BILINEAR SOBOLEV-POINCARÉ INEQUALITIES AND LEIBNIZ-TYPE RULES

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Abstract. The dual purpose of this article is to establish bilinear Poincaré-type estimates associated to an approximation of the identity and to explore the connections between bilinear pseudo-differential operators and bilinear potential-type operators. The common underlying theme in both topics is their applications to Leibniz-type rules in Sobolev and Campanato-Morrey spaces under Sobolev scaling.

1. Introduction

Leibniz-type rules quantify the regularity of a product of functions in terms of the regularity of its factors. In this sense, Leibniz-type rules are represented by inequalities of the form

$$\|fg\|_Z \lesssim \|f\|_{X_1}\|g\|_{Y_1} + \|f\|_{X_2}\|g\|_{Y_2}, \tag{1.1}$$

where X_1, X_2, Y_1, Y_2, and Z are appropriate functional spaces. Along these lines, perhaps the better-known Leibniz-type rules correspond to the fractional Leibniz rules, pioneered by Kato-Ponce [33], Christ-Weinstein [16] and Kenig-Ponce-Vega [34] in their work on PDEs, where the spaces X_1, X_2, Y_1, Y_2, and Z belong to the scale of Sobolev spaces $W^{s,p}$; namely,

$$\|fg\|_{W^{s,q}} \lesssim \|f\|_{W^{s,p_1}}\|g\|_{L^{p_2}} + \|f\|_{L^{p_1}}\|g\|_{W^{s,p_2}}, \tag{1.2}$$

where $s \geq 0$ and

$$\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} \quad \text{with } 1 < p_1, p_2 < \infty. \tag{1.3}$$

The estimates (1.2) follow as a consequence of boundedness properties on products of Lebesgue spaces of bilinear Coifman-Meyer multipliers ([18, 19, 30]): If $\sigma \in \mathcal{C}^\infty(\mathbb{R}^{2n})$ satisfies

$$|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi,\eta)| \leq C_{\alpha,\beta}(|\xi|+|\eta|)^{-(|\alpha|+|\beta|)}, \quad \xi,\eta \in \mathbb{R}^n, \quad \alpha,\, \beta \in \mathbb{N}_0^n, \tag{1.4}$$

and

$$T_\sigma(f,g)(x) := \int_{\mathbb{R}^{2n}} \sigma(\xi,\eta)\hat{f}(\xi)\hat{g}(\eta)e^{ix\cdot(\xi+\eta)}\, d\xi\, d\eta, \qquad x \in \mathbb{R}^n, f, g \in \mathcal{S}(\mathbb{R}^n),$$

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then T_σ is bounded from $L^{p_1} \times L^{p_2}$ into L^q, where p_1, p_2, and q conform to the Hölder scaling (1.3). Then, inequalities (1.2) are obtained from this result after observing that, by frequency decoupling, the identity

$$J^s(fg)(x) = T_{\sigma_1}(J^s f, g)(x) + T_{\sigma_2}(f, J^s g)(x), \tag{1.5}$$

holds true for some bilinear symbols σ_1 and σ_2 satisfying (1.4) and

$$\widehat{J^s(h)}(\xi) := (1 + |\xi|^2)^{s/2}\, \hat{h}(\xi), \quad \xi \in \mathbb{R}^n, s > 0, h \in \mathcal{S}(\mathbb{R}^n).$$

Two immediate conclusions can be derived from this approach. First, since the symbol $\sigma_0 \equiv 1$ satisfies (1.4) and yields, through T_{σ_0}, the product of two functions, the Hölder scaling (1.3) occurs naturally. Second, the identity (1.5) can be exploited to produce Leibniz-type rules (1.1) involving function spaces that interact well with the Bessel potentials J_s (for example, Besov and Triebel-Lizorkin spaces) provided that mapping properties for bilinear multipliers T_σ are established for such spaces. Indeed, implementations of this program (see, for instance, [11, 14, 15, 31, 36, 44]), produce Besov, Triebel-Lizorkin, and mixed Besov-Lebesgue Leibniz-type rules.

A Littlewood-Paley-free path towards Leibniz-type rules was introduced in [42] in the scales of Campanato-Morrey spaces. In this context, the role of the identity (1.5) is played by the inequality

$$|f(x)g(x) - f_B g_B| \lesssim \mathcal{I}_1(|\nabla f|\chi_B, |g|\chi_B) + \mathcal{I}_1(|f|\chi_B, |\nabla g|\chi_B), \quad x \in B, \tag{1.6}$$

where $B \subset \mathbb{R}^n$ is a ball, $f, g \in \mathcal{C}^1(B)$, $\mathcal{I}_1$ is a bilinear potential operator, and $f_B := \frac{1}{|B|}\int_B f(x)\,dx$. Inequality (1.6) arises as a bilinear interpretation of the linear inequality

$$|f(x) - f_B| \lesssim I_1(|\nabla f|\chi_B), \quad x \in B,\ f \in \mathcal{C}^1(B), \tag{1.7}$$

where I_1 denotes the Riesz potential of order 1. Inequality (1.7) is usually referred to as a *representation formula* (for the oscillation $|f(x) - f_B|$). In the linear setting, representation formulas and Poincaré inequalities imply embedding of Campanato-Morrey spaces (see, for instance, [40, 41] for such embeddings in the Carnot-Carathéodory framework). As proved in [42], via (1.6), the bilinear analogs to these embeddings come in the form of Campanato-Morrey Leibniz-type rules. More precisely, in the scale of Campanato-Morrey spaces ($\mathcal{L}^{p,\lambda}(w)$ and $L^{q,\lambda}(w)$ below), a typical weighted Leibniz-type rule takes the form (see [42])

$$\|fg\|_{L^{q,\lambda}(w)} \lesssim \|\nabla f\|_{\mathcal{L}^{p_1,\lambda_1}(u)}\|g\|_{\mathcal{L}^{p_2,\lambda_2}(v)} + \|f\|_{\mathcal{L}^{p_1,\lambda_1}(u)}\|\nabla g\|_{\mathcal{L}^{p_2,\lambda_2}(v)}, \tag{1.8}$$

for (a large class of) weights u, v, w and indices $q, \lambda, p_1, \lambda_1$, p_2, and λ_2. In the unweighted case, the natural scaling for (1.8) turns out to be the *bilinear Sobolev scaling*

$$\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{n} \quad \text{with } 1 < p_1, p_2 < \infty. \tag{1.9}$$

From (1.6), it now becomes apparent that the prevailing tools for obtaining inequalities (1.8) rely on boundedness properties of suitable bilinear potential-type operators. Thus, in the scale of Campanato-Morrey spaces, bilinear potential-type operators play the role that paraproducts and the bilinear Coifman-Meyer multipliers play in the proofs of the Sobolev-based Leibniz-type rules (1.2) and their Besov and Triebel-Lizorkin counterparts. Accordingly, the time-frequency Fourier-based tools in the latter are replaced by real-analysis methods in the former.

The purpose of this article is twofold. On the one hand, we further develop time-frequency and real-analysis approaches that allow to prove new Leibniz-type rules in Sobolev and Campanato-Morrey spaces. We do so by separately examining the mapping properties of bilinear pseudo-differential and potential operators, and by later combining them via the inequalities

$$|T_\sigma(f,g)| \lesssim \mathcal{B}_s(|f|,|g|) \quad \text{and} \quad |T_\tau(f,g)| \lesssim \mathcal{I}_s(|f|,|g|), \quad f,g \in \mathcal{S}(\mathbb{R}^n), \tag{1.10}$$

where $\mathcal{B}_s$ is the bilinear fractional integral of order s, introduced and studied in [28] and [35], $\mathcal{I}_s$ is the bilinear Riesz potential of order s introduced in [35], and σ and τ belong to standard classes of bilinear symbols of order $-s$. As a consequence of these newly-found bonds between bilinear pseudo-differential and potential operators, we obtain, among others, Sobolev-based fractional Leibniz rules such as

$$\|fg\|_{W^{m,q}} \lesssim \|f\|_{W^{s+m,p_1}}\|g\|_{L^{p_2}} + \|f\|_{L^{p_1}}\|g\|_{W^{s+m,p_2}}, \quad m \geq 0, \tag{1.11}$$

under the bilinear Sobolev scaling

$$\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{s}{n} \quad \text{with } 1 < p_1, p_2 < \infty,\ s \in (0, 2n). \tag{1.12}$$

The relation (1.12) sheds additional light onto the balance between integrability and smoothness built into inequalities of the type (1.2).

On the other hand, we explore the behavior of the bilinear oscillation $|f(x)g(x) - f_B g_B|$ when the mean-value operator is replaced by an approximation of the identity $\{S_t\}$. Our exposition includes the case of the infinitesimal generator L of an analytic semigroup $\{S_t\}_{t>0}$ on $L^2(\mathbb{R}^n)$ (i.e. $S_t = e^{-tL}$) whose kernel $p_t(x,y)$ has fast-enough off-diagonal decay. The quantity $S_t f = e^{-tL} f$ can be thought of as an average version of f at the scale t and plays the role of f_B for some $t = t_B$, when defining function spaces, such as BMO_L and H^1_L, which better capture properties of the solutions to $Lu = 0$; see for instance [22, 32]. In the linear case, the study of Sobolev-Poincaré inequalities associated to the oscillation $|f - S_{t_B} f|$ has been successfully carried out in [2], yielding the so-called *generalized* or *expanded* Sobolev-Poincaré inequalities; namely,

$$\left(\frac{1}{|B|}\int_B |f - S_{t_B} f|^q\right)^{1/q} \lesssim \sum_{k\in\mathbb{N}_0} \alpha_k\, r(2^k B) \left(\frac{1}{|2^k B|}\int_{2^k B} |\nabla f|^p\right)^{1/p}, \tag{1.13}$$

for suitable choices of indices $1 < p < q$ and sequences $\{\alpha_k\} \subset [0,\infty)$. As described in [2], the presence of the series expansion on the right-hand side of (1.13) accounts for the lack of localization of the approximation of the identity $\{S_t\}$. Also, inequalities like (1.13) are shown to be weaker than the classical ones (Proposition 4.2 in [1]) (and even more, some of them can be proved in some situations where classical inequalities fail, [2]), but are strong enough to sustain Calderón-Zygmund decompositions, see [1]. In the present article, we study bilinear oscillations of the type $|fg - S_{t_B} f S_{t_B} g|$, use them to define *bilinear Campanato-Morrey* spaces associated to

$\{S_t\}$ and estimate them to produce *bilinear generalized Sobolev-Poincaré inequalities* (i.e., bilinear versions of (1.13)) and associated (weighted) Leibniz-type rules.

The article is organized as follows. In Section 2 we recall some definitions and known results on boundedness properties of bilinear fractional integrals in weighted and unweighted Lebesgue spaces that will be useful for our proofs. Throughout the paper, we use upper-case letters to label theorems corresponding to known results (such as Theorem A below) while we use single numbers (with no reference to the section) for theorems, propositions and corollaries that are new and proved in this article.

In Section 3 we establish bilinear Poincaré-type inequalities in the Euclidean setting associated to a general approximation of the identity $\{S_t\}$. These bilinear Poincaré-type inequalities arise as a natural bilinear counterpart of the so-called *(linear) pseudo-Poincaré inequalities* (see [2, 20, 51, 52, 53]). Following the outline above, our proof consists in establishing a bilinear representation formula tailored to the semigroup $\{S_t\}$, which, as expected, turns out to be an expanded version of (1.6), see Theorem 3. This bilinear representation formula involves logarithmic perturbations of the bilinear fractional integral used in [42], whose kernels are proved to still satisfy appropriate growth conditions that guarantee boundedness of the operator on products of weighted Lebesgue spaces. In Section 4 we present applications to Leibniz-type rules for products of functions in Campanato-Morrey spaces associated to a semigroup. In Section 5 we point out relevant extensions to the contexts of doubling Riemannian manifolds and Carnot groups.

In Section 6 we close the circle of ideas developed in Sections 1-3 by relating bilinear pseudo-differential operators of the type

$$T_\sigma(f,g)(x)=\int_{\mathbb{R}^{2n}} e^{ix(\xi+\eta)}\sigma(x,\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)d\xi d\eta, \quad f,g\in\mathcal{S}(\mathbb{R}^n),\ x\in\mathbb{R}^n, \tag{1.14}$$

where now σ is an x-dependent bilinear symbol in a standard class, and bilinear potential operators. In particular, from estimates on the bilinear kernel of T_σ in (1.14), we are able to prove (1.10) and subsequently deduce mapping properties under the bilinear Sobolev scaling (1.12). These mapping properties supply a number of new Leibniz-type rules, see Corollary 11 and Remark 6.3.

The bilinear Poincaré estimates introduced in [42] rely on the oscillation of the pointwise product of two functions (i.e. T_σ with $\sigma\equiv 1$); in turn, they give rise to bilinear Sobolev inequalities of the form

$$\|fg\|_{L^q}\lesssim\|\nabla f\|_{L^{p_1}}\|g\|_{L^{p_2}}+\|f\|_{L^{p_1}}\|\nabla g\|_{L^{p_2}}, \tag{1.15}$$

for exponents $p_1,p_2>1$ and $q>0$ satisfying the Sobolev relation (1.9). These results correspond to the limit of bilinear Poincaré inequalities on balls, by making the radius of the ball tend to infinity. We direct the reader to [42, 46, 47] for other versions of (1.15), including weights and higher order derivatives in the context of Hörmander vector fields. The method presented in Section 6 further substantiates inequalities of the type (1.15) under Sobolev scaling and unifies their study in the language of bilinear pseudo-differential operators.

2. Bilinear fractional integrals and their boundedness properties on weighted Lebesgue spaces

Given a weight w defined on $\mathbb{R}^n$ and $p>0$, the notation L^p_w will be used to refer to the weighted Lebesgue space of all functions $f:\mathbb{R}^n\to\mathbb{C}$ such that $\|f\|^p_{L^p_w}:=\int_{\mathbb{R}^n}|f(x)|^p w(x)\,dx<\infty$, when $w\equiv 1$ we will simply write L^p.

If w_1, w_2 are weights defined on $\mathbb{R}^n$, $1<p_1,\ p_2<\infty$, $q>0$, and $w:=w_1^{q/p_1}w_2^{q/p_2}$, we say that (w_1,w_2) satisfies the $A_{(p_1,p_2),q}$ condition (or that (w_1,w_2) belongs to the class $A_{(p_1,p_2),q}$) if

$$[(w_1,w_2)]_{A_{(p_1,p_2),q}}:=\sup_B\Big(\frac{1}{|B|}\int_B w(x)\,dx\Big)\prod_{j=1}^{2}\Big(\frac{1}{|B|}\int_B w_j(x)^{1-p_j'}\,dx\Big)^{\frac{q}{p_j'}}<\infty,$$

where the supremum is taken over all Euclidean balls $B\subset\mathbb{R}^n$ and $|B|$ denotes the Lebesgue measure of B.

