

Phase operators, phase states and vector phase states for SU_3 and $SU_{2,1}$

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Abstract

This paper focuses on phase operators, phase states and vector phase states for the sl_3 Lie algebra. We introduce a one-parameter generalized oscillator algebra $\mathcal{A}_\kappa(2)$ which provides a unified scheme for dealing with su_3 (for $\kappa < 0$), $su_{2,1}$ (for $\kappa > 0$) and $h_4 \otimes h_4$ (for $\kappa = 0$) symmetries. Finite- and infinite-dimensional representations of $\mathcal{A}_\kappa(2)$ are constructed for $\kappa < 0$ and $\kappa \geq 0$, respectively. Phase operators associated with $\mathcal{A}_\kappa(2)$ are defined and temporally stable phase states (as well as vector phase states) are constructed as eigenstates of these operators. Finally, we discuss a relation between quantized phase states and a quadratic discrete Fourier transform and show how to use these states for constructing mutually unbiased bases.

1 INTRODUCTION AND MOTIVATIONS

It is well known that defining an Hermitian (or unitary, by means of an exponentiation trick) phase operator for the Fock space of the isotropic harmonic oscillator and, more generally, for an infinite-dimensional Hilbert space is not an easy problem.^[21] Pegg and Barnett^[20] solved it by replacing the oscillator algebra by a truncated oscillator algebra and thus were able to give a description of the phase properties of quantum states for the single modes of the electromagnetic field. In this spirit, Vourdas^[23] introduced phase operators and phase states (i.e., eigenvectors of a phase operator) for su_2 and $su_{1,1}$; for the $su_{1,1}$ Lie algebra, he noticed that the infinite-dimensional character of the representation space prevents to define a unitary phase operator. Phase operators and phase states for other symmetries were also studied. In particular, Klimov *et al.*^[15] obtained phase states for some specific representations of su_3 .

Recently, a generalized oscillator algebra \mathcal{A}_κ , depending on a real parameter κ , was introduced to cover the cases of Lie algebras su_2 (for $\kappa < 0$) and $su_{1,1}$ (for $\kappa > 0$) as well as Weyl-Heisenberg algebra h_4 (for $\kappa = 0$).^[3, 5] Temporally stable phase states were defined as eigenstates of phase operators for finite-dimensional ($\kappa < 0$) and infinite-dimensional representations ($\kappa \geq 0$) of the \mathcal{A}_κ algebra.^[5] In the finite-dimensional case, corresponding either to $\kappa < 0$ or to $\kappa \geq 0$ with truncation, temporally stable phase states proved to be useful for deriving mutually unbiased bases.^[3, 5] Such bases play an important role in quantum information and quantum cryptography.

In this paper, we introduce an algebra, noted $\mathcal{A}_\kappa(2)$, which generalizes the \mathcal{A}_κ algebra. For $\kappa < 0$, this new algebra is similar to that considered in the seminal work of Palev^[18, 19] in the context of A_n -statistics. The $\mathcal{A}_\kappa(2)$ algebra allows to give an unified treatment of algebras su_3 (for $\kappa < 0$), $su_{2,1}$ (for $\kappa > 0$) and $h_4 \otimes h_4$ (for $\kappa = 0$). When we started this work, our aim was to study in an unified way: (i) phase operators for su_3 , $su_{2,1}$ and $h_4 \otimes h_4$ and (ii) the corresponding phase states. We discovered, for $\kappa < 0$, that phase states can be defined only for partitions of the relevant Hilbert spaces and that a global definition of *phase states* requires the introduction of *vector phase states*, a concept that is closely related to that of vector coherent states. The notion of vector coherent states was strongly investigated by Hecht^[8] and Zhang *et al.*^[29] at the end of the nineties. This notion was subsequently developed in Refs. [1, 2, 7, 22] with applications to quantum dynamical systems presenting degeneracies. In particular, the authors of Ref. [2] defined a vectorial generalization of the Gazeau-Klauder coherent states^[7] leading to vector coherent states. Recently, this notion of vector coherent states was extensively investigated

(see the works in Refs. [1, 22]).

This paper is organized as follows. The $\mathcal{A}_\kappa(2)$ generalized algebra is introduced in Section 2. We then define a quantum system associated with this algebra and generalizing the two-dimensional harmonic oscillator. In Section 3, phase operators and temporally stable vector phase states for the $\mathcal{A}_\kappa(2)$ algebra with $\kappa < 0$ are constructed. The phase operators and the corresponding temporally stable phase states for $\mathcal{A}_\kappa(2)$ with $\kappa \geq 0$ are presented in Section 4. Section 5 deals with a truncation of the $\mathcal{A}_\kappa(2)$ algebra, with $\kappa \geq 0$, necessary in order to get unitary phase operators. In Section 6, we show how a quantization of the temporality parameter occurring in the phase states for $\mathcal{A}_\kappa(2)$ with $\kappa < 0$ can lead to mutually unbiased bases.

2 GENERALIZED OSCILLATOR ALGEBRA $\mathcal{A}_\kappa(2)$

2.1 The algebra

We first define the $\mathcal{A}_\kappa(2)$ algebra. This algebra is generated by six linear operators a_i^- , a_i^+ and N_i with $i = 1, 2$ satisfying the commutation relations

$$[a_i^-, a_i^+] = I + \kappa(N_1 + N_2 + N_i), \quad [N_i, a_j^\pm] = \pm \delta_{i,j} a_i^\pm, \quad i, j = 1, 2 \quad (1)$$

and

$$[a_i^\pm, a_j^\pm] = 0, \quad i \neq j, \quad (2)$$

complemented by the triple relations

$$[a_i^\pm, [a_i^\pm, a_j^\mp]] = 0, \quad i \neq j. \quad (3)$$

In Eq. (1), I denotes the identity operator and κ is a deformation parameter assumed to be real.

Note that the \mathcal{A}_κ algebra introduced in Ref. [5] formally follows from $\mathcal{A}_\kappa(2)$ by omitting the relation $[a_2^-, a_2^+] = I + \kappa(N_1 + 2N_2)$ and by taking

$$a_2^- = a_2^+ = N_2 = 0, \quad a_1^- = a^-, \quad a_1^+ = a^+, \quad N_1 = N$$

in the remaining definitions of $\mathcal{A}_\kappa(2)$. Therefore, generalized oscillator algebra \mathcal{A}_κ in Ref. [5] should logically be noted $\mathcal{A}_\kappa(1)$.

For $\kappa = 0$, the $\mathcal{A}_0(2)$ algebra is nothing but the algebra for a two-dimensional isotropic harmonic oscillator and thus corresponds to two commuting copies of the Weyl-Heisenberg algebra h_4 .

For $\kappa \neq 0$, the $\mathcal{A}_\kappa(2)$ algebra resembles the algebra associated with the so-called A_n -statistics (for $n = 2$) which was introduced by Palev^[18] and further studied from the microscopic point of view by Palev and Van der Jeugt.^[19] In this respect, let us recall that A_n -statistics is described by the sl_{n+1} Lie algebra generated by n pairs of creation and annihilation operators (of the type of the a_i^+ and a_i^- operators above) satisfying usual commutation relations and triple commutation relations. Such a presentation of sl_{n+1} is along the lines of the Jacobson approach according to which the A_n Lie algebra can be defined by means of $2n$, rather than $n(n+2)$, generators satisfying commutation relations and triple commutation relations.^[10] These $2n$ Jacobson generators correspond to n pairs of creation and annihilation operators. In our case, the $\mathcal{A}_\kappa(2)$ algebra for $\kappa \neq 0$, with two pairs of Jacobson generators ((a_i^+, a_i^-) for $i = 1, 2$), can be identified to the Lie algebras su_3 for $\kappa < 0$ and $su_{2,1}$ for $\kappa > 0$. This can be seen as follows.

Let us define a new pair (a_3^+, a_3^-) of operators in terms of the two pairs (a_1^+, a_1^-) and (a_2^+, a_2^-) of creation and annihilation operators through

$$a_3^+ = [a_2^+, a_1^-], \quad a_3^- = [a_1^+, a_2^-]. \quad (4)$$

Following the trick used in Ref. [3] for the $\mathcal{A}_\kappa(1)$ algebra, let us introduce the operators

$$E_{+\alpha} = \frac{1}{\sqrt{|\kappa|}} a_\alpha^+, \quad E_{-\alpha} = \frac{1}{\sqrt{|\kappa|}} a_\alpha^-, \quad \alpha = 1, 2, 3$$

$$H_1 = \frac{1}{2\kappa} [I + \kappa(2N_1 + N_2)], \quad H_2 = \frac{1}{2\kappa} [I + \kappa(2N_2 + N_1)]$$

with $\kappa \neq 0$. It can be shown that the set $\{E_{\pm\alpha}; H_i : \alpha = 1, 2, 3; i = 1, 2\}$ spans su_3 for $\kappa < 0$ and $su_{2,1}$ for $\kappa > 0$.

