Forward equations for option prices in semimartingale models

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Abstract

We derive a forward partial integro-differential equation for prices of call options in a model where the dynamics of the underlying asset under the pricing measure is described by a -possibly discontinuous- semimartingale. This result generalizes Dupire's forward equation to a large class of non-Markovian models with jumps.

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Contents

Since the seminal work of Black, Scholes and Merton [7, 30] partial differential equations (PDE) have been used as a way of characterizing and efficiently computing option prices. In the Black-Scholes-Merton model and various extensions of this model which retain the Markov property of the risk factors, option prices can be characterized in terms of solutions to a backward PDE, whose variables are time (to maturity) and the value of the underlying asset. The use of backward PDEs for option pricing has been extended to cover options with path-dependent and early exercise features, as well as to multifactor models (see e.g. [1]). When the underlying asset exhibit jumps, option prices can be computed by solving an analogous partial integro-differential equation (PIDE) [2, 14].

A second important step was taken by Dupire [15, 16, 18] who showed that when the underlying asset is assumed to follow a diffusion process

$$dS_t = S_t \sigma(t, S_t) dW_t$$

prices of call options (at a given date t_0) solve a *forward* PDE in the strike and maturity variables:

$$\frac{\partial C_{t_0}}{\partial T}(T,K) = -r(T)K\frac{\partial C_{t_0}}{\partial K}(T,K) + \frac{K^2\sigma(T,K)^2}{2}\frac{\partial^2 C_{t_0}}{\partial K^2}(T,K)$$

on $[t_0,\infty[\times]0,\infty[$ with the initial condition: $\forall K>0 \quad C_{t_0}(t_0,K) = (S_{t_0}-K)_+$. This forward equation allows to price call options with various strikes and maturities on the same underlying asset, by solving a *single* partial differential equation. Dupire's forward equation also provides useful insights into the *inverse problem* of calibrating diffusion models to observed call and put option prices [6].

Given the theoretical and computational usefulness of the forward equation, there have been various attempts to extend Dupire's forward equation to other types of options and processes, most notably to Markov processes with jumps [2, 10, 12, 26, 9]. Most of these constructions use the Markov property of the underlying process in a crucial way (see however [27]).

As noted by Dupire [17], the forward PDE holds in a more general context than the backward PDE: even if the (risk-neutral) dynamics of the underlying asset is not necessarily Markovian, but described by a continuous Brownian martingale

$$dS_t = S_t \delta_t dW_t$$

then call options still verify a forward PDE where the diffusion coefficient is given by the local (or effective) volatility function $\sigma(t,S)$ given by

$$\sigma(t,S) = \sqrt{E[\delta_t^2 | S_t = S]}$$

This method is linked to the "Markovian projection" problem: the construction of a Markov process which mimicks the marginal distributions of a martingale [5, 23, 29]. Such "mimicking processes" provide a method to extend the Dupire equation to non-Markovian settings.

We show in this work that the forward equation for call prices holds in a more general setting, where the dynamics of the underlying asset is described by a - possibly discontinuous - semimartingale. Our parametrization of the price dynamics is general, allows for stochastic volatility and does *not* assume jumps to be independent or driven by a Lévy process, although it includes these cases. Also, our derivation does not require ellipticity or non-degeneracy of the diffusion coefficient. The result is thus applicable to various stochastic volatility models with jumps, pure jump models and point process models used in equity and credit risk modeling.

Our result extends the forward equation from the original diffusion setting of Dupire [16] to various examples of non-Markovian and/or discontinuous processes and implies previous derivations of forward equations [2, 10, 9, 12, 16, 17, 26, 28] as special cases. Section 2 gives examples of forward PIDEs obtained in various settings: time-changed Lévy processes, local Lévy models and point processes used in portfolio default risk modeling. In the case where the underlying risk factor follows, an Itô process or a Markovian jump-diffusion driven by a Lévy process, we retrieve previously known forms of the forward equation. In this case, our approach gives a rigorous derivation of these results under precise assumptions in a unified framework. In some cases, such as index options (Sec. 2.5) or CDO expected tranche notionals (Sec. 2.6), our method leads to a new, more general form of the forward equation valid for a larger class of models than previously studied [3, 12, 34].

The forward equation for call options is a PIDE in one (spatial) dimension, regardless of the number of factor driving the underlying asset. It may thus be used as a method for reducing the dimension of the problem. The case of index options (Section 2.5) in a multivariate jump-diffusion model illustrates how the forward equation projects a high dimensional pricing problem into a one-dimensional state equation.

1 Forward PIDEs for call options

1.1 General formulation of the forward equation

Consider a (strictly positive) price process S whose dynamics under the pricing measure $\mathbb{P}$ is given by a stochastic volatility model with jumps:

$$S_T = S_0 + \int_0^T r(t) S_{t^-} dt + \int_0^T S_{t^-} \delta_t dW_t + \int_0^T \int_{-\infty}^{+\infty} S_{t^-}(e^y - 1)\tilde{M}(dt\ dy) \quad (1)$$

where $r(t) > 0$ represents a (deterministic) bounded discount rate, δ_t the (random) volatility process and M is an integer-valued random measure with compensator $\mu(dt\, dy; \omega) = m(t, dy, \omega)\, dt$, representing jumps in the log-price, and $\tilde{M} = M - \mu$ is the compensated random measure associated to M (see [13] for further background). Both the volatility δ_t and $m(t, dy)$, which represents the intensity of jumps of size y at time t, are allowed to be stochastic. In particular, we do *not* assume the jumps to be driven by a Lévy process or a process with independent increments.

We assume the following conditions:

Assumption 1 (Full support). *For every t,* $\mathrm{supp}(S_t) = [0, \infty[$.

Assumption 2 (Integrability condition).

$$\forall T > 0, \quad \mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \delta_t^2\, dt + \int_0^T dt \int_{\mathbb{R}} (e^y - 1)^2 m(t, dy)\right)\right] < \infty \qquad \text{(H)}$$

The value $C_{t_0}(T, K)$ at time t_0 of a call option with expiry $T > t_0$ and strike $K > 0$ is given by

$$C_{t_0}(T, K) = e^{-\int_{t_0}^T r(t)\,dt} E^{\mathbb{P}}[\max(S_T - K, 0)|\mathcal{F}_{t_0}] \qquad (2)$$

As argued in Section 1.2, under Assumption (H), the expectation in (2) is finite.

Our main result is the following:

Theorem 1 (Forward PIDE for call options). *Let ψ_t be the exponential double tail of the compensator $m(t, dy)$*

$$\psi_t(z) = \begin{cases} \int_{-\infty}^z dx\ e^x \int_{-\infty}^x m(t, du) & z < 0 \\ \int_z^{+\infty} dx\ e^x \int_x^{\infty} m(t, du) & z > 0 \end{cases} \qquad (3)$$

and define, for $t \in [t_0, \infty[, z > 0$,

$$\begin{cases} \sigma(t, z) & = \sqrt{\mathbb{E}\left[\delta_t^2 | S_{t^-} = z\right]}; \\ \chi_{t,y}(z) & = \mathbb{E}\left[\psi_t\left(z\right) | S_{t-} = y\right] \end{cases} \qquad (4)$$

Under assumption (H)*, the call option price $(T, K) \mapsto C_{t_0}(T, K)$, as a function of maturity and strike, is a solution (in the sense of distributions) of the partial integro-differential equation:*

$$\frac{\partial C_{t_0}}{\partial T} = -r(T)K\frac{\partial C_{t_0}}{\partial K} + \frac{K^2\sigma(T, K)^2}{2}\frac{\partial^2 C_{t_0}}{\partial K^2} + \int_0^{+\infty} y\frac{\partial^2 C_{t_0}}{\partial K^2}(T, dy)\,\chi_{T,y}\left(\ln\left(\frac{K}{y}\right)\right) \qquad (5)$$

on $[t_0, \infty[\times]0, \infty[$ with the initial condition: $\forall K > 0 \quad C_{t_0}(t_0, K) = (S_{t_0} - K)_+$.

Remark 1. *Recall that $f : [t_0, \infty[\times]0, \infty[\mapsto \mathbb{R}$ is a solution of* (5) *in the sense of distributions if for any test function $\varphi \in C_0^\infty(]0, \infty[, \mathbb{R})$ and for any $T \geq t_0$,*

$$\int_0^\infty dK\varphi(K)\left[-\frac{\partial f}{\partial T} - r(T)K\frac{\partial f}{\partial K} + \frac{K^2\sigma(T, K)^2}{2}\frac{\partial^2 f}{\partial K^2} + \int_0^{+\infty} y\frac{\partial^2 f}{\partial K^2}(T, dy)\,\chi_{T,y}\left(\ln\left(\frac{K}{y}\right)\right)\right] = 0$$

where $C_0^\infty(]0, \infty[, \mathbb{R})$ is the set of infinitely differentiable functions with compact support in $]0, \infty[$. This notion of generalized solution allows to separate the discussion of existence of solutions from the discussion of their regularity (which may be delicate, see [14]).

Remark 2. *The discounted asset price*

$$\hat{S}_T = e^{-\int_0^T r(t)dt} S_T,$$

is the stochastic exponential of the martingale U defined by

$$U_T = \int_0^T \delta_t \, dW_t + \int_0^T \int (e^y - 1)\tilde{M}(dt\,dy).$$

Under assumption (H)*, we have*

$$\forall T > 0, \quad \mathbb{E}\left[\exp\left(\frac{1}{2}\langle U, U\rangle_T^d + \langle U, U\rangle_T^c\right)\right] < \infty$$

and [32, Theorem 9] implies that $(\hat{S}_T)$ is a $\mathbb{P}$-martingale.

The form of the integral term in (5) may seem different from the integral term appearing in backward PIDEs [14, 25]. The following lemma expresses $\chi_{T,y}(z)$ in a more familiar formin terms of call payoffs:

Lemma 1. *Let $n(t, dz, y, \omega)\, dt$ be a random measure on $[0, T] \times \mathbb{R} \times \mathbb{R}^+$ verifying*

$$\forall t \in [0, T], \quad \int_{-\infty}^{\infty} (e^z \wedge |z|^2) n(t, dz, y, \omega) < \infty \qquad \text{a.s.}$$

Then the exponential double tail $\chi_{t,y}(z)$ of n, defined as

$$\chi_{t,y}(z) = \begin{cases} \int_{-\infty}^{z} dx\ e^x \int_{-\infty}^{x} n(t, du, y) & z < 0 \\ \int_{z}^{+\infty} dx\ e^x \int_{x}^{\infty} n(t, du, y) & z > 0 \end{cases} \tag{6}$$

verifies

$$\int_{\mathbb{R}} [(ye^z - K)^+ - e^z(y - K)^+ - K(e^z - 1)1_{\{y>K\}}] n(t, dz, y) = y\, \chi_{t,y}\left(\ln\left(\frac{K}{y}\right)\right)$$

Proof. Let $K, T > 0$. Then:

$$\begin{aligned} & \int_{\mathbb{R}} [(ye^z - K)^+ - e^z(y - K)^+ - K(e^z - 1)1_{\{y>K\}}] n(t, dz, y) \\ = \quad & \int_{\mathbb{R}} [(ye^z - K)1_{\{z>\ln(\frac{K}{y})\}} - e^z(y - K)1_{\{y>K\}} - K(e^z - 1)1_{\{y>K\}}] n(t, dz, y) \\ = \quad & \int_{\mathbb{R}} [(ye^z - K)1_{\{z>\ln(\frac{K}{y})\}} + (K - ye^z)1_{\{y>K\}}] n(t, dz, y). \end{aligned}$$

- If $K \geq y$, then

$$\begin{aligned} & \int_{\mathbb{R}} 1_{\{K\geq y\}}[(ye^z - K)1_{\{z>\ln(\frac{K}{y})\}} + (K - ye^z)1_{\{y>K\}}] n(t, dz, y) \\ = \quad & \int_{\ln(\frac{K}{y})}^{+\infty} y(e^z - e^{\ln(\frac{K}{y})})\, n(t, dz, y). \end{aligned}$$

- If $K < y$, then

$$
\begin{aligned}
&\int_{\mathbb{R}} 1_{\{K<y\}}[(ye^z - K)1_{\{z>\ln(\frac{K}{y})\}} + (K - ye^z)1_{\{y>K\}}]n(t,dz,y)\\
= & \int_{\ln(\frac{K}{y})}^{+\infty} [(ye^z - K) + (K - ye^z)]n(t,dz,y) + \int_{-\infty}^{\ln(\frac{K}{y})} [K - ye^z]n(t,dz,y)\\
= & \int_{-\infty}^{\ln(\frac{K}{y})} y(e^{\ln(\frac{K}{y})} - e^z)n(t,dz,y).
\end{aligned}
$$

Using integration by parts, $\chi_{t,y}$ can be equivalently expressed as

$$
\chi_{t,y}(z) = \begin{cases} \int_{-\infty}^{z}(e^z - e^u)\, n(t,du,y) & z<0 \\ \int_{z}^{\infty}(e^u - e^z)\, n(t,du,y) & z>0 \end{cases}
$$

Hence:

$$
\int_{\mathbb{R}} [(ye^z - K)^+ - e^z(y-K)^+ - K(e^z-1)1_{\{y>K\}}]n(t,dz,y) = y\,\chi_{t,y}\left(\ln\left(\frac{K}{y}\right)\right).
$$

□

1.2 Derivation of the forward equation

In this section we present a proof of Theorem 1 using the Tanaka-Meyer formula for semimartingales [24, Theorem 9.43] under assumption (H).