The classes $A_{(p_1,p_2),q}$ are inspired in the classes of weights $A_{p,q}$, $1\le p,\ q<\infty$, defined by Muckenhoupt and Wheeden in [48] to study weighted norm inequalities for the fractional integral: a weight u defined on $\mathbb{R}^n$ is in the class $A_{p,q}$ if

$$\sup_B\Big(\frac{1}{|B|}\int_B u^{\frac{q}{p}}\,dx\Big)\Big(\frac{1}{|B|}\int_B u^{(1-p')}\,dx\Big)^{\frac{q}{p'}}<\infty.$$

The classes $A_{(p_1,p_2),q}$ for $1/q=1/p_1+1/p_2$ were introduced in [39] to study characterizations of weights for boundedness properties of certain bilinear maximal functions and bilinear Calderón-Zygmund operators in weighted Lebesgue spaces. As shown in [46], the classes $A_{(p_1,p_2),q}$ characterize the weights rendering analogous bounds for bilinear fractional integral operators .

Remark 2.1. If (w_1,w_2) satisfies the $A_{(p_1,p_2),q}$ condition then $w=w_1^{q/p_1}w_2^{q/p_2}$ and $w_i^{1-p_i'}$, $i=1,2$, are A_∞ weights as shown in [39, Theorem 3.6] and [46, Theorem 3.4].

For $\alpha>0$, we consider bilinear fractional integral operators on $\mathbb{R}^n$ of order $\alpha>0$ defined by

$$\mathcal{B}_\alpha(f,g)(x):=\int_{\mathbb{R}^n}\frac{f(x-s_1y)g(x-s_2y)}{|y|^{n-\alpha}}\,dy,\quad x\in\mathbb{R}^n. \tag{2.16}$$

$$\mathcal{I}_\alpha(f,g)(x):=\int_{\mathbb{R}^{2n}}\frac{f(y)g(z)}{(|x-y|+|x-z|)^{2n-\alpha}}\,dydz,\quad x\in\mathbb{R}^n, \tag{2.17}$$

where $s_1\ne s_2$ are nonzero real numbers. In the following theorem we summarize results concerning boundedness properties on weighted and unweighted Lebesgue spaces for the operators $\mathcal{B}_\alpha$ and $\mathcal{I}_\alpha$, which will be useful in some of our proofs.

Theorem A. *In* $\mathbb{R}^n$:

(a) [35, 46] *Let* $\alpha\in(0,2n)$, $1<p_1,p_2<\infty$ *and* $q>0$ *such that* $\frac{1}{q}=\frac{1}{p_1}+\frac{1}{p_2}-\frac{\alpha}{n}$. *Then* $\mathcal{I}_\alpha$ *is bounded from* $L^{p_1}_{w_1}\times L^{p_2}_{w_2}$ *into* L^q_w *for* $w:=w_1^{q/p_1}w_2^{q/p_2}$ *and pairs of weights* (w_1,w_2) *satisfying the* $A_{(p_1,p_2),q}$ *condition.*

(b) [28, 29, 35] *Let* $\alpha \in (0, n)$, $1 < p_1, p_2 < \infty$ *and* $q > 0$ *such that* $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n}$. *Then* $\mathcal{B}_\alpha$ *is bounded from* $L^{p_1} \times L^{p_2}$ *into* L^q.

(c) [Remark 2.2] *Let* $\alpha \in (0, n)$, $1 < p_1, p_2 < \infty$ *such that* $1/p := 1/p_1 + 1/p_2 < 1$ *and* $q > 1$ *such that* $1/q = 1/p - \alpha/n$. *Then* $\mathcal{B}_\alpha$ *is bounded from* $L^{p_1}_{w_1} \times L^{p_2}_{w_2}$ *into* L^q_w *for* $w := w_1^{q/p_1} w_2^{q/p_2}$ *and weights* w_1, w_2 *in* $A_{p,q}$.

Remark 2.2. Part (c) of Theorem A follows from the following observations. Muckenhoupt and Wheeden [48] showed that the linear fractional integral operator

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(x-y)}{|y|^{n-\alpha}}\, dy$$

satisfies

$$\left(\int_{\mathbb{R}^n} |I_\alpha f(x)|^q u^{\frac{q}{p}}\, dx\right)^{1/q} \le C \left(\int_{\mathbb{R}^n} |f(x)|^p u\, dx\right)^{1/p}$$

for $1/q = 1/p - \alpha/n$, $u \in A_{p,q}$ and p, $q > 1$. Using p and q as in the statement of part (c) of Theorem A, let $r = p_1/p$ and $s = p_2/p$, so that $r, s > 1$ and $1/r + 1/s = 1$. By Hölder's inequality

$$|\mathcal{B}_\alpha(f,g)| \lesssim I_\alpha(|f|^r)^{1/r} I_\alpha(|g|^s)^{1/s},$$

and

$$\begin{aligned}\left(\int_{\mathbb{R}^n} |\mathcal{B}_\alpha(f,g)|^q w\right)^{1/q} &\le \left(\int_{\mathbb{R}^n} I_\alpha(|f|^r)^{q/r} I_\alpha(|g|^s)^{q/s} w_1^{\frac{q}{p_1}} w_2^{\frac{q}{p_2}}\right)^{1/q}\\ &\le \left(\int_{\mathbb{R}^n} I_\alpha(|f|^r)^q w_1^{\frac{q}{p}}\, dx\right)^{1/qr} \left(\int_{\mathbb{R}^n} I_\alpha(|g|^s)^q w_2^{\frac{q}{p}}\right)^{1/sq}.\end{aligned}$$

Using the result of Muckenhoupt and Wheeden, the last inequality is bounded by

$$C\left(\int_{\mathbb{R}^n} |f|^{rp} w_1\right)^{1/rp} \left(\int_{\mathbb{R}^n} |g|^{sp} w_2\right)^{1/sp} = C\left(\int_{\mathbb{R}^n} |f|^{p_1} w_1\right)^{1/p_1} \left(\int_{\mathbb{R}^n} |g|^{p_2} w_2\right)^{1/p_2},$$

which is the desired result.

Multilinear potential operators, of which $\mathcal{I}_\alpha$ is a particular case, were studied in [42] in the context of spaces of homogeneous type. We now briefly recall some those results, as they will be used in the proofs in the next sections.

Let (X, ρ, μ) be a *space of homogenous type* in the sense of Coifman-Weiss [17]. That is, X is a non-empty set, ρ is a quasi-metric defined on X that satisfies the quasi-triangle inequality

$$\rho(x,y) \le \kappa(\rho(x,z) + \rho(z,y)), \qquad x, y, z \in X, \tag{2.18}$$

for some $\kappa \ge 1$, and μ is a Borel measure on X (with respect to the topology defined by ρ) such that there exists a constant $L_0 \ge 0$ verifying

$$0 < \mu(B_\rho(x, 2r) \le L_0\, \mu(B_\rho(x,r)) < \infty \tag{2.19}$$

for all $x \in X$ and $0 < r < \infty$, and where $B_\rho(x,r) = \{y \in X : \rho(x,y) < r\}$ is the ρ-ball of center x and radius r. Given a ball $B = B_\rho(x,r)$ and $\theta > 0$ we will usually write $r(B)$ to denote the radius r and θB to denote $B_\rho(x, \theta r)$. In the Euclidean setting, this is, when $X = \mathbb{R}^n$, ρ is Euclidean distance and μ is Lebesgue measure, we use the notation $B(x,r)$ instead of $B_\rho(x,r)$.

The measure μ is said to satisfy the *reverse doubling property* if for every $\eta > 1$ there are constants $c(\eta) > 0$ and $\gamma > 0$ such that

$$\frac{\mu(B_\rho(x_1,r_1))}{\mu(B_\rho(x_2,r_2))} \geq c(\eta) \left(\frac{r_1}{r_2}\right)^\gamma, \tag{2.20}$$

whenever $B_\rho(x_2,r_2) \subset B_\rho(x_1,r_1)$, x_1, $x_2 \in X$ and $0 < r_1$, $r_2 \leq \eta\, \mathrm{diam}_\rho(\mathrm{X})$.

We consider bilinear potential operators of the form

$$\mathcal{T}(f,g)(x) = \int_{X^2} f(y)g(z)K(x,y,z)\, d\mu(y)d\mu(z), \tag{2.21}$$

where the kernel K is the restriction of a nonnegative continuous kernel $\tilde{K}(x_1,x_2,y,z)$ (i.e. $K(x,y,z) = \tilde{K}(x,x,y,z)$ for $(x,y,z) \in X \times X \times X$) that satisfies the following *growth conditions*: for every $c > 1$ there exists $C > 1$ such that

$$\begin{aligned} \tilde{K}(x_1,x_2,y,z) &\leq C\tilde{K}(v,w,y,z) \quad \text{if } \rho(v,y)+\rho(w,z) \leq c\,(\rho(x_1,y)+\rho(x_2,z)), \text{ and} \\ \tilde{K}(x_1,x_2,y,z) &\leq C\tilde{K}(y,z,v,w) \quad \text{if } \rho(y,v)+\rho(z,w) \leq c\,(\rho(x_1,y)+\rho(x_2,z)). \end{aligned} \tag{2.22}$$

Following [54], the functional φ associated to K is defined by

$$\varphi(B) := \sup\{K(x,y,z) : (x,y,z) \in B \times B \times B, \rho(x,y)+\rho(x,z) \geq c\, r(B)\}$$

for a sufficiently small positive constant c and for B a ρ ball such that $r(B) \leq \eta\, \mathrm{diam}_\rho(\mathrm{X})$, for some fixed $\eta > 1$. The functional φ associated to K will be assumed to satisfy the following property: there exists $\delta > 0$ such that for all $C_1 > 1$ there exists $C_2 > 0$ such that

$$\varphi(B')\mu(B')^2 \leq C_2 \left(\frac{r(B')}{r(B)}\right)^\delta \varphi(B)\mu(B)^2 \tag{2.23}$$

for all balls $B' \subset B$, with $r(B')$, $r(B) < C_1\, \mathrm{diam}_\rho(\mathrm{X})$.

We note that, in the Euclidean setting, the operator $\mathcal{I}_\alpha$ defined in (2.17) has kernel and associated functional given, respectively, by

$$K(x,y,z) = \frac{1}{(|x-y|+|x-z|)^{2n-\alpha}} \qquad \text{and} \qquad \varphi(B) \sim r(B)^{\alpha-2n},$$

and both satisfy (2.22) and (2.23).

Theorem B ([42])**.** *Suppose that $1 < p_1, p_2 \leq \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{2} < p \leq q < \infty$. Let (X,ρ,μ) be a space of homogeneous type that satisfies the reverse doubling property (2.20) and let K be a kernel such that (2.22) holds with φ satisfying (2.23). Furthermore, let u, v_k, $k = 1,2$ be weights defined on X that satisfy condition (2.24) if $q > 1$ or condition (2.25) if $q \leq 1$, where*

$$\sup_{B\ \rho\text{-ball}} \varphi(B)\mu(B)^{\frac{1}{q}+\frac{1}{p_1{}'}+\frac{1}{p_2'}} \left(\frac{1}{\mu(B)}\int_B u^{qt} d\mu\right)^{1/qt} \prod_{j=1}^{2} \left(\frac{1}{\mu(B)}\int_B v_i^{-tp_i'} d\mu\right)^{1/tp_i'} < \infty, \tag{2.24}$$

for some $t > 1$,

(2.25)
$$\sup_{B\ \rho\text{-ball}} \varphi(B)\mu(B)^{\frac{1}{q}+\frac{1}{p_1'}+\frac{1}{p_2'}} \left(\frac{1}{\mu(B)}\int_B u^q d\mu\right)^{1/q} \prod_{j=1}^{2}\left(\frac{1}{\mu(B)}\int_B v_i^{-tp_i'} d\mu\right)^{1/tp_i'} < \infty,$$

for some $t > 1$, *with the supremum taken over* ρ*-balls with* $r(B) \lesssim diam_\rho(X)$. *Then there exists a constant* C *such that*

$$\left(\int_X (|\mathcal{T}(f_1, f_2)|u)^q \, d\mu\right)^{1/q} \le C \prod_{k=1}^{2}\left(\int_X (|f_k|v_k)^{p_k} \, d\mu\right)^{1/p_k}$$

for all $(f_1, f_2) \in L^{p_1}_{v_1^{p_1}}(X) \times L^{p_2}_{v_2^{p_2}}(X)$. *The constant* C *depends only on the constants appearing in* (2.18), (2.19), (2.20), (2.22), (2.23), (2.24) *and* (2.25).

Remark 2.3. A careful examination of the proof of Theorem B yields

$$\|\mathcal{T}\|_{\mathrm{op}} \lesssim \frac{C}{1 - D^{-\delta}}\mathcal{W},$$

where $\mathcal{W}$ is the constant from (2.24) or (2.25), $C = \max(C_1, C_2)$ and $\delta > 0$ are the constants from (2.23), and $D > 1$ is a structural constant.

Remark 2.4. In the Euclidean setting, consider weights w_1, $w_2 \in A_{(p_1,p_2),q}$ and $w = w_1^{q/p_1} w_2^{q/p_2}$, for some $1 < p_1$, $p_2 < \infty$, $0 < \frac{1}{q} < \frac{1}{p_1} + \frac{1}{p_2}$ and suppose that $\mathcal{T}$ is an operator of the form (2.21) such that

$$\sup_B \varphi(B)|B|^{\frac{1}{q}+\frac{1}{p_1'}+\frac{1}{p_2'}} \sim \sup_B \varphi(B) r(B)^{\frac{n}{q}+\frac{n}{p_1'}+\frac{n}{p_2'}} < \infty.$$

It then follows that $u := w^{\frac{1}{q}}$ and $v_k := w_k^{\frac{1}{p_k}}$, $k = 1, 2$, satisfy (2.24) and (2.25). Indeed, the second factor in (2.24) is given by

$$\sup_{x\in\mathbb{R}^n, r>0} \left(\frac{1}{|B(x,r)|}\int_{B(x,r)} w^t dx\right)^{1/qt} \prod_{j=1}^{2}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} w_i^{-\frac{t}{p_i-1}} dx.\right)^{1/tp_i'} \tag{2.26}$$

Since w, $w_1^{-\frac{1}{p_1-1}}$, $w_2^{-\frac{1}{p_2-1}}$ are A_∞ weights (see Remark 2.1), there exists $t > 1$ such that (2.26) is bounded by

$$\sup_B \left(\frac{1}{|B|}\int_B w \, dx\right)^{1/q} \prod_{j=1}^{2}\left(\frac{1}{|B|}\int_B w_i^{-\frac{1}{p_i-1}} dx\right)^{1/p_i'} = [(w_1, w_2)]_{A_{(p_1,p_2),q}} < \infty,$$

where finiteness is due to (w_1, w_2) satisfying the $A_{(p_1,p_2),q}$ condition. A similar reasoning applies to (2.25).

