9) in the case when $f_{k,n}\equiv f$ do not depend on k,n (in such case, one has $\phi_\tau \equiv f$, too). We have

$$\sigma_n^2 = n^{-1}\sum_{k,k'=1}^{n} \mathrm{E}f(\boldsymbol{Y}_n(k))\,f(\boldsymbol{Y}_n(k')) = \int_0^1 F_n(\tau)\mathrm{d}\tau, \tag{3.24}$$

where

$$F_n(\tau) := \sum_{j=1-[n\tau]}^{n-[n\tau]} \mathrm{E}f(\boldsymbol{Y}_n([n\tau]))\,f(\boldsymbol{Y}_n([n\tau]+j)). \tag{3.25}$$

Condition (3.8) implies that

$$\mathrm{E}f(\boldsymbol{Y}_n([n\tau]))\,f(\boldsymbol{Y}_n([n\tau]+j)) \to \mathrm{E}f(\boldsymbol{W}_\tau(0))\,f(\boldsymbol{W}_\tau(j))$$

for each $j\in\mathbb{Z}$ as $n\to\infty$. From (3.3) and with the inequality of previous part (i), it exists $C>0$ such that

$$|\mathrm{E}f(\boldsymbol{Y}_n([n\tau]))\,f(\boldsymbol{Y}_n([n\tau]+j))\Big| \le C|\rho(j)|^m, \tag{3.26}$$

and $\sum_{j\in\mathbb{Z}}|\rho(j)|^m<\infty$. Hence, from Lebesgue Theorem,

$$F_n(\tau) = \sum_{j\in\mathbb{Z}} \mathbf{1}_{j\in\{1-[n\tau],\cdots,n-[n\tau]\}}\mathrm{E}f(\boldsymbol{Y}_n([n\tau]))\,f(\boldsymbol{Y}_n([n\tau]+j)) \underset{n\to\infty}{\longrightarrow} \sum_{j\in\mathbb{Z}}\mathrm{E}f(\boldsymbol{W}_\tau(0))\,f(\boldsymbol{W}_\tau(j)).$$

The dominated convergence theorem allows also to pass the limit under the integral, thereby proving (3.6) with σ^2 given in (3.9) in the case $f_{k,n}\equiv f$.

To end the proof, consider the general case of $f_{k,n}$ as in (iii). Let $Z_{n,\epsilon}$ be defined as in (3.15). Note relation (3.16) holds as its proof does not use (3.6). In part (ii), we used (3.6) to prove (3.17). Now we want to prove (3.17) using (3.8) instead of (3.6). This will suffice for the proof of (iii), as the remaining argument is the same as in part (ii).

Consider the variance $\sigma_{n,\epsilon}^2 = \mathrm{E}Z_{n,\epsilon}^2$ of $Z_{n,\epsilon}$ defined in (3.15):

$$\sigma_{n,\epsilon}^2 = n^{-1}\left(\sum_{0\le i\le M}\mathrm{E}D_i^2 + 2\sum_{0\le i<j\le M}\mathrm{E}D_iD_j\right),$$

where

$$D_i \ := \sum_{k/n\in[\tau_i,\tau_{i+1})} \phi_{\tau_i,\epsilon}\left(\boldsymbol{Y}_n(k)\right).$$

Let us show that for ϵ, M fixed, and as $n\to\infty$,

$$\mathrm{E}D_iD_j \quad = \quad o(n) \qquad (i\neq j), \tag{3.27}$$

$$n^{-1}\mathrm{E}D_i^2 \quad \underset{n\to\infty}{\longrightarrow} \quad \int_{\tau_i}^{\tau_{i+1}} \sum_{j\in\mathbb{Z}} \mathrm{E}\phi_{\tau_i,\epsilon}\left(\boldsymbol{W}_\tau(0)\right)\phi_{\tau_i,\epsilon}\left(\boldsymbol{W}_\tau(j)\right)\mathrm{d}\tau. \tag{3.28}$$