Proof. We first note that, by replacing $\mathbb{P}$ by the conditional measure $\mathbb{P}_{|\mathcal{F}_{t_0}}$ given $\mathcal{F}_{t_0}$, we may replace the conditional expectation in (2) by an expectation with respect to the marginal distribution $p_T^S(dy)$ of S_T under $\mathbb{P}_{|\mathcal{F}_{t_0}}$. Thus, without loss of generality, we set $t_0 = 0$ in the sequel and consider the case where $\mathcal{F}_0$ is the σ-algebra generated by all $\mathbb{P}$-null sets and we denote $C_0(T,K) \equiv C(T,K)$ for simplicity. (2) can be expressed as

$$
C(T,K) = e^{-\int_0^T r(t)\,dt} \int_{\mathbb{R}^+} (y-K)^+ \; p_T^S(dy). \tag{7}
$$

By differentiating with respect to K, we obtain:

$$
\begin{aligned}
&\frac{\partial C}{\partial K}(T,K) = -e^{-\int_0^T r(t)\,dt} \int_K^\infty p_T^S(dy) = -e^{-\int_0^T r(t)\,dt}\mathbb{E}\left[1_{\{S_T>K\}}\right],\\
&\frac{\partial^2 C}{\partial K^2}(T,dy) = e^{-\int_0^T r(t)\,dt} p_T^S(dy).
\end{aligned} \tag{8}
$$

Let $L_t^K = L_t^K(S)$ be the semimartingale local time of S at K under $\mathbb{P}$ (see [24, Chapter 9] or [33, Ch. IV] for definitions). For $h > 0$, applying the Tanaka-

Meyer formula to $(S_t - K)^+$ between T and $T+h$, we have

$$\begin{aligned}(S_{T+h} - K)^+ = (S_T - K)^+ + \int_T^{T+h} 1_{\{S_{t-}>K\}} dS_t + \frac{1}{2}(L^K_{T+h} - L^K_T) \\ + \sum_{T<t\leq T+h} (S_t - K)^+ - (S_{t-} - K)^+ - 1_{\{S_{t-}>K\}} \Delta S_t.\end{aligned} \tag{9}$$

As noted in Remark 2, the integrability condition (H) implies that the discounted price $\hat{S}_t = e^{-\int_0^t r(s)\,ds} S_t = \mathcal{E}(U)_t$ is a martingale under $\mathbb{P}$. So (1) can be expressed as $dS_t = r(t)S_{t-}dt + d\hat{S}_t$ and

$$\int_T^{T+h} 1_{\{S_{t-}>K\}} dS_t = \int_T^{T+h} 1_{\{S_{t-}>K\}} d\hat{S}_t + \int_T^{T+h} r(t) S_{t-} 1_{\{S_{t-}>K\}} dt$$

where the first term is a martingale. Taking expectations, we get:

$$\begin{aligned}& e^{\int_0^{T+h} r(t)\,dt} C(T+h, K) - e^{\int_0^T r(t)\,dt} C(T, K) \\ = \; & \mathbb{E}\left[\int_T^{T+h} r(t) S_t\, 1_{\{S_{t-}>K\}} dt + \frac{1}{2}(L^K_{T+h} - L^K_T)\right] \\ + \; & \mathbb{E}\left[\sum_{T<t\leq T+h} (S_t - K)^+ - (S_{t-} - K)^+ - 1_{\{S_{t-}>K\}} \Delta S_t\right].\end{aligned}$$

Noting that $S_{t-} 1_{\{S_{t-}>K\}} = (S_{t-} - K)^+ + K 1_{\{S_{t-}>K\}}$, we obtain

$$\mathbb{E}\left[\int_T^{T+h} r(t) S_{t-} 1_{\{S_{t-}>K\}} dt\right] = \int_T^{T+h} r(t) e^{\int_0^t r(s)\,ds} \left[C(t,K) - K\frac{\partial C}{\partial K}(t,K)\right] dt$$

using Fubini's theorem and (8). As for the jump term,

$$\begin{aligned}& \mathbb{E}\left[\sum_{T<t\leq T+h} (S_t - K)^+ - (S_{t-} - K)^+ - 1_{\{S_{t-}>K\}} \Delta S_t\right] \\ = \; & \mathbb{E}\left[\int_T^{T+h} dt \int m(t,dx)\, (S_{t-}e^x - K)^+ - (S_{t-} - K)^+ - 1_{\{S_{t-}>K\}} S_{t-}(e^x - 1)\right] \\ = \; & \mathbb{E}\Big[\int_T^{T+h} dt \int m(t,dx) \big((S_{t-}e^x - K)^+ - (S_{t-} - K)^+ \\ & \quad - (S_{t-} - K)^+(e^x - 1) - K 1_{\{S_{t-}>K\}}(e^x - 1)\big)\Big] \\ = \; & \int_T^{T+h} dt \mathbb{E}\Big[\int m(t,dx) \big((S_{t-}e^x - K)^+ \\ & \quad - e^x (S_{t-} - K)^+ - K 1_{\{S_{t-}>K\}}(e^x - 1)\big)\Big]\end{aligned}$$

Applying Lemma 1 to the random measure m we obtain:

$$\int m(t,dx)\big((S_{t-}e^x - K)^+ - e^x(S_{t-}-K)^+ - K1_{\{S_{t-}>K\}}(e^x-1)\big) = S_{t-}\,\psi_{t,S_{t-}}\left(\ln\left(\frac{K}{S_{t-}}\right)\right)$$

holds, leading to:

$$\begin{aligned}
&\mathbb{E}\left[\sum_{T<t\leq T+h}(S_t-K)^+ - (S_{t-}-K)^+ - 1_{\{S_{t-}>K\}}\Delta S_t\right]\\
=\ & \int_T^{T+h} dt\,\mathbb{E}\left[S_{t-}\,\psi_{t,S_{t-}}\left(\ln\left(\frac{K}{S_{t-}}\right)\right)\right]\\
=\ & \int_T^{T+h} dt\,\mathbb{E}\left[S_{t-}\mathbb{E}\left[\psi_{t,S_{t-}}\left(\ln\left(\frac{K}{S_{t-}}\right)\right)|S_{t-}\right]\right]\\
=\ & \int_T^{T+h} dt\,\mathbb{E}\left[S_{t-}\chi_{t,S_{t-}}\left(\ln\left(\frac{K}{S_{t-}}\right)\right)\right] \qquad (10)
\end{aligned}$$

Let $\varphi \in C_0^\infty(]0,\infty[)$ be an infinitely differentiable function with compact support. The occupation time formula (see [24, Theorem 9.46]) yields:

$$\int_0^{+\infty} dK\,\varphi(K)(L_{T+h}^K - L_T^K) = \int_T^{T+h}\varphi(S_{t-})d[S]_t^c = \int_T^{T+h} dt\,\varphi(S_{t-})S_{t-}^2\delta_t^2$$

$(\hat{S}_T)$ is a martingale, hence : $\mathbb{E}[S_T] < \infty$. Since $(S_{T+h}-K)^+ < S_{T+h}$, $(S_T-K)^+ < S_T$, $|\sum_{T<t\leq T+h}(S_t-K)^+ - (S_{t-}-K)^+ - 1_{\{S_{t-}>K\}}\Delta S_t| < 3S_t$ and $\mathbb{E}\left[\int_T^{T+h}1_{\{S_{t-}>K\}}dS_t\right] < \infty$ then (65) leads to : $\mathbb{E}\left[L_{T+h}^K - L_T^K\right] < \infty$; furthermore, since φ is bounded and has compact support, one may take expectations on both sides and apply Fubini's theorem to obtain:

$$\begin{aligned}
\mathbb{E}\left[\int_0^\infty dK\varphi(K)(L_{T+h}^K - L_T^K)\right] &= \mathbb{E}\left[\int_T^{T+h}\varphi(S_{t-})S_{t-}^2\delta_t^2 dt\right]\\
&= \int_T^{T+h} dt\,\mathbb{E}\left[\varphi(S_{t-})S_{t-}^2\delta_t^2\right]\\
&= \int_T^{T+h} dt\,\mathbb{E}\left[\mathbb{E}\left[\varphi(S_{t-})S_{t-}^2\delta_t^2|S_{t-}\right]\right]\\
&= \mathbb{E}\left[\int_T^{T+h} dt\,\varphi(S_{t-})S_{t-}^2\sigma(t,S_{t-})^2\right]\\
&= \int_0^\infty\int_T^{T+h}\varphi(K)K^2\sigma(t,K)^2 p_t^S(dK)\,dt\\
&= \int_T^{T+h} dt\,e^{\int_0^t r(s)\,ds}\int_0^\infty\varphi(K)K^2\sigma(t,K)^2\frac{\partial^2 C}{\partial K^2}(t,dK)
\end{aligned}$$

where the last line is obtained by using (8). Gathering together all the terms, we obtain:

$$\begin{aligned}
&\int_0^\infty dK\,\varphi(K)\left[e^{\int_0^{T+h} r(t)\,dt} C(T+h,K) - e^{\int_0^T r(t)\,dt} C(T,K)\right] \\
= &\int_T^{T+h} dt\, r(t)\, e^{\int_0^t r(s)\,ds} \int_0^\infty dK\,\varphi(K)[C(t,K) - K\frac{\partial C}{\partial K}(t,K)] \\
+ &\int_T^{T+h} dt\, e^{\int_0^t r(s)\,ds} \int_0^\infty \frac{\varphi(K)}{2} K^2 \sigma(t,K)^2 \frac{\partial^2 C}{\partial K^2}(t,dK) \\
+ &\int_T^{T+h} dt \int_0^\infty dK\,\varphi(K)\,\mathbb{E}\left[S_{t-}\ \chi_{t,S_{t-}}\left(\ln\left(\frac{K}{S_{t-}}\right)\right)\right] \qquad (11)
\end{aligned}$$

Dividing by h and taking the limit $h \to 0$ yields:

$$\begin{aligned}
&\int_0^\infty dK \varphi(K) e^{\int_0^T r(t)\,dt}\left[\frac{\partial C}{\partial T}(T,K) + r(T)C(T,K)\right] \\
= &\int_0^\infty dK \varphi(K) e^{\int_0^T r(t)\,dt}\left[r(T)C(T,K) - r(T)K\frac{\partial C}{\partial K}(T,K)\right] \\
+ &\int_0^\infty \frac{\varphi(K)}{2} K^2 \sigma(t,K)^2 e^{\int_0^T r(t)\,dt} \frac{\partial^2 C}{\partial K^2}(T,dK) \\
+ &\int_0^\infty dK\,\varphi(K)\mathbb{E}\left[S_{t-}\ \chi_{T,S_{T-}}\left(\ln\left(\frac{K}{S_{T-}}\right)\right)\right] \\
= &\int_0^\infty dK \varphi(K) e^{\int_0^T r(t)\,dt}\left[r(T)C(T,K) - r(T)K\frac{\partial C}{\partial K}(T,K)\right] \\
+ &\int_0^\infty \frac{\varphi(K)}{2} K^2 \sigma(t,K)^2 e^{\int_0^T r(t)\,dt} \frac{\partial^2 C}{\partial K^2}(T,dK) \\
+ &\int_0^\infty dK\,\varphi(K) e^{\int_0^T r(t)\,dt} \int_0^{+\infty} y \frac{\partial^2 C}{\partial K^2}(T,dy) \chi_{T,y}\left(\ln\left(\frac{K}{y}\right)\right) \qquad (12)
\end{aligned}$$

Since this equality holds for any $\varphi \in C_0^\infty(]0,\infty[,\mathbb{R})$, $C(.,.)$ is a solution of

$$\begin{aligned}
\frac{\partial C}{\partial T}(T,K) \quad = \quad &\frac{K^2\sigma(t,K)^2}{2}\frac{\partial^2 C}{\partial K^2}(T,K) - r(T)K\frac{\partial C}{\partial K}(T,K) \\
+ &\int_0^{+\infty} y \frac{\partial^2 C}{\partial K^2}(T,dy)\chi_{T,y}\left(\ln\left(\frac{K}{y}\right)\right)
\end{aligned}$$

in the sense of distributions on $[0,T]\times]0,\infty[$. □

1.3 Uniqueness of solutions of the forward PIDE

Theorem 1 shows that the call price $(T,K) \mapsto C_{t_0}(T,K)$ solves the forward PIDE (5). Uniqueness of the solution of such PIDEs has has been shown using analytical methods [4, 21] under various types of conditions on the coefficients . We give below a direct proof of uniqueness for (5) using a probabilistic method, under explicit conditions which cover most examples of models used in finance.