The last two remarks imply the following

Corollary 1. *In the* n*-dimensional Euclidean setting, consider weights* w_1, $w_2 \in A_{(p_1,p_2),q}$ *and* $w = w_1^{q/p_1} w_2^{q/p_2}$, *for some* $1 < p_1$, $p_2 < \infty$, $0 < \frac{1}{q} < \frac{1}{p_1} + \frac{1}{p_2}$. *Suppose that*

$\mathcal{T}$ is an operator of the form (2.21) *such that its kernel satisfies* (2.22)*, the associated functional φ satisfies* (2.23)*, and*

$$\sup_B \varphi(B) r(B)^{\frac{n}{q}+\frac{n}{p_1{}'}+\frac{n}{p_2'}} < \infty.$$

Then there exists a constant A such that

$$\left(\int_{\mathbb{R}^n} |\mathcal{T}(f_1, f_2)|^q w \, dx\right)^{1/q} \leq A \prod_{k=1}^{2} \left(\int_{\mathbb{R}^n} |f_k|^{p_k} w_k \, dx\right)^{1/p_k}$$

for all $(f_1, f_2) \in L^{p_1}_{w_1} \times L^{p_2}_{w_1}$. The constant A satisfies

$$A \leq c \sup_B \varphi(B) r(B)^{\frac{n}{q}+\frac{n}{p_1{}'}+\frac{n}{p_2'}},$$

where c depends only on $[(w_1, w_2)]_{A_{(p_1,p_2),q}}$ and other absolute constants.

3. Bilinear Poincaré-type inequalities relative to an approximation of the identity

An *approximation of the identity* of order $m > 0$ in $\mathbb{R}^n$ is a collection of operators $\mathcal{S} := \{S_t\}_{t>0}$ acting on functions defined on $\mathbb{R}^n$,

$$S_t f(x) = \int_{\mathbb{R}^n} p_t(x, y) f(y) \, dy, \quad x \in \mathbb{R}^n,$$

such that for each $t > 0$ the kernels p_t satisfy $\int_{\mathbb{R}^n} p_t(x, y) \, dy = 1$ for all x and the scaled Poisson bound

$$|p_t(x, y)| \leq t^{-n/m} \gamma\left(\frac{|x - y|}{t^{1/m}}\right), \qquad x, y \in \mathbb{R}^n, \tag{3.27}$$

where $\gamma : [0, \infty) \to [0, \infty)$ is a bounded, decreasing function for which

$$\lim_{r\to\infty} r^{2n+\varepsilon} \gamma(r) = 0, \qquad \text{for some } \varepsilon > 0. \tag{3.28}$$

As examples, it is well-known that if a sectorial operator L generates a holomorphic semigroup $\{e^{-zL}\}_z$ whose kernels satisfy suitable pointwise bounds, then $S_t = e^{-tL}$ gives rise to an approximation of the identity. The resolvents $S_t = (1 + tL)^{-M}$ or $S_t = 1 - (1 - e^{-tL})^N$ can be considered as well. We refer the reader to [22] and [45] for more details concerning holomorphic functional calculus. Other examples can be built on a second-order divergence form operator $L = -\mathrm{div}(A\nabla)$ with an elliptic matrix-valued function A. Since L is maximal accretive, it admits a bounded H_∞-calculus on $L^2(\mathbb{R}^n)$. Moreover, when A has real entries or when the dimension $n \in \{1, 2\}$, then the operator L generates an analytic semigroup on L^2 with a heat kernel satisfying Gaussian upper-bounds.

The main result of this section is the following:

Theorem 2. *Let $\mathcal{S} := \{S_t\}_{t>0}$ and $\mathcal{S}' := \{t\partial_t S_t\}_{t>0}$ be approximations of the identity in $\mathbb{R}^n$ of order $m > 0$ and constant ε in* (3.28)*, $1 < p_1, p_2 < \infty$, $q > 0$, and $0 < \alpha <$*

$\min\{1,\varepsilon\}$ *such that* $\frac{1}{q}=\frac{1}{p_1}+\frac{1}{p_2}-\frac{1-\alpha}{n}$. *If* (w_1,w_2) *satisfy the* $A_{(p_1,p_2),q}$ *condition and* $w:=w_1^{q/p_1}w_2^{q/p_2}$, *then there exists a constant* C *such that for all Euclidean balls* B

$$\begin{aligned}&\left\|fg-S_{r(B)^m}(f)S_{r(B)^m}(g)\right\|_{L^q_w(B)}\\&\quad\le C\,r(B)^\alpha\sum_{l\ge 0}2^{-l(\varepsilon-\alpha)}\left[\|\nabla f\|_{L^{p_1}_{w_1}(2^{l+1}B)}\|g\|_{L^{p_2}_{w_2}(2^{l+1}B)}+\|f\|_{L^{p_1}_{w_1}(2^{l+1}B)}\|\nabla g\|_{L^{p_2}_{w_2}(2^{l+1}B)}\right].\end{aligned}$$

Remark 3.1. It is possible to consider two collections of operators $\mathcal{S}^1:=\{S^1_t\}_{t>0}$ and $\mathcal{S}^2:=\{S^2_t\}_{t>0}$, then the proof of Theorem 2 holds true when estimating the oscillation $\|fg-S^1_{r(B)^m}(f)S^2_{r(B)^m}(g)\|_{L^q_w(B)}$.

Remark 3.2. Note that condition (3.28) assumes exponent $2n+\varepsilon$ rather than $n+\varepsilon$. This is quite natural in our context since the proof of Theorem 2 involves the semigroup $\mathcal{P}_t:=S_t\otimes S_t$ which is expected to have decay for $2n$-dimensional variables.

Remark 3.3. The scaling of the result in Theorem 2 is in accordance with the classical situation corresponding to $\alpha=0$ and obtained in [42]. More precisely, a particular case of [42, Theorem 1] reads

$$\|fg-f_Bg_B\|_{L^q_w(B)}\le C\left(\|\nabla f\|_{L^{p_1}_{w_1}(B)}\|g\|_{L^{p_2}_{w_2}(B)}+\|f\|_{L^{p_1}_{w_1}(B)}\|\nabla g\|_{L^{p_2}_{w_2}(B)}\right)\tag{3.29}$$

for $\frac{1}{q}=\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{n}$, $(w_1,w_2)\in A_{(p_1,p_1),q}$ and $w=w_1^{q/p_1}w_2^{q/p_2}$. Hölder's inequality and the conditions on the weights imply

$$\|fg-f_Bg_B\|_{L^q_w(B)}\le C\,r(B)^\alpha\left(\|\nabla f\|_{L^{p_1}_{w_1}(B)}\|g\|_{L^{p_2}_{w_2}(B)}+\|f\|_{L^{p_1}_{w_1}(B)}\|\nabla g\|_{L^{p_2}_{w_2}(B)}\right)$$

for $\frac{1}{q}=\frac{1}{p_1}+\frac{1}{p_2}-\frac{1-\alpha}{n}$, $(w_1,w_2)\in A_{(p_1,p_1),q}$ and $w=w_1^{q/p_1}w_2^{q/p_2}$.

For instance, let p_1, p_2, q, α, w_1, w_2, w be as in the statement of Theorem 2. Define $\bar q$ by $\frac{1}{\bar q}=\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{n}=\frac{1}{q}-\frac{\alpha}{n}$ and assume that $\bar q>0$ and that the pair (w_1,w_2) is in $A_{(p_1,p_2),\bar q}$. Setting $\bar w=w_1^{\bar q/p_1}w_2^{\bar q/p_2}$ and using Hölder's inequality and (3.29) we obtain

$$\begin{aligned}\|fg-f_Bg_B\|_{L^q_w(B)}&\le\left(\int_B\bar w\left(w_1^{\frac{q-\bar q}{p_1}}w_2^{\frac{q-\bar q}{p_2}}\right)^{(\frac{\bar q}{q})'}\right)^{\frac{1}{q(\bar q/q)'}}\|fg-f_Bg_B\|_{L^{\bar q}_{\bar w}(B)}\\&\lesssim r(B)^\alpha\left(\|\nabla f\|_{L^{p_1}_{w_1}(B)}\|g\|_{L^{p_2}_{w_2}(B)}+\|f\|_{L^{p_1}_{w_1}(B)}\|\nabla g\|_{L^{p_2}_{w_2}(B)}\right).\end{aligned}$$

Note, however, that Theorem 2 does not include the case $\alpha=0$.

Remark 3.4. Since we do not require spatial regularity on the kernels p_t in (3.27), our results can be extended to every subset of $\mathbb{R}^n$ (not necessarily Lipschitz) by considering truncations as used in [23].

Our proof of Theorem 2 is based on an appropriate representation formula for the bilinear oscillations associated to the approximation of the identity and the boundedness properties of operators studied in [42]. We present the details in the next two subsections.

3.1. **Representation formula.** We start by introducing the collection of bilinear operators that shape our representation formula. For a ball $B \subset \mathbb{R}^n$, the operator $\mathcal{J}_B$ is defined as

$$\mathcal{J}_B(f_1, f_2)(x) := \int_{B\times B} K(x,(a,b)) f_1(a) f_2(b)\, da\, db \quad x \in B, \tag{3.30}$$

with kernel

$$K(x,(a,b)) := \frac{1}{(|x-a|+|x-b|)^{2n-1}} \log\left(\frac{8\, r(B)}{|x-a|+|x-b|}\right), \qquad x,\, a,\, b \in B.$$

Theorem 3 (Bilinear representation formula)**.** *Let $\mathcal{S} = \{S_t\}_{t>0}$ and $\mathcal{S}' = \{t\partial_t S_t\}_{t>0}$ be approximations of the identity in $\mathbb{R}^n$ of order $m > 0$ and constant ε in (3.28). There exists a constant $C > 0$ such that for every ball $B \subset \mathbb{R}^n$ and $x \in B$,*

$$\begin{aligned}&\left|f(x)g(x) - S_{r(B)^m}(f)(x) S_{r(B)^m}(g)(x)\right|\\ &\leq C \sum_{l\geq 0} 2^{-l\varepsilon}(l+1)\left[\mathcal{J}_{2^{l+1}B}(|\nabla f|\chi_{2^{l+1}B}, |g|\chi_{2^{l+1}B})(x) + \mathcal{J}_{2^{l+1}B}(|f|\chi_{2^{l+1}B}, |\nabla g|\chi_{2^{l+1}B})(x)\right].\end{aligned}$$

Remark 3.5. As mentioned in the Introduction, since the approximation operator $S_{r(B)^m}$ is not a local operator, we cannot expect perfectly localized estimates as for the "classical" Poincaré inequality.

Proof. We consider the operator on $\mathbb{R}^{2n}$ given by $\mathcal{P}_t := S_t \otimes S_t$, that is,

$$\mathcal{P}_t(F)(x,x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} p_t(x,y) p_t(x,z) F(y,z)\, dy dz.$$

For given functions f and g defined on $\mathbb{R}^n$, let $F(y,z) := f(y)g(z)$. Fix B of radius $r(B)$, $x \in B$ and for each $t \in (0, r(B)^m)$ let B_t be the ball of radius $t^{1/m}$ centered at $x \in \mathbb{R}^n$. Then

$$\begin{aligned} F(x,x) - \mathcal{P}_{r(B)^m}(F)(x,x) &= -\int_0^{r(B)^m} t\partial_t \mathcal{P}_t(F)(x,x)\, \frac{dt}{t}\\ &= -\int_0^{r(B)^m} t\partial_t \mathcal{P}_t(F - F_{B_t\times B_t})(x,x)\, \frac{dt}{t},\end{aligned}$$

where we used that $F_{B_t\times B_t} = f_{B_t} g_{B_t}$ is a constant and $\partial_t S_t(1) = 0$ for all $t > 0$. The pointwise bounds (3.27) for the kernels $p_t(x, y)$ give

$$
\begin{aligned}
&|t\partial_t \mathcal{P}_t(F - F_{B_t\times B_t})(x,x)| \\
&\quad \lesssim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} t^{-\frac{2n}{m}} \left(1+\frac{|x-y|}{t^{\frac{1}{m}}}\right)^{-2n-\varepsilon} \left(1+\frac{|x-z|}{t^{\frac{1}{m}}}\right)^{-2n-\varepsilon} |f(y)g(z) - f_{B_t}g_{B_t}|\,dydz \\
&\quad \lesssim \iint\limits_{B_t\times B_t} t^{-\frac{2n}{m}} \left(1+\frac{|x-y|}{t^{\frac{1}{m}}}\right)^{-2n-\varepsilon} \left(1+\frac{|x-z|}{t^{\frac{1}{m}}}\right)^{-2n-\varepsilon} |f(y)g(z) - f_{B_t}g_{B_t}|\,dydz \\
&\quad + \sum_{l\in\mathbb{N}} \iint\limits_{C_l(B_t\times B_t)} t^{-\frac{2n}{m}} \left(1+\frac{|x-y|}{t^{\frac{1}{m}}}\right)^{-2n-\varepsilon} \left(1+\frac{|x-z|}{t^{\frac{1}{m}}}\right)^{-2n-\varepsilon} |f(y)g(z) - f_{B_t}g_{B_t}|\,dydz \\
&\quad =: I_0(f,g,t)(x) + \sum_{l\in\mathbb{N}} I_l(f,g,t)(x),
\end{aligned}
$$

where for $l \geq 1$, $C_l(B_t \times B_t)$ denotes the annulus

$$C_l(B_t \times B_t) := 2^l(B_t \times B_t) \setminus 2^{l-1}(B_t \times B_t).$$

We now proceed to estimating each of the terms $I_l(f, g, t)$, $l \geq 0$.