2.2 Representation of $\mathcal{A}_\kappa(2)$

We now look for a Hilbertian representation of the $\mathcal{A}_\kappa(2)$ algebra on a Hilbert-Fock space \mathcal{F}_κ of dimension d with d finite or infinite. Let

$$\{|n_1, n_2\rangle : n_1, n_2 = 0, 1, 2, \dots\}$$

be an orthonormal basis of \mathcal{F}_κ with

$$\langle n_1, n_2 | n'_1, n'_2 \rangle = \delta_{n_1, n'_1} \delta_{n_2, n'_2}.$$

Number operators N_1 and N_2 are supposed to be diagonal in this basis, i.e.,

$$N_i |n_1, n_2\rangle = n_i |n_1, n_2\rangle, \quad i = 1, 2 \quad (5)$$

while the action of the creation and annihilation operators a_1^\pm and a_2^\pm is defined by

$$a_1^+ |n_1, n_2\rangle = \sqrt{F_1(n_1 + 1, n_2)} e^{-i[H(n_1+1, n_2) - H(n_1, n_2)]\varphi} |n_1 + 1, n_2\rangle, \quad (6)$$

$$a_1^- |n_1, n_2\rangle = \sqrt{F_1(n_1, n_2)} e^{+i[H(n_1, n_2) - H(n_1-1, n_2)]\varphi} |n_1 - 1, n_2\rangle, \quad a_1^- |0, n_2\rangle = 0 \quad (7)$$

and

$$a_2^+ |n_1, n_2\rangle = \sqrt{F_2(n_1, n_2 + 1)} e^{-i[H(n_1, n_2+1) - H(n_1, n_2)]\varphi} |n_1, n_2 + 1\rangle, \quad (8)$$

$$a_2^- |n_1, n_2\rangle = \sqrt{F_2(n_1, n_2)} e^{+i[H(n_1, n_2) - H(n_1, n_2-1)]\varphi} |n_1, n_2 - 1\rangle, \quad a_2^- |n_1, 0\rangle = 0. \quad (9)$$

In Eqs. (6)-(9), φ is an arbitrary real parameter and the positive valued functions $F_1 : \mathbb{N}^2 \rightarrow \mathbb{R}_+$, $F_2 : \mathbb{N}^2 \rightarrow \mathbb{R}_+$ and $H : \mathbb{N}^2 \rightarrow \mathbb{R}_+$ are such that

$$H = F_1 + F_2.$$

It is a simple matter of calculation to check that (5)-(9) generate a representation of the $\mathcal{A}_\kappa(2)$ algebra defined by (1)-(3) provided that $F_1(n_1, n_2)$ and $F_2(n_1, n_2)$ satisfy the recurrence relations

$$F_1(n_1 + 1, n_2) - F_1(n_1, n_2) = 1 + \kappa(2n_1 + n_2), \quad F_1(0, n_2) = 0 \quad (10)$$

$$F_2(n_1, n_2 + 1) - F_2(n_1, n_2) = 1 + \kappa(2n_2 + n_1), \quad F_2(n_1, 0) = 0. \quad (11)$$

The solutions of Eqs. (10) and (11) are

$$F_i(n_1, n_2) = n_i[1 + \kappa(n_1 + n_2 - 1)], \quad i = 1, 2. \quad (12)$$

To ensure that the structure functions F_1 and F_2 be positive definite, we must have

$$1 + \kappa(n_1 + n_2 - 1) > 0, \quad n_1 + n_2 > 0, \quad (13)$$

a condition to be discussed according to the sign of κ . In the representation of $\mathcal{A}_\kappa(2)$ defined by Eqs. (5)-(13), creation (annihilation) operators a_i^+ (a_i^-) and number operators N_i satisfy the Hermitian conjugation relations

$$(a_i^-)^\dagger = a_i^+, \quad (N_i)^\dagger = N_i, \quad i = 1, 2$$

as for the two-dimensional oscillator.

The d dimension of the representation space \mathcal{F}_κ can be deduced from condition (13). Two cases need to be considered according to as $\kappa \geq 0$ or $\kappa < 0$.

- For $\kappa \geq 0$, Eq. (13) is trivially satisfied so that the d dimension of \mathcal{F}_κ is infinite. This is well known in the case $\kappa = 0$ which corresponds to a two-dimensional isotropic harmonic oscillator. For $\kappa > 0$, the representation corresponds to the symmetric discrete (infinite-dimensional) irreducible representation of the $SU_{2,1}$ group.
- For $\kappa < 0$, there exists a finite number of states satisfying condition (13). Indeed, we have

$$n_1 + n_2 = 0, 1, \dots, E\left(-\frac{1}{\kappa}\right),$$

where $E(x)$ denotes the integer part of x . In the following, we shall take $-1/\kappa$ integer when $\kappa < 0$. Consequently, for $\kappa < 0$ the d dimension of the finite-dimensional space \mathcal{F}_κ is

$$d = \frac{1}{2}(k+1)(k+2), \quad k = -\frac{1}{\kappa} \in \mathbb{N}^*. \quad (14)$$

We know that the dimension $d(\lambda, \mu)$ of the irreducible representation (λ, μ) of SU_3 is given by

$$d(\lambda, \mu) = \frac{1}{2}(\lambda+1)(\mu+1)(\lambda+\mu+2), \quad \lambda \in \mathbb{N}, \quad \mu \in \mathbb{N}.$$

Therefore, the finite-dimensional representation of $\mathcal{A}_\kappa(2)$ defined by (5)-(13) with $-1/\kappa = k \in \mathbb{N}^*$ corresponds to the irreducible representation $(0, k)$ or its adjoint $(k, 0)$ of SU_3 .

2.3 Generalized oscillator Hamiltonian

Since the $\mathcal{A}_\kappa(2)$ algebra can be viewed as an extension of the two-dimensional oscillator algebra, it is natural to consider the $a_1^+ a_1^- + a_2^+ a_2^-$ operator as an Hamiltonian associated with $\mathcal{A}_\kappa(2)$. The action of this operator on the space \mathcal{F}_κ is given by

$$\begin{aligned} (a_1^+ a_1^- + a_2^+ a_2^-) |n_1, n_2\rangle &= [F_1(n_1, n_2) + F_2(n_1, n_2)] |n_1, n_2\rangle \\ &= (n_1 + n_2) [1 + \kappa(n_1 + n_2 - 1)] |n_1, n_2\rangle \end{aligned}$$

or

$$(a_1^+ a_1^- + a_2^+ a_2^-) |n_1, n_2\rangle = H(n_1, n_2) |n_1, n_2\rangle.$$

Thus, the $a_1^+ a_1^- + a_2^+ a_2^-$ Hamiltonian can be written

$$a_1^+ a_1^- + a_2^+ a_2^- = H,$$

with

$$H \equiv H(N_1, N_2) = (N_1 + N_2)[1 + \kappa(N_1 + N_2 - 1)]$$

modulo its action on \mathcal{F}_κ .

The H Hamiltonian is clearly a nonlinear extension of the Hamiltonian for the two-dimensional isotropic harmonic oscillator. The eigenvalues

$$\lambda_n = n[1 + \kappa(n - 1)], \quad n = n_1 + n_2, \quad n_1 \in \mathbb{N}, \quad n_2 \in \mathbb{N}$$

of H can be reduced for $\kappa = 0$ to the energies n of the two-dimensional oscillator (up to additive and multiplicative constants). For $\kappa \neq 0$, the degeneracy of the λ_n level is $n + 1$ and coincides with the degeneracy of the n level corresponding to $\kappa = 0$.

3 PHASE OPERATORS AND PHASE STATES FOR $\mathcal{A}_\kappa(2)$ WITH $\kappa < 0$

3.1 Phase operators in finite dimension

3.1.1 The E_{1d} and E_{2d} phase operators

For $\kappa < 0$ the finite-dimensional space \mathcal{F}_κ is spanned by the basis

$$\{|n_1, n_2\rangle : n_1, n_2 \text{ ranging} \mid n_1 + n_2 \leq k\}.$$

This space can be partitioned as

$$\mathcal{F}_\kappa = \bigoplus_{l=0}^k \mathcal{A}_{\kappa,l},$$

where $\mathcal{A}_{\kappa,l}$ is spanned by

$$\{|n, l\rangle : n = 0, 1, \dots, k - l\}.$$

We have

$$\dim \mathcal{A}_{\kappa,l} = k - l + 1$$

so that (6)-(7) must be completed by

$$a_1^+ |k - l, l\rangle = 0,$$

which can be deduced from the calculation of $\langle k-l, l | a_1^- a_1^+ | k-l, l \rangle$. The operators a_1^+ and a_1^- leave each subspace $\mathcal{A}_{\kappa, l}$ invariant. Then, it is convenient to write

$$a_1^\pm = \sum_{l=0}^k a_1^\pm(l),$$

with the actions

$$\begin{aligned} a_1^+(l)|n, l'\rangle &= \delta_{l, l'} \sqrt{F_1(n+1, l)} e^{-i[H(n+1, l) - H(n, l)]\varphi} |n+1, l\rangle, \\ a_1^+(l)|k-l', l'\rangle &= 0, \\ a_1^-(l)|n, l'\rangle &= \delta_{l, l'} \sqrt{F_1(n, l)} e^{+i[H(n, l) - H(n-1, l)]\varphi} |n-1, l\rangle, \\ a_1^-(l)|0, l'\rangle &= 0, \end{aligned}$$

which show that $a_1^+(l)$ and $a_1^-(l)$ leave $\mathcal{A}_{\kappa, l}$ invariant.