Here, (3.28) follows from the argument in the beginning of the proof of (iii), as $\phi_{\tau_i,\epsilon}$ does not depend on k,n. Relation (3.27) is implied by the following computations. Using the Hermitian rank of functions $\phi_{\tau_i,\epsilon}$, for $i<j$ one obtains

$$\begin{aligned}\left|\mathrm{E}\phi_{\tau_i,\epsilon}\left(\boldsymbol{Y}_n([n\tau_i]+k)\right)\phi_{\tau_j,\epsilon}\left(\boldsymbol{Y}_n([n\tau_j]+\ell)\right)\right| &\leq C\|\phi_{\tau_i,\epsilon}\|\cdot\|\phi_{\tau_j,\epsilon}\| \max_{1\leq p,q\leq\nu}\left|r_n^{(p,q)}([n\tau_i]+k,[n\tau_j]+\ell)\right|^m \\ &\leq C\|\phi_{\tau_i,\epsilon}\|\cdot\|\phi_{\tau_j,\epsilon}\|\left|\rho([n\tau_j]-[n\tau_i]+\ell-k)\right|^m.\end{aligned}$$

Therefore, for $i<j$, and ε small enough,

$$\begin{aligned}\left|\mathrm{E}D_iD_j\right| &\leq C\max_{\tau\in[0,1]}\|\phi_\tau\|^2 \sum_{k=0}^{[\tau_{i+1}n]-[\tau_in]} \sum_{\ell=0}^{[\tau_{j+1}n]-[\tau_jn]} \left|\rho([n\tau_j]-[n\tau_i]+\ell-k)\right|^m \\ &\leq C\max_{\tau\in[0,1]}\|\phi_\tau\|^2\sum_{k=1}^{n}k\left|\rho(k)\right|^m \ = \ o(n)\end{aligned}$$

since $\sum_{k=1}^n k\left|\rho(k)\right|^m \leq \sqrt{n}\sum_{1\leq k\leq\sqrt{n}}\left|\rho(k)\right|^m + n\sum_{k>\sqrt{n}}\left|\rho(k)\right|^m = o(n)$. Thus, (3.27) is proved. From (3.27), (3.28) it follows that for any $\epsilon>0$

$$\lim_{n\to\infty}\sigma_{n,\epsilon}^2 \ = \ \bar\sigma_\epsilon^2 \ := \ \sum_{i=0}^{M}\int_{\tau_i}^{\tau_{i+1}}\sum_{j\in\mathbb{Z}}\mathrm{E}\phi_{\tau_i,\epsilon}\left(\boldsymbol{W}_\tau(0)\right)\phi_{\tau_i,\epsilon}\left(\boldsymbol{W}_\tau(j)\right)\mathrm{d}\tau.$$

Consider the difference $\bar\sigma_\epsilon^2-\sigma^2=\sum_{i=0}^M\int_{\tau_i}^{\tau_{i+1}}\sum_{j\in\mathbb{Z}}\Theta_{M,\epsilon}(\tau,j)\mathrm{d}\tau$, where

$$\begin{aligned}|\Theta_{M,\epsilon}(\tau,j)| &= |\mathrm{E}\phi_{\tau_i,\epsilon}(\boldsymbol{W}_\tau(0))\phi_{\tau_i,\epsilon}(\boldsymbol{W}_\tau(j))-\mathrm{E}\phi_\tau(\boldsymbol{W}_\tau(0))\phi_\tau(\boldsymbol{W}_\tau(j))| \\ &\leq |\mathrm{E}\left(\phi_{\tau_i,\epsilon}(\boldsymbol{W}_\tau(0))-\phi_\tau(\boldsymbol{W}_\tau(0))\right)\phi_{\tau_i,\epsilon}(\boldsymbol{W}_\tau(j))| \ + \ |\mathrm{E}\left(\phi_{\tau_i,\epsilon}\left(\boldsymbol{W}_\tau(j)\right)-\phi_\tau\left(\boldsymbol{W}_\tau(j)\right)\right)\phi_\tau\left(\boldsymbol{W}_\tau(0)\right)| \\ &\leq \|\phi_{\tau_i,\epsilon}-\phi_\tau\|\left(\|\phi_{\tau_i,\epsilon}\|+\|\phi_\tau\|\right).\end{aligned} \tag{3.29}$$