Define, for $u \in \mathbb{R}, t \in [0,T[, z > 0$ the measure $n(t,du,z)$ by

$$\begin{aligned} n(t,[u,\infty[,z) &= -e^{-u}\frac{\partial}{\partial u}\left[\chi_{t,z}(u)\right] \quad u>0 \\ n(t,]-\infty,u],z) &= e^{-u}\frac{\partial}{\partial u}\left[\chi_{t,z}(u)\right] \quad u<0 \end{aligned} \tag{13}$$

Throughout this section, we make the following assumption: and

Assumption 3.

$$\forall T>0, \forall B\in\mathcal{B}(\mathbb{R})-\{0\}, \quad (t,z)\to\sigma(t,z), \quad (t,z)\to n(t,B,z)$$

are continuous in $z \in \mathbb{R}^+$, uniformly in $t \in [0,T]$ and

$$\exists K_T>0, \forall (t,z)\in[0,T]\times\mathbb{R}^+,\ |\sigma(t,z)| + \int_{\mathbb{R}} (1\wedge |z|^2)\, n(t,du,z) \le K_T \quad (H')$$

Note that (H') implies our previous assumption (H).

Theorem 2. *Under Assumption 3, if*

$$\begin{aligned}
&\text{either} \quad (i) \quad \forall R>0\ \forall t\in[0,T[, \quad \inf_{\{0\le z\le R\}} \sigma(t,z) > 0 \\
&\text{or} \quad (ii) \quad \sigma(t,z)\equiv 0 \quad \text{and } \exists\beta\in]0,2[, \exists C>0, \forall R>0, \forall (t,z)\in[0,T[\times[0,R], \\
&\qquad \forall f\in C_0^0(\mathbb{R}-\{0\},\mathbb{R}_+), \qquad \int \left(n(t,du,z) - \frac{C\,du}{|u|^{1+\beta}}\right) f(u) \ge 0 \\
&\qquad \exists K'_{T,R}>0, \int_{\{|u|\le 1\}} |u|^{\beta} \left(n(t,du,z) - \frac{C\,du}{|u|^{1+\beta}}\right) dt \le K'_{T,R} \\
&\text{and} \quad (iii) \quad \lim_{R\to\infty} \int_0^T \sup_{z\in\mathbb{R}^+} n\left(t,\{|u|\ge R\},z\right) dt = 0
\end{aligned}$$

then the call option price $(T,K) \mapsto C_{t_0}(T,K)$, as a function of maturity and strike, is the unique solution (in the sense of distributions) of the partial integro-differential equation (5) on $[t_0,\infty[\times]0,\infty[$ with the initial condition:
$\forall K>0 \quad C_{t_0}(t_0,K) = (S_{t_0}-K)_+$.

The proof uses the uniqueness of the solution of the forward Kolmogorov equation associated to a certain integro-differential operator. We start with the following result, which has some independent interest:

Proposition 1. *Define for $t \in [0,T]$ and $f \in C_0^\infty(\mathbb{R})$, the integro-differential operator L_t given by*

$$\begin{aligned} L_t f(x) &= r(t)xf'(x) + \frac{x^2\sigma(t,x)^2}{2} f''(x) \\ &\quad + \int_{\mathbb{R}} \left[f(t,xe^y) - f(t,x) - x(e^y-1).f'(x)\right] n(t,dy,x) \end{aligned} \tag{14}$$

Under Assumption 3, if either conditions (i) or (ii) and (iii) of Theorem 2 hold, then for each x_0 in $\mathbb{R}^+$, there exists a unique family $(p_t(x_0,dy), t\geq 0)$ of bounded measures such that

$$\forall g\in\mathcal{C}_0^\infty(]0,\infty[,\mathbb{R}),\qquad \int g(y)\frac{dp}{dt}(x_0,dy)=\int p_t(x_0,dy)L_t g(y)\qquad p_0(x_0,.)=\epsilon_{x_0} \tag{15}$$

where ϵ_{x_0} is the point mass at x_0. Furthermore, $p_t(x_0,.)$ is a probability measure on $[0,\infty[$.

Proof. Denote by $(X_t)_{t\in[0,T]}$ the canonical process on $D([0,T],\mathbb{R}_+)$. Under assumptions (i) (or (ii)) and (iii), L_t verifies Assumptions 1–4 in [31] and by [31, Theorem 1], the martingale problem for $(L_t)_{t\in[0,T]}$ is well-posed: for any $x_0\in\mathbb{R}$, $s\in[0,T[$, there exists a unique probability measure $\mathbb{Q}_{s,x_0}$ on $D([0,T],\mathbb{R}_+)$ such that $\mathbb{Q}_{s,x_0}(X_s=x_0)=1$ and for $f\in\mathcal{C}_0^\infty(\mathbb{R}^+)$:

$$f(X_t)-f(x_0)-\int_s^t L_u f(X_u)\,du$$

is a $\mathbb{Q}_{s,x_0}$-martingale. Furthermore, (X_t) is a Markov process under $\mathbb{Q}_{x_0}$, and $(P_t)_{t\in[0,T]}$ defined by

$$\forall f\in\mathcal{C}_b^0(\mathbb{R}^+)\quad P_t f(x_0)=\mathbb{E}^{\mathbb{Q}_{x_0}}\left[f(X_t)\right] \tag{16}$$

is a (non-homogeneous) positive strongly continuous contraction semigroup on $\mathcal{C}_b^0(\mathbb{R}^+)$ [19, Chapter 1].
If $p_t(x_0,dy)$ denotes the law of (X_t) starting from x_0 under $\mathbb{Q}$, the martingale property shows that $p_t(x_0,dy)$ satisfies the equation (15) that we simply rewrites after integration with respect to time t:

$$\int p_t(x_0,dy)g(y)=g(x_0)+\int_0^t\int p_s(x_0,dy)L_s g(y)\,ds \tag{17}$$

This solution of (15) is in particular positive with mass 1.
To show uniqueness, let $f\in C_0^\infty(\mathbb{R}^+)$ and $\gamma\in\mathcal{C}^1([0,T])$ and consider the non-time dependent operator A mapping functions of the form $(t,x)\in[0,T]\times\mathbb{R}\to f(x)\gamma(t)$, which will be denoted $\mathcal{C}_0^\infty(\mathbb{R}^+)\otimes\mathcal{C}^1([0,T])$, into :

$$A(f\gamma)(t,x)=\gamma(t)L_t f(x)+f(x)\gamma'(t) \tag{18}$$

Using [19, Theorem 7.1 and Theorem 10.1,Chapter 4]), uniqueness holds for the martingale problem associated to the operator L on $C_0^\infty(\mathbb{R}^+)$ if and only if uniqueness holds for the martingale problem associated to the A on $C_0^\infty(\mathbb{R}^+)\otimes\mathcal{C}^1([0,T])$. For any x_0 in $\mathbb{R}^+$, if $(X,\mathbb{Q}_{x_0})$ is a solution of the martingale problem L, then the law of $\eta_t=(t,X_t)$ is a solution of the martingale problem for A: for any $f\in\mathcal{C}_0^\infty(\mathbb{R}^+)$ and $\gamma\in\mathcal{C}([0,T])$:

$$\int p_t(x_0,dy)f(y)\gamma(t)=f(x_0)\gamma(0)+\int_0^t\int p_s(x_0,dy)A(f\gamma)(s,y)\,ds \tag{19}$$

Assume there exists a measure $q_t(dy)$ such that $q_0(dy) = \epsilon_{x_0}(dy)$ solution of (17), then after integration by parts:

$$\int q_t(dy) f(y)\gamma(t) = f(x_0)\gamma(0) + \int_0^t \int q_s(dy) A(f\gamma)(s,y)\, ds \tag{20}$$

holds.
Define, for t in $[0,T]$, $g \in \mathcal{C}_0^\infty(\mathbb{R}^+) \otimes \mathcal{C}^1([0,T])$

$$\begin{aligned} P_t g(x_0) &= \int p_t(x_0, dy) g(t,y) \\ Q_t g &= \int q_t(dy) g(t,y) \end{aligned}$$

Given (19) and (20), for all $\epsilon > 0$:

$$\begin{aligned} P_t(f\gamma)(x_0) - P_\epsilon(f\gamma)(x_0) = \int_\epsilon^t \int p_u(x_0, dy) A(f\gamma)(u,y)\, du = \int_\epsilon^t P_u(A(f\gamma))(x_0)\, du \\ Q_t(f\gamma) - Q_\epsilon(f\gamma) = \int_\epsilon^t \int q_u(dy) A(f\gamma)(u,y)\, du = \int_\epsilon^t Q_u(A(f\gamma))\, du \end{aligned} \tag{21}$$

Since the functions $t \to \sigma(t,.)$, and $t \to n(t,B,.)$ for any $B \in \mathcal{B}(\mathbb{R}) - \{0\}$ are bounded in t on $[0,T]$, it implies that for any fixed $f \in \mathcal{C}_0^\infty(\mathbb{R}^+)$ and any fixed $\gamma \in \mathcal{C}^1([0,T])$, $t \to Q_t A(f\gamma)$ and $t \to P_t A(f\gamma)(x_0)$ are bounded on $[0,T]$ and shows that $Q_t.$ ans $P_t.(x_0)$ are weakly right-continuous in t on $\mathcal{C}_0^\infty(\mathbb{R}^+) \otimes \mathcal{C}^1([0,T])$, i.e, for $T \ge t' \ge t$:

$$\lim_{t' \to t} P_{t'}(f\gamma)(x_0) = P_t(f\gamma)(x_0) \qquad \lim_{t' \to t} Q_{t'}(f\gamma) = Q_t(f\gamma)$$

Fix $\lambda > 0$, we have

$$\begin{aligned} \lambda \int_0^\infty e^{-\lambda t} P_t(f\gamma)(x_0)\, dt &= f(x_0)\gamma(0) + \lambda \int_0^\infty e^{-\lambda t} \int_0^t P_s(A(f\gamma))(x_0)\, ds\, dt \\ &= f(x_0)\gamma(0) + \lambda \int_0^\infty e^{-\lambda t} \left(\int_s^\infty e^{-\lambda t} dt \right) P_s(A(f\gamma))(x_0)\, ds \\ &= f(x_0)\gamma(0) + \int_0^\infty e^{-\lambda s} P_s(A(f\gamma))(x_0)\, ds \end{aligned}$$

Consequently,

$$\int_0^\infty e^{-\lambda t} P_t(\lambda - A)(f\gamma)(x_0)\, dt = f(x_0)\gamma(0) = \int_0^\infty e^{-\lambda t} Q_t(\lambda - A)(f\gamma)\, dt \tag{22}$$

Since P_t is a positive strongly continuous contraction semigroup on $\mathcal{C}_b^0(\mathbb{R}^+)$ for the operator L on $\mathcal{C}_0^\infty(\mathbb{R}^+)$, one can easily show that it holds for the operator A on the domain $\mathcal{C}_0^\infty(\mathbb{R}^+) \otimes \mathcal{C}^1([0,T])$. Applying the Hille-Yosida theory (see [19, Proposition 2.1 and Theorem 2.6]), for all $\lambda > 0$, $\mathcal{R}(\lambda - A) = \mathcal{C}_b^0(\mathbb{R}^+ \times [0,T])$, where $\mathcal{R}(\lambda - A)$ denotes the image of $\mathcal{C}_0^\infty(\mathbb{R}^+) \otimes \mathcal{C}^1([0,T])$ by the mapping $g \to (\lambda - A)g$. Hence, since (22) holds then for all h in $\mathcal{C}_b^0(\mathbb{R}^+ \times [0,T])$:

$$\int_0^\infty e^{-\lambda t}\, P_t h\,(x_0)\, dt = \int_0^\infty e^{-\lambda t}\, Q_t h\, dt \tag{23}$$

Since $\mathcal{C}_b^0(\mathbb{R}^+ \times [0,T])$ is separating (see [19, Proposition 4.4, Chapter 3]), $P_t.(x_0)$ and $Q_t.$ are weakly right-continuous and (22) holds for any $\lambda > 0$, the flows $q_t(dy)$ and $p_t(x_0, dy)$ are the same on $\mathcal{C}_b^0(\mathbb{R}^+ \times [0,T])$ and obviously on $\mathcal{C}_b^0(\mathbb{R}^+)$. This ends the proof. □