Let us now define the E_{1d} operator by

$$E_{1d}|n_1, n_2\rangle = e^{i[H(n_1, n_2) - H(n_1-1, n_2)]\varphi} |n_1-1, n_2\rangle, \quad 0 \leq n_1 + n_2 \leq k, \quad n_1 \neq 0$$

and

$$E_{1d}|0, n_2\rangle = e^{i[H(0, n_2) - H(k-n_2, n_2)]\varphi} |k-n_2, n_2\rangle, \quad 0 \leq n_2 \leq k, \quad n_1 = 0.$$

Thus, it is possible to write

$$a_1^- = E_{1d} \sqrt{F_1(N_1, N_2)} \Leftrightarrow a_1^+ = \sqrt{F_1(N_1, N_2)} (E_{1d})^\dagger. \quad (15)$$

The E_{1d} operator can be developed as

$$E_{1d} = \sum_{l=0}^k E_{1d}(l),$$

with

$$E_{1d}(l)|n, l'\rangle = \delta_{l, l'} e^{i[H(n, l) - H(n-1, l)]\varphi} |n-1, l\rangle, \quad n \neq 0, \quad (16)$$

$$E_{1d}(l)|0, l'\rangle = \delta_{l, l'} e^{i[H(0, l) - H(k-l, l)]\varphi} |k-l, l\rangle, \quad n = 0. \quad (17)$$

Operator $E_{1d}(l)$ leaves $\mathcal{A}_{\kappa, l}$ invariant and satisfies

$$E_{1d}(l)(E_{1d}(l'))^\dagger = (E_{1d}(l'))^\dagger E_{1d}(l) = \delta_{l, l'} \sum_{n=0}^{k-l} |n, l\rangle \langle n, l|.$$

Consequently, we obtain

$$E_{1d}(E_{1d})^\dagger = (E_{1d})^\dagger E_{1d} = \sum_{l=0}^k (E_{1d}(l))^\dagger E_{1d}(l) = \sum_{l=0}^k \sum_{n=0}^{k-l} |n, l\rangle \langle n, l| = I,$$

which shows that E_{1d} is unitary. Therefore, Eq. (15) constitutes a polar decomposition of a_1^- and a_1^+ .

Similar developments can be obtained for a_2^- and a_2^+ . We limit ourselves to the main results concerning the decomposition

$$a_2^- = E_{2d} \sqrt{F_2(N_1, N_2)} \Leftrightarrow a_2^+ = \sqrt{F_2(N_1, N_2)} (E_{2d})^\dagger.$$

In connection with this decomposition, we use the partition

$$\mathcal{F}_\kappa = \bigoplus_{l=0}^k \mathcal{B}_{\kappa, l},$$

where the $\mathcal{B}_{\kappa, l}$ subspace, of dimension $k - l + 1$, is spanned by the basis

$$\{|l, n\rangle : n = 0, 1, \dots, k - l\}.$$

We can write

$$E_{2d} = \sum_{l=0}^k E_{2d}(l),$$

where the $E_{2d}(l)$ operator satisfies

$$E_{2d}(l)|l', n\rangle = \delta_{l, l'} e^{i[H(l, n) - H(l, n-1)]\varphi} |l, n-1\rangle, \quad n \neq 0, \quad (18)$$

$$E_{2d}(l)|l', 0\rangle = \delta_{l, l'} e^{i[H(l, 0) - H(l, k-l)]\varphi} |l, k-l\rangle, \quad n = 0, \quad (19)$$

and

$$E_{2d}(l)(E_{2d}(l'))^\dagger = (E_{2d}(l'))^\dagger E_{2d}(l) = \delta_{l, l'} \sum_{n=0}^{k-l} |l, n\rangle \langle l, n|.$$

This yields

$$E_{2d}(E_{2d})^\dagger = (E_{2d})^\dagger E_{2d} = \sum_{l=0}^k (E_{2d}(l))^\dagger E_{2d}(l) = I$$

and the operator E_{2d} , like E_{1d} , is unitary.

3.1.2 The E_{3d} phase operator

Let us go back to the pair (a_3^+, a_3^-) of operators defined by (4) in terms of the pairs (a_1^+, a_1^-) and (a_2^+, a_2^-) . The action of a_3^+ and a_3^- on \mathcal{F}_κ follows from (6)-(9). We get

$$\begin{aligned} a_3^+ |n_1, n_2\rangle &= -\kappa \sqrt{n_1(n_2 + 1)} |n_1 - 1, n_2 + 1\rangle, \\ a_3^- |n_1, n_2\rangle &= -\kappa \sqrt{(n_1 + 1)n_2} |n_1 + 1, n_2 - 1\rangle. \end{aligned}$$

From Eqs. (3) and (4), it is clear that the two pairs (a_1^+, a_1^-) and (a_2^+, a_2^-) commute when $\kappa = 0$. We thus recover that the $\mathcal{A}_0(2)$ algebra corresponds to a two-dimensional harmonic oscillator.

Here, it is appropriate to use the partition

$$\mathcal{F}_\kappa = \bigoplus_{l=0}^k \mathcal{C}_{\kappa, l}, \quad (20)$$

where the subspace $\mathcal{C}_{\kappa, l}$, of dimension $l + 1$ (but not $k - l + 1$ as for $\mathcal{A}_{\kappa, l}$ and $\mathcal{B}_{\kappa, l}$), spanned by the basis

$$\{|l - n, n\rangle : n = 0, 1, \dots, l\}$$

is left invariant by a_3^+ and a_3^- . Following the same line of reasoning as for E_{1d} and E_{2d} , we can associate an operator E_{3d} with the ladder operators a_3^+ and a_3^- . We take operator E_{3d} associated with the partition (20) such that

$$a_3^- = E_{3d} \sqrt{F_3(N_1, N_2)} \Leftrightarrow a_3^+ = \sqrt{F_3(N_1, N_2)} (E_{3d})^\dagger,$$

where

$$\sqrt{F_3(N_1, N_2)} = -\kappa \sqrt{(N_1 + 1)N_2}.$$

The E_{3d} operator reads

$$E_{3d} = \sum_{l=0}^k E_{3d}(l),$$

where $E_{3d}(l)$ can be taken to satisfy

$$E_{3d}(l) |l' - n, n\rangle = \delta_{l, l'} |l - n + 1, n - 1\rangle, \quad n \neq 0,$$

$$E_{3d}(l) |l', 0\rangle = \delta_{l, l'} |0, l\rangle, \quad n = 0.$$

Finally, we have

$$E_{3d}(l)(E_{3d}(l'))^\dagger = (E_{3d}(l'))^\dagger E_{3d}(l) = \delta_{l,l'} \sum_{n=0}^l |l-n, n\rangle \langle l-n, n|.$$

As a consequence, we obtain

$$E_{3d}(E_{3d})^\dagger = (E_{3d})^\dagger E_{3d} = \sum_{l=0}^k (E_{3d}(l))^\dagger E_{3d}(l) = I,$$

a result that reflects the unitarity property of E_{3d} .

3.1.3 The E_d phase operator

Operators $E_{1d}(l)$, $E_{2d}(l)$ and $E_{3d}(l)$, defined for $\kappa < 0$ as components of the operators E_{1d} , E_{2d} and E_{3d} , leave invariant the sets $\mathcal{A}_{\kappa,l}$, $\mathcal{B}_{\kappa,l}$ and $\mathcal{C}_{\kappa,l}$, respectively. Therefore, operators E_{1d} , E_{2d} and E_{3d} do not connect all elements of \mathcal{F}_κ , i.e., a given element of \mathcal{F}_κ cannot be obtained from repeated applications of E_{1d} , E_{2d} and E_{3d} on an arbitrary element of \mathcal{F}_κ .

We now define a new operator E_d which can connect (by means of repeated applications) any couple of elements in the d -dimensional space \mathcal{F}_κ corresponding to $\kappa < 0$. Let this global operator be defined via the action

$$E_d|n, l\rangle = e^{i[H(n,l)-H(n-1,l)]\varphi}|n-1, l\rangle, \quad n = 1, 2, \dots, k-l, \quad l = 0, 1, \dots, k \quad (21)$$

and the boundary actions

$$E_d|0, l\rangle = e^{i[H(0,l)-H(k-l+1,l-1)]\varphi}|k-l+1, l-1\rangle, \quad l = 1, 2, \dots, k \quad (22)$$

$$E_d|0, 0\rangle = e^{i[H(0,0)-H(0,k)]\varphi}|0, k\rangle. \quad (23)$$

The E_d operator is obviously unitary.