Using uniform continuity of $\phi_\tau, \tau\in[0,1]$ (in the sense of $\mathbb{L}^2-$norm continuity), we obtain that the right-hand side of (3.29) can be made arbitrarily small by choosing M (= the number of partition intervals of $[0,1]$) and $t(\epsilon)$ (= the truncation level of Hermite expansion) sufficiently large, uniformly in $\tau\in[0,1]$ and $j\in\mathbb{Z}$. On the other hand, $|\Theta_{M,\epsilon}(\tau,j)|\leq C\sup_{\tau\in[0,1]}\|\phi_\tau\|^2|\rho(j)|^m$ by Arcones' inequality, c.f. (3.26). Therefore $|\Theta_{M,\epsilon}(\tau,j)|$ is dominated by a summable function uniformly in M,ϵ. Now, (3.17) follows by an application of Lebesgue theorem. This proves part (iii) and Theorem 1 too. ☐

4 Applications of Theorem 1

4.1 Limit theorems for the IR statistic of Gaussian processes

This application was developed in Bardet and Surgailis (2011). Let $(X_t)_{t\in\mathbb{R}}$ be a Gaussian process admitting a tangent process (which is a self-similar process with parameter $H(t)$) and consider the Increment Ratio (IR) statistic

$$R^{2,n}(X) \quad := \quad \frac{1}{n-2}\sum_{k=0}^{n-3}\frac{\left|\Delta_k^{2,n}X+\Delta_{k+1}^{2,n}X\right|}{|\Delta_k^{2,n}X|+|\Delta_{k+1}^{2,n}X|},$$

with $\Delta_k^{2,n} X = X_{(k+2)/n} - 2\, X_{(k+1)/n} + X_{k/n}$ and the convention $\frac{0}{0} := 1$. Then,

$$R^{2,n}(X) - \mathrm{E} R^{2,n}(X) = \frac{1}{n-2} \sum_{k=0}^{n-3} \tilde{\eta}_n(k),$$

where $\tilde{\eta}_n(k) := \eta_n(k) - \mathrm{E}\eta_n(k)$ and $\eta_n(k) := \ \psi(\Delta_k^{2,n} X, \Delta_{k+1}^{2,n} X)$, $\psi(x,y) = |x+y|/(|x|+|y|)$ are nonlinear functions of Gaussian vectors $(\Delta_k^{2,n} X, \Delta_{k+1}^{2,n} X) \in \mathbb{R}^2$ having the Hermite rank 2. Write $\eta_n(k)$ as a (bounded) function in standardized Gaussian variables:

$$\eta_n(k) \ = \ f_{k,n}\left(\boldsymbol{Y}_n(k)\right),$$

where $\boldsymbol{Y}_n(k) = (Y_n^{(1)}(k), Y_n^{(2)}(k)) \in \mathbb{R}^2$,

$$\begin{aligned}
Y_n^{(1)}(k) \ &:= \ \frac{\Delta_k^{2,n} X}{\sigma_{2,n}(k)}, \\
Y_n^{(2)}(k) \ &:= \ -\frac{\Delta_k^{2,n} X}{\sigma_{2,n}(k)} \frac{\rho_{2,n}(k)}{\sqrt{1-\rho_{2,n}^2(k)}} + \frac{\Delta_{k+1}^{2,n} X}{\sigma_{2,n}(k+1)} \frac{1}{\sqrt{1-\rho_{2,n}^2(k)}}, \\
\text{and} \quad f_{k,n}\big(x^{(1)}, x^{(2)}\big) \ &:= \ \psi\left(x^{(1)}, \frac{\sigma_{2,n}(k+1)}{\sigma_{2,n}(k)} \Big(\rho_{2,n}(k) x^{(1)} + \sqrt{1-\rho_{2,n}^2(k)}\, x^{(2)}\Big)\right),
\end{aligned}$$