We can now study the uniqueness of the forward $PIDE$ (5) and prove Theorem 2

Proof. of Theorem 2.
If one decomposes L_t into a differential and an integral component:

$$\begin{aligned} L_t &= A_t + B_t \\ A_t f(y) &= r(t) y f'(y) + \frac{y^2 \sigma(t,y)^2}{2} f''(y) \\ B_t f(y) &= \int_{\mathbb{R}} [f(ye^z) - f(y) - y(e^z - 1) f'(y)] n(t, dz, y) \end{aligned}$$

then using the fact that $y \frac{\partial}{\partial y}(y-x)^+ = x 1_{\{y>x\}} + (y-x)_+ = y\, 1_{\{y>x\}}$ and $\frac{\partial^2}{\partial y^2}(y-x)^+ = \epsilon_x(y)$ where ϵ_x is a unit mass at x, we obtain

$$A_t (y-x)^+ = r(t) y\, 1_{\{y>x\}} + \frac{y^2 \sigma(t,y)^2}{2} \epsilon_x(y)$$

and

$$\begin{aligned} B_T (y-x)^+ &= \int_{\mathbb{R}} [(ye^z - x)^+ - (y-x)^+ - (e^z - 1)\left(x 1_{\{y>x\}} + (y-x)^+\right)] n(t, dz, y) \\ &= \int_{\mathbb{R}} [(ye^z - x)^+ - e^z (y-x)^+ - x(e^z - 1) 1_{\{y>x\}}] n(t, dz, y) \end{aligned}$$

Then, using Lemma 1 for the random measure $n(t, dz, y)$ and $\psi_{t,y}$ its exponential double tail:

$$B_t (y-x)^+ = y \psi_{t,y} \left(\ln \left(\frac{x}{y} \right) \right)$$

Hence, the following identity holds:

$$L_t (y-x)^+ = r(t) \left(x 1_{\{y>x\}} + (y-x)_+ \right) + \frac{y^2 \sigma(t,y)^2}{2} \epsilon_x(y) + y \psi_{t,y} \left(\ln \left(\frac{x}{y} \right) \right) \tag{24}$$

Let $f : [t_0, \infty[\times]0, \infty[\mapsto \mathbb{R}$ be a solution in the sense of distributions of (5) with the initial condition : $f(0, x) = (S_0 - x)^+$. Integration by parts yields

$$\begin{aligned}
& \int_0^\infty \frac{\partial^2 f}{\partial x^2}(t, dy) L_t (y-x)^+ \\
= \quad & \int_0^\infty \frac{\partial^2 f}{\partial x^2}(t, dy) \left(r(t)(x 1_{\{y>x\}} + (y-x)_+) + \frac{y^2 \sigma(t,y)^2}{2} \epsilon_x(y) + y \psi_{t,y} \left(\ln \left(\frac{x}{y} \right) \right) \right) \\
= \quad & -r(t) x \int_0^\infty \frac{\partial^2 f}{\partial x^2}(t, dy) 1_{\{y>x\}} + r(t) \int_0^\infty \frac{\partial^2 f}{\partial x^2}(t, dy)(y-x)^+ \\
+ \quad & \frac{x^2 \sigma(t,x)^2}{2} \frac{\partial^2 f}{\partial x^2} + \int_0^\infty \frac{\partial^2 f}{\partial x^2}(t, dy) y \psi_{t,y} \left(\ln \left(\frac{x}{y} \right) \right) \\
= \quad & -r(t) x \frac{\partial f}{\partial x} + r(t) f(t,x) + \frac{x^2 \sigma(t,x)^2}{2} \frac{\partial^2 f}{\partial x^2} + \int_0^\infty \frac{\partial^2 f}{\partial x^2}(t, dy)\, y\, \psi_{t,y} \left(\ln \left(\frac{x}{y} \right) \right)
\end{aligned}$$

Hence given (5), the following identity holds:

$$\frac{\partial f}{\partial t}(t,x) = -r(t) f(t,x) + \int_0^\infty \frac{\partial^2 f}{\partial x^2}(t, dy) L_t (y-x)^+ \tag{25}$$

or equivalently after integration with respect to time t:

$$e^{\int_0^t r(s)\,ds} f(t,x) - f(0,x) = \int_0^\infty e^{\int_0^t r(s)\,ds} \frac{\partial^2 f}{\partial x^2}(t, dy) L_t (y-x)^+ \tag{26}$$

After integration by parts, one shows that:

$$f(t,x) = \int_0^\infty \frac{\partial^2 f}{\partial x^2}(t, dy)(y-x)^+ \tag{27}$$

Hence (25) rewrites:

$$\int_0^\infty e^{\int_0^t r(s)\,ds} \frac{\partial^2 f}{\partial x^2}(t, dy)(y-x)^+ - (S_0 - x)^+ = \int_0^t \int_0^\infty e^{\int_0^s r(u)\,du} \frac{\partial^2 f}{\partial x^2}(s, dy) L_s (y-x)^+ \, ds \tag{28}$$

Define $q_t(dy) \equiv e^{\int_0^t r(s)\,ds} \frac{\partial^2 f}{\partial x^2}(t, dy)$, we have $q_0(dy) = \epsilon_{S_0}(dy) = p_0(S_0, dy)$. Take g in $\mathcal{C}_0^\infty(]0, \infty[, \mathbb{R})$, after integration by parts, one shows that:

$$g(y) = \int_0^\infty g''(z)(y-z)^+ \, dz \tag{29}$$

Replacing in $\int_{\mathbb{R}} g(y)q_t(dy)$ and using (28)

$$\begin{aligned}
\int_0^\infty g(y)q_t(dy) &\equiv \int_0^\infty g(y)\, e^{\int_0^t r(s)\,ds}\, \frac{\partial^2 f}{\partial x^2}(t,dy) \\
&= \int_0^\infty g''(z) \int_0^\infty e^{\int_0^t r(s)\,ds}\, \frac{\partial^2 f}{\partial x^2}(t,dy)(y-z)^+\,dz \\
&= \int_0^\infty g''(z)(S_0-z)^+\,dz + \int_0^\infty g''(z) \int_0^t \int_0^\infty e^{\int_0^s r(u)\,du} \frac{\partial^2 f}{\partial x^2}(s,dy) L_s (y-z)^+\,dz \\
&= g(S_0) + \int_0^t \int_0^\infty e^{\int_0^s r(u)\,du} \frac{\partial^2 f}{\partial x^2}(s,dy) L_s [\int_0^\infty g''(z)(y-z)^+\,dz] \\
&= g(S_0) + \int_0^t \int_0^\infty q_s(dy) L_s g(y)\,ds
\end{aligned}$$

This is the equation (17). Proposition 1 yields to $e^{\int_0^t r(s)\,ds}\,\frac{\partial^2 f}{\partial x^2}(t,dy)$ is unique that is $e^{\int_0^t r(s)\,ds}\,\frac{\partial^2 f}{\partial x^2}(t,dy) \equiv p_t(S_0,dy)$ (with the notations in Proposition 1)and leads to the uniqueness of the solution of the forward PIDE (5). □

2 Examples

We now give various examples of pricing models for which Theorem 1 allows to retrieve or generalize previously known forms of forward pricing equations.

2.1 Itô processes

When (S_t) is an Itô process i.e. when the jump part is absent, the forward equation (5) reduces to the Dupire equation [16]. In this case our result reduces to the following:

Proposition 2 (Dupire PDE). *Consider the price process* (S_t) *whose dynamics under the pricing measure* $\mathbb{P}$ *is given by:*

$$S_T = S_0 + \int_0^T r(t) S_t dt + \int_0^T S_t \delta_t dW_t \tag{30}$$

Define

$$\sigma(t,z) = \sqrt{\mathbb{E}\left[\delta_t^2 | S_t = z\right]}$$

If

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \delta_t^2\,dt\right)\right] < \infty \qquad a.s. \quad (A_{1a})$$

the call option price (2) *is a solution (in the sense of distributions) of the partial differential equation:*

$$\frac{\partial C_{t_0}}{\partial T} = -r(T) K \frac{\partial C_{t_0}}{\partial K} + \frac{K^2 \sigma(T,K)^2}{2}\,\frac{\partial^2 C_{t_0}}{\partial K^2} \tag{31}$$

on $[t_0,\infty[\times]0,\infty[$ *with the initial condition:* $\forall K>0 \quad C_{t_0}(t_0,K)=(S_{t_0}-K)_+$.

Notice in particular that this result does not require a non-degeneracy condition on the diffusion term.

Proof. It is sufficient to take $\mu\equiv 0$ in (1) then equivalently in (5). We leave the end of the proof to the reader. □

2.2 Markovian jump-diffusion models

Another important particular case in the literature is the case of a Markov jump-diffusion driven by a Poisson random measure. Andersen and Andreasen [2] derived a forward PIDE in the situation where the jumps are driven by a compound Poisson process with time-homogeneous Gaussian jumps. We will now show here that Theorem 1 implies the PIDE derived in [2], given here in a more general context allowing for a time- and state-dependent Lévy measure, as well as infinite number of jumps per unit time ("infinite jump activity").

Proposition 3 (Forward PIDE for jump diffusion model)**.** *Consider the price process* S *whose dynamics under the pricing measure* $\mathbb{P}$ *is given by:*

$$S_t=S_0+\int_0^T r(t)S_{t-}dt+\int_0^T S_{t-}\sigma(t,S_{t-})dB_t+\int_0^T\int_{-\infty}^{+\infty}S_{t-}(e^y-1)\tilde{N}(dtdy) \tag{32}$$

where B_t *is a Brownian motion and* N *a Poisson random measure on* $[0,T]\times\mathbb{R}$ *with compensator* $\nu(dz)\,dt$*,* $\tilde{N}$ *the associated compensated random measure. Assume:*

$$\begin{cases}\sigma(.,.) \quad \text{is bounded} & (A'_{1a})\\ \int_{\{|y|>1\}} e^{2y}\nu(dy)<\infty & (A'_{2a})\end{cases}$$

Then the call option price

$$C_{t_0}(T,K)=e^{-\int_{t_0}^T r(t)\,dt}E^{\mathbb{P}}[\max(S_T-K,0)|\mathcal{F}_{t_0}]$$

is a solution (in the sense of distributions) of the partial integro-differential equation:

$$\begin{aligned}\frac{\partial C_{t_0}}{\partial T}&=-r(T)K\frac{\partial C_{t_0}}{\partial K}+\frac{K^2\sigma(T,K)^2}{2}\frac{\partial^2 C_{t_0}}{\partial K^2}\\&+\int_{\mathbb{R}}\nu(dz)\,e^z\left[C_{t_0}(T,Ke^{-z})-C_{t_0}(T,K)-K(e^{-z}-1)\frac{\partial C_{t_0}}{\partial K}\right]\end{aligned} \tag{33}$$

on $[t_0,\infty[\times]0,\infty[$ *with the initial condition:* $\forall K>0 \quad C_{t_0}(t_0,K)=(S_{t_0}-K)_+$.