By making the identification

$$\Phi_{\frac{1}{2}l(2k-l+3)+n} \equiv |n, l\rangle, \quad n = 0, 1, \dots, k-l, \quad l = 0, 1, \dots, k,$$

the set

$$\{\Phi_j : j = 0, 1, \dots, d-1\}$$

constitutes a basis for the d -dimensional Fock space \mathcal{F}_κ . Then, the various sets $\mathcal{A}_{\kappa,l}$ can be rewritten as

$$\begin{aligned}\mathcal{A}_{\kappa,0} &: \{\Phi_0, \Phi_1, \dots, \Phi_{k-1}, \Phi_k\} \\ \mathcal{A}_{\kappa,1} &: \{\Phi_{k+1}, \Phi_{k+2}, \dots, \Phi_{2k}\} \\ &\vdots \\ \mathcal{A}_{\kappa,k} &: \{\Phi_{d-1}\}.\end{aligned}$$

Repeated applications of E_d on the vectors Φ_j with $j = 0, 1, \dots, d-1$ can be summarized by the following cyclic sequence

$$E_d : \Phi_{d-1} \mapsto \Phi_{d-2} \mapsto \dots \mapsto \Phi_1 \mapsto \Phi_0 \mapsto \Phi_{d-1} \mapsto \text{etc.}$$

The E_d operator thus makes it possible to move inside each $\mathcal{A}_{\kappa,l}$ set and to connect the various sets according to the sequence

$$E_d : \mathcal{A}_{\kappa,k} \rightarrow \mathcal{A}_{\kappa,k-1} \rightarrow \dots \rightarrow \mathcal{A}_{\kappa,0} \rightarrow \mathcal{A}_{\kappa,k} \rightarrow \text{etc.}$$

Similar results hold for the partitions of \mathcal{F}_κ in $\mathcal{B}_{\kappa,l}$ or $\mathcal{C}_{\kappa,l}$ subsets.

3.2 Phase states in finite dimension

3.2.1 Phase states for $E_{1d}(l)$ and $E_{2d}(l)$

We first derive the eigenstates of $E_{1d}(l)$. For this purpose, let us consider the eigenvalue equation

$$E_{1d}(l)|z_l\rangle = z_l|z_l\rangle, \quad |z_l\rangle = \sum_{n=0}^{k-l} a_n z_l^n |n, l\rangle, \quad z_l \in \mathbb{C}.$$

Using definition (16)-(17), we obtain the following recurrence relation for the coefficients a_n

$$a_n = e^{-i[H(n,l)-H(n-1,l)]\varphi} a_{n-1}, \quad n = 1, 2, \dots, k-l$$

with

$$a_0 = e^{-i[H(0,l)-H(k-l,l)]\varphi} a_{k-l}$$

and the condition

$$(z_l)^{k-l+1} = 1.$$

Therefore, we get

$$a_n = e^{-i[H(n,l)-H(0,l)]\varphi} a_0, \quad n = 0, 1, \dots, k-l$$

and the complex variable z_l is a root of unity given by

$$z_l = q_l^m, \quad m = 0, 1, \dots, k-l,$$

where

$$q_l = \exp\left(\frac{2\pi i}{k-l+1}\right) \quad (24)$$

is reminiscent of the deformation parameter used in the theory of quantum groups. The a_0 constant can be obtained, up to a phase factor, from the normalization condition $\langle z_l | z_l \rangle = 1$. We take

$$a_0 = \frac{1}{\sqrt{k-l+1}} e^{-iH(0,l)\varphi}, \quad (25)$$

where the phase factor is chosen in order to ensure temporal stability of the $|z_l\rangle$ state. Finally, we arrive at the following normalized eigenstates of $E_{1d}(l)$

$$|z_l\rangle \equiv |l, m, \varphi\rangle = \frac{1}{\sqrt{k-l+1}} \sum_{n=0}^{k-l} e^{-iH(n,l)\varphi} q_l^{mn} |n, l\rangle. \quad (26)$$

The $|l, m, \varphi\rangle$ states are labeled by the parameters $l \in \{0, 1, \dots, k\}$, $m \in \mathbb{Z}/(k-l+1)\mathbb{Z}$ and $\varphi \in \mathbb{R}$. They satisfy

$$E_{1d}(l)|l, m, \varphi\rangle = e^{i\theta_m} |l, m, \varphi\rangle, \quad \theta_m = m \frac{2\pi}{k-l+1}, \quad m = 0, 1, \dots, k-l, \quad (27)$$

which shows that $E_{1d}(l)$ is a phase operator.

The phase states $|l, m, \varphi\rangle$ have remarkable properties:

- They are temporally stable with respect to the evolution operator associated with the H Hamiltonian. In other words, they satisfy

$$e^{-iHt} |l, m, \varphi\rangle = |l, m, \varphi + t\rangle$$

for any value of the real parameter t .

- For fixed φ and l , they satisfy the equiprobability relation

$$|\langle n, l | l, m, \varphi \rangle| = \frac{1}{\sqrt{k-l+1}}$$

and the property

$$\sum_{m=0}^{k-l} |l, m, \varphi\rangle \langle l, m, \varphi| = \sum_{n=0}^{k-l} |n, l\rangle \langle n, l|.$$

- The overlap between two phase states $|l', m', \varphi'\rangle$ and $|l, m, \varphi\rangle$ reads

$$\langle l, m, \varphi | l', m', \varphi' \rangle = \delta_{l,l'} \frac{1}{k-l+1} \sum_{n=0}^{k-l} q_l^{\rho(m-m', \varphi-\varphi', n)},$$

where

$$\rho(m-m', \varphi-\varphi', n) = -(m-m')n + \frac{k-l+1}{2\pi}(\varphi-\varphi')H(n, l)$$

with q_l defined in (24). As a particular case, for fixed φ we have the orthonormality relation

$$\langle l, m, \varphi | l', m', \varphi \rangle = \delta_{l,l'} \delta_{m,m'}.$$

However, not all temporally stable phase states are orthogonal.

Similar results can be derived for the $E_{2d}(l)$ operator by exchanging the roles played by n and l . It is enough to mention that the $|z_l\rangle$ eigenstates of $E_{2d}(l)$ can be taken in the form

$$|z_l\rangle \equiv |l, m, \varphi\rangle = \frac{1}{\sqrt{k-l+1}} \sum_{n=0}^{k-l} e^{-iH(l,n)\varphi} q_l^{mn} |l, n\rangle$$

and present properties identical to those of the states in (26).

3.2.2 Phase states for $E_{3d}(l)$

The eigenstates of the $E_{3d}(l)$ operator are given by

$$E_{3d}(l)|w_l\rangle = w_l|w_l\rangle, \quad |w_l\rangle = \sum_{n=0}^l c_n w_l^n |l-n, n\rangle, \quad w_l \in \mathbb{C}.$$

The use of (18)-(19) leads to the recurrence relation

$$c_{n+1} = c_n, \quad n = 0, 1, \dots, l-1$$

with the condition

$$c_0 = c_l(w_l)^{l+1}.$$

It follows that

$$c_n = c_0, \quad n = 0, 1, \dots, l$$

and the w_l eigenvalues satisfy

$$(w_l)^{l+1} = 1.$$

Therefore, the admissible values for w_l are

$$w_l = \omega_l^m, \quad m = 0, 1, \dots, l,$$

with

$$\omega_l = \exp\left(\frac{2\pi i}{l+1}\right).$$

As a result, the normalized eigenstates of $E_{3d}(l)$ can be taken in the form

$$|w_l\rangle \equiv ||l, m, \varphi\rangle\rangle = \frac{1}{\sqrt{l+1}} e^{-iH(0,l)\varphi} \sum_{n=0}^l \omega_l^{mn} |l-n, n\rangle. \quad (28)$$

The $||l, m, \varphi\rangle\rangle$ states depend on the parameters $l \in \{0, 1, \dots, k\}$, $m \in \mathbb{Z}/(l+1)\mathbb{Z}$ and $\varphi \in \mathbb{R}$. They satisfy

$$E_{3d}(l) ||l, m, \varphi\rangle\rangle = e^{i\theta_m} ||l, m, \varphi\rangle\rangle, \quad \theta_m = m \frac{2\pi}{l+1}, \quad (29)$$

so that $E_{3d}(l)$ is a phase operator.

For fixed l , the set $\{||l, m, 0\rangle\rangle : m = 0, 1, \dots, l\}$, corresponding to $\varphi = 0$, follows from the set $\{|l-n, n\rangle : n = 0, 1, \dots, l\}$ by making use of a (quantum) discrete Fourier transform.^[25] Note that for $\varphi = 0$, the $||l, m, 0\rangle\rangle$ phase states have the same form as the phase states for SU_2 derived by Vourdas.^[23] In the case where $\varphi \neq 0$, the $||l, m, \varphi\rangle\rangle$ phase states for $E_{3d}(l)$ satisfy properties similar to those of the $|l, m, \varphi\rangle$ phase states for $E_{1d}(l)$ and for $E_{2d}(l)$ modulo the substitutions $(n, l) \rightarrow (l-n, n)$, $q_l \rightarrow \omega_l$ and $k-l \rightarrow l$.