where $\sigma_{2,n}^2(k),\ \rho_{2,n}(k)$ are defined by

$$\sigma_{2,n}^2(k) := \mathrm{E}\Big[\left(\Delta_k^{2,n} X\right)^2\Big], \qquad \rho_{2,n}(k) \ := \ \frac{\mathrm{E}\big[\Delta_k^{2,n} X \ \Delta_{k+1}^{2,n} X\big]}{\sigma_{2,n}(k)\sigma_{2,n}(k+1)}.$$

Thus, the asymptotic behavior of $\frac{1}{n-2}\sum_{k=0}^{n-3} f_{k,n}\left(\boldsymbol{Y}_n(k)\right)$ provides the one of $R^{2,n}(X)$. Then, if X satisfies additional conditions (especially on its convergence to its tangent process and the asymptotic behavior of the covariances of $\Delta_j^{2,n} X$ and $\Delta_k^{2,n} X$), $f_{k,n} \underset{n\to\infty}{\longrightarrow} f$ where these functions have Hermite rank 2 and Theorem 1 can be applied to establish that $\sqrt{n}\big(R^{2,n}(X) - \int_0^1 \Lambda(H(t))\, dt\big) \underset{n\to\infty}{\xrightarrow{\mathcal{D}}} \mathcal{N}(0,\sigma^2)$ with explicit function Λ and σ^2. Moreover another application of Lemma 1 provides the almost sure consistency of $R^{2,n}(X)$, *i.e.* $R^{2,n}(X) \underset{n\to\infty}{\xrightarrow{a.s.}} \int_0^1 \Lambda(H(t))\, dt$.

Such results can be applied to fractional Brownian motions but as well to multifractional Brownian motions (without stationary properties). More details can be seen in Bardet and Surgailis (2011).

4.2 A central limit theorem for functions of locally stationary Gaussian processes

Using an adaptation of Dahlhaus and Polonik (2006, 2009), we will say that $(X_{t,n})_{1\leq t\leq n,\, n\in\mathbb{N}^*}$ is a locally stationary Gaussian process if

$$X_{t,n} := \sum_{j\in\mathbb{Z}} a_{t,n}(j)\, \varepsilon_{t-j}, \quad \text{for all } 1 \leq t \leq n,\ n \in \mathbb{N}^*,$$

where $(\varepsilon_k)_{k\in\mathbb{Z}}$ is a sequence of independent standardized Gaussian variables and for $1 \leq t \leq n$, $n \in N^*$ the sequences $(a_{t,n}(j))_{j\in\mathbb{Z}}$ are such that there exist $K \geq 0$ and $\kappa > 0$ satisfying for all $n \in \mathbb{N}^*$ and $j \in \mathbb{Z}$,

$$\max_{1\leq t\leq n} |a_{t,n}(j)| \leq \frac{K}{u_j}, \qquad \text{with } u_j = \max(1, |j|^{\alpha-1}) \text{ for } j \in \mathbb{Z} \tag{4.1}$$

with $\alpha < 1/2$ and such that there exist functions $u \in (0,1] \mapsto a(u,j) \in \mathbb{R}$ satisfying:

$$\sup_{u\in(0,1]} |a(u,j)| \leq \frac{K}{u_j} \tag{4.2}$$

$$\max_{1\leq t\leq n}\big|a(t/n,j)-a_{t,n}(j)\big|\leq \frac{K}{n}\,\frac{1}{u_j} \tag{4.3}$$

$$\sup_{(u,v)\in[0,1]^2}\Big|\frac{a(u,j)-a(v,j)}{u-v}\Big|\leq \frac{K}{u_j} \tag{4.4}$$