Proof. As in the proof of Theorem 1, by replacing $\mathbb{P}$ by the conditional measure $\mathbb{P}_{\mathcal{F}_{t_0}}$ given $\mathcal{F}_{t_0}$, we may replace the conditional expectation in (2) by an expectation with respect to the marginal distribution $p^S_T(dy)$ of S_T under $\mathbb{P}_{|\mathcal{F}_{t_0}}$. Thus, without loss of generality, we put $t_0=0$ in the sequel, consider the case where $\mathcal{F}_0$

is the σ-algebra generated by all $\mathbb{P}$-null sets and we denote $C_0(T,K) \equiv C(T,K)$ for simplicity.
By differentiating (2) in the sense of distributions with respect to K, we obtain:

$$\frac{\partial C}{\partial K}(T,K) = -e^{-\int_0^T r(t)\,dt} \int_K^\infty p_T^S(dy), \qquad \frac{\partial^2 C}{\partial K^2}(T,dy) = e^{-\int_0^T r(t)\,dt} p_T^S(dy). \quad (34)$$

In this particular case, $m(t,dz)\,dt \equiv \nu(dz)\,dt$ and ψ_t is simply defined by:

$$\psi_t(z) \equiv \psi(z) = \begin{cases} \int_{-\infty}^z dx\; e^x \int_{-\infty}^x \nu(du) & z < 0 \\ \int_z^{+\infty} dx\; e^x \int_x^\infty \nu(du) & z > 0 \end{cases}$$

Then (4) yields $\chi_{t,y}(z) \equiv \chi(z) = \psi(z)$. Let now focus on the term

$$\int_0^{+\infty} y \frac{\partial^2 C}{\partial K^2}(T,dy)\, \chi\left(\ln\left(\frac{K}{y}\right)\right)$$

in (5). Applying Lemma 1:

$$\begin{aligned} & \int_0^{+\infty} y \frac{\partial^2 C}{\partial K^2}(T,dy)\, \chi\left(\ln\left(\frac{K}{y}\right)\right) \\ = & \int_0^\infty e^{-\int_0^T r(t)\,dt} p_T^S(dy) \int_{\mathbb{R}} [(ye^z - K)^+ - e^z(y-K)^+ - K(e^z - 1)1_{\{y>K\}}]\nu(dz) \\ = & \int_{\mathbb{R}} e^z \int_0^\infty e^{-\int_0^T r(t)\,dt} p_T^S(dy)[(y - Ke^{-z})^+ - (y-K)^+ - K(1-e^{-z})1_{\{y>K\}}]\,\nu(dz) \\ = & \int_{\mathbb{R}} e^z \left[C(T,Ke^{-z}) - C(T,K) - K(e^{-z}-1)\frac{\partial C}{\partial K}\right] \nu(dz) \qquad (35) \end{aligned}$$

This ends the proof. □

2.3 Pure jump processes

We now consider price processes with no Brownian component. Assumption (H) then reduces to

$$\forall T > 0, \quad \mathbb{E}\left[\exp\left(\int_0^T dt \int (e^y - 1)^2 m(t,\,dy)\right)\right] < \infty \qquad (A_{2a})$$

and the forward equation for call option becomes

$$\frac{\partial C}{\partial T} + r(T) K \frac{\partial C}{\partial K} = \int_0^{+\infty} y \frac{\partial^2 C}{\partial K^2}(T,dy)\, \chi_{T,y}\left(\ln\left(\frac{K}{y}\right)\right) \qquad (36)$$

It is convenient to use the change of variable: $v = \ln y, k = \ln K$. Define, $c(k,T) = C(e^k,T)$. Then one can write this PIDE as:

$$\frac{\partial c}{\partial T} + r(T)\frac{\partial c}{\partial k} = \int_{-\infty}^{+\infty} e^{2(v-k)} \left(\frac{\partial^2 c}{\partial k^2} - \frac{\partial c}{\partial k}\right)(T,dv)\, \chi_{T,v}(k-v) \qquad (37)$$

where $\chi_{T,v}$ is defined by:

$$\chi_{T,v}(z) = \mathbb{E}\left[\psi_T(z)|S_{T-} = e^v\right]$$

with:

$$\psi_T(z) = \begin{cases} \int_{-\infty}^{z} dx\ e^x \int_{-\infty}^{x} m(T,du) & z < 0 \\ \int_{z}^{+\infty} dx\ e^x \int_{x}^{\infty} m(T,du) & z > 0 \end{cases}$$

In the case, considered in [9], where the Lévy density m_Y has a deterministic separable form:

$$m_Y(t,dz,y)\,dt = \alpha(y,t)\,k(z)\,dz\,dt \tag{38}$$

Equation (37) allows us to recover[1] equation (14) in [9]:

$$\frac{\partial c}{\partial T} + r(T)\frac{\partial c}{\partial k} = \int_{-\infty}^{+\infty} \kappa(k-v)e^{2(v-k)}\alpha(e^v,T)\left(\frac{\partial^2 c}{\partial k^2} - \frac{\partial c}{\partial k}\right)(T,dv)$$

where κ is defined as the exponential double tail of $k(u)\,du$, i.e:

$$\kappa(z) = \begin{cases} \int_{-\infty}^{z} dx\ e^x \int_{-\infty}^{x} k(u)\,du & z < 0 \\ \int_{z}^{+\infty} dx\ e^x \int_{x}^{\infty} k(u)\,du & z > 0 \end{cases}$$

The right hand side can be written as a convolution of distributions:

$$\frac{\partial c}{\partial T} + r(T)\frac{\partial c}{\partial k} = [a_T(.)\ \left(\frac{\partial^2 c}{\partial k^2} - \frac{\partial c}{\partial k}\right)] * g \qquad \text{where} \tag{39}$$

$$g(u) = e^{-2u}\kappa(u) \qquad a_T(u) = \alpha(e^u,T) \tag{40}$$

Therefore, it implies that from the knowledge of $c(.,.)$ and a choice for $\kappa(.)$ we can recover a_T hence $\alpha(.,.)$. As noted by Carr et al. [9], this equation is analogous to the Dupire formula for diffusions: it enables to "invert" the structure of the jumps–represented by α– from the cross-section of option prices. Note that, like the Dupire formula, this inversion involves a double deconvolution/differentiation of c which illustrates the ill-posedness of the inverse problem.

2.4 Time changed Lévy processes

Time changed Lévy processes were proposed in [8] in the context of option pricing. Consider the price process S whose dynamics under the pricing measure $\mathbb{P}$ is given by:

$$S_t \equiv e^{\int_0^t r(u)\,du}\,X_t \qquad X_t = \exp\left(L_{\Theta_t}\right) \qquad \Theta_t = \int_0^t \theta_s ds \tag{41}$$

where L_t is a Lévy process with characteristic triplet (b, σ^2, ν), N its jump measure and (θ_t) is a locally bounded positive semimartingale. We assume L

[1]Note however that the equation given in [9] does not seem to be correct: it involves the double tail of $k(z)\,dz$ instead of the exponential double tail.

and θ are $\mathcal{F}_t$-adapted.
$X_t \equiv e^{-\int_0^t r(u)\,du} S_t$ is a martingale under the pricing measure $\mathbb{P}$ if $\exp(L_t)$ is a martingale which requires the following condition on the characteristic triplet of (L_t):

$$b + \frac{1}{2}\sigma^2 + \int_{\mathbb{R}} (e^z - 1 - z\,1_{\{|z|\le 1\}})\nu(dy) = 0 \tag{42}$$

Define the value $C_{t_0}(T,K)$ at time t_0 of the call option with expiry $T > t_0$ and strike $K > 0$ of the stock price (S_t):

$$C_{t_0}(T,K) = e^{-\int_0^T r(t)\,dt} E^{\mathbb{P}}[\max(S_T - K, 0)|\mathcal{F}_{t_0}] \tag{43}$$

Proposition 4. *Define*

$$\alpha(t,x) = E[\theta_t | X_{t-} = x]$$

and χ the exponential double tail of $\nu(du)$

$$\chi(z) = \begin{cases} \int_{-\infty}^{z} dx\ e^x \int_{-\infty}^{x} \nu(du) & z < 0 \\ \int_{z}^{+\infty} dx\ e^x \int_{x}^{\infty} \nu(du) & z > 0 \end{cases} \tag{44}$$

Assume $\beta = \frac{1}{2}\sigma^2 + \int_{\mathbb{R}} (e^y - 1)^2 \nu(dy) < \infty$ holds and

$$\mathbb{E}\left[\exp\left(\beta\Theta_T\right)\right] < \infty \tag{45}$$

Then the call option price $C_{t_0} : (T,K) \mapsto C_{t_0}(T,K)$ at date t_0, as a function of maturity and strike, is a solution (in the sense of distributions) of the partial integro-differential equation:

$$\begin{aligned} \frac{\partial C}{\partial T} &= -r\alpha(T,K)K\frac{\partial C}{\partial K} + \frac{K^2\alpha(T,K)\sigma^2}{2}\,\frac{\partial^2 C}{\partial K^2} \\ &\quad + \int_0^{+\infty} y\frac{\partial^2 C}{\partial K^2}(T,dy)\,\alpha(T,y)\,\chi\left(\ln\left(\frac{K}{y}\right)\right) \end{aligned} \tag{46}$$

on $[t,\infty[\times]0,\infty[$ with the initial condition: $\forall K > 0 \quad C_{t_0}(t_0,K) = (S_{t_0} - K)_+$.

Proof. Using [5, Theorem 4], (L_{Θ_t}) writes

$$\begin{aligned} L_{\Theta_t} &= L_0 + \int_0^t \sigma\sqrt{\theta_s} dB_s + \int_0^t b\theta_s ds \\ &+ \int_0^t \theta_s \int_{|z|\le 1} z\tilde{N}(ds\,dz) + \int_0^t \theta_s \int_{\{|z|>1\}} zN(ds\,dz) \end{aligned}$$

where N is an integer-valued random measure with compensator $\nu(dz)\,dt$, $\tilde{N}$ its compensated random measure. Applying the Itô formula yields

$$\begin{aligned}
X_t &= X_0 + \int_0^t X_{s-} dL_{T_s} + \frac{1}{2}\int_0^t X_{s-}\sigma^2\theta_s\, ds + \sum_{s\le t} X_s - X_{s-} - X_{s-}\Delta L_{T_s} \\
&= X_0 + \int_0^t X_{s-}\left[b\theta_s + \frac{1}{2}\sigma^2\theta_s\right] ds + \int_0^t X_{s-}\sigma\sqrt{\theta_s}\, dB_s \\
&+ \int_0^t X_{s-}\theta_s \int_{\{|z|\le 1\}} z\tilde{N}(ds\, dz) + \int_0^t X_{s-}\theta_s \int_{\{|z|>1\}} zN(ds\, dz) \\
&+ \int_0^t \int_{\mathbb{R}} X_{s-}\theta_s (e^z - 1 - z) N(ds\, dz)
\end{aligned}$$

Under our assumptions, $\int (e^z - 1 - z\, 1_{\{|z|\le 1\}})\nu(dz) < \infty$, hence:

$$\begin{aligned}
X_t &= X_0 + \int_0^t X_{s-}\left[b\theta_s + \frac{1}{2}\sigma^2\theta_s + \int_{\mathbb{R}} (e^z - 1 - z\, 1_{\{|z|\le 1\}})\theta_s\nu(dz)\right] ds + \int_0^t X_{s-}\sigma\sqrt{\theta}\, dB_s \\
&+ \int_0^t \int_{\mathbb{R}} X_{s-}\theta_s (e^z - 1)\tilde{N}(ds\, dz) \\
&= X_0 + \int_0^t X_{s-}\sigma\sqrt{\theta_s}\, dB_s + \int_0^t \int_{\mathbb{R}} X_{s-}\theta_s (e^z - 1)\tilde{N}(ds\, dz)
\end{aligned}$$

and (S_t) may be expressed as:

$$S_t = S_0 + \int_0^t S_{s-} r(s)\, ds + \int_0^t S_{s-}\sigma\sqrt{\theta_s}\, dB_s + \int_0^t \int_{\mathbb{R}} S_{s-}\theta_s (e^z - 1)\tilde{N}(ds\, dz)$$

Assumption (45) implies that assumption (H) of Theorem 1 and (S_t) is now in the suitable form (1) to apply Theorem 1, which yields the result. □

2.5 Index options in a multivariate jump-diffusion model

Consider a multivariate model with d assets:

$$S_T^i = S_0^i + \int_0^T r(t) S_{t^-}^i dt + \int_0^T S_{t^-}\delta_t^i dW_t^i + \int_0^T \int_{\mathbb{R}^d} S_{t^-}^i (e^{y_i} - 1)\tilde{N}(dt\, dy)$$

where δ^i is an adapted process taking values in $\mathbb{R}$ representing the volatility of asset i, W is a d-dimensional Wiener process, N is a Poisson random measure on $[0,T]\times\mathbb{R}^d$ with compensator $\nu(dy)\, dt$, $\tilde{N}$ denotes its compensated random measure.
The Wiener processes W^i are correlated: for all $1 \le (i,j) \le d$, $\langle W^i, W^j\rangle_t = \rho_{i,j} t$, with $\rho_{ij} > 0$ and $\rho_{ii} = 1$.
An index is defined as a weighted sum of the asset prices:

$$I_t = \sum_{i=1}^d w_i S_t^i$$

The value $C_{t_0}(T,K)$ at time t_0 of an index call option with expiry $T > t_0$ and strike $K > 0$ is given by

$$C_{t_0}(T,K) = e^{-\int_{t_0}^T r(t)\,dt} E^{\mathbb{P}}[\max(I_T - K, 0)|\mathcal{F}_{t_0}] \tag{47}$$

The following result is a generalization the forward PIDE studied by Avellaneda et al. [3] for the diffusion case:

Theorem 3. *Forward PIDE for index options. Assume*

$$\begin{cases} \forall T > 0 \quad \mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \|\delta_t\|^2\,dt\right)\right] < \infty & (A_{1b}) \\ \int_{\mathbb{R}^d}(1 \wedge \|y\|)\,\nu(dy) < \infty \quad a.s. \quad (A_{2b}) \\ \int_{\{\|y\|>1\}} e^{2\|y\|}\nu(dy) < \infty \quad a.s. \quad (A_{3b}) \end{cases} \tag{48}$$

Define

$$\eta_t(z) = \begin{cases} \int_{-\infty}^z dx\ e^x \int_{\mathbb{R}^d} 1_{\ln\left(\frac{\sum_{1\le i\le d-1} w_i S^i_{t-} e^{y_i}}{I_{t-}}\right) \le x}\,\nu(dy) \quad z < 0 \\ \int_z^\infty dx\ e^x \int_{\mathbb{R}^d} 1_{\ln\left(\frac{\sum_{1\le i\le d-1} w_i S^i_{t-} e^{y_i}}{I_{t-}}\right) \ge x}\,\nu(dy) \quad z > 0 \end{cases} \tag{49}$$

and

$$\sigma(t,z) = \frac{1}{z}\sqrt{\mathbb{E}\left[\left(\sum_{i,j=1}^d w_i w_j \rho_{ij}\,\delta^i_t \delta^j_t\, S^i_{t-} S^j_{t-}\right)|I_{t^-} = z\right]}; \tag{50}$$

$$\chi_{t,y}(z) = \mathbb{E}\left[\eta_t\left(z\right)|I_{t-} = y\right] \tag{51}$$

The index call price $(T,K) \mapsto C_{t_0}(T,K)$, as a function of maturity and strike, is a solution (in the sense of distributions) of the partial integro-differential equation:

$$\frac{\partial C}{\partial T} = -r(T)K\frac{\partial C}{\partial K} + \frac{\sigma(T,K)^2}{2}\frac{\partial^2 C}{\partial K^2} + \int_0^{+\infty} y\frac{\partial^2 C}{\partial K^2}(T,dy)\,\chi_{T,y}\left(\ln\left(\frac{K}{y}\right)\right) \tag{52}$$

on $[t_0,\infty[\times]0,\infty[$ *with the initial condition:* $\forall K > 0 \quad C_{t_0}(t_0,K) = (I_{t_0} - K)_+$.