The bound for $I_0(f, g, t)$. Notice that for all $y, z \in B_t$,

$$|f(y)g(z) - f_{B_t}g_{B_t}| \lesssim \iint_{B_t\times B_t} \frac{|\nabla f(a)||g(b)| + |f(a)||\nabla g(b)|}{(|y-a|+|z-b|)^{2n-1}}\,dadb. \tag{3.31}$$

Indeed, the usual representation formula for a linear oscillation in $(\mathbb{R}^n)^2$ gives

$$|F(y,z) - F_{B_t\times B_t}| \leq C\int_{B_t\times B_t} \frac{|\nabla F(a,b)|}{|(y,z)-(a,b)|^{2n-1}} dadb$$

which yields (3.31). Hence, we get

$$
\begin{aligned}
I_0(f,g,t)(x) &\lesssim \iint\limits_{B_t\times B_t} t^{-2n/m}|f(y)g(z) - f_{B_t}g_{B_t}|\,dydz \\
&\lesssim \iint\limits_{B_t\times B_t} (|\nabla f(a)||g(b)| + |f(a)||\nabla g(b)|) I(a,b,t)\,da\,db,
\end{aligned}
$$

where

$$I(a,b,t) := \iint\limits_{B_t\times B_t} \frac{t^{-2n/m}}{(|y-a|+|z-b|)^{2n-1}}\,dy\,dz.$$

For $a, b \in B_t$, we have

$$I(a,b,t) \leq \iint\limits_{\substack{|y-a|\leq 2t^{1/m}\\ |z-b|\leq 2t^{1/m}}} \frac{1}{(|y-a|+|z-b|)^{2n-1}} \frac{dy}{t^{n/m}} \frac{dz}{t^{n/m}}$$

$$\lesssim \int_0^{2t^{1/m}} \int_0^{2t^{1/m}} \frac{1}{(u+v)^{2n-1}} u^{n-1} v^{n-1} \frac{du}{t^{n/m}} \frac{dv}{t^{n/m}}$$

$$\lesssim t^{(-2n+1)/m} \int_0^1 \int_0^1 \frac{u^{n-1} v^{n-1}}{(u+v)^{2n-1}} du dv \lesssim t^{(-2n+1)/m}, \tag{3.32}$$

where the last integral is controlled by separately estimating for $v \geq u$ and for $u \geq v$. We conclude that, for $a, b \in B_t$,

$$\iint\limits_{B_t \times B_t} \frac{1}{(|y-a|+|z-b|)^{2n-1}} \frac{dy}{t^{n/m}} \frac{dz}{t^{n/m}} \lesssim t^{(-2n+1)/m} \lesssim \frac{1}{(|x-a|+|x-b|)^{2n-1}}, \tag{3.33}$$

and therefore

$$I_0(f,g,t)(x) \lesssim \iint\limits_{B_t \times B_t} \frac{|\nabla f(a)||g(b)| + |f(a)||\nabla g(b)|}{(|x-a|+|x-b|)^{2n-1}} \, da \, db.$$

Integration with respect to the variable $t \in (0, r(B)^m)$ yields,

$$\int_0^{r(B)^m} I_0(f,g,t)(x) \frac{dt}{t} \lesssim \int_0^{r(B)^m} \iint\limits_{B_t \times B_t} \frac{|\nabla f(a)||g(b)| + |f(a)||\nabla g(b)|}{(|x-a|+|x-b|)^{2n-1}} \, da \, db \, \frac{dt}{t}$$

$$\lesssim \int_0^{r(B)^m} \iint\limits_{\substack{|x-a|\leq t^{1/m}\\ |x-b|\leq t^{1/m}}} \frac{|\nabla f(a)||g(b)| + |f(a)||\nabla g(b)|}{(|x-a|+|x-b|)^{2n-1}} \, da \, db \, \frac{dt}{t}$$

$$\lesssim \iint\limits_{2B \times 2B} \int\limits_{\substack{0\leq t\leq r(B)^m\\ |x-a|\leq t^{1/m}\\ |x-b|\leq t^{1/m}}} \frac{|\nabla f(a)||g(b)| + |f(a)||\nabla g(b)|}{(|x-a|+|x-b|)^{2n-1}} \frac{dt}{t} \, da \, db$$

$$\lesssim \iint\limits_{2B \times 2B} \frac{|\nabla f(a)||g(b)| + |f(a)||\nabla g(b)|}{(|x-a|+|x-b|)^{2n-1}} \log\left(1 + \frac{r(B)^m}{\max\{|x-a|^m, |x-b|^m\}}\right) da \, db$$

$$\lesssim \iint\limits_{2B \times 2B} \frac{|\nabla f(a)||g(b)| + |f(a)||\nabla g(b)|}{(|x-a|+|x-b|)^{2n-1}} \log\left(\frac{16 r(B)}{|x-a|+|x-b|}\right) da \, db$$

$$\lesssim \mathcal{J}_{2B}(|\nabla f|, |g|)(x) + \mathcal{J}_{2B}(|f|, |\nabla g|)(x),$$

where the operator $\mathcal{J}_{2B}$ was defined in (3.30). It remains to treat the terms $I_l(f,g,t)(x)$ with $l \geq 1$.

The bound for $I_l(f,g,t)$ with $l \geq 1$. Recall that I_l is given by

$$I_l(f,g,t)(x) := \iint_{C_l(B_t\times B_t)} t^{-2n/m}\left(1+\frac{|x-y|}{t^{\frac{1}{m}}}\right)^{-2n-\varepsilon}\left(1+\frac{|x-z|}{t^{\frac{1}{m}}}\right)^{-2n-\varepsilon} \times |f(y)g(z)-f_{B_t}g_{B_t}|\,dydz,$$

where $B_t = B(x,t^{1/m})$ (and therefore $x\in B_t$) and $C_l(B_t\times B_t) := 2^l(B_t\times B_t)\backslash 2^{l-1}(B_t\times B_t)$. We have to estimate the oscillation $|f(y)g(z)-f_{B_t}g_{B_t}|$, with $(y,z)\in C_l(B_t\times B_t)$, for which we consider the intermediate averages as follows:

$$|f(y)g(z)-f_{B_t}g_{B_t}| \leq |f(y)g(z)-f_{2^lB_t}g_{2^lB_t}| + \sum_{k=0}^{l-1}\left|f_{2^{k+1}B_t}g_{2^{k+1}B_t}-f_{2^kB_t}g_{2^kB_t}\right|.$$

For all $k\in 0,...,l-1$, we use

$$\begin{aligned}|f_{2^{k+1}B_t}g_{2^{k+1}B_t}-f_{2^kB_t}g_{2^kB_t}| &\lesssim (2^kt^{1/m})^{-2n}\iint_{2^kB_t\times 2^kB_t}|f(u)g(v)-f_{2^{k+1}B_t}g_{2^{k+1}B_t}|dudv\\ &\lesssim (2^kt^{1/m})^{-2n}\iint_{2^{k+1}B_t\times 2^{k+1}B_t}|f(u)g(v)-f_{2^{k+1}B_t}g_{2^{k+1}B_t}|dudv.\end{aligned}$$

As done in (3.31) applied to the ball $2^{k+1}B_t$, we obtain that for $(u,v)\in 2^{k+1}B_t\times 2^{k+1}B_t$

$$|f(u)g(v)-f_{2^{k+1}B_t}g_{2^{k+1}B_t}| \lesssim \iint_{2^{k+1}B_t\times 2^{k+1}B_t}\frac{|\nabla f(a)||g(b)|+|f(a)||\nabla g(b)|}{(|u-a|+|v-b|)^{2n-1}}\,dadb.$$

Proceeding as in (3.33), by replacing the ball B_t with $2^{k+1}B_t$, we have that for $(a,b)\in 2^{k+1}B_t\times 2^{k+1}B_t$ and $(u,v)\in 2^{k+1}B_t\times 2^{k+1}B_t$,

$$\iint_{2^{k+1}B_t\times 2^{k+1}B_t}\frac{2^{-2kn}t^{-2n/m}dydz}{(|u-a|+|v-b|)^{2n-1}} \lesssim (2^kt^{1/m})^{-(2n-1)} \lesssim (|x-a|+|x-b|)^{1-2n}. \tag{3.34}$$

Combining everything we have

$$\begin{aligned}&|f_{2^{k+1}B_t}g_{2^{k+1}B_t}-f_{2^kB_t}g_{2^kB_t}|\\ &\lesssim \frac{2^{2kn}t^{2n/m}}{(2^kt^{1/m})^{2n}}\iint_{2^{k+1}B_t\times 2^{k+1}B_t}\frac{|\nabla f(a)||g(b)|+|f(a)||\nabla g(b)|}{(|x-a|+|x-b|)^{2n-1}}\,dadb\\ &\lesssim \iint_{2^{k+1}B_t\times 2^{k+1}B_t}\frac{|\nabla f(a)||g(b)|+|f(a)||\nabla g(b)|}{(|x-a|+|x-b|)^{2n-1}}\,dadb.\end{aligned}$$

We conclude that for $(y,z) \in 2^{l+1}(B_t \times B_t)$ (actually for any y and z)

$$
\begin{aligned}
|f(y)g(z) - f_{B_t}g_{B_t}| &\lesssim |f(y)g(z) - f_{2^lB_t}g_{2^lB_t}| \\
&\quad + \sum_{k=0}^{l-1} \iint\limits_{2^{k+1}B_t \times 2^{k+1}B_t} \frac{|\nabla f(a)||g(b)| + |f(a)||\nabla g(b)|}{(|x-a|+|x-b|)^{2n-1}}\, dadb. \qquad (3.35)
\end{aligned}
$$

Consequently,

$$I_l(f,g,t)(x) \lesssim I_l^1(f,g,t)(x) + I_l^2(f,g,t)(x)$$

with

$$
\begin{aligned}
I_l^1(f,g,t)(x) := \iint\limits_{C_l(B_t\times B_t)} &\left[\left(1+\frac{|x-y|}{t^{\frac{1}{m}}}\right)\left(1+\frac{|x-z|}{t^{\frac{1}{m}}}\right)\right]^{-2n-\varepsilon} \\
&\times |f(y)g(z) - f_{2^lB_t}g_{2^lB_t}| \frac{dydz}{t^{2n/m}}
\end{aligned}
$$

and

$$
\begin{aligned}
I_l^2(f,g,t)(x) := \sum_{k=0}^{l} \iint\limits_{C_l(B_t\times B_t)} &\left[\iint\limits_{2^{k+1}B_t\times 2^{k+1}B_t} \left(1+\frac{|x-y|}{t^{\frac{1}{m}}}\right)^{-2n-\varepsilon}\left(1+\frac{|x-z|}{t^{\frac{1}{m}}}\right)^{-2n-\varepsilon}\right. \\
&\left.\frac{|\nabla f(a)||g(b)| + |f(a)||\nabla g(b)|}{(|x-a|+|x-b|)^{2n-1}}\, dadb\right] \frac{dydz}{t^{2n/m}}.
\end{aligned}
$$

The first term $I_l^1(f,g,t)(x)$ can be estimated in the same way as the quantity $I_0(f,g,t)$ by replacing B_t with 2^lB_t. Since $(y,z) \in C_l(B_t \times B_t)$ and $x \in B_t$, the term

$$\left(1+\frac{|x-y|}{t^{\frac{1}{m}}}\right)^{-2n-\varepsilon}\left(1+\frac{|x-z|}{t^{\frac{1}{m}}}\right)^{-2n-\varepsilon}$$

provides an extra factor $2^{-l(\varepsilon+2n)}$ which partially compensates the normalization coefficient 2^{2ln}. So we have

$$
\begin{aligned}
\int_0^{r(B)^m} I_l^1(f,g,t)(x)\frac{dt}{t} \lesssim 2^{-l\varepsilon}\Big[&\mathcal{J}_{2^{l+1}B}(|\nabla f|\chi_{2^{l+1}B}, |g|\chi_{2^{l+1}B})(x) \\
&+ \mathcal{J}_{2^{l+1}B}(|f|\chi_{2^{l+1}B}, |\nabla g|\chi_{2^{l+1}B})(x)\Big].
\end{aligned}
$$

We now study the term related to $I_l^2(f,g,t)(x)$. Since $x \in B_t$,

$$\iint\limits_{C_l(B_t\times B_t)} \left(1+\frac{|x-y|}{t^{\frac{1}{m}}}\right)^{-2n-\varepsilon}\left(1+\frac{|x-z|}{t^{\frac{1}{m}}}\right)^{-2n-\varepsilon} \frac{dydz}{t^{2n/m}} \lesssim 2^{-l(\varepsilon+n)}.$$

Integrating in the variable $t \in (0, r(B)^m)$, we obtain

$$\begin{aligned}
&\int_0^{r(B)^m} I_l^2(f,g,t)(x)\,\frac{dt}{t}\\
&\quad\lesssim \int_0^{r(B)^m} 2^{-l(\varepsilon+n)} \sum_{k=0}^{l-1} \iint_{2^{k+1}B_t\times 2^{k+1}B_t} \frac{|\nabla f(a)||g(b)|+|f(a)||\nabla g(b)|}{(|x-a|+|x-b|)^{2n-1}} dadb\,\frac{dt}{t}\\
&\quad\lesssim l2^{-l(\varepsilon+n)} \iint_{2^lB\times 2^lB} \left(\int_{\substack{0\le t\le r(B)^m\\ |x-a|\le 2^l t^{1/m}\\ |x-b|\le 2^l t^{1/m}}} \frac{dt}{t}\right) \frac{|\nabla f(a)||g(b)|+|f(a)||\nabla g(b)|}{(|x-a|+|x-b|)^{2n-1}} dadb\\
&\quad\lesssim l2^{-l(\varepsilon+n)} \iint_{2^lB\times 2^lB} \frac{|\nabla f(a)||g(b)|+|f(a)||\nabla g(b)|}{(|x-a|+|x-b|)^{2n-1}}\\
&\qquad\qquad\qquad \times \log\left(1+\frac{r(B)}{2^{-l}(|x-a|+|x-b|)}\right) dadb\\
&\quad\lesssim l2^{-l(\varepsilon+n)} \iint_{2^lB\times 2^lB} \frac{|\nabla f(a)||g(b)|+|f(a)||\nabla g(b)|}{(|x-a|+|x-b|)^{2n-1}} \log\left(\frac{8\cdot 2^{l+1}r(B)}{(|x-a|+|x-b|)}\right) dadb\\
&\quad\lesssim l2^{-l(\varepsilon+n)} \Big[\mathcal{J}_{2^{l+1}B}(|\nabla f|\chi_{2^{l+1}B}, |g|\chi_{2^{l+1}B})(x) + \mathcal{J}_{2^{l+1}B}(|f|\chi_{2^{l+1}B}, |\nabla g|\chi_{2^{l+1}B})(x)\Big].
\end{aligned}$$

Having obtained pointwise estimates both for $I_0(f,g,t)$ and $I_l(f,g,t)$, we can now conclude the proof of the theorem.