Remark 1 *In Dahlhaus and Polonik (2006, 2009), only the short-memory case was considered and for any $1\leq t\leq n$, the sequence $(a_{t,n}(j))_{j\in\mathbb{Z}}\in\ell^1$. Here Condition (4.1) allows the long-memory case and $(a_{t,n}(j))_{j\in\mathbb{Z}}\in\ell^2$ is only required. It was also such the case in Roueff and Von Sachs (2010) where similar conditions than (4.1) and (4.2) are provided in terms of spectral density. However the property of local stationarity is more general in Dahlhaus and Polonik (2006, 2009) because the parameter curves are allowed to have jumps and Conditions (4.3) and (4.4) are replaced by*

$$\sup_{\substack{0\,\leq\, x_0\,<\,\ldots\,<\,x_m\,\leq\,1\\ m\,\in\,\mathbb{N}^*}}\sum_{k=1}^m|a(x_k,j)-a(x_{k-1},j)|\leq\frac{K}{u_j}\quad \textit{and}\quad \sup_{j\in\mathbb{Z}}\sum_{t=1}^n|a_{t,n}(j)-a(\frac{t}{n},j)|\leq K.$$

Let $f_{t,n}\in\mathbb{L}_0^2(\boldsymbol{Z})$ $(n\geq 1, 1\leq t\leq n)$, with $\boldsymbol{Z}$ a standardized Gaussian vector $\mathbb{R}^d$-valued, be a triangular array of functions all having Hermite rank at least $m\in\mathbb{N}^*$. Assume that there exists a $\mathbb{L}_0^2(\boldsymbol{X})-$valued continuous function $f_\tau,\tau\in[0,1]$, such that $\sup_{\tau\in[0,1]}\|f_{[\tau n],n}-f_\tau\|^2\underset{n\to\infty}{\longrightarrow}0$. Let $0\leq i_1<\cdots<i_d\in\mathbb{N}^d$ and $\boldsymbol{X}_{t,n}:=\big(X_{t+i_1,n},\cdots,X_{t+i_d,n}\big)$ for $t=1,\cdots,n-i_d$ and $n>i_d$. If $m>(1-2\alpha)^{-1}$ then,

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n-i_d}f_{t,n}(\boldsymbol{X}_{t,n})\ \underset{n\to\infty}{\overset{\mathcal{D}}{\longrightarrow}}\ \mathcal{N}\big(0\,,\,\sigma^2\big) \tag{4.5}$$

with

$$\sigma^2=\int_0^1 d\tau\,\Big(\sum_{j\in\mathbb{Z}}\mathrm{E}\big[f_\tau\big(\sum_{k\in\mathbb{Z}}a(\tau,k)\varepsilon_{-k}\big)\,f_\tau\big(\sum_{k\in\mathbb{Z}}a(\tau,k)\varepsilon_{j-k}\big)\big]\Big). \tag{4.6}$$

Proof. Let $\Sigma_{t,n}=\mathrm{Cov}\big(\boldsymbol{X}_{t,n}\big)=\big(\mathrm{E}\big[X_{t+i_p,n}\cdot X_{t+i_q,n}\big]\big)_{1\leq p,q\leq d}$. Then for $\tau\in[0,1]$,

$$\begin{aligned}\big(\mathrm{E}\big[X_{[n\tau]+i_p,n}\cdot X_{[n\tau]+i_q,n}\big]\big)_{1\leq p,q\leq d}\quad &= \quad\big(\sum_{k\in\mathbb{Z}}a_{[n\tau]+i_p,n}(i_q-i_p+k)a_{[n\tau]+i_q,n}(k)\big)_{1\leq p,q\leq d}\\ &\underset{n\to\infty}{\longrightarrow}\quad\big(\sum_{k\in\mathbb{Z}}a(\tau,i_q-i_p+k)a(\tau,k)\big)_{1\leq p,q\leq d}=:\Sigma_\tau,\end{aligned} \tag{4.7}$$

using Lebesgue Theorem and assumptions on sequences $(a_{t,n}(j))$. Now for $\boldsymbol{x}\in\mathbb{R}^d$, define $h_{t,n}(\boldsymbol{x}):=f_{t,n}\big((\Sigma_{t,n})^{1/2}\,\boldsymbol{x}\big)$. From assumptions on $(f_{t,n})$ and (4.7),

$$\sup_{\tau\in[0,1]}\|h_{[\tau n],n}-h_\tau\|^2\underset{n\to\infty}{\longrightarrow}0,\quad\text{with}\quad h_\tau(\boldsymbol{x})=f_\tau(\Sigma_\tau\,\boldsymbol{x}).$$