Proof. $(B_t)_{t\ge0}$ defined by

$$dB_t = \frac{\sum_{i=1}^d w_i S^i_{t-}\delta^i_t dW^i_t}{\left(\sum_{i,j=1}^d w_i w_j \rho_{ij}\,\delta^i_t\delta^j_t\, S^i_{t-}S^j_{t-}\right)^{1/2}}$$

is a continuous local martingale with quadratic variation t: by Lévy's theorem,

B is a Brownian motion. Hence I may be decomposed as

$$\begin{aligned} I_T &= \sum_{i=1}^d w_i S_0^i + \int_0^T r(t) I_{t-}\, dt + \int_0^T \left(\sum_{i,j=1}^d w_i w_j \rho_{ij}\, \delta_t^i \delta_t^j\, S_{t-}^i S_{t-}^j \right)^{\frac{1}{2}} dB_t \\ &\quad + \int_0^T \int_{\mathbb{R}^d} \sum_{i=1}^d w_i S_{t-}^i (e^{y_i} - 1) \tilde{N}(dt\, dy) \end{aligned} \tag{53}$$

The essential part of the proof consists in rewriting (I_t) in the suitable form (1) to apply Theorem 1. Applying the Itô formula to $\ln(I_T)$ yields:

$$\begin{aligned} & \ln(I_T) - \ln(I_0) \\ = & \int_0^T \Big[r(t) - \frac{1}{2I_{t-}^2} \sum_{i,j=1}^d w_i w_j \rho_{ij}\, \delta_t^i \delta_t^j\, S_{t-}^i S_{t-}^j \\ & \qquad - \int \big(\frac{\sum_{1\le i\le d} w_i S_{t-}^i e^{y_i}}{I_{t-}} - 1 - \ln\big(\frac{\sum_{1\le i\le d} w_i S_{t-}^i e^{y_i}}{I_{t-}}\big)\big)\nu(dy) \Big]\, dt \\ + & \int_0^T \frac{1}{I_{t-}} \left(\sum_{i,j=1}^d w_i w_j \rho_{ij}\, \delta_t^i \delta_t^j\, S_{t-}^i S_{t-}^j \right)^{\frac{1}{2}} dB_t \\ + & \int_0^T \int \ln\left(\frac{\sum_{1\le i\le d} w_i S_{t-}^i e^{y_i}}{I_{t-}} \right) \tilde{N}(dt\, dy) \end{aligned}$$

The last equality is obtained since
$\int \big(\frac{\sum_{1\le i\le d} w_i S_{t-}^i e^{y_i}}{I_{t-}} - 1 - \ln\big(\frac{\sum_{1\le i\le d} w_i S_{t-}^i e^{y_i}}{I_{t-}}\big)\big)\nu(dy) < \infty$: using the convexity property of the logarithm (one recalls that $\sum_{1\le i\le d} \frac{w_i S_{t-}^i}{I_{t-}} = 1$), and the Hölder inequality:

$$\left| \ln\left(\frac{\sum_{1\le i\le d} w_i S_{t-}^i e^{y_i}}{I_{t-}} \right) \right| \le \left| \sum_{1\le i\le d} \frac{w_i S_{t-}^i}{I_{t-}} y_i \right| \le \sum_{1\le i\le d} |y_i| \le \|y\|,$$

hence the functions $y \to \ln\left(\frac{\sum_{1\le i\le d} w_i S_{t-}^i e^{y_i}}{I_{t-}} \right)$ and $y \to \frac{\sum_{1\le i\le d} w_i S_{t-}^i e^{y_i}}{I_{t-}}$ are integrable with respect to $\nu(dy)$ by assumptions (A_{2b}) and (A_{3b}). We furthermore observe that

$$\begin{aligned} & \int 1 \wedge \left| \ln\left(\frac{\sum_{1\le i\le d} w_i S_{t-}^i e^{y_i}}{I_{t-}} \right) \right| \nu(dy) < \infty \quad a.s. \\ & \int_0^T \int_{\{\|y\|>1\}} e^{2\left|\ln\left(\frac{\sum_{1\le i\le d} w_i S_{t-}^i e^{y_i}}{I_{t-}} \right)\right|} \nu(dy)\, dt < \infty \quad a.s. \end{aligned} \tag{54}$$

Similarly, since (A_{2b}) and (A_{3b}), for all $1 \leq i \leq d$, $\int \left(e^{y_i} - 1 - 1_{\{|y_i|\leq 1\}} y_i\right) \nu(dy) < \infty$ and $\ln(S_T^i)$ rewrites:

$$\begin{aligned}
\ln(S_T^i) &= \ln(S_0^i) + \int_0^T \left(r(t) - \frac{1}{2}(\delta_t^i)^2 - \int \left(e^{y_i} - 1 - 1_{\{|y_i|\leq 1\}} y_i\right) \nu(dy)\right) dt \\
&+ \int_0^T \delta_t^i \, dW_t^i + \int_0^T \int y_i \, \tilde{N}(dt\, dy)
\end{aligned}$$

Define the d-dimensional martingale $W_t = (W_t^1, \cdots, W_t^{d-1}, B_t)$. For all $1 \leq i, j \leq d-1$, $\langle W^i, W^j \rangle_t = \rho_{i,j} t$;

$$\langle W^i, B \rangle_t = \frac{\sum_{j=1}^d w_j \rho_{ij} S_{t-}^j \delta_t^j}{\left(\sum_{i,j=1}^d w_i w_j \rho_{ij}\, \delta_t^i \delta_t^j\, S_{t-}^i S_{t-}^j\right)^{1/2}}\, t$$

Define

$$\Theta_t = \begin{pmatrix}
1 & \cdots & \rho_{1,d-1} & \frac{\sum_{j=1}^d w_j \rho_{1j} S_{t-}^j \delta_t^j}{\left(\sum_{i,j=1}^d w_i w_j \rho_{ij}\, \delta_t^i \delta_t^j\, S_{t-}^i S_{t-}^j\right)^{1/2}} \\
\vdots & \ddots & \vdots & \vdots \\
\rho_{d-1,1} & \cdots & 1 & \frac{\sum_{j=1}^d w_j \rho_{d-1,j} S_{t-}^j \delta_t^j}{\left(\sum_{i,j=1}^d w_i w_j \rho_{ij}\, \delta_t^i \delta_t^j\, S_{t-}^i S_{t-}^j\right)^{1/2}} \\
\frac{\sum_{j=1}^d w_j \rho_{1,j} S_{t-}^j \delta_t^j}{\left(\sum_{i,j=1}^d w_i w_j \rho_{ij}\, \delta_t^i \delta_t^j\, S_{t-}^i S_{t-}^j\right)^{1/2}} & \cdots & \frac{\sum_{j=1}^d w_j \rho_{d-1,j} S_{t-}^j \delta_t^j}{\left(\sum_{i,j=1}^d w_i w_j \rho_{ij}\, \delta_t^i \delta_t^j\, S_{t-}^i S_{t-}^j\right)^{1/2}} & 1
\end{pmatrix}$$

There exists a standard Brownian motion (Z_t) such that $W_t = AZ_t$ where A is a $d \times d$ matrix verifying

$$\Theta = {}^t A\, A.$$

Define $X_T \equiv \left(\ln(S_T^1), \cdots, \ln(S_T^{d-1}), \ln(I_T)\right)$;

$$\delta = \begin{pmatrix}
\delta_t^1 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & \delta_t^{d-1} & 0 \\
0 & \cdots & 0 & \frac{1}{I_{t-}} \left(\sum_{i,j=1}^d w_i w_j \rho_{ij}\, \delta_t^i \delta_t^j\, S_{t-}^i S_{t-}^j\right)^{\frac{1}{2}}
\end{pmatrix}$$

$$\beta_t = \begin{pmatrix}
r(t) - \frac{1}{2}(\delta_t^1)^2 - \int (e^{y_1} - 1 - y_1)\, \nu(dy) \\
\vdots \\
r(t) - \frac{1}{2}(\delta_t^{d-1})^2 - \int (e^{y_{d-1}} - 1 - y_{d-1})\, \nu(dy) \\
r(t) - \frac{1}{2I_{t-}^2} \sum_{i,j=1}^d w_i w_j \rho_{ij}\, \delta_t^i \delta_t^j\, S_{t-}^i S_{t-}^j - \int \left(\frac{\sum_{1\leq i\leq d} w_i S_{t-}^i e^{y_i}}{I_{t-}} - 1 - \ln\left(\frac{\sum_{1\leq i\leq d} w_i S_{t-}^i e^{y_i}}{I_{t-}}\right)\right) \nu(dy)
\end{pmatrix}$$

and

$$\psi_t(y) = \begin{pmatrix}
y_1 \\
\vdots \\
y_{d-1} \\
\ln\left(\frac{\sum_{1\leq i\leq d} w_i S_{t-}^i e^{y_i}}{I_{t-}}\right)
\end{pmatrix}$$

then (X_T) may be expressed as:

$$X_T = X_0 + \int_0^T \beta_t \, dt + \int_0^T \delta_t A \, dZ_t + \int_0^T \int_{\mathbb{R}^d} \psi_t(y) \, \tilde{N}(dt\, dy) \qquad (55)$$

For all t in $[0,T]$,for all $y \in \psi_t(\mathbb{R}^d)$, if one defines

$$\phi_t(y) = \left(y_1, \cdots, y_{d-1}, \ln\left(\frac{e^{y_d} I_{t-} - \sum_{1\le i\le d-1} w_i S^i_{t-} e^{y_i}}{w_d S^d_{t-}} \right) \right)$$

then ϕ is the left inverse ϕ of ψ that is:

$$\phi_t(\omega, \psi_t(\omega, y)) = y.$$

Observe that $\psi_t(.,0) = 0$, ϕ is predictable, and $\phi_t(\omega,.)$ is differentiable on $Im(\psi_t)$ with Jacobian matrix $\nabla_y \phi_t(y)$:

$$(\nabla_y \phi_t(y)) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ \frac{-e^{y_1} w_1 S^1_{t-}}{e^{y_d} I_{t-} - \sum_{1\le i\le d-1} w_i S^i_{t-} e^{y_i}} & \cdots & \frac{-e^{y_{d-1}} w_{d-1} S^{d-1}_{t-}}{e^{y_d} I_{t-} - \sum_{1\le i\le d-1} w_i S^i_{t-} e^{y_i}} & \frac{e^{y_d} I_{t-}}{e^{y_d} I_{t-} - \sum_{1\le i\le d-1} w_i S^i_{t-} e^{y_i}} \end{pmatrix}$$

(ψ, ν) satisfies [5, Assumption (H3)]: since (A_{2b}), for all T and for all t in $[0,T]$,

$$\begin{aligned} & \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} (1 \wedge \|\psi_t(.,y)\|^2) \, \nu(dy) \, dt \right] \\ = \; & \int_0^T \int_{\mathbb{R}^d} 1 \wedge \left(y_1^2 + \cdots + y_{d-1}^2 + \ln\left(\frac{\sum_{1\le i\le d} w_i S^i_{t-} e^{y_i}}{I_{t-}} \right)^2 \right) \nu(dy) \, dt \\ \le \; & \int_0^T \int_{\mathbb{R}^d} 1 \wedge (2\|y\|^2) \, \nu(dy) \, dt < \infty \end{aligned}$$