3.2.3 Phase states for E_d

We are now in a position to derive the eigenstates of the E_d operator. They are given by the following eigenvalue equation

$$E_d|\psi\rangle = \lambda|\psi\rangle, \quad (30)$$

where

$$|\psi\rangle = \sum_{l=0}^k \sum_{n=0}^{k-l} C_{n,l}|n, l\rangle. \quad (31)$$

Introducing (31) into (30) and using the definition in (21)-(23) of the E_d operator, a straightforward but long calculation leads to following recurrence relations

$$C_{n+1,l}e^{i[H(n+1,l)-H(n,l)]\varphi} = \lambda C_{n,l} \quad (32)$$

$$C_{0,l+1}e^{i[H(0,l+1)-H(k-l,l)]\varphi} = \lambda C_{k-l,l} \quad (33)$$

for $l = 0, 1, \dots, k-1$. For $l = k$, we have

$$C_{0,0}e^{i[H(0,0)-H(0,k)]\varphi} = \lambda C_{0,k}. \quad (34)$$

(Note that (33) with $l = k$ yields (34) if $C_{0,k+1}$ is identified to $C_{0,0}$.) From the recurrence relation (32), it is easy to get

$$C_{n,l} = \lambda^n e^{-i[H(n,l)-H(0,l)]\varphi} C_{0,l}, \quad (35)$$

which, for $n = k-l$, gives

$$C_{k-l,l} = \lambda^{k-l} e^{-i[H(k-l,l)-H(0,l)]\varphi} C_{0,l} \quad (36)$$

in terms of $C_{0,l}$. By introducing (36) into (33), we obtain the recurrence relation

$$C_{0,l+1}e^{i[H(0,l+1)-H(0,l)]\varphi} = \lambda^{k-l+1} C_{0,l} \quad (37)$$

that completely determines the $C_{0,l}$ coefficients and subsequently the $C_{n,l}$ coefficients owing to (35). Indeed, the iteration of Eq. (37) gives

$$C_{0,l} = \lambda^{\frac{1}{2}l(2k-l+3)} e^{-i[H(0,l)-H(0,0)]\varphi} C_{0,0}. \quad (38)$$

By combining (35) with (38), we finally obtain

$$C_{n,l} = \lambda^{\frac{1}{2}l(2k-l+3)+n} e^{-iH(n,l)\varphi} C_{0,0}. \quad (39)$$

Note that for $l = k$ ($\Rightarrow n = 0$), Eq. (39) becomes

$$C_{0,k} = \lambda^{\frac{1}{2}k(k+3)} e^{-iH(0,k)\varphi} C_{0,0}. \quad (40)$$

The introduction of (40) in (34) produces the condition

$$\lambda^d = 1.$$

Consequently, the λ eigenvalues are

$$\lambda = \exp\left(\frac{2\pi i}{d}m\right), \quad m = 0, 1, \dots, d-1.$$

Finally, the normalized eigenvectors of the E_d operator read

$$|\psi\rangle \equiv |m, \varphi\rangle = \frac{1}{\sqrt{d}} \sum_{l=0}^k q^{\frac{1}{2}ml(2k-l+3)} \sum_{n=0}^{k-l} q^{mn} e^{-iH(n,l)\varphi} |n, l\rangle, \quad (41)$$

where

$$q = \exp\left(\frac{2\pi i}{d}\right). \quad (42)$$

The $|m, \varphi\rangle$ states are labeled by the parameters $m \in \mathbb{Z}/d\mathbb{Z}$ and $\varphi \in \mathbb{R}$. They satisfy

$$E_d |m, \varphi\rangle = e^{i\theta_m} |m, \varphi\rangle, \quad \theta_m = m\frac{2\pi}{d}, \quad m = 0, 1, \dots, d-1.$$

As a conclusion, E_d is a unitary phase operator.

The $|m, \varphi\rangle$ phase states satisfy interesting properties:

- They are temporally stable under time evolution, i.e.,

$$e^{-iHt} |m, \varphi\rangle = |m, \varphi + t\rangle$$

for any value of the real parameter t .

- For fixed φ , they satisfy the relation

$$|\langle n, l | m, \varphi \rangle| = \frac{1}{\sqrt{d}}$$

and the closure property

$$\sum_{m=0}^{d-1} |m, \varphi\rangle \langle m, \varphi| = \sum_{l=0}^k \sum_{n=0}^{k-l} |n, l\rangle \langle n, l| = I.$$

- The overlap between two phase states $|m', \varphi'\rangle$ and $|m, \varphi\rangle$ reads

$$\langle m, \varphi | m', \varphi' \rangle = \frac{1}{d} \sum_{l=0}^k \sum_{n=0}^{k-l} q^{\tau(m'-m, \varphi-\varphi', n, l)},$$

where

$$\tau(m' - m, \varphi - \varphi', n, l) = (m' - m) \left[\frac{1}{2} l (2k - l + 3) + n \right] + \frac{d}{2\pi} (\varphi - \varphi') H(n, l)$$

with q defined in (42). As a particular case, we have the orthonormality relation

$$\langle m, \varphi | m', \varphi' \rangle = \delta_{m, m'}.$$

However, the temporally stable phase states are not all orthogonal.

3.2.4 The $k = 1$ particular case

To close Section 3.2, we now establish a contact with the results of Klimov *et al.*^[15] which correspond to $k = 1$ (i.e., $\kappa = -1$). In this particular case, the \mathcal{F}_κ Fock space is three-dimensional ($d = 3$). It corresponds to the representation space of SU_3 relevant for ordinary quarks and antiquarks in particle physics and for qutrits in quantum information. For the purpose of comparison, we put

$$|\phi_1\rangle \equiv |0, 0\rangle, \quad |\phi_2\rangle \equiv |1, 0\rangle, \quad |\phi_3\rangle \equiv |0, 1\rangle.$$

Then, the operators E_{13} , E_{23} , E_{33} and E_3 assume the form

$$\begin{aligned} E_{13} &= e^{i\varphi} |\phi_1\rangle\langle\phi_2| + e^{-i\varphi} |\phi_2\rangle\langle\phi_1| + |\phi_3\rangle\langle\phi_3| \\ E_{23} &= e^{i\varphi} |\phi_1\rangle\langle\phi_3| + e^{-i\varphi} |\phi_3\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2| \\ E_{33} &= |\phi_2\rangle\langle\phi_3| + |\phi_3\rangle\langle\phi_2| + |\phi_1\rangle\langle\phi_1| \\ E_3 &= e^{i\varphi} |\phi_1\rangle\langle\phi_2| + |\phi_2\rangle\langle\phi_3| + e^{-i\varphi} |\phi_3\rangle\langle\phi_1|. \end{aligned}$$

Operators E_{13} , E_{23} and E_{33} have a form similar to that of the phase operators

$$\begin{aligned} \hat{E}_{12} &= |\phi_1\rangle\langle\phi_2| - |\phi_2\rangle\langle\phi_1| + |\phi_3\rangle\langle\phi_3| \\ \hat{E}_{13} &= |\phi_1\rangle\langle\phi_3| - |\phi_3\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2| \\ \hat{E}_{23} &= |\phi_2\rangle\langle\phi_3| - |\phi_3\rangle\langle\phi_2| + |\phi_1\rangle\langle\phi_1| \end{aligned}$$

introduced in Ref. [15] in connection with qutrits. Although the E_{13} , E_{23} and E_{33} operators derived in the present work cannot be deduced from the \hat{E}_{12} , \hat{E}_{13} and \hat{E}_{23} operators

of Ref. [15] by means of similarity transformations, the two sets of operators are equivalent in the sense that their action on the $|\phi_1\rangle$, $|\phi_2\rangle$ and $|\phi_3\rangle$ vectors are identical up to phase factors. In addition, in the case where we do not take into account the spectator state ($|\phi_3\rangle$, $|\phi_2\rangle$ or $|\phi_1\rangle$ for E_{13} , E_{23} or E_{33} , respectively), our SU_3 phase operators are reduced to SU_2 phase operators which present the same periodicity condition (i.e., their square is the identity operator) as the SU_2 phase operators of Ref. [23]. In the $\varphi = 0$ case, our SU_2 phase states turn out to be identical to the phase states derived by Vourdas.^[23] Finally, note that the E_3 (and, more generally, E_d) operator is new; it has no equivalent in Ref. [15].

3.3 Vector phase states in finite dimension

We have now the necessary tools for introducing vector phase states associated with the unitary phase operators E_{1d} , E_{2d} and E_{3d} . We give below a construction similar to the one discussed in Ref. [2].

3.3.1 Vector phase states for E_{1d} and E_{2d}

To define vector phase states, we introduce the $(k+1) \times (k+1)$ -matrix

$$\mathbf{Z} = \text{diag}(z_0, z_1, \dots, z_k), \quad z_l = q_l^m$$

and the $(k+1) \times 1$ -vector

$$[n, l] = \begin{pmatrix} 0 \\ \vdots \\ |n, l\rangle \\ \vdots \\ 0 \end{pmatrix},$$

where the $|n, l\rangle$ entry appears on the l -th line (with $l = 0, 1, \dots, k$). Then, let us define

$$[l, m, \varphi] = \frac{1}{\sqrt{k-l+1}} \sum_{n=0}^{k-l} e^{-iH(n,l)\varphi} \mathbf{Z}^n [n, l]. \quad (43)$$

From Eq. (26), we have

$$[l, m, \varphi] = \begin{pmatrix} 0 \\ \vdots \\ |l, m, \varphi\rangle \\ \vdots \\ 0 \end{pmatrix}, \quad (44)$$

where $|l, m, \varphi\rangle$ occurs on the l -th line.