Thus we are going to apply Theorem 1 part *(iii)* (using also its notation) to the array of functions $(h_{t,n}(\boldsymbol{x}))$ and $\boldsymbol{Y}_n(k)=(\Sigma_{t,n})^{-1/2}\boldsymbol{X}_{t,n}$. Obvious computations show that with $\boldsymbol{W}_\tau(j)=\Sigma_\tau^{-1/2}\left(\sum_{k\in\mathbb{Z}}a(\tau,k)\varepsilon_{j+i_p-k}\right)_{1\leq p\leq d}$ one obtains the required relation $\big(\boldsymbol{Y}_n([n\tau]+j)\big)_{-J\leq j\leq J}\ \underset{n\to\infty}{\overset{\mathcal{D}}{\longrightarrow}}\ (\boldsymbol{W}_\tau(j))_{-J\leq j\leq J}$. Then the expression (4.6) of the asymptotic variance can be deduced.

It still needs to check the condition (3.3) of Theorem 1. Using (4.7), for all $1\leq p,q\leq d$, n large enough and $1\leq j,k\leq n-i_d$,

$$\begin{aligned}r_n^{(p,q)}(j,k)\quad&=\quad\mathrm{E}\big[X_{j+i_p,n}\cdot X_{k+i_q,n}\big]\\ &=\quad\sum_{\ell\in\mathbb{Z}}a_{j+i_p,n}(i_q-i_p+\ell+k-j)a_{k+i_q,n}(\ell)\end{aligned}$$

$$\begin{aligned}
\Longrightarrow \big|r_n^{(p,q)}(j,k)\big| &\leq 2\sum_{\ell\in\mathbb{Z}}\big|a(\frac{j+i_p}{n},i_q-i_p+k-j+\ell)a(\frac{j+i_q}{n},\ell)\big| \\
&\leq 2\sum_{\ell\in\mathbb{Z}}\sup_{\tau\in(0,1]}|a(\tau,i_q-i_p+k-j+\ell)|\sup_{\tau\in(0,1]}|a(\tau,\ell)| \\
&\leq \rho(k-j),
\end{aligned}$$

with $\rho(k-j)=2\max_{-i_d\leq s\leq i_d}\Big\{\sum_{\ell\in\mathbb{Z}}\sup_{\tau\in(0,1]}|a(\tau,k-j+\ell)|\sup_{\tau\in(0,1]}|a(\tau,s+\ell)|\Big\}$. But Condition (4.2) implies, with $k>0$ such that $k-i_d>0$:

$$|\rho(k)| \leq 2K^2\max_{-i_d\leq s\leq i_d}\Big\{\sum_{\ell\in\mathbb{Z}}\frac{1}{u_\ell\,u_{\ell+k-s}}\Big\}\leq\begin{cases} K^2C(\alpha)\,k^{2\alpha-1} & \text{if } 0<\alpha<1/2\\ 4K^2\,\log(k)\,k^{2\alpha-1} & \text{if } \alpha=0\\ K^2C(\alpha)\,k^{\alpha-1} & \text{if } \alpha<0\end{cases}$$

Therefore, from the condition $m(1-2\alpha)>1$, one deduces $\sum_{k\in\mathbb{Z}}|\rho(k)|^m<\infty$ and the central limit theorem (4.5) holds. ☐

Note that the condition $m(1-2\alpha)>1$ was already obtained in case of stationary Gaussian long memory process in Taqqu (1975). The central limit theorem (4.5) can typically be applied to provide the asymptotic behavior of variance, covariance,..., of locally stationary long memory processes.

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