Define ν_ϕ, the image of ν by ϕ:

$$\forall B \in \mathcal{B}(\mathbb{R}^d - \{0\}) \subset \psi_t(\mathbb{R}^d) \quad \nu_\phi(\omega, t, B) = \nu(\phi_t(\omega, B)) \qquad (56)$$

Applying [5, Lemma 2], X_T may be expressed as:

$$X_T = X_0 + \int_0^T \beta_t \, dt + \int_0^T \delta_t A \, dZ_t + \int_0^T \int y \, \tilde{M}(dt\, dy)$$

where M is an integer-valued random measure (resp. $\tilde{M}$ its compensated random measure) with compensator

$$\mu(\omega; dt\; dy) = m(t, dy; \omega) \, dt$$

defined via its density with respect to ν_ϕ:

$$\begin{aligned}\frac{d\mu}{d\nu_\phi}(\omega,t,y) &= 1_{\{\psi_t(\mathbb{R}^d)\}}(y)\,|\det\nabla_y\phi_t|(y)\\ &= 1_{\{\psi_t(\mathbb{R}^d)\}}(y)\left|\frac{e^{y_d}I_{t-}}{e^{y_d}I_{t-}-\sum_{1\le i\le d-1}w_iS^i_{t-}e^{y_i}}\right|\end{aligned}$$

Considering now the d-th component of X_T, one obtains the semimartingale decomposition of $\ln(I_t)$:

$$\begin{aligned}&\ln(I_T)-\ln(I_0)\\ =&\int_0^T\Bigg(r(t)-\frac{1}{2I_{t-}^2}\left(\sum_{i,j=1}^d w_iw_j\rho_{ij}\,\delta_t^i\delta_t^j\,S^i_{t-}S^j_{t-}\right)\\ &\qquad-\int\left(\frac{\sum_{1\le i\le d}w_iS^i_{t-}e^{y_i}}{I_{t-}}-1-\ln\left(\frac{\sum_{1\le i\le d}w_iS^i_{t-}e^{y_i}}{I_{t-}}\right)\right)\nu(dy)\Bigg)\,dt\\ +&\int_0^T\frac{1}{I_{t-}}\left(\sum_{i,j=1}^d w_iw_j\rho_{ij}\,\delta_t^i\delta_t^j\,S^i_{t-}S^j_{t-}\right)^{\frac12}dB_t+\int_0^T\int y\,\tilde K(dt\,dy)\end{aligned}$$

where K is an integer-valued random measure on $[0,T]\times\mathbb{R}$ with compensator $k(t,dy)\,dt$ defined by: $\forall B\in\mathcal{B}(\mathbb{R}-\{0\})$,

$$k(t,B)=\mu(t,\mathbb{R}^{d-1}\times B)\tag{57}$$

and $\tilde K$ its compensated random measure. Let compute k:

$$\begin{aligned}k(t,B) &= \int_{\mathbb{R}^{d-1}\times B}\mu(t,dy)=\int_{\mathbb{R}^{d-1}\times B}1_{\{\psi_t(\mathbb{R}^d)\}}(y)\,|\det\nabla_y\phi_t|(y)\,\nu_\phi(t,dy)\\ &= \int_{\mathbb{R}^{d-1}\times B\cap\psi_t(\mathbb{R}^d)}|\det\nabla_y\phi_t|(\psi_t(y))\,\nu(dy)\\ &= \int_{\left\{y\in\mathbb{R}^d-\{0\},\ln\left(\frac{\sum_{1\le i\le d-1}w_iS^i_{t-}e^{y_i}}{I_{t-}}\right)\in B\right\}}\nu(dy)\end{aligned}$$

In particular, the exponential double tail of $k(t,dy)$ which we denote $\eta_t(z)$

$$\eta_t(z)=\begin{cases}\int_{-\infty}^z dx\;e^xk(t,]-\infty,x]) & z<0\\ \int_z^{+\infty}dx\;e^xk(t,[x,\infty[) & z>0\end{cases}$$

is given by (49). To conclude (I_t) writes:

$$I_T=I_0+\int_0^T r(t)I_{t-}\,dt+\int_0^T\left(\sum_{i,j=1}^d w_iw_j\rho_{ij}\,\delta_t^i\delta_t^j\,S^i_{t-}S^j_{t-}\right)^{\frac12}dB_t+\int_0^T\int_{\mathbb{R}^d}(e^y-1)\,I_{t-}\tilde K(dt\,dy)$$

The normalized volatility of I_t satisfies for all t in $[0,T]$,

$$\frac{\sum_{i,j=1}^d w_i w_j \rho_{ij}\, \delta_t^i \delta_t^j\, S_{t-}^i S_{t-}^j}{I_{t-}^2} \le \sum_{i,j=1}^d \rho_{ij}\, \delta_t^i \delta_t^j$$

and

$$\left| \ln\left(\frac{\sum_{1\le i\le d} w_i S_{t-}^i e^{y_i}}{I_{t-}} \right) \right| \le \|y\|.$$

Hence:

$$\begin{aligned}
& \frac{1}{2}\int_0^T \frac{\sum_{i,j=1}^d w_i w_j \rho_{ij}\, \delta_t^i \delta_t^j\, S_{t-}^i S_{t-}^j}{I_{t-}^2}\, dt + \int_0^T \int (e^y - 1)^2 k(t,\, dy)\, dt \\
= \;& \frac{1}{2}\int_0^T \frac{\sum_{i,j=1}^d w_i w_j \rho_{ij}\, \delta_t^i \delta_t^j\, S_{t-}^i S_{t-}^j}{I_{t-}^2}\, dt \\
+ \;& \int_0^T \int_{\mathbb{R}^d} \left(\frac{\sum_{1\le i\le d-1} w_i S_{t-}^i e^{y_i} + w_d S_{t-}^d e^y}{I_{t-}} - 1 \right)^2 \nu(dy_1, \cdots, dy_{d-1}, dy)\, dt \\
\le \;& \frac{1}{2}\sum_{i,j=1}^d \rho_{ij}\, \delta_t^i \delta_t^j + \int_0^T \int_{\mathbb{R}^d} (e^{\|y\|} - 1)^2 \nu(dy_1, \cdots, dy_{d-1}, dy)\, dt
\end{aligned}$$

Using assumptions (A_{1b}), (A_{2b}) and (A_{3b}), the last inequality implies that (I_t) satisfies (H). Hence (I_t) is now in a suitable form to apply Theorem 1, which yields the result. □

2.6 Forward equations for CDO pricing

Portfolio credit derivatives such as CDOs or index default swaps are derivatives whose payoff depends on the total loss L_t due to defaults in a reference portfolio of obligors. Reduced-form top-down models of portfolio default risk [20, 22, 34, 11, 35] represent the default losses of a portfolio as a *marked point process* $(L_t)_{t\ge 0}$ where the jump times represents credit events in the portfolio and the jump sizes ΔL_t represent the portfolio loss upon a default event. Marked point processes with random intensities are increasingly used as ingredients in such models [20, 22, 28, 34, 35].

In all such models the loss process (represented as a fraction of the portfolio notional) may be represented as

$$L_t = \int_0^t \int_0^1 x\, M(ds\, dx)$$

where $M(dt\, dx)$ is an integer-valued random measure whith compensator $\mu(dt\, dx; \omega) = m(t, dx; \omega)\, dt$. Assume furthermore:

$$\int_0^1 x\, m(t, dx) < \infty \tag{58}$$

so that L_t rewrites

$$L_t = \int_0^t \int_0^1 x \, \left(m(s, dx)\, ds + \tilde{M}(ds\, dx) \right)$$

where

$$\int_0^t \int_0^1 x \, \tilde{M}(ds\, dx)$$

is a $\mathbb{P}$-martingale.
The compensator $\mu(dt\, dx; \omega)$ has finite mass

$$\lambda_t(\omega) = \int_0^1 m(t, dx; \omega)$$

N_t represents the number of defaults and λ represents the default intensity i.e. the (random) jump intensity of the point process $N_t = M([0,t] \times [0,1])$. Denote by $T_1 \leq T_2 \leq ..$ the jump times of N. The *marked point process* L may also be represented as

$$L_t = \sum_{k=1}^{N_t} Z_k$$

where the "mark" Z_k taking values in $[0,1]$ is distributed according to

$$F_t(dx; \omega) = \frac{m_X(t, dx; \omega)}{\lambda_t(\omega)}$$

Note that the percentage loss L_t belongs to $[0,1]$, so $\Delta L_t \in [0, 1 - L_{t-}]$. For the equity tranche $[0, K]$, we define the expected tranche notional at maturity T as

$$C_{t_0}(T, K) = \mathbb{E}[(K - L_T)_+ | \mathcal{F}_{t_0}] \qquad (59)$$

As noted in [11], the prices of portfolio credit derivatives such as CDO tranches only depend on the loss process through the expected tranche notionals. Therefore, if one is able to compute $C_{t_0}(T, K)$ then one is able to compute the values of all CDO tranches at date t_0. In the case of a loss process with constant loss increment, Cont and Savescu [12] derived a forward equation for the expected tranche notional. The following result generalizes the forward equation derived by Cont and Savescu [12] to a more general setting which allows for random, dependent loss sizes and possible dependence between the loss given default and the default intensity:

Proposition 5 (Forward equation for expected tranche notionals)**.** *Define the integer-valued random measure $M_Y(dt\, dy)$ with compensator $m_Y(t, dy, z)\, dt$ defined by :*

$$\forall A \in \mathcal{B}([0,1]), \qquad m_Y(t, A, z) = E[m_X(t, A, .) | L_{t-} = z] \qquad (60)$$

and the effective default intensity

$$\lambda^Y(t, z) = \int_0^{1-z} m_Y(t, dy, z) \qquad (61)$$

The expected tranche notional $(T,K) \mapsto C_{t_0}(T,K)$, as a function of maturity and strike, is a solution of the partial integro-differential equation:

$$\frac{\partial C}{\partial T} = -\int_0^K \frac{\partial^2 C}{\partial K^2}(T,dy)\left[\int_0^{K-y}(K-y-z)\,m_Y(T,dz,y)-(K-y)\lambda^Y(T,y)\right] \tag{62}$$

on $[t_0,\infty[\times]0,1[$ *with the initial condition:* $\forall K\in[0,1] \quad C_{t_0}(t_0,K)=(L_{t_0}-K)_+$.

Proof. By replacing $\mathbb{P}$ by the conditional measure $\mathbb{P}_{|\mathcal{F}_0}$ given $\mathcal{F}_0$, we may replace the conditional expectation in (59) by an expectation with respect to the marginal distribution $p_T(dy)$ of L_T under $\mathbb{P}_{|\mathcal{F}_{t_0}}$. Thus, without loss of generality, we put $t_0=0$ in the sequel and consider the case where $\mathcal{F}_0$ is the σ-algebra generated by all $\mathbb{P}$-null sets.
(59) can be expressed as

$$C(T,K)=\int_{\mathbb{R}^+}(K-y)^+\;p_T(dy) \tag{63}$$

By differentiating with respect to K, we get:

$$\frac{\partial C}{\partial K}=\int_0^K p_T(dy)=\mathbb{E}\left[1_{\{L_{t-}\le K\}}\right] \qquad \frac{\partial^2 C}{\partial K^2}(T,dy)=p_T(dy) \tag{64}$$

For $h>0$, applying the Tanaka-Meyer formula to $(K-L_t)^+$ between T and $T+h$, we have

$$\begin{aligned}(K-L_{T+h})^+ &= (K-L_T)^+ - \int_T^{T+h} 1_{\{L_{t-}\le K\}}dL_t \\ &+ \sum_{T<t\le T+h}\left[(K-L_t)^+-(K-L_{t-})^+ + 1_{\{L_{t-}\le K\}}\Delta L_t\right].\end{aligned} \tag{65}$$

Taking expectations, we get:

$$\begin{aligned}C(T+h,K)-C(T,K) &= \mathbb{E}\left[\int_T^{T+h} dt\,1_{\{L_{t-}\le K\}}\int_0^{1-L_{t-}} x\,m(t,dx)\right] \\ &+ \mathbb{E}\left[\sum_{T<t\le T+h}(K-L_t)^+-(K-L_{t-})^+ + 1_{\{L_{t-}\le K\}}\Delta L_t\right].\end{aligned}$$