We shall refer the states (44) to as vector phase states. In this matrix presentation, it is useful to associate the matrix

$$\mathbf{E}_{1d} = \text{diag}(E_{1d}(0), E_{1d}(1), \dots, E_{1d}(k))$$

with the unitary phase operator E_{1d} . It is easy to check that \mathbf{E}_{1d} satisfies the matrix eigenvalue equation

$$\mathbf{E}_{1d}[l, m, \varphi] = e^{i\theta_m}[l, m, \varphi], \quad \theta_m = m \frac{2\pi}{k-l+1}$$

(cf. Eq. (27)).

Other properties of vector phase states $[l, m, \varphi]$ can be deduced from those of phase states $|l, m, \varphi\rangle$. For instance, we obtain

- The temporal stability condition

$$e^{-iHt}[l, m, \varphi] = [l, m, \varphi + t]$$

for t real.

- The closure relation

$$\bigoplus_{l=0}^k \sum_{m=0}^{k-l} [l, m, \varphi][l, m, \varphi]^\dagger = \mathbf{I}_d,$$

where \mathbf{I}_d is the unit matrix of dimension $d \times d$ with d given by (14).

Similar vector phase states can be obtained for E_{2d} by permuting the n and l quantum numbers occurring in the derivation of the vector phase states for E_{1d} .

3.3.2 Vector phase states for E_{3d}

Let us define the diagonal matrix of dimension $(k+1) \times (k+1)$

$$\mathbf{W} = \text{diag}(w_0, w_1, \dots, w_k), \quad w_l = \omega_l^m$$

and the column vector of dimension $(k+1) \times 1$

$$[[n-l, n]] = \begin{pmatrix} 0 \\ \vdots \\ |l-n, n\rangle \\ \vdots \\ 0 \end{pmatrix},$$

where the $|l - n, n\rangle$ state occurs on the l -th line (with $l = 0, 1, \dots, k$). By defining

$$[[l, m, \varphi]] = \frac{1}{\sqrt{l+1}} e^{-iH(l,0)\varphi} \sum_{n=0}^l \mathbf{W}^n [[l - n, n]],$$

we obtain

$$[[l, m, \varphi]] = \begin{pmatrix} 0 \\ \vdots \\ ||l, m, \varphi\rangle\rangle \\ \vdots \\ 0 \end{pmatrix}, \quad (45)$$

where the $||l, m, \varphi\rangle\rangle$ phase state appears on the l -th line.

Equation (45) defines vector phase states associated with the E_{3d} phase operator. These states satisfy the eigenvalue equation

$$\mathbf{E}_{3d} [[l, m, \varphi]] = e^{i\theta_m} [[l, m, \varphi]], \quad \theta_m = m \frac{2\pi}{l+1},$$

where

$$\mathbf{E}_{3d} = \text{diag} (E_{3d}(0), E_{3d}(1), \dots, E_{3d}(k)).$$

The $[[l, m, \varphi]]$ vector phase states satisfy properties which can be deduced from those of the $[l, m, \varphi]$ vector phase states owing to simple correspondence rules.

4 PHASE OPERATORS AND PHASE STATES FOR $\mathcal{A}_\kappa(2)$ WITH $\kappa \geq 0$

4.1 Phase operators in infinite dimension

In the case $\kappa \geq 0$, we can decompose the Jacobson operators a_i^- and a_i^+ as

$$a_i^- = E_{i\infty} \sqrt{F_i(N_1, N_2)}, \quad a_i^+ = \sqrt{F_i(N_1, N_2)} (E_{i\infty})^\dagger, \quad i = 1, 2, \quad (46)$$

where

$$E_{1\infty} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} e^{i[H(n_1+1, n_2) - H(n_1, n_2)]\varphi} |n_1, n_2\rangle \langle n_1 + 1, n_2| \quad (47)$$

$$E_{2\infty} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} e^{i[H(n_1, n_2+1) - H(n_1, n_2)]\varphi} |n_1, n_2\rangle \langle n_1, n_2 + 1|. \quad (48)$$

The operators $E_{i\infty}$, $i = 1, 2$, satisfy

$$E_{1\infty} (E_{1\infty})^\dagger = I, \quad (E_{1\infty})^\dagger E_{1\infty} = I - \sum_{n_2=0}^{\infty} |0, n_2\rangle \langle 0, n_2| \quad (49)$$

$$E_{2\infty} (E_{2\infty})^\dagger = I, \quad (E_{2\infty})^\dagger E_{2\infty} = I - \sum_{n_1=0}^{\infty} |n_1, 0\rangle \langle n_1, 0|. \quad (50)$$

Equations (49) and (50) show that $E_{i\infty}$, $i = 1, 2$, are not unitary operators.

In a similar way, operators a_3^+ and a_3^- can be rewritten

$$a_3^- = -\kappa E_{3\infty} \sqrt{(N_1 + 1)N_2}, \quad a_3^+ = -\kappa \sqrt{(N_1 + 1)N_2} (E_{3\infty})^\dagger,$$

where

$$E_{3\infty} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} |n_1 + 1, n_2\rangle \langle n_1, n_2 + 1|.$$

The $E_{3\infty}$ operator is not unitary since

$$E_{3\infty} (E_{3\infty})^\dagger = I - \sum_{n_2=0}^{\infty} |0, n_2\rangle \langle 0, n_2|, \quad (E_{3\infty})^\dagger E_{3\infty} = I - \sum_{n_1=0}^{\infty} |n_1, 0\rangle \langle n_1, 0|,$$

to be compared with (49) and (50).

The $E_{3\infty}$ operator is not independent of $E_{1\infty}$ and $E_{2\infty}$. Indeed, it can be expressed as

$$E_{3\infty} = (E_{1\infty})^\dagger E_{2\infty}, \quad (51)$$

a relation of central importance for deriving its eigenvalues (see Section 4.2).

4.2 Phase states in infinite dimension

It is easy to show that operators $E_{1\infty}$ and $E_{2\infty}$ commute. Hence, that they can be simultaneously diagonalized. In this regard, let us consider the eigenvalue equations

$$E_{1\infty} |z_1, z_2\rangle = z_1 |z_1, z_2\rangle, \quad E_{2\infty} |z_1, z_2\rangle = z_2 |z_1, z_2\rangle, \quad (z_1, z_2) \in \mathbb{C}^2, \quad (52)$$

where

$$|z_1, z_2\rangle = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} D_{n_1, n_2} |n_1, n_2\rangle.$$

By using the definitions of the nonunitary phase operators (47) and (48), it is easy to check from the eigenvalue equations (52) that the complex coefficients D_{n_1, n_2} satisfy the following recurrence relations

$$D_{n_1+1, n_2} e^{iH(n_1+1, n_2)\varphi} = z_1 D_{n_1, n_2} e^{iH(n_1, n_2)\varphi} \quad (53)$$

$$D_{n_1, n_2+1} e^{iH(n_1, n_2+1)\varphi} = z_2 D_{n_1, n_2} e^{iH(n_1, n_2)\varphi}, \quad (54)$$

which lead to

$$D_{n_1, n_2} = e^{-iH(n_1, n_2)\varphi} z_1^{n_1} z_2^{n_2} D_{0,0}.$$

It follows that the normalized common eigenstates of the operators $E_{1\infty}$ and $E_{2\infty}$ are given by

$$|z_1, z_2\rangle = \sqrt{(1 - |z_1|^2)(1 - |z_2|^2)} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} z_1^{n_1} z_2^{n_2} e^{-iH(n_1, n_2)\varphi} |n_1, n_2\rangle$$

on the domain $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\}$. Following the method developed in Refs. [24, 26] for the Lie algebra $su_{1,1}$ and in Ref. [5] for the algebra $\mathcal{A}_\kappa(1)$, we define the states

$$|\theta_1, \theta_2, \varphi\rangle = \lim_{z_1 \rightarrow e^{i\theta_1}} \lim_{z_2 \rightarrow e^{i\theta_2}} \frac{1}{\sqrt{(1 - |z_1|^2)(1 - |z_2|^2)}} |z_1, z_2\rangle,$$

where $\theta_1, \theta_2 \in [-\pi, +\pi]$. We thus get

$$|\theta_1, \theta_2, \varphi\rangle = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} e^{in_1\theta_1} e^{in_2\theta_2} e^{-iH(n_1, n_2)\varphi} |n_1, n_2\rangle.$$

These states, defined on $S^1 \times S^1$, turn out to be phase states since we have

$$E_{1\infty} |\theta_1, \theta_2, \varphi\rangle = e^{i\theta_1} |\theta_1, \theta_2, \varphi\rangle, \quad E_{2\infty} |\theta_1, \theta_2, \varphi\rangle = e^{i\theta_2} |\theta_1, \theta_2, \varphi\rangle.$$

Hence, the operators $E_{i\infty}$, $i = 1, 2$, are (nonunitary) phase operators.

The main properties of the $|\theta_1, \theta_2, \varphi\rangle$ states are the following.

- They are temporally stable in the sense that

$$e^{-iHt} |\theta_1, \theta_2, \varphi\rangle = |\theta_1, \theta_2, \varphi + t\rangle,$$

with t real.