End of the proof of Theorem 3. Using the estimates for $I_0(f,g,t)$, $I_l^1(f,g,t)$ and $I_l^2(f,g,t)$, we finally obtain that

$$\begin{aligned}
&\big|f(x)g(x) - S_{r(B)^m}(f)(x)S_{r(B)^m}(g)(x)\big|\\
&\lesssim \sum_{l\ge 0} 2^{-l\varepsilon}(1+l2^{-ln})\\
&\times \Big[\mathcal{J}_{2^{l+1}B}(|\nabla f|\chi_{2^{l+1}B}, |g|\chi_{2^{l+1}B})(x) + \mathcal{J}_{2^{l+1}B}(|f|\chi_{2^{l+1}B}, |\nabla g|\chi_{2^{l+1}B})(x)\Big]\\
&\lesssim \sum_{l\ge 0} 2^{-l\varepsilon}\Big[\mathcal{J}_{2^{l+1}B}(|\nabla f|\chi_{2^{l+1}B}, |g|\chi_{2^{l+1}B})(x) + \mathcal{J}_{2^{l+1}B}(|f|\chi_{2^{l+1}B}, |\nabla g|\chi_{2^{l+1}B})(x)\Big].
\end{aligned}$$

□

3.2. **Boundedness properties of the operator $\mathcal{J}_B$.** Boundedness properties of the operators $\mathcal{J}_B$ follow from results for multilinear potential operators in the context of spaces of homogeneous type studied in [42]. We use those results, which were recalled in Section 2, to prove the following proposition.

Proposition 4. *Let $p_1, p_2 > 1$, $q > 0$, $0 < \alpha \le 1$ and $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{1-\alpha}{n}$. If (w_1, w_2) belongs to the class $A_{(p_1,p_2),q}$ then the operator $\mathcal{J}_B$ defined in (3.30) satisfies*

$$\|\mathcal{J}_B\|_{L^{p_1}_{w_1}(B)\times L^{p_2}_{w_2}(B)\to L^q_w(B)} \lesssim [r(B)]^\alpha,$$

with a constant uniform in B.

Proof. Following the results in [42], we work in the space of homogeneous type $(B, |\cdot - \cdot|, dx)$ noting that the constants in (2.18), (2.19), (2.20) are independent of B.

We will consider the kernel

$$\tilde{K}((x,y),(a,b)) := \frac{1}{(|x-a|+|y-b|)^{2n-1}} \log\left(\frac{8\,r(B)}{|x-a|+|y-b|}\right), \qquad x,\, y,\, a,\, b \in B$$

and check that $\tilde{K}$ satisfies (2.22) and (2.23). For condition (2.22), note that for any $c>1$ the function $h(t) = \frac{1}{t^{2n-1}}\log(\frac{8r(B)}{t})$ satisfies $h(t) \le Ch(t')$ if $t' \le c\,t$ and $t,\, t' \le 4r(B)$, for some $C>0$ independent of B. Regarding condition (2.23), recall that the ball with center $x \in B$ and radius $r>0$ in the space $(B, |\cdot - \cdot|, dx)$ is $B(x,r)\cap B$ where $B(x,r)$ is the Euclidean ball in $\mathbb{R}^n$ of radius r centered at x. Since for $x\in B$ and $r \lesssim r(B)$, $|B(x,r)\cap B| \sim |B(x,r)| = c_n\, r^n$, we then have to prove that there exists $\delta>0$ such that given $C_1>1$ there is $C_2>0$ independent of B for which

$$\frac{\varphi(B_1\cap B)}{\varphi(B_2\cap B)} \le C_2 \left(\frac{r_2}{r_1}\right)^{2n-\delta},$$

for all balls $B_i = B(x_i, r_i)$, $x_i \in B$, $r_i \le C_1 r(B)$, $B_1\cap B \subset B_2 \cap B$, where

$$\varphi(B_i\cap B) = \sup\{K(x,a,b) : x,\, a,\, b \in B_i\cap B,\, |x-a|+|x-b| \ge c\, r_i\}$$

for some fixed positive small constant c and $i = 1, 2$. We have $\varphi(B_i \cap B) = (\frac{1}{cr_i})^{2n-1}\log(\frac{8r(B)}{cr_i})$ which gives

$$\frac{\varphi(B_1\cap B)}{\varphi(B_2\cap B)} \sim \left(\frac{r_2}{r_1}\right)^{2n-1} \frac{\log\left(\frac{8r(B)}{cr_1}\right)}{\log\left(\frac{8r(B)}{cr_2}\right)} \lesssim \left(\frac{r_2}{r_1}\right)^{2n-\delta}, \qquad 0<\delta<1,$$

since $\frac{\log(t')}{\log(t)} \lesssim (\frac{t'}{t})^{\gamma}$ for $2 \le t \le t'$ and $0<\gamma<1$.

We now check that the assumptions on the weights w_1, w_2, and w imply (2.24) if $q>1$ and (2.25) if $q \le 1$ with $u = w^{1/q} = w_1^{1/p_1} w_2^{1/p_2}$, $v_k = w_k^{1/p_k}$, $k=1,2$. This means that we have to prove that there exists $t>1$ such that

(3.36)
$$\sup_Q \varphi(Q)|Q|^{\frac{1}{q}+\frac{1}{p_1{}'}+\frac{1}{p_2'}} \left(\frac{1}{|Q|}\int_Q w^t dx\right)^{1/qt} \prod_{j=1}^{2}\left(\frac{1}{|Q|}\int_Q w_i^{-\frac{t}{p_i-1}} dx\right)^{1/tp_i'} < \infty, \quad q>1,$$

and

(3.37)
$$\sup_Q \varphi(Q)|Q|^{\frac{1}{q}+\frac{1}{p_1{}'}+\frac{1}{p_2'}} \left(\frac{1}{|Q|}\int_Q w dx\right)^{1/q} \prod_{j=1}^{2}\left(\frac{1}{|Q|}\int_Q w_i^{-\frac{t}{p_i-1}} dx\right)^{1/tp_i'} < \infty, \quad q \le 1,$$

where the sup is taken over all balls Q in the space $(B, |\cdot - \cdot|, dx)$ with $r(Q) \lesssim r(B)$. The proofs follow using the same ideas as in Remark 2.4 . Let Q be a ball in the space $(B, |\cdot - \cdot|, dx)$ with $r(Q) \lesssim r(B)$; then $Q = B\cap B(x,r)$ for some $x\in B$ and $r>0$, $r(Q) = r \lesssim r(B)$ and $|Q| \sim |B(x,r)|$. Moreover, using the relation between

p_1, p_2, q and α as in the statement of the proposition,

$$\varphi(Q)|Q|^{\frac{1}{q}+\frac{1}{p_1{}'}+\frac{1}{p_2'}} \sim \frac{1}{r(Q)^{2n-1}} \log\left(\frac{8r(B)}{c\,r(Q)}\right) r(Q)^{2n-1+\alpha}$$
$$= r(Q)^{\alpha} \log\left(\frac{8r(B)}{c\,r(Q)}\right) \lesssim r(B)^{\alpha}.$$

In addition, the second factor in (3.36) is bounded by

$$\sup_{x\in\mathbb{R}^n, r>0} \left(\frac{1}{|B(x,r)|}\int_{B(x,r)} w^t dx\right)^{1/qt} \prod_{j=1}^{2}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} w_i^{-\frac{t}{p_i-1}} dx\right)^{1/tp_i'}. \tag{3.38}$$

Since w, $w_1^{-\frac{1}{p_1-1}}$, $w_2^{-\frac{1}{p_2-1}}$ are A_∞ weights (see Remark 2.1), there exists $t>1$ such that (3.38) is bounded by

$$\sup_{x\in\mathbb{R}^n, r>0} \left(\frac{1}{|B(x,r)|}\int_{B(x,r)} w\, dx\right)^{1/q} \prod_{j=1}^{2}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} w_i^{-\frac{1}{p_i-1}} dx\right)^{1/p_i'} < \infty,$$

where finiteness is due to (w_1, w_2) satisfying the $A_{(p_1,p_2),q}$ condition. A similar reasoning applies to (3.37). We conclude that (3.36) and (3.37) are bounded by a multiple (independent of B) of $r(B)^\alpha$.

By Theorem B and Remark 2.3 we have that $\mathcal{J}_B$ is bounded from $L^{p_1}_{w_1}(B)\times L^{p_2}_{w_2}(B)$ into $L^q_w(B)$ and the operator norm is bounded by a multiple (uniform on B) of $r(B)^\alpha$. □

3.3. **Proof of Theorem 2.** Let p_1, p_2, q, w_1, w_2 and w be as in the statement of Theorem 2. By Proposition 4 we have

$$\left\|\mathcal{J}_{2^l B,1}\right\|_{L^{p_1}_{w_1}(B)\times L^{p_2}_{w_2}(B)\to L^q_w(B)} \lesssim \left[2^l r(B)\right]^{\alpha},$$

uniformly in B and $l\geq 0$, this and Theorem 3 imply

$$\left\|fg - S_{r(B)^m}(f)S_{r(B)^m}(g)\right\|_{L^q_w(B)}$$
$$\lesssim \sum_{l\geq 0} 2^{-l\varepsilon}2^{\alpha(l+1)} r(B)^\alpha \left[\|\nabla f\|_{L^{p_1}_{w_1}(2^{l+1}B)}\|g\|_{L^{p_2}_{w_2}(2^{l+1}B)} + \|f\|_{L^{p_1}_{w_1}(2^{l+1}B)}\|\nabla g\|_{L^{p_2}_{w_2}(2^{l+1}B)}\right],$$

which concludes the proof of Theorem 2. □

Applying an analogous proof to that of Theorem 2, we obtain the following result:

Theorem 5. *Under the same assumptions of Theorem 2,*

$$\left\|fg - S_{r(B)^m}\left[S_{r(B)^m}(f)S_{r(B)^m}(g)\right]\right\|_{L^q_w(B)}$$
$$\leq C\, r(B)^\alpha \sum_{l\geq 0} 2^{-l(\varepsilon-\alpha)} \left[\|\nabla f\|_{L^{p_1}_{w_1}(2^{l+1}B)}\|g\|_{L^{p_2}_{w_2}(2^{l+1}B)} + \|f\|_{L^{p_1}_{w_1}(2^{l+1}B)}\|\nabla g\|_{L^{p_2}_{w_2}(2^{l+1}B)}\right].$$

We will leave it to the reader to check the details for the fact that the proof of Theorem 2 still holds after noting that a similar representation formula can be used as we can write

$$fg - S_{r(B)^m}\left[S_{r(B)^m}(f)S_{r(B)^m}(g)\right] = -\int_0^{r(B)^m} t\partial_t S_t\left[\mathcal{P}_t(F)\right]\frac{dt}{t},$$

since $t\partial_t S_t\left[\mathcal{P}_t\right]$ satisfies the same estimates as $t\partial_t\mathcal{P}_t$ and the cancellation property $t\partial_t S_t\left[\mathcal{P}_t(\mathbf{1})\right] = 0$.

4. Leibniz-type rules in Campanato-Morrey spaces associated to generalized approximations of identity

In this section we apply Theorem 2 to prove a Leibniz-type rule of the form (1.1) where the spaces X_1, X_2, Y_1, Y_2 belong to the scale of the classical Campanato-Morrey spaces and the space Z quantifies the oscillation $|fg - S_{r(B)^m}(f)S_{r(B)^m}(f)|$ of the product fg in $L^q(B)$ where $B \subset \mathbb{R}^n$ is a Euclidean ball in $\mathbb{R}^n$ (compare to (1.8)). In this context, it will become clear how, as announced in the Introduction, the bilinear potential operators introduced in Section 2 play the role that paraproducts and the bilinear Coifman-Meyer multipliers play in the proofs of the Sobolev-based Leibniz-type rules (1.2).

Next, we recall the definition of the classical Campanato-Morrey spaces and introduce notions of bilinear Campanato-Morrey spaces associated to approximations of the identity and semigroups.

For $p > 0$ and $\lambda \geq 0$ we say that $f \in L^1_{loc}(\mathbb{R}^n)$ belongs to the *Campanato-Morrey space* $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ if

$$\|f\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)} := \sup_{B\subset\mathbb{R}^n}\frac{1}{|B|^\lambda}\left(\frac{1}{|B|}\int_B |f(x)|^p\,dx\right)^{\frac{1}{p}} \tag{4.39}$$

is finite. For $f, g \in L^1(\mathbb{R}^n)$ we say that the pair (f,g) belongs to the *bilinear Campanato-Morrey space* $L^{p,\lambda}_{\mathcal{S}\otimes\mathcal{S}}(\mathbb{R}^n)$ associated to an approximation of the identity $\mathcal{S} = \{S_t\}_{t>0}$ of order $m > 0$ if

$$\|(f,g)\|_{L^{p,\lambda}_{\mathcal{S}\otimes\mathcal{S}}(\mathbb{R}^n)} := \sup_{B\subset\mathbb{R}^n}\frac{1}{|B|^\lambda}\left(\frac{1}{|B|}\int_B |f(x)g(x) - S_{r(B)^m}(f)(x)S_{r(B)^m}(g)(x)|^p\,dx\right)^{\frac{1}{p}} \tag{4.40}$$

is finite. We use the notation $\mathcal{S}\otimes\mathcal{S}$ to signify that the oscillation in question coincides with the tensorial oscillation $|(f\otimes g)(x,y) - (S\otimes S)_{r(B)^m}(f\otimes g)(x,y)|$, for $x, y \in B$, restricted to the diagonal $x = y$. These new spaces $L^{p,\lambda}_{\mathcal{S}\otimes\mathcal{S}}(\mathbb{R}^n)$ arise as natural bilinear counterparts to the Campanato-Morrey spaces $L^{p,\lambda}_{\mathcal{S}}(\mathbb{R}^n)$ associated to $\mathcal{S}$ introduced by Duong and Yan in [22, 23] and further studied in [21], [24] [55]. In this case, $f \in L^{p,\lambda}_{\mathcal{S}}(\mathbb{R}^n)$ if

$$\|f\|_{L^{p,\lambda}_{\mathcal{S}}(\mathbb{R}^n)} := \sup_{B\subset\mathbb{R}^n}\frac{1}{|B|^\lambda}\left(\frac{1}{|B|}\int_B |f(x) - S_{r(B)^m}(f)(x)|^p\,dx\right)^{\frac{1}{p}} < \infty. \tag{4.41}$$

Theorem 6. *Let $\mathcal{S} := \{S_t\}_{t>0}$ and $\mathcal{S}' := \{t\partial_t S_t\}_{t>0}$ be approximations of the identity of order $m > 0$ in $\mathbb{R}^n$ and constant ε in (3.28), $1 < p_1, p_2 < \infty$, $0 < \alpha < \min(\varepsilon, 1)$ and $q > 0$ such that $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{1-\alpha}{n}$. Given $\lambda_1, \lambda_2 \geq 0$ set $\lambda = \frac{1}{n} + \lambda_1 + \lambda_2$ and assume that $\varepsilon > n\left(\lambda + \frac{1}{q}\right)$. Then there exists a structural constant $C > 0$ such that the following Leibniz-type rule holds true*