First:

$$
\begin{aligned}
& \mathbb{E}\left[\int_T^{T+h} dt\, 1_{\{L_{t-}\leq K\}} \int_0^{1-L_{t-}} x\, m(t,dx)\right] \\
= & \int_T^{T+h} dt\, \mathbb{E}\left[1_{\{L_{t-}\leq K\}} \int_0^{1-L_{t-}} x\, m(t,dx)\right] \\
= & \int_T^{T+h} dt\, \mathbb{E}\left[\mathbb{E}\left[1_{\{L_{t-}\leq K\}} \int_0^{1-L_{t-}} x\, m(t,dx)\Big| L_{t-}\right]\right] \\
= & \int_T^{T+h} dt\, \mathbb{E}\left[1_{\{L_{t-}\leq K\}} \int_0^{1-L_{t-}} x\, m_Y(t,dx,L_yt-)\right] \\
= & \int_T^{T+h} dt \int_0^K p_T(dy) \left(\int_0^{1-y} x\, m_Y(t,dx,y)\right)
\end{aligned}
$$

As for the jump term,

$$
\begin{aligned}
& \mathbb{E}\left[\sum_{T<t\leq T+h} (K-L_t)^+ - (K-L_{t-})^+ + 1_{\{L_{t-}\leq K\}}\Delta L_t\right] \\
= & \mathbb{E}\left[\int_T^{T+h} dt \int_0^{1-L_{t-}} m(t,dx)\, \left((K-L_{t-}-x)^+ - (K-L_{t-})^+ + 1_{\{L_{t-}\leq K\}}x\right)\right] \\
= & \int_T^{T+h} dt\, \mathbb{E}\left[\int_0^{1-L_{t-}} m(t,dx) \left((K-L_{t-}-x)^+ - (K-L_{t-})^+ + 1_{\{L_{t-}\leq K\}}x\right)\right] \\
= & \int_T^{T+h} dt\, \mathbb{E}\left[\mathbb{E}\left[\int_0^{1-L_{t-}} m(t,dx) \left((K-L_{t-}-x)^+ - (K-L_{t-})^+ + 1_{\{L_{t-}\leq K\}}x\right) \Big| L_{t-}\right]\right] \\
= & \int_T^{T+h} dt\, \mathbb{E}\left[\int_0^{1-L_{t-}} m_Y(t,dx,L_{t-}) \left((K-L_{t-}-x)^+ - (K-L_{t-})^+ + 1_{\{L_{t-}\leq K\}}x\right)\right] \\
= & \int_T^{T+h} dt \int_0^1 p_T(dy) \int_0^{1-y} m_Y(t,dx,y) \left((K-y-x)^+ - (K-y)^+ + 1_{\{y\leq K\}}x\right)
\end{aligned}
$$

But

$$
\begin{aligned}
& \int_0^1 p_T(dy) \int_0^{1-y} m_Y(t,dx,y) \left((K-y-x)^+ - (K-y)^+ + 1_{\{y\leq K\}}x\right) \\
= & \int_0^K p_T(dy) \int_0^{1-y} m_Y(t,dx,y) \left((K-y-x)1_{\{K-y>x\}} - (K-y-x)\right) \\
= & \int_0^K p_T(dy) \int_{K-y}^{1-y} m_Y(t,dx,y)(K-y-x)
\end{aligned}
$$

Gathering together all the terms, we obtain:

$$
\begin{aligned}
& [C(T+h,K)-C(T,K)] \\
= & \int_T^{T+h} dt \int_0^K p_T(dy)\left(\int_0^{1-y} x\, m_Y(t,dx,y)\right) \\
+ & \int_T^{T+h} dt \int_0^K p_T(dy)\left(\int_{K-y}^{1-y} m_Y(t,dx,y)(K-y-x)\right) \\
= & \int_T^{T+h} dt \int_0^K p_T(dy)\left(-\int_0^{K-y} m_Y(t,dx,y)(K-y-x)+(K-y)\lambda^Y(T,y)\right)
\end{aligned}
$$

Dividing by h and taking the limit $h \to 0$ yields:

$$
\begin{aligned}
\frac{\partial C}{\partial T} &= -\int_0^K p_T(dy)\left[\int_0^{K-y}(K-y-x)\, m_Y(T,dx,y)-(K-y)\lambda^Y(T,y)\right] \\
&= -\int_0^K \frac{\partial^2 C}{\partial K^2}(T,dy)\left[\int_0^{K-y}(K-y-x)\, m_Y(T,dx,y)-(K-y)\lambda^Y(T,y)\right]
\end{aligned}
$$

□

In [12], loss given default (i.e. the jump size of L) is assumed constant $\delta=(1-R)/n$: the marks Z_k are then deterministic and equal to $\delta: L_t=\delta N_t$ and one can compute $C(T,K)$ using the law of N_t. Setting $t_0=0$ and assuming as above that $\mathcal{F}_{t_0}$ onl

$$
C(T,K)=\mathbb{E}[(K-L_T)^+]=\mathbb{E}[(k\,\delta-L_T)^+]=\delta\mathbb{E}[(k-N_T)^+]\equiv \delta\, C_k(T) \quad (66)
$$

The compensator of L_t is $\lambda_t\,\epsilon_\delta(dz)\,dt$, where $\epsilon_\delta(dz)$ is the point mass at the point δ. The effective compensator becomes: $m_Y(t,dz,y)=E[\lambda_t|L_{t-}=y]\,\epsilon_\delta(dz)\,dt=\lambda^Y(t,y)\,\epsilon_\delta(dz)$ and the effective default intensity is $\lambda^Y(t,y)=E[\lambda_t|L_{t-}=y]$.

If we set $y=j\delta$ then : $\lambda^Y(t,j\delta)=E[\lambda_t|L_{t-}=j\delta]=E[\lambda_t|N_{t-}=j]=a_j(t)$ and $p_t(dy)=\sum_{j=0}^n q_j(t)\epsilon_{j\delta}(dy)$ with the notations in [12].
Let us focus on (62) in this case. We recall from the proof of Proposition 5 that:

$$
\begin{aligned}
\frac{\partial C}{\partial T}(T,k\delta) &= \int_0^1 p_T(dy)\, H_T.\,(k\delta-y)^+ \\
&= \int_0^1 p_T(dy)\int_0^{1-y}\left[(k\delta-y-z)^+-(k\delta-y)^+\right]\lambda^Y(T,y)\,\epsilon_\delta(dz) \\
&= \int_0^1 p_T(dy)\,\lambda^Y(T,y)\left[(k\delta-y-\delta)^+-(k\delta-y)^+\right]1_{\{\delta<1-y\}} \\
&= -\delta\sum_{j=0}^n q_j(T)\,a_j(T)\,1_{\{j\le k-1\}} \qquad (67)
\end{aligned}
$$

This expression can be simplified as in [12, Proposition 2], leading to the forward equation

$$\begin{aligned}\frac{\partial C_k(T)}{\partial T} &= a_k(T)C_{k-1}(T) - a_{k-1}(T)C_k(T) - \sum_{j=1}^{k-2} C_j(T)[a_{j+1}(T) - 2a_j(T) + a_{j-1}(T)] \\ &= [a_k(T) - a_{k-1}(T)]C_{k-1}(T) - \sum_{j=1}^{k-2} (\nabla^2 a)_j C_j(T) - a_{k-1}(T)[C_k(T) - C_{k-1}(T)]\end{aligned}$$

Hence we recover [12, Proposition 2] as a special case of Proposition 5.

References

[1] Y. Achdou and O. Pironneau, *Computational methods for option pricing*, Frontiers in Applied Mathematics, SIAM, 2005.

[2] L. Andersen and J. Andreasen, *Jump diffusion models: Volatility smile fitting and numerical methods for pricing*, Review of Derivatives Research, 4 (2000), pp. 231–262.

[3] M. Avellaneda, D. Boyer-Olson, J. Busca, and P. Friz, *Application of large deviation methods to the pricing of index options in finance*, C. R. Math. Acad. Sci. Paris, 336 (2003), pp. 263–266.

[4] G. Barles and C. Imbert, *Second-order elliptic integro-differential equations: viscosity solutions theory revisited*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 25 (2008), pp. 567–585.

[5] A. Bentata and R. Cont, *Mimicking the marginal distributions of a semimartingale*, working paper, arXiv:0910.3992v2 [math.PR], 2009.

[6] H. Berestycki, J. Busca, and I. Florent, *Asymptotics and calibration of local volatility models*, Quantitative Finance, 2 (2002), pp. 61–69.

[7] F. Black and M. Scholes, *The pricing of options and corporate liabilities*, Journal of Political Economy, 3 (1973), p. 637654.

[8] P. Carr, H. Geman, D. B. Madan, and M. Yor, *Stochastic volatility for Lévy processes*, Math. Finance, 13 (2003), pp. 345–382.

[9] ——, *From local volatility to local Lévy models*, Quant. Finance, 4 (2004), pp. 581–588.

[10] P. Carr and A. Hirsa, *Why be backward? Forward equations for American options*, RISK, (2003).

[11] R. Cont and A. Minca, *Recovering portfolio default intensities implied by CDO tranches*, Working Paper 2008-01, Columbia University, 2008.

[12] R. Cont and I. Savescu, *Forward equations for portfolio credit derivatives*, in Frontiers in Quantitative Finance: Credit Risk and Volatility Modeling, R. Cont, ed., Wiley, 2008, ch. 11, pp. 269–288.

[13] R. Cont and P. Tankov, *Financial modelling with jump processes*, CRC Press, 2004.

[14] R. Cont and E. Voltchkova, *Integro-differential equations for option prices in exponential Lévy models*, Finance and Stochastics, 9 (2005), pp. 299–325.

[15] B. Dupire, *Model art*, Risk, 6 (1993), pp. 118–120.

[16] ——, *Pricing with a smile*, Risk, 7 (1994), pp. 18–20.

[17] B. Dupire, *A unified theory of volatility*, working paper, Paribas, 1996.

[18] B. Dupire, *Pricing and hedging with smiles*, in Mathematics of Derivative Securities, M. Dempster and S. Pliska, eds., Cambridge University Press, 1997, pp. 103–111.

[19] S. N. Ethier and T. G. Kurtz, *Markov Processes: Characterization And Convergence*, Wiley, 1986.

[20] D. Filipovic and T. Schmidt, *Pricing and hedging of CDOs: A top down approach*, working paper, SSRN, 2009.

[21] M. G. Garroni and J. L. Menaldi, *Second order elliptic integro-differential problems*, vol. 430 of Chapman & Hall/CRC Research Notes in Mathematics, Chapman & Hall/CRC, Boca Raton, FL, 2002.

[22] K. Giesecke, *Portfolio credit risk: top down vs. bottom up approaches*, in Frontiers in Quantitative Finance: credit risk and volatility modeling, R. Cont, ed., Wiley, 2008, ch. 10, pp. 251–265.

[23] I. Gyöngy, *Mimicking the one-dimensional marginal distributions of processes having an Itô differential*, Probab. Theory Relat. Fields, 71 (1986), pp. 501–516.

[24] S. W. He, J. G. Wang, and J. A. Yan, *Semimartingale theory and stochastic calculus*, Kexue Chubanshe (Science Press), Beijing, 1992.

[25] N. Hilber, N. Reich, C. Schwab, and C. Winter, *Numerical methods for Lévy processes*, Finance and Stochastics, 13 (2009), pp. 471–500.

[26] B. Jourdain, *Stochastic flows approach to Dupire's formula*, Finance and Stochastics, 11 (2007), pp. 521–535.

[27] F. Klebaner, *Option price when the stock is a semimartingale*, Electron. Comm. Probab., 7 (2002), pp. 79–83 (electronic).

[28] A. V. Lopatin and T. Misirpashaev, *Two-dimensional markovian model for dynamics of aggregate credit loss*, in Advances in Econometrics, J.-P. Fouque, T. B. Fomby, and K. Solna, eds., vol. 22, Emerald Group Publishing, 2008, pp. 243–274.

[29] D. Madan and M. Yor, *Making Markov martingales meet marginals*, Bernoulli, 8 (2002), pp. 509–536.

[30] R. Merton, *Theory of rational option pricing*, Bell Journal of Economics, 4 (1973), pp. 141–183.

[31] R. Mikulevičius and H. Pragarauskas, *On the martingale problem associated with integro-differential operators*, "Mokslas", Vilnius, 1990.

[32] P. Protter and K. Shimbo, *No arbitrage and general semimartingales.* Ethier, Stewart N. (ed.) et al., Markov processes and related topics: A Festschrift for Thomas G. Kurtz. Beachwood, OH. Institute of Mathematical Statistics Collections 4, 267-283, 2008.

[33] P. E. Protter, *Stochastic integration and differential equations*, Springer-Verlag, Berlin, 2005. Second edition.

[34] P. Schönbucher, *Portfolio losses and the term structure of loss transition rates: a new methodology for the pricing of portfolio credit derivatives*, working paper, 2005.

[35] J. Sidenius, V. Piterbarg, and L. Andersen, *A new framework for dynamic credit portfolio loss modeling*, International Journal of Theoretical and Applied Finance, 11 (2008), pp. 163 – 197.