- They are not normalized and not orthogonal. However, for fixed φ , they satisfy the closure relation

$$\frac{1}{(2\pi)^2} \int_{-\pi}^{+\pi} d\theta_1 \int_{-\pi}^{+\pi} d\theta_2 |\theta_1, \theta_2, \varphi\rangle \langle \theta_1, \theta_2, \varphi| = I.$$

In view of Eq. (51), we have

$$E_{3\infty} |\theta_1, \theta_2, \varphi\rangle = e^{i(\theta_2 - \theta_1)} |\theta_1, \theta_2, \varphi\rangle,$$

so that the $|\theta_1, \theta_2, \varphi\rangle$ states are common eigenstates to $E_{1\infty}$, $E_{2\infty}$ and $E_{3\infty}$.

To close, a comparison is in order. For $\varphi = 0$, the $|\theta_1, \theta_2, 0\rangle$ states have the same form as the phase states derived in Ref. [6] which present the closure property but are not temporally stable.

5 TRUNCATED GENERALIZED OSCILLATOR ALGEBRA

For $\kappa \geq 0$ the \mathcal{F}_κ Hilbert space associated with $\mathcal{A}_\kappa(2)$ is infinite-dimensional and it is thus impossible to define a unitary phase operator. On the other hand, for $\kappa < 0$ the \mathcal{F}_κ space is finite-dimensional and there is no problem to define unitary phase operators. Therefore, for $\kappa \geq 0$ it is appropriate to truncate the \mathcal{F}_κ space in order to get a subspace $\mathcal{F}_{\kappa,\sigma}$ of dimension $(\sigma + 1)(\sigma + 2)/2$ with σ playing the role of k . Then, it will be possible to define unitary phase operators and vector phase vectors for the $\mathcal{F}_{\kappa,\sigma}$ truncated space with $\kappa \geq 0$. To achieve this goal, we shall adapt the truncation procedure discussed in Ref. [20] for the h_4 Weyl-Heisenberg algebra and in Ref. [3, 5] for the $\mathcal{A}_\kappa(1)$ algebra with $\kappa \geq 0$.

The restriction of infinite-dimensional space \mathcal{F}_κ ($\kappa \geq 0$) to finite-dimensional space $\mathcal{F}_{\kappa,\sigma}$ with basis

$$\{|n_1, n_2\rangle : n_1, n_2 \text{ ranging} \mid n_1 + n_2 \leq \sigma\}$$

can be done by means of the projection operator

$$\Pi_\sigma = \sum_{n_1=0}^{\sigma} \sum_{n_2=0}^{\sigma-n_1} |n_1, n_2\rangle \langle n_1, n_2| = \sum_{n_2=0}^{\sigma} \sum_{n_1=0}^{\sigma-n_2} |n_1, n_2\rangle \langle n_1, n_2|.$$

Let us then define the four new ladder operators

$$b_i^\pm = \Pi_\sigma a_i^\pm \Pi_\sigma, \quad i = 1, 2.$$

They can be rewritten as

$$b_1^+ = (b_1^-)^\dagger = \sum_{n_2=0}^{\sigma-1} \sum_{n_1=0}^{\sigma-n_2-1} \sqrt{F_1(n_1+1, n_2)} e^{-i[H(n_1+1, n_2)-H(n_1, n_2)]\varphi} |n_1+1, n_2\rangle \langle n_1, n_2|$$

$$b_2^+ = (b_2^-)^\dagger = \sum_{n_1=0}^{\sigma-1} \sum_{n_2=0}^{\sigma-n_1-1} \sqrt{F_2(n_1, n_2+1)} e^{-i[H(n_1, n_2+1)-H(n_1, n_2)]\varphi} |n_1, n_2+1\rangle \langle n_1, n_2|$$

A straightforward calculation shows that the action of b_1^\pm on \mathcal{F}_κ is given by

$$b_1^+ |n_1, n_2\rangle = \sqrt{F_1(n_1+1, n_2)} e^{-i[H(n_1+1, n_2)-H(n_1, n_2)]\varphi} |n_1+1, n_2\rangle$$

for $n_1 + n_2 = 0, 1, \dots, \sigma - 1$

$$b_1^+ |\sigma - n_2, n_2\rangle = 0 \quad \text{for } n_2 = 0, 1, \dots, \sigma$$

$$b_1^+ |n_1, n_2\rangle = 0 \quad \text{for } n_1 + n_2 = \sigma, \sigma + 1, \sigma + 2, \dots$$

and

$$b_1^- |n_1, n_2\rangle = \sqrt{F_1(n_1, n_2)} e^{+i[H(n_1, n_2)-H(n_1-1, n_2)]\varphi} |n_1-1, n_2\rangle$$

for $n_1 \neq 0$ and $n_2 = 0, 1, \dots, \sigma - 1$

$$b_1^- |0, n_2\rangle = 0 \quad \text{for } n_2 = 0, 1, \dots, \sigma$$

$$b_1^- |n_1, n_2\rangle = 0 \quad \text{for } n_1 + n_2 = \sigma + 1, \sigma + 2, \sigma + 3, \dots$$

Similarly, we have

$$b_2^+ |n_1, n_2\rangle = \sqrt{F_2(n_1, n_2+1)} e^{-i[H(n_1, n_2+1)-H(n_1, n_2)]\varphi} |n_1, n_2+1\rangle$$

for $n_1 + n_2 = 0, 1, \dots, \sigma - 1$

$$b_2^+ |n_1, \sigma - n_1\rangle = 0 \quad \text{for } n_1 = 0, 1, \dots, \sigma$$

$$b_2^+ |n_1, n_2\rangle = 0 \quad \text{for } n_1 + n_2 = \sigma, \sigma + 1, \sigma + 2, \dots$$

and

$$b_2^- |n_1, n_2\rangle = \sqrt{F_2(n_1, n_2)} e^{+i[H(n_1, n_2)-H(n_1, n_2-1)]\varphi} |n_1, n_2-1\rangle$$

for $n_2 \neq 0$ and $n_1 = 0, 1, \dots, \sigma - 1$

$$b_2^- |n_1, 0\rangle = 0 \quad \text{for } n_1 = 0, 1, \dots, \sigma$$

$$b_2^- |n_1, n_2\rangle = 0 \quad \text{for } n_1 + n_2 = \sigma + 1, \sigma + 2, \sigma + 3, \dots$$

Therefore, the action of operators b_i^\pm ($i = 1, 2$) on $\mathcal{F}_{\kappa, \sigma}$ with $\kappa \geq 0$ is similar to that of a_i^\pm ($i = 1, 2$) on \mathcal{F}_κ with $\kappa < 0$.

We may ask what is the algebra generated by operators b_i^\pm and N_i ($i = 1, 2$)? Indeed, the latter operators satisfy the following algebraic relations when acting on the $\mathcal{F}_{\kappa,\sigma}$ space

$$\begin{aligned} [b_1^-, b_1^+] &= I + \kappa(2N_1 + N_2) - \sum_{l=0}^{\sigma} F_1(\sigma - l + 1, l) |\sigma - l, l\rangle \langle \sigma - l, l| \\ [b_2^-, b_2^+] &= I + \kappa(2N_2 + N_1) - \sum_{l=0}^{\sigma} F_2(l, \sigma - l + 1) |l, \sigma - l\rangle \langle l, \sigma - l| \\ [N_i, b_j^\pm] &= \pm \delta_{i,j} b_i^\pm, \quad i, j = 1, 2 \\ [b_i^\pm, b_j^\pm] &= 0, \quad [b_i^\pm, [b_i^\pm, b_j^\mp]] = 0, \quad i \neq j. \end{aligned}$$

Operators b_i^\pm and N_i ($i = 1, 2$) acting on $\mathcal{F}_{\kappa,\sigma}$ generate an algebra, noted $\mathcal{A}_{\kappa,\sigma}(2)$. The $\mathcal{A}_{\kappa,\sigma}(2)$ algebra generalizes $\mathcal{A}_{\kappa,s}(1)$ which results from the truncation of the $\mathcal{A}_\kappa(1)$ algebra.^[5] By using the trick to pass from $\mathcal{A}_\kappa(2)$ to $\mathcal{A}_\kappa(1)$, see section 2.1, we get $\mathcal{A}_{\kappa,s-1}(2) \rightarrow \mathcal{A}_{\kappa,s}(1)$. The $\mathcal{A}_{\kappa,s}(1)$ truncated algebra gives in turn the Pegg-Barnett truncated algebra^[20] when $\kappa \rightarrow 0$.