(4.42)

$$\|(f,g)\|_{L^{q,\lambda}_{\mathcal{S}\otimes\mathcal{S}}(\mathbb{R}^n)} \leq C\left(\|\nabla f\|_{\mathcal{L}^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|_{\mathcal{L}^{p_2,\lambda_2}(\mathbb{R}^n)} + \|f\|_{\mathcal{L}^{p_1,\lambda_1}(\mathbb{R}^n)}\|\nabla g\|_{\mathcal{L}^{p_2,\lambda_2}(\mathbb{R}^n)}\right).$$

Proof. From Theorem 2 we have

$$\begin{aligned}&\left\|fg - S_{r(B)^m}(f)S_{r(B)^m}(g)\right\|_{L^q(B)}\\ &\quad\lesssim r(B)^\alpha \sum_{l\geq 0} 2^{-l(\varepsilon-\alpha)}\left(\|\nabla f\|_{L^{p_1}(2^l B)}\|g\|_{L^{p_2}(2^l B)} + \|f\|_{L^{p_1}(2^l B)}\|\nabla g\|_{L^{p_2}(2^l B)}\right).\end{aligned}$$

By writing

$$\|\nabla f\|_{L^{p_1}(2^l B)} = |2^l B|^{\lambda_1+\frac{1}{p_1}}\frac{1}{|2^l B|^{\lambda_1}}\left(\frac{1}{|2^l B|}\int_{2^l B}|\nabla f|^{p_1}\right)^{\frac{1}{p_1}} \leq |2^l B|^{\lambda_1+\frac{1}{p_1}}\|\nabla f\|_{\mathcal{L}^{p_1,\lambda_1}(\mathbb{R}^n)}$$

and

$$\|g\|_{L^{p_2}(2^l B)} = |2^l B|^{\lambda_2+\frac{1}{p_2}}\frac{1}{|2^l B|^{\lambda_2}}\left(\frac{1}{|2^l B|}\int_{2^l B}|g|^{p_2}\right)^{\frac{1}{p_2}} \leq |2^l B|^{\lambda_2+\frac{1}{p_2}}\|g\|_{\mathcal{L}^{p_2,\lambda_2}(\mathbb{R}^n)},$$

and similarly with $\|f\|_{L^{p_1}(2^l B)}$ and $\|\nabla g\|_{L^{p_2}(2^l B)}$, and by setting $s := \lambda_1 + \lambda_2 + \frac{1}{p_1} + \frac{1}{p_2}$, we obtain

$$\begin{aligned}&r(B)^\alpha \sum_{l\geq 0} 2^{-l(\varepsilon-\alpha)}\left(\|\nabla f\|_{L^{p_1}(2^l B)}\|g\|_{L^{p_2}(2^l B)} + \|f\|_{L^{p_1}(2^l B)}\|\nabla g\|_{L^{p_2}(2^l B)}\right)\\ &\leq |B|^{\frac{\alpha}{n}+s}\sum_{l\geq 0} 2^{-l(\varepsilon-\alpha-ns)}\left(\|\nabla f\|_{\mathcal{L}^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|_{\mathcal{L}^{p_2,\lambda_2}(\mathbb{R}^n)} + \|f\|_{\mathcal{L}^{p_1,\lambda_1}(\mathbb{R}^n)}\|\nabla g\|_{\mathcal{L}^{p_2,\lambda_2}(\mathbb{R}^n)}\right)\\ &\leq C|B|^{\frac{\alpha}{n}+s}\left(\|\nabla f\|_{\mathcal{L}^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|_{\mathcal{L}^{p_2,\lambda_2}(\mathbb{R}^n)} + \|f\|_{\mathcal{L}^{p_1,\lambda_1}(\mathbb{R}^n)}\|\nabla g\|_{\mathcal{L}^{p_2,\lambda_2}(\mathbb{R}^n)}\right),\end{aligned}$$

since $\alpha + ns = n\left(\frac{\alpha}{n} + \frac{1}{p_1} + \frac{1}{p_2} + \lambda_1 + \lambda_2\right) = n\left(\lambda + \frac{1}{q}\right) < \varepsilon$. Consequently, using that $\lambda + \frac{1}{q} = \frac{\alpha}{n} + s$,

$$\begin{aligned}&\frac{1}{|B|^\lambda}\left(\frac{1}{|B|}\int_B |f(x)g(x) - S_{r(B)^m}(f)(x)S_{r(B)^m}(g)(x)|^q\,dx\right)^{\frac{1}{q}}\\ &\qquad = \frac{1}{|B|^{\frac{\alpha}{n}+s}}\left\|fg - S_{r(B)^m}(f)S_{r(B)^m}(g)\right\|_{L^q(B)}\\ &\qquad \leq C\left(\|\nabla f\|_{\mathcal{L}^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|_{\mathcal{L}^{p_2,\lambda_2}(\mathbb{R}^n)} + \|f\|_{\mathcal{L}^{p_1,\lambda_1}(\mathbb{R}^n)}\|\nabla g\|_{\mathcal{L}^{p_2,\lambda_2}(\mathbb{R}^n)}\right),\end{aligned}$$

and (4.42) follows. □

In relation with (4.41), we define another suitable notion of Campanato-Morrey spaces associated to an approximation of the identity $\mathcal{S} = \{S_t\}$: a function f belongs to $\tilde{L}^{p,\lambda}_{\mathcal{S}}(\mathbb{R}^n)$ if

$$\|f\|_{\tilde{L}^{p,\lambda}_{\mathcal{S}}(\mathbb{R}^n)} := \sup_{B\subset\mathbb{R}^n} \inf_{h\in L^1_{loc}} \frac{1}{|B|^\lambda}\left(\frac{1}{|B|}\int_B |f(x) - S_{r(B)^m}(h)(x)|^p\,dx\right)^{\frac{1}{p}} < \infty,$$

where the supremum is taken over all Euclidean balls $B \subset \mathbb{R}^n$. Then, we have the following Leibniz-type rule:

Theorem 7. *Let $\mathcal{S} := \{S_t\}_{t>0}$ and $\mathcal{S}' := \{t\partial_t S_t\}_{t>0}$ be approximations of the identity in $\mathbb{R}^n$ of order $m > 0$ and constant ε in (3.28), $1 < p_1, p_2 < \infty$, $0 < \alpha < \min(\varepsilon, 1)$ and $q > 0$ such that $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{1-\alpha}{n}$. Given $\lambda_1, \lambda_2 \geq 0$ set $\lambda = \frac{1}{n} + \lambda_1 + \lambda_2$ and assume that $\varepsilon > n\left(\lambda + \frac{1}{q}\right)$. Then there exists a structural constant $C > 0$ such that the following Leibniz-type rule holds true*

$$\|fg\|_{\tilde{L}^{q,\lambda}_{\mathcal{S}}(\mathbb{R}^n)} \leq C\left(\|\nabla f\|_{\mathcal{L}^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|_{\mathcal{L}^{p_2,\lambda_2}(\mathbb{R}^n)} + \|f\|_{\mathcal{L}^{p_1,\lambda_1}(\mathbb{R}^n)}\|\nabla g\|_{\mathcal{L}^{p_2,\lambda_2}(\mathbb{R}^n)}\right). \tag{4.43}$$

The proof follows by estimating the norm

$$\sup_{B\subset\mathbb{R}^n} \inf_{h\in L^1_{loc}} \frac{1}{|B|^\lambda}\left(\frac{1}{|B|}\int_B |f(x)g(x) - S_{r(B)^m}(h)(x)|^p\,dx\right)^{\frac{1}{p}}$$

with $h = S_{r(B)^m}(f)S_{r(B)^m}(g)$ and following the arguments in Theorem 6 by invoking Theorem 5 instead of Theorem 2.

5. Extensions to doubling Riemannian manifolds and Carnot groups

5.1. **Doubling Riemannian manifolds.** Let $(M, \rho, d\mu)$ be a doubling Riemannian manifold, this is a space of homogeneous type with a gradient vector field ∇ (e.g. a complete Riemannian manifold with nonnegative Ricci curvature).

An approximation of the identity of order $m > 0$ in M is a collection of operators $\mathcal{S} := \{S_t\}_{t>0}$ acting on functions defined on M,

$$S_t f(x) = \int_M p_t(x,y) f(y)\,d\mu(y),$$

such that for each $t > 0$ the kernels p_t satisfy $\int_M p_t(x,y)\,d\mu(y) = 1$ for all x and the *scaled Poisson bound*

$$|p_t(x,y)| \leq \mu(B_\rho(x, t^{1/m}))^{-1}\,\gamma\left(\frac{\rho(x,y)}{t^{1/m}}\right), \tag{5.44}$$

where $\gamma : [0,\infty) \to [0,\infty)$ is a bounded, decreasing function such that

$$\lim_{r\to\infty} r^{2n+\varepsilon}\gamma(r) = 0, \qquad \text{for some } \varepsilon > 0. \tag{5.45}$$

Theorem 8. *Assume (M, ρ, μ) is a doubling Riemannian manifold. Let $\mathcal{S} := \{S_t\}_{t>0}$ and $\mathcal{S}' := \{t\partial_t S_t\}_{t>0}$ be approximations of the identity in M of order $m > 0$ and* *constant* ε *in* (5.45), $1 < p_1, p_2 < \infty$, $q > 0$, *and* $0 < \alpha < \min\{1, \varepsilon\}$ *such that* $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{1-\alpha}{n}$. *Then there exists a constant* C *such that for all balls* $B \subset M$

$$\begin{aligned} &\left\| fg - S_{r(B)^m}(f) S_{r(B)^m}(g) \right\|_{L^q(B)} \\ &\quad \leq C\, r(B)^\alpha \sum_{l \geq 0} 2^{-l(\varepsilon-\alpha)} \left[\|\nabla f\|_{L^{p_1}(2^{l+1}B)} \|g\|_{L^{p_2}(2^{l+1}B)} + \|f\|_{L^{p_1}(2^{l+1}B)} \|\nabla g\|_{L^{p_2}(2^{l+1}B)} \right]. \end{aligned}$$

The proof of this theorem follows from that of Theorem 2 after minor modifications. The Leibniz rules in Campanato/Morrey spaces, obtained in Section 4, can be extended to this framework as well.

5.2. **Carnot groups.** In this section we provide a description of how to extend our results of section 3 in the context of Carnot groups. Let Ω be an open connected subset of $\mathbb{R}^n$ and $\mathbf{X} = \{X_k\}_{k=1}^M$ be a family of infinitely differentiable vector fields with values in $\mathbb{R}^n$. We identify X_k with the first order differential operator acting on continuously differentiable functions defined on Ω by the formula

$$X_k f(x) = X_k(x) \cdot \nabla f(x), \quad k = 1, \cdots, M,$$

and we set $\mathbf{X}f = (X_1 f, X_2 f, \cdots, X_M f)$ and

$$|\mathbf{X}f(x)| = \left(\sum_{k=1}^M |X_k f(x)|^2 \right)^{1/2}, \quad x \in \Omega.$$

Given two vector fields X_i and X_j define the commutator or Lie bracket by $[X_i, X_j] = X_i X_j - X_j X_i$. We will assume that $\mathbf{X}$ satisfies Hörmander's condition in Ω; that is, there is some finite positive integer M_0 such that the commutators of the vector fields in $\mathbf{X}$ up to length M_0 span $\mathbb{R}^n$ at each point of Ω.

Suppose that $\mathbf{X} = \{X_k\}_{k=1}^M$ satisfies Hörmander's condition in Ω. Let $C_{\mathbf{X}}$ be the family of absolutely continuous curves $\zeta : [a, b] \to \Omega$, $a \leq b$, such that there exist measurable functions $c_j(t)$, $a \leq t \leq b$, $j = 1, \cdots, M$, satisfying $\sum_{j=1}^M c_j(t)^2 \leq 1$ and $\zeta'(t) = \sum_{j=1}^M c_j(t) Y_j(\zeta(t))$ for almost every $t \in [a, b]$. If x, $y \in \Omega$ define

$$\rho(x, y) = \inf\{T > 0 : \text{ there exists } \zeta \in C_{\mathbf{X}} \text{ with } \zeta(0) = x \text{ and } \zeta(T) = y\}.$$

The function ρ is in fact a metric in Ω called the Carnot-Carathéodory metric on Ω associated to $\mathbf{X}$. For details about the geometry of Carnot-Carathéodory spaces see Nagel-Stein-Wainger [50].

Let $\mathbb{G}$ be a Lie group on $\mathbb{R}^n$, that is a group law on $\mathbb{R}^n$ such that the map $(x, y) \mapsto xy^{-1}$ is C^∞. The Lie algebra associated to $\mathbb{G}$, denoted $\mathfrak{g}$, is the collection of all left invariant vector fields on $\mathbb{G}$. A *Carnot group* is a Lie group whose Lie algebra admits a stratification

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_l,$$

where $[V_1, V_i] = \text{span}\{[Y, Z] : Y \in V_1, Z \in V_i\} = V_{i+1}$, $i = 1, \cdots, l-1$, and $[V_1, V_i] = \{0\}$ for $i \geq l$. A basis for V_1 generates the whole Lie algebra. We will often denote this family as $\{X_1, \ldots, X_{n_1}\}$ and refer to it as a family of generators for the Carnot group. In particular, a system of generators $\{X_1, \ldots, X_{n_1}\}$ satisfies Hörmander's condition, and hence we have the notion of a Carnot-Caratheodory metric.

Set $n_i = \dim(V_i)$, then $n = n_1 + \cdots + n_l$, and the number $Q = \sum_{i=1}^{l} i n_i$ is called the homogeneous dimension of $\mathbb{G}$. The dilation operators

$$\delta_\lambda x = (\lambda x^{(1)}, \lambda^2 x^{(2)}, \ldots, \lambda^l x^{(l)}) \qquad x^{(i)} \in \mathbb{R}^{n_i}$$

form automorphisms of $\mathbb{G}$ for each $\lambda > 0$. Furthermore, if B is a metric ball of radius $r(B)$ with respect to the Carnot-Carathéodory metric then $|B| = c\, r(B)^Q$, which shows that $(\mathbb{R}^n, \rho, \text{Lebesgue measure})$ is a space of homogenous type. We refer the reader to [12] for more information about analysis on Carnot groups.

An approximation of the identity of order $m > 0$ in $\mathbb{G}$ is a collection of operators $\mathcal{S} := \{S_t\}_{t>0}$ acting on functions defined on $\mathbb{R}^n$,

$$S_t f(x) = \int_{\mathbb{R}^n} p_t(x,y) f(y)\, dy,$$

such that for each $t > 0$ the kernels p_t satisfy $\int_{\mathbb{R}^n} p_t(x,y)\, dy = 1$ for all x and the *scaled Poisson bound*

$$|p_t(x,y)| \le t^{-Q/m} \gamma\Big(\frac{\rho(x,y)}{t^{1/m}}\Big),$$

where $\gamma : [0,\infty) \to [0,\infty)$ is a bounded, decreasing function such that

$$\lim_{r\to\infty} r^{2Q+\varepsilon}\gamma(r) = 0, \quad \text{for some } \varepsilon > 0.$$

Theorem 9. *Suppose $\mathbb{G}$ is a homogeneous Carnot group of dimension Q with generators $\mathbf{X} = \{X_1, \ldots, X_{n_1}\}$ and ρ is the Carnot-Carathéodory metric on $\mathbb{R}^n$ associated to $\mathbf{X}$. Suppose further that $\mathcal{S} = \{S_t\}_{t>0}$ and $\mathcal{S}' = \{t\partial_t S_t\}_{t>0}$ are approximations of the identity in $\mathbb{G}$ of order m and ε as given above. If $p_1, p_2 > 1$, $0 < \alpha < \min(\varepsilon, 1)$ and $q > 0$ are such that $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{1-\alpha}{Q}$, then, for every ρ-ball B,*

$$\begin{aligned} &\big\| fg - S_{r(B)^m}(f) S_{r(B)^m}(g) \big\|_{L^q(B)} \\ &\quad \lesssim r(B)^\alpha \sum_{l\ge 0} 2^{-l(\varepsilon-\alpha)}(l+1) \left[\|\mathbf{X}f\|_{L^{p_1}(2^{l+1}B)} \|g\|_{L^{p_2}(2^{l+1}B)} + \|f\|_{L^{p_1}(2^{l+1}B)} \|\mathbf{X}g\|_{L^{p_2}(2^{l+1}B)} \right]. \end{aligned}$$

Sketch of Proof. We will take the same approach as the proof of Theorem 2. The multilinear representation formula is given by

$$\begin{aligned} &|f(x)g(x) - S_{r(B)^m} f(x) S_{r(B)^m} g(x)| \\ &\qquad \lesssim \sum_{l\ge 0} 2^{-l(\varepsilon-Q)} [\mathcal{J}_{2^{l+1}B}(|\mathbf{X}f|, |g|)(x) + \mathcal{J}_{2^{l+1}B}(|f|, |\mathbf{X}g|)(x). \end{aligned} \tag{5.46}$$

where

$$\mathcal{J}_B(f,g)(x) = \iint_{B\times B} \frac{f(y)g(z)}{(\rho(x,y)+\rho(x,z))^{2Q-1}} \log\Big(\frac{cr(B)}{\rho(x,y)+\rho(x,z)}\Big)\, dydz \qquad x \in B$$

and B is a ball in $\mathbb{R}^n$ with respect to the metric ρ. The operator $\mathcal{J}_B$ satisfies the necessary growth bounds on its kernel and hence

$$\|\mathcal{J}_B\|_{L^{p_1}(B)\times L^{p_2}(B)\to L^q(B)} \lesssim [r(B)]^\alpha. \tag{5.47}$$

The inequalities (5.46) and (5.47) prove the desired result. The proof of inequality (5.46) follows that of Theorem 3 with the Euclidean distance replaced by $\rho(x,y)$ and the dimension n replaced by Q. We just highlight the analog to inequality (3.33),

$$\iint_{B_t\times B_t} \frac{1}{(\rho(y,a)+\rho(z,b))^{2Q-1}}\,dydz \lesssim \frac{1}{(\rho(x,a)+\rho(x,b))^{2Q-1}}. \tag{5.48}$$

Let $B = B_\rho$ be a ball in $\mathbb{R}^n$ with respect to the metric ρ, $x\in B$, $r(B)$ be the radius of B. Suppose $0<t<r(B)^m$ and $a,b\in B_t = B_\rho(x,t^{1/m})$ then

$$\begin{aligned}\iint_{B_t\times B_t} \frac{1}{(\rho(y,a)+\rho(z,b))^{2Q-1}}\,dydz &\lesssim \iint_{B_\rho(a,2t^{1m})\times B_\rho(b,2t^{1/m})} \frac{1}{(\rho(y,a)+\rho(z,b))^{2Q-1}}\,dydx\\ &\lesssim \sum_{k\geq 0}\iint_{D_k} \frac{1}{(\rho(y,a)+\rho(z,b))^{2Q-1}}\,dydx\end{aligned}$$

where

$$D_k := \{(y,z) : 2^{-k}t^{1/m} \leq \rho(a,y) < 2^{-k+1}t^{1/m}, 2^{-k}t^{1/m}\leq \rho(b,z) < 2^{-k+1}t^{1/m}\}.$$

We continue estimating each term in the series

$$\begin{aligned}&\iint_{D_k} \frac{1}{(\rho(y,a)+\rho(z,b))^{2Q-1}}\,dydz\\ &\lesssim (2^k t^{-1/m})^{2Q-1}|B_\rho(a,2^{-k+1}t^{1/m})|\cdot|B_\rho(b,2^{-k+1}t^{1/m})|\\ &\lesssim 2^{-k}t^{1/m}\end{aligned}$$

which leads to

$$\begin{aligned}\iint_{B_t\times B_t} \frac{1}{(\rho(y,a)+\rho(z,b))^{2Q-1}}\,dydz &\lesssim \iint_{B_\rho(a,2t^{1m})\times B_\rho(b,2t^{1/m})} \frac{1}{(\rho(y,a)+\rho(z,b))^{2Q-1}}\,dydx\\ &\lesssim t^{1/m}\\ &\lesssim \frac{1}{(\rho(x,a)+\rho(x,b))^{2Q-1}}\end{aligned}$$

This estimate contributes to the first term on the right side of inequality (5.46), the other terms are obtained in a similar manner.

□

6. Boundedness of bilinear pseudodifferential operators under Sobolev scaling

Let $BS^m_{\rho,\delta}(\mathbb{R}^n)$ and $BS^m_{\rho,\delta;\theta}(\mathbb{R}^n)$, where $m\in\mathbb{R}$, $0\leq\delta\leq\rho\leq 1$, $\theta\in(0,\pi)$, be the classes of symbols $\sigma\in C^\infty(\mathbb{R}^{3n})$ satisfying,

$$\left|\partial_x^\alpha\partial_\xi^\beta\partial_\eta^\gamma\sigma(x,\xi,\eta)\right| \leq C_{\alpha,\beta,\gamma}\,(1+|\xi|+|\eta|)^{m-\rho(|\beta|+|\gamma|)+\delta|\alpha|}, \tag{6.49}$$

respectively,

$$\left|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x,\xi,\eta)\right| \leq C_{\alpha,\beta,\gamma}\ (1+|\xi-\tan(\theta)\,\eta|)^{m-\rho(|\beta|+|\gamma|)+\delta|\alpha|}, \tag{6.50}$$

for all x, ξ, $\eta \in \mathbb{R}^n$, all multi-indices α, β, $\gamma \in \mathbb{N}_0$ and some constants $C_{\alpha,\beta,\gamma}$, with the convention that $\theta = \frac{\pi}{2}$ corresponds to decay in terms of $1+|\eta|$. We will use the notation $B\dot{S}^m_{1,\delta}(\mathbb{R}^n)$ and $B\dot{S}^m_{1,0;\theta}(\mathbb{R}^n)$ for the homogeneous versions of the above classes, defined by replacing $1+|\xi|+|\eta|$ by $|\xi|+|\eta|$ and $1+|\xi-\tan(\theta)\,\eta|$ by $|\xi-\tan(\theta)\,\eta|$ in (6.49) and (6.50), respectively.

These classes can be regarded as the bilinear counterparts to the linear Hörmander classes $S^m_{\rho,\delta}(\mathbb{R}^n)$ (and their homogeneous analogs $\dot{S}^m_{\rho,\delta}(\mathbb{R}^n)$) which consists of symbols $\sigma \in C^\infty(\mathbb{R}^{2n})$ such that

$$\left|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)\right| \leq C_{\alpha,\beta}\ (1+|\xi|)^{m-\rho|\beta|+\delta|\alpha|},$$

for all x, $\xi \in \mathbb{R}^n$, all multiindices α, β, and some constants $C_{\alpha,\beta}$.

Our results assume symbols in the classes $BS^m_{1,\delta}(\mathbb{R}^n)$ or $B\dot{S}^m_{1,\delta}(\mathbb{R}^n)$, as well as those symbols in $BS^m_{1,\delta;\theta}(\mathbb{R}^n)$ or $B\dot{S}^m_{1,\delta;\theta}(\mathbb{R}^n)$ of the form

$$\sigma(x,\xi,\eta) = \sigma_0(x,\xi-\tan(\theta)\,\eta), \tag{6.51}$$

where $\sigma_0 \in S^m_{1,\delta}(\mathbb{R}^n)$ or $\dot{S}^m_{1,\delta}(\mathbb{R}^n)$, respectively.

The bilinear operators with symbols in $BS^0_{1,\delta}(\mathbb{R}^n)$ or $BS^0_{1,0;\theta}(\mathbb{R})$ generalize the product of functions and, as such, enjoy boundedness properties on Lebesgue spaces with indices related by Hölder's relation. The class $BS^0_{1,0}$ was introduced by Coifman and Meyer ([18, 19]). If $\sigma \in BS^0_{1,\delta}(\mathbb{R}^n)$, $0 \leq \delta < 1$, then T_σ is actually a bilinear Calderón-Zygmund operator, and therefore it satisfies a rich variety of other boundedness properties including weighted inequalities (consult [3], [5] [39], [43]). For a number of properties for the Hörmander classes $BS^m_{\rho,\delta}(\mathbb{R}^n)$, including symbolic calculus and boundedness properties of the associated bilinear operators, see [3, 5, 6, 7, 8] and references therein. The classes $BS^0_{1,0;\theta}(\mathbb{R}^n)$, of which the symbol of the bilinear Hilbert transform is a prototype when $n = 1$, give rise to operators with symbols and kernels that are more singular and that do not fall into the scope of the bilinear Calderón-Zygmund theory. The classes $BS^m_{\rho,\delta;\theta}(\mathbb{R}^n)$ were first introduced in [6] inspired by their x-independent versions which originated in the work of Lacey and Thiele on the bilinear Hilbert transform in [37, 38] and were extensively studied in [25, 26, 27, 49] and continued in [4, 9, 10, 11].

We summarize in the following theorem a few of the results referenced above.

Theorem C (Boundedness on Lebesgue spaces under Hölder scaling)**.** *Let* p_1, $p_2 \in (1,\infty]$ *and* q *defined by* $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2}$.

(a) If $\sigma \in BS^0_{1,\delta}(\mathbb{R}^n)$ *with* $0 \leq \delta < 1$ *then* T_σ *is bounded from* $L^{p_1}_{w_1} \times L^{p_2}_{w_2}$ *into* L^q_w *for every pair of weights* (w_1, w_2) *satisfying the* $A_{(p_1,p_2),q}$ *condition and* $w := w_1^{q/p_1} w_2^{q/p_2}$.

(b) If $\sigma \in BS^0_{1,0;\theta}(\mathbb{R})$ for some $\theta \in (0,\pi) \setminus \{\pi/2, 3\pi/4\}$ and $\frac{1}{p_1} + \frac{1}{p_2} < \frac{3}{2}$ then T_σ is bounded from $L^{p_1} \times L^{p_2}$ into L^q. Moreover, there is $\sigma \in BS^0_{1,0;3\pi/4}(\mathbb{R})$ (respectively, $BS^0_{1,0;\pi/2}(\mathbb{R})$) such that T_σ is unbounded from $L^{p_1} \times L^{p_2}$ into L^q.

(c) If $\sigma \in BS^0_{1,\delta;\theta}(\mathbb{R})$ for some $\theta \in (0,\pi) \setminus \{\pi/2, 3\pi/4\}$, $0 \le \delta < 1$, is of the form (6.51) and $\frac{1}{p_1} + \frac{1}{p_2} < \frac{3}{2}$ then T_σ is bounded from $L^{p_1} \times L^{p_2}$ into L^q provided that $T(1,1) \in BMO$ for T equal to T_σ and its two adjoints.

In this section we prove boundedness properties on Lebesgue spaces for bilinear pseudodifferential operator with symbols of negative order where the indices relation is now dictated by the Sobolev scaling. More precisely, the main result in this section is the following:

Theorem 10. *Suppose $n \in \mathbb{N}$ and consider exponents $p_1, p_2 \in (1,\infty)$ and q, $s > 0$ such that*

$$\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{s}{n}. \tag{6.52}$$

(a) If $s \in (0, 2n)$, $0 \le \delta < 1$, and $\sigma \in BS^{-s}_{1,\delta}(\mathbb{R}^n) \cup \dot{BS}^{-s}_{1,\delta}(\mathbb{R}^n)$ then T_σ is bounded from $L^{p_1}_{w_1} \times L^{p_2}_{w_2}$ into L^q_w for every pair of weights (w_1, w_2) satisfying the $A_{(p_1,p_2),q}$ condition and $w := w_1^{q/p_1} w_2^{q/p_2}$.

(b) If $s \in (0,n)$, $\theta \in (0,\pi) \setminus \{\pi/2, 3\pi/4\}$, $0 \le \delta < 1$ and $\sigma \in BS^{-s}_{1,\delta;\theta}(\mathbb{R}^n) \cup \dot{BS}^{-s}_{1,\delta;\theta}(\mathbb{R}^n)$ is of the form (6.51) then the bilinear operator T_σ is bounded from $L^{p_1} \times L^{p_2}$ into L^q. If in addition $\frac{1}{p} := \frac{1}{p_1} + \frac{1}{p_2} < 1$, then T_σ is bounded from $L^{p_1}_{w_1} \times L^{p_2}_{w_2}$ into L^q_w for weights w_1, w_2 in the class $A_{p,q}$ and $w := w_1^{q/p_1} w_2^{q/p_2}$.