As a conclusion, the action of b_i^\pm ($i = 1, 2$) on the complement of $\mathcal{F}_{\kappa,\sigma}$ with respect to \mathcal{F}_κ leads to the null vector while the action of these operators on the $\mathcal{F}_{\kappa,\sigma}$ space with $\kappa \geq 0$ is the same as the action of a_i^\pm ($i = 1, 2$) on the \mathcal{F}_κ space with $\kappa < 0$ modulo some evident changes of notations. It is thus possible to apply the procedure developed for \mathcal{F}_κ space with $\kappa < 0$ in order to obtain unitary phase operators on $\mathcal{F}_{\kappa,\sigma}$ with $\kappa \geq 0$ and the corresponding vector phase states. The derivation of the vector phase states for the $\mathcal{A}_{\kappa,\sigma}(2)$ truncated algebra can be done simply by replacing k by σ . In this respect, the σ truncation index can be compared to the k quenching index (or Chen index) used for characterizing the finite-dimensional representation $(0, k)$ or $(k, 0)$ of SU_3 .^[28]

6 APPLICATION TO MUTUALLY UNBIASED BASES

We now examine the possibility to produce specific bases, known as mutually unbiased bases (MUBs) in quantum information, for finite-dimensional Hilbert spaces from the phase states of $E_{1d}(l)$, $E_{2d}(l)$ and $E_{3d}(l)$. Let us recall that two distinct orthonormal bases

$$\{|a\alpha\rangle : \alpha = 0, 1, \dots, N - 1\}$$

and

$$\{|b\beta\rangle : \beta = 0, 1, \dots, N - 1\}$$

of the N -dimensional Hilbert spaces \mathbb{C}^N are said to be unbiased if and only if

$$\forall \alpha = 0, 1, \dots, N-1, \quad \forall \beta = 0, 1, \dots, N-1 : |\langle a\alpha | b\beta \rangle| = \frac{1}{\sqrt{N}}$$

(cf. Refs. [9, 16, 17, 27]).

We begin with the $|l, m, \varphi\rangle$ phase states associated with the $E_{1d}(l)$ phase operator (see (26) and (27)). In Eq. (26), l can take the values $0, 1, \dots, k$. Let us put $l = 0$ and switch to the notations

$$k \equiv N-1, \quad m \equiv \alpha, \quad |n, 0\rangle \equiv |N-1-n\rangle$$

(with $\alpha, n = 0, 1, \dots, N-1$) for easy comparison with some previous works. Then, Eq. (26) becomes

$$|0, \alpha, \varphi\rangle = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \exp \left[-\frac{i}{N-1} n(N-n)\varphi + \frac{2\pi i}{N} n\alpha \right] |N-1-n\rangle. \quad (55)$$

For $\varphi = 0$, Eq. (55) describes a (quantum) discrete Fourier transform^[25] that allows to pass from the set $\{|N-1-n\rangle : n = 0, 1, \dots, N-1\}$ of cardinal N to the set $\{|0, \alpha, 0\rangle : \alpha = 0, 1, \dots, N-1\}$ of cardinal N too. In the special case where φ is quantized as

$$\varphi = -\pi \frac{N-1}{N} a, \quad a = 0, 1, \dots, N-1, \quad (56)$$

equation (55) leads to

$$|0, \alpha, \varphi\rangle \equiv |a\alpha\rangle = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} q_0^{n(N-n)a/2+n\alpha} |N-1-n\rangle, \quad (57)$$

where

$$q_0 = \exp \left(\frac{2\pi i}{N} \right).$$

Equation (57) with $a \neq 0$ corresponds to a (quantum) quadratic discrete Fourier transform.^[11, 12, 13] In this regard, note that the $|a\alpha\rangle$ state in (57) can be identified with the $|a\alpha; r\rangle$ state with $r = 0$ discussed recently in the framework of the quadratic discrete Fourier transform.^[14] Following Ref. [14], we consider the set

$$B_N = \{|N-1-n\rangle : n = 0, 1, 2, \dots, N-1\} = \{|n\rangle : n = 0, 1, 2, \dots, N-1\}$$

as an orthonormal basis for the N -dimensional Hilbert space. This basis is called computational basis in quantum information. Then, the sets

$$B_{0a} = \{ |a\alpha\rangle : \alpha = 0, 1, 2, \dots, N-1 \}, \quad a = 0, 1, 2, \dots, N-1$$

constitute N new orthonormal bases of the space. The B_{0a} basis is a special case, corresponding to $r = 0$, of the B_{ra} bases derived in Ref. [14] from a polar decomposition of the su_2 Lie algebra. The overlap between two bases B_{0a} and B_{0b} is given by

$$\langle a\alpha | b\beta \rangle = \frac{1}{N} \sum_{n=0}^{N-1} q_0^{n(N-n)(b-a)/2+n(\beta-\alpha)},$$

a relation which can be expressed in term of the generalized Gauss sum^[4]

$$S(u, v, w) = \sum_{k=0}^{|w|-1} e^{i\pi(uk^2+vk)/w}.$$

In fact, we obtain

$$\langle a\alpha | b\beta \rangle = \frac{1}{N} S(u, v, w), \quad (58)$$

with

$$u = a - b, \quad v = -(a - b)N - 2(\alpha - \beta), \quad w = N.$$

In the case where N is a prime integer, the calculation of $S(u, v, w)$ in (58) yields

$$|\langle a\alpha | b\beta \rangle| = \frac{1}{\sqrt{N}}, \quad a \neq b, \quad \alpha, \beta = 0, 1, \dots, N-1, \quad N \text{ prime.} \quad (59)$$

On the other hand, it is evident that

$$|\langle n | a\alpha \rangle| = \frac{1}{\sqrt{N}}, \quad n, \alpha = 0, 1, \dots, N-1 \quad (60)$$

holds for any strictly positive value of N . As a result, Eqs. (59) and (60) shows that bases B_N and B_{0a} with $a = 0, 1, \dots, N-1$ provide a complete set of $N+1$ MUBs when N is a prime integer.

A similar result can be derived by quantizing, according to (56), the φ parameter occurring in the eigenstates of $E_{2d}(0)$.

The form of the $E_{3d}(l)$ phase operator being different from those of $E_{1d}(l)$ and $E_{2d}(l)$, we proceed in a different way for obtaining MUBs from the $||l, m, \varphi\rangle\rangle$ eigenstates of

$E_{3d}(l)$ (see (28) and (29)). We put $\varphi = 0$ in (28) and apply the $e^{-iF_3(N_1, N_2)\varphi}$ operator on the resultant state. This gives

$$e^{-iF_3(N_1, N_2)\varphi} ||l, m, 0\rangle\rangle = \frac{1}{\sqrt{l+1}} \sum_{n=0}^l \exp \left[-i \frac{1}{k^2} n(l+1-n)\varphi \right] \omega_l^{mn} |l-n, n\rangle.$$

For the sake of comparison, we introduce

$$l \equiv N-1, \quad m \equiv \alpha, \quad \omega_{N-1} \equiv \exp \left(\frac{2\pi i}{N} \right), \quad |l-n, n\rangle \equiv |N-1-n\rangle$$

and we quantize φ via

$$\varphi = -\pi \frac{k^2}{N} a, \quad a = 0, 1, \dots, N-1.$$

Hence, the vector

$$e^{-iF_3(N_1, N_2)\varphi} ||l, m, 0\rangle\rangle \equiv |a\alpha\rangle$$

reads

$$|a\alpha\rangle = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \omega_{N-1}^{n(N-n)a/2+n\alpha} |N-1-n\rangle, \quad (61)$$

which bears the same form as (57). Consequently for N a prime integer, Eq. (61) generates N MUBs B_{0a} with $a = 0, 1, \dots, N-1$ which together with the computational basis B_N form a complete set of $N+1$ MUBs.

7 CONCLUDING REMARKS

The main results of this work are the following.

The su_3 , $su_{2,1}$ and $h_4 \otimes h_4$ algebras can be described in an unified way via the introduction of the $\mathcal{A}_\kappa(2)$ algebra. A quantum system with a quadratic spectrum (for $\kappa \neq 0$) is associated with $\mathcal{A}_\kappa(2)$; for $\kappa = 0$, this system coincides with the two-dimensional isotropic harmonic oscillator.

In the case $\kappa < 0$, the unitary phase operators (E_{1d} , E_{2d} and E_{3d}) defined in this paper generalize those constructed in Ref. [15] for an su_3 three-level system (corresponding to $d = 3$); they give rise to new phase states, namely, vector phase states which are eigenstates obtained along lines similar to those developed in Ref. [2, 22] for obtaining a vectorial generalization of the coherent states introduced in Ref. [7]. Still for $\kappa < 0$, a

new type of unitary phase operator (E_d) can be defined ; its specificity is to span all vectors of the d -dimensional representation space of $\mathcal{A}_\kappa(2)$ from any vector of the space.

In the case $\kappa \geq 0$, it is possible to define nonunitary phase operators. They can be turned to unitary phase operators by truncating (to some finite but arbitrarily large order) the representation space of $\mathcal{A}_\kappa(2)$. This leads to a truncated generalized oscillator algebra ($\mathcal{A}_{\kappa,\sigma}(2)$) that can be reduced to the Pegg-Barnett truncated oscillator algebra^[20] through an appropriate limiting process where $\kappa \rightarrow 0$.

Among the various properties of the phase states and vector phase states derived for $\kappa < 0$ and $\kappa \geq 0$, the property of temporal stability is essential. It has no equivalent in Ref. [23]. In last analysis, this property results from the introduction of a phase factor (φ) in the action of the annihilation and creation operators of $\mathcal{A}_\kappa(2)$. As an unexpected result, the quantization of this phase factor allows to derive mutually unbiased bases from temporally stable phase states for $\kappa < 0$. This is a further evidence that “phases do matters after all”^[17] and are important in quantum mechanics.

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