Proof. We start with the proof of part (a). Let $0 \le \delta < 1$, $s \in (0, 2n)$, and $\sigma \in BS^{-s}_{1,\delta}(\mathbb{R}^n) \cup \dot{BS}^{-s}_{1,\delta}(\mathbb{R}^n)$. The results will follow from part (a) of Theorem A once we have proved that the operator T_σ is controlled by the bilinear fractional integral $\mathcal{I}_s$ as defined in (2.17). T_σ is given by the spatial representation

$$T_\sigma(f,g)(x) = \iint_{\mathbb{R}^n \times \mathbb{R}^n} k(x, x-y, x-z) f(y) g(z) dy dz$$

where the kernel k is defined by

$$k(x,u,v) := \widehat{\sigma(x,\cdot,\cdot)}(u,v).$$

We will prove that,

$$|k(x,u,v)| \lesssim \frac{1}{(|u|+|v|)^{2n-s}}, \qquad \text{uniformly in } x, \tag{6.53}$$

which gives

$$|T_\sigma(f,g)(x)| \lesssim \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(y)||g(z)|}{(|x-y|+|x-z|)^{2n-s}} dy dz = \mathcal{I}_s(|f|,|g|)(x),$$

and therefore the boundedness properties of T_σ follow from part (a) of Theorem A.

Let $\Psi(\xi,\eta)$ be a smooth function in $\mathbb{R}^{2n}$ supported on the annulus $1 \leq |(\xi,\eta)| \leq 2$, and such that

$$\int_0^\infty \Psi(t\xi, t\eta)\frac{dt}{t} = 1, \quad (\xi,\eta) \neq (0,0).$$

So for each scale $t > 0$, we have to estimate $\widehat{\Psi(t\cdot)\sigma(x,\cdot)}$. Integration by parts gives

$$\left|\widehat{\Psi(t\cdot)\sigma(x,\cdot)}(u,v)\right| \lesssim \frac{t^{-2n+s}}{\left(1+t^{-1}|(u,v)|\right)^N}$$

for every large enough integer N. The factor t^s comes from the extra decay $(1+|\xi|+|\eta|)^{-s} \lesssim t^s$ when taking into account the support of Ψ. Then, integration over $t \in (0,\infty)$ yields

$$\begin{aligned} |k(x,u,v)| &\lesssim \int_0^\infty \left|\widehat{\Psi(t\cdot)\sigma(x,\cdot)}(u,v)\right| \frac{dt}{t} \lesssim \int_0^\infty \frac{t^{-2n+s}}{\left(1+t^{-1}|(u,v)|\right)^N}\frac{dt}{t} \\ &\lesssim |(u,v)|^{-2n+s} \int_0^\infty \frac{t^{2n-s}}{(1+t)^N}\frac{dt}{t} \lesssim |(u,v)|^{-2n+s}, \end{aligned}$$

which ends the proof of (6.53).

We now turn to the proof of part (b) of the theorem. If $s \in (0,n)$, $\theta \in (0,\pi) \setminus \{\pi/2, 3\pi/4\}$ and $\sigma \in BS^{-s}_{1,\delta;\theta}(\mathbb{R}^n) \cup B\dot{S}^{-s}_{1,\delta;\theta}(\mathbb{R}^n)$ is of the form $\sigma(x,\xi,\eta) = \sigma_0(x, \xi - \tan(\theta)\,\eta)$ with $\sigma_0 \in S^{-s}_{1,\delta}(\mathbb{R}^n)$ or $\sigma_0 \in \dot{S}^{-s}_{1,\delta}(\mathbb{R}^n)$ as appropriate, we consider the following spatial representation for T_σ :

$$T_\sigma(f,g)(x) = \int_{\mathbb{R}^n} k(x,y) f(x+y) g(x - \tan(\theta)\, y) dy$$

where the kernel k is defined by

$$k(x,y) := \widehat{\sigma_0(x,\cdot)}(y).$$

Following the same reasoning as above, we obtain

$$|k(x,y)| \lesssim |y|^{s-n}, \quad \text{uniformly in } x,$$

and therefore

$$|T_\sigma(f,g)| \lesssim \mathcal{B}_s(f,g),$$

with $\mathcal{B}_s$ defined in (2.16). The result then follows from parts (b) and (c) of Theorem A. □

Remark 6.1. We note that pointwise decay properties of the kernels (and their derivatives) of pseudodifferential operators with symbols in the Hörmander classes have been studied in [5, Theorem 5.1]. In particular, it is proved there that if $\sigma \in BS^{-s}_{1,\delta}(\mathbb{R}^n)$, then (6.53) holds.

As usual, it is possible to get "off-diagonal decays" (which are stronger than global estimates) for the inhomogeneous symbols: for every ball B of radius 1,

$$\|T_\sigma(f,g)\|_{L^q(B)} \lesssim \left(\sum_{k\geq 0} 2^{-kN}\|f\|_{L^{p_1}(2^kB)}\right)\left(\sum_{k\geq 0} 2^{-kN}\|g\|_{L^{p_2}(2^kB)}\right)$$

for a large enough integer N. Indeed, the case $k = 0$ follows from Theorem 10 and the fast decreasing coefficients, $k > 0$, from the fast decay of the bilinear kernel, due to the inhomogeneous condition of regularity (see Theorem 5.1 in [5]).

In the following, we consider the inhomogeneous and homogeneous Sobolev spaces for indeces $\alpha \in \mathbb{R}$ and $1 < p < \infty$,

$$W^{\alpha,p}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \mathcal{F}^{-1}((1+|\xi|^2)^{\alpha/2}\hat{f}(\xi)) \in L^p(\mathbb{R}^n)\}$$

and

$$\dot{W}^{\alpha,p}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \mathcal{F}^{-1}(|\xi|^\alpha \hat{f}(\xi)) \in L^p(\mathbb{R}^n)\},$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform.

Corollary 11 (Leibniz-type rules)**.** *Let $n \in \mathbb{N}$ and consider exponents $p_1, p_2 \in (1,\infty)$ and q, $s > 0$ such that $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{s}{n}$.*

(a) If $s \in (0, 2n)$, $0 \le \delta < 1$, and $\sigma \in BS^m_{1,\delta}(\mathbb{R}^n)$ for some $m \ge -s$ then

$$\|T_\sigma(f,g)\|_{L^q} \lesssim \|f\|_{W^{m+s,p_1}}\|g\|_{L^{p_2}} + \|f\|_{L^{p_1}}\|g\|_{W^{m+s,p_2}}.$$

(b) If $n \in \mathbb{N}$, $s \in (0, 2n)$, $0 \le \delta < 1$, and $\sigma \in \dot{BS}^m_{1,\delta}(\mathbb{R}^n)$ for some $m \ge -s$ then

$$\|T_\sigma(f,g)\|_{L^q} \lesssim \|f\|_{\dot{W}^{m+s,p_1}}\|g\|_{L^{p_2}} + \|f\|_{L^{p_1}}\|g\|_{\dot{W}^{m+s,p_2}}.$$

(c) If $s \in (0, n)$, $\theta \in (0,\pi) \setminus \{\pi/2, 3\pi/4\}$ and $\sigma \in BS^m_{1,\delta;\theta}(\mathbb{R}^n)$ for some $m \ge -s$ and $0 \le \delta < 1$ is of the form (6.51) then

$$\|T_\sigma(f,g)\|_{L^q} \lesssim \|f\|_{W^{m+s,p_1}}\|g\|_{L^{p_2}} + \|f\|_{L^{p_1}}\|g\|_{W^{m+s,p_2}}.$$

(d) If $s \in (0, n)$, $\theta \in (0,\pi) \setminus \{\pi/2, 3\pi/4\}$ and $\sigma \in \dot{BS}^m_{1,\delta;\theta}(\mathbb{R}^n)$ for some $m \ge -s$ and $0 \le \delta < 1$ is of the form (6.51) then

$$\|T_\sigma(f,g)\|_{L^q} \lesssim \|f\|_{\dot{W}^{m+s,p_1}}\|g\|_{L^{p_2}} + \|f\|_{L^{p_1}}\|g\|_{\dot{W}^{m+s,p_2}}.$$

Proof. Parts (a) and (c) of Corollary 11 follow from Theorem 10 and composition with Bessel potentials of order $m+s$, along the lines of [6, Theorem 2] (see also [31, Theorem 1.4] and [5, Corollary 8]). Let σ be a symbol in one of the classes indicated in the statements of parts (a) or (c) of Corollary 11 and consider $\phi \in C^\infty(\mathbb{R})$ such that $0 \le \phi \le 1$, $\mathrm{supp}(\phi) \subset [-2, 2]$ and $\phi(r) + \phi(1/r) = 1$ on $[0,\infty)$, then, the symbols σ_1 and σ_2 defined by

$$\sigma_1(x,\xi,\eta) = \sigma(x,\xi,\eta)\phi\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right)(1+|\eta|^2)^{-(m+s)/2}$$

and

$$\sigma_2(x,\xi,\eta) = \sigma(x,\xi,\eta)\phi\left(\frac{1+|\eta|^2}{1+|\xi|^2}\right)(1+|\xi|^2)^{-(m+s)/2}$$

are symbols of order $-s$ in the corresponding classes, and the operators T_σ, T_{σ_1}, and T_{σ_2} are related through

$$T_\sigma(f,g) = T_{\sigma_1}(J^{m+s}f, g) + T_{\sigma_2}(f, J^{m+s}g),$$

where J^{m+s} denotes the linear Fourier multiplier with symbol $(1+|\xi|^2)^{(m+s)/2}$.

Parts (b) and (d) of Corollary 11 follow in the same way using the operators with Fourier multiplier $(|\xi| + |\eta|)^{m+s}$. □

Remark 6.2. The case $s=0$ in Theorem 10 for $\sigma \in BS^0_{1,\delta}(\mathbb{R}^n)$ or for special cases of $\sigma \in BS^0_{1,\delta;\theta}(\mathbb{R})$ is contained in Theorem C; results for $s=0$ remain unknown for the general case of $BS^0_{1,\delta;\theta}(\mathbb{R})$ and for the classes $BS^0_{1,\delta;\theta}(\mathbb{R}^n)$ with $n>1$. Another result in the spirit of Corollary 11 is given in [6, Propositions 3 and Remark 7], where it is proven that if $\sigma \in BS^0_{1,0}(\mathbb{R}^n)$, then T_σ is bounded from $W^{s,p_1}(\mathbb{R}^n) \times W^{s,p_2}(\mathbb{R}^n)$ into $W^{s,q}(\mathbb{R}^n)$ for p_1, $p_2>1$, $q>0$, $\frac{1}{p_1}+\frac{1}{p_2}<1$, $\frac{1}{q}=\frac{1}{p_1}+\frac{1}{p_2}-\frac{s}{n}$ and s a nonnegative number smaller than or equal to the largest integer strictly smaller than $\min(\frac{n}{p_1},\frac{n}{p_2})$.

Remark 6.3. We emphasize the following particular cases of Corollary 11. Let $n \in \mathbb{N}$, $s \in (0,2n)$, $p_1,p_2 \in (1,\infty)$ and $q>0$ such that $\frac{1}{q}=\frac{1}{p_1}+\frac{1}{p_2}-\frac{s}{n}$.

- **Fractional Leibniz rule under Sobolev scaling.** If $m>0$
$$\|fg\|_{W^{m,q}} \lesssim \|f\|_{W^{m+s,p_1}}\|g\|_{L^{p_2}}+\|f\|_{L^{p_1}}\|g\|_{W^{m+s,p_2}}.$$
- **Paraproduct estimates under Sobolev scaling.** Consider a radial, real-valued function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that $\hat\varphi(\xi)=1$ for $|\xi|\le 1$ and $\varphi(\xi)=0$ for $|\xi|\ge 3/2$. Let ψ be given by $\hat\psi(\xi)=\hat\varphi(\xi/2)-\hat\varphi(\xi)$. For $f\in L^1(\mathbb{R}^n)$ we set
$$S_j(f):=\varphi_j * f \qquad \text{and} \qquad \Delta_j(f):=S_{j+1}(f)-S_j(f),$$
where $\varphi_j(x)=2^{jn}\varphi(2^jx)$, $j\in\mathbb{Z}$. We also define $\psi_j(x):=2^{jn}\psi(2^jx)$ and note that $\mathrm{supp}(\widehat{\psi_j})\subset\{\xi : 2^j\le|\xi|\le 3\,2^j\}$. For f, $g\in\mathcal{S}(\mathbb{R}^n)$ we define the Bony paraproduct of f and g by
$$\Pi(f,g):=\sum_{j\in\mathbb{Z}}\Delta_j(f)S_{j-1}(g).$$
Straightforward computations show that $fg=\Pi(f,g)+\Pi(g,f)+\sum_{m=-1}^{1}R_m(f,g)$, where $R_m(f,g)=\sum_{j\in\mathbb{Z}}\Delta_j(f)\Delta_{j+m}(g)$ for $m=-1,0,1$.
The symbol σ of the paraproduct Π is x-independent,
$$\Pi(f,g)(x)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\sigma(\xi,\eta)\hat f(\eta)\hat g(\xi)e^{ix(\xi+\eta)}\,d\eta\,d\xi,$$
is given by
$$\sigma(\xi,\eta)=\sum_{j\in\mathbb{Z}}\widehat{\psi_j}(\xi)\widehat{\varphi_{j-1}}(\eta),$$
and belongs to the class $\dot{BS}^0_{1,0}$. As a consequence of Corollary 11, we have
$$\|\Pi(f,g)\|_{L^q}\lesssim\|f\|_{\dot W^{s,p_1}}\|g\|_{L^{p_2}}+\|f\|_{L^{p_1}}\|g\|_{\dot W^{s,p_2}}.$$
- **Lowering the exponents for linear embeddings.** It is well-known that in $\mathbb{R}^n$, for $s\in(0,1)$, $W^{s,p}$ is continuously embedded into L^q as soon as $p<d/s$ and $q\ge 1$ with
$$\frac{1}{q}=\frac{1}{p}-\frac{s}{d}.$$
By the previous approach, we get bilinear analogs: indeed we have proved that $(f,g)\to fg$ is continuous from $W^{s,p_1}\times W^{s,p_2}$ into L^q as soon as $p<d/s$ (where p is the harmonic mean value of p_1,p_2) and $q>1/2$. It is then possible to use this bilinear approach to give extensions of the linear inequalities for $q<1$.

Proposition 12. *Let consider $s \in (0,1)$ and $p = t/2 < d/s$ and $q \leq 1$ with*

$$\frac{1}{q} = \frac{1}{p} - \frac{s}{d} = \frac{2}{t} - \frac{s}{d}.$$

Then for every nonnegative smooth function h, we have

$$\|h\|_{L^q} \lesssim \|h^{1/2}\|^2_{W^{s,t}} = \|h^{1/2}\|^2_{W^{s,2p}}.$$

Proof. We just write $h = h^{1/2}h^{1/2}$ and apply the bilinear inequalities to the functions $f = g = h^{1/2}$ with the exponents $p_1 = p_2 = t$. □

Such inequalities are of interest since they allow for an exponent $q \leq 1$. To do that we have to pay the cost of estimating the regularity of $\sqrt{h}$.

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