

A remark on Besov spaces interpolation over the 2-adic group

Diego Chamorro

April 19, 2011

Abstract

Motivated by a recent result which identifies in the special setting of the 2-adic group the Besov space $\dot{B}_1^{1,\infty}(\mathbb{Z}_2)$ with $BV(\mathbb{Z}_2)$, the space of function of bounded variation, we study in this article some functional relationships between Besov spaces.

Keywords: Besov spaces, interpolation, p -adic groups.

MSC 2010: 22E35, 46E35

1 Introduction

The starting point of this article is given by the following inequality proved by A. Cohen, W. Dahmen, I. Daubechies & R. De Vore in [4]. For a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $f \in BV \cap \dot{B}_\infty^{-1,\infty}$ we have

$$\|f\|_{L^2}^2 \leq C \|f\|_{BV} \|f\|_{\dot{B}_\infty^{-1,\infty}} \quad (1)$$

Here BV denotes the space of functions of bounded variation and $\dot{B}_\infty^{-1,\infty}$ stands for an homogeneous Besov space. In the article [3], we proved that in the special setting of the 2-adic group \mathbb{Z}_2 , the space $BV(\mathbb{Z}_2)$ can be identified to the Besov space $\dot{B}_1^{1,\infty}(\mathbb{Z}_2)$ and therefore, inequality (1) becomes

$$\|f\|_{L^2}^2 \leq C \|f\|_{\dot{B}_1^{1,\infty}} \|f\|_{\dot{B}_\infty^{-1,\infty}} \quad (2)$$

Note that the previous estimate is *false* in \mathbb{Z}_2 , see [3] for a counterexample. The identification between these two functional spaces and the consequences on the inequality (1) are very surprising in the sense that these estimates *depend* on the underlying group structure: compare the topological properties of \mathbb{R}^n to the totally discontinuous setting of \mathbb{Z}_2 .

However, one may think that the Besov norm $\|\cdot\|_{\dot{B}_1^{1,\infty}}$ in the right hand side of (2) is *too* small to achieve the inequality. Thus, it is a natural question to study the validity of (2) if we replace this norm by a *bigger* one (just think on the inclusion of Besov spaces $\dot{B}_1^{1,q} \subset \dot{B}_1^{1,\infty}$ valid for $q \geq 1$). The answer to this question is given by the next result

Theorem 1 *If $f : \mathbb{Z}_2 \rightarrow \mathbb{R}$ is a function such that $f \in \dot{B}_1^{1,q} \cap \dot{B}_\infty^{-1,\infty}(\mathbb{Z}_2)$ with $q > 2$, then the following inequality is false:*

$$\|f\|_{L^2}^2 \leq C \|f\|_{\dot{B}_1^{1,q}} \|f\|_{\dot{B}_\infty^{-1,\infty}} \quad (3)$$

This is the main theorem of this article and we will construct a counterexample in the section 4 below, but before, it would be interesting to compare inequality (3) to the general estimates given by the interpolation theory¹.

¹see the book [2] for more details.

Indeed, following this general theory, we can obtain inequalities of the form

$$\|f\|_{L^2}^2 \leq C \|f\|_{\dot{B}_{p_0}^{s_0, q_0}} \|f\|_{\dot{B}_{p_1}^{s_1, q_1}} \quad (4)$$

for some special values of the real parameters $s_0, s_1, p_0, p_1, q_0, q_1$.

Perhaps the most popular case is given by the real method: set $p_0 = p_1 = p$, fix $0 < \theta < 1$ and suppose $s_0 \neq s_1$ with the relationship $s = (1 - \theta)s_0 + \theta s_1$. We obtain the following expression

$$(\dot{B}_p^{s_0, q_0}, \dot{B}_p^{s_1, q_1})_{\theta, q} = \dot{B}_p^{s, q}$$

which gives us the estimate

$$\|f\|_{\dot{B}_p^{s, q}} \leq C \|f\|_{\dot{B}_p^{s_0, q_0}}^{1-\theta} \|f\|_{\dot{B}_p^{s_1, q_1}}^{\theta} \quad (5)$$

It is very important to remark that in this particular case no relationship between q_0, q_1 and q is asked. Obviously, inequality (3) can not be obtained from (5), since $p_0 \neq p_1$.

The case when $p_0 \neq p_1$ is more restrictive and following the complex method we have for $1 \leq p_0, q_0 \leq +\infty$ and $1 \leq p_1, q_1 < +\infty$ the formula

$$[\dot{B}_{p_0}^{s_0, q_0}, \dot{B}_{p_1}^{s_1, q_1}]_{\theta} = \dot{B}_p^{s, q}$$

which gives us an estimate of the type (4) with $s = (1 - \theta)s_0 + \theta s_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$. Note that we have in this case a relationship between q_0, q_1 and q . Again, this method can not be applied to inequality (3).

It seems of course that inequality (3) cannot be obtained by an simple interpolation argument -actually this inequality is false in \mathbb{R}^n -, but what it would make it plausible in the setting of \mathbb{Z}_2 is the special relationship between inequalities (1) and (2) and this is the main reason why theorem 1 is relevant.

The plan of the article is the following. In section 2 we recall some properties of the p -adic spaces, in section 3 we give the definition of Besov spaces over the 2-adic group \mathbb{Z}_2 and in section 4 we prove theorem 1.

2 p -adic groups

Our main reference here are the books [10], [8] and [1] where more details concerning the topological structure of the p -adic groups can be found.

We write $a|b$ when a divide b or, equivalently, when b is a multiple of a . Let p be any prime number, for $0 \neq x \in \mathbb{Z}$, we define the p -adic valuation of x by $\gamma(x) = \max\{r : p^r | x\} \geq 0$ and, for any rational number $x = \frac{a}{b} \in \mathbb{Q}$, we write $\gamma(x) = \gamma(a) - \gamma(b)$. Furthermore if $x = 0$, we agree to write $\gamma(0) = +\infty$.

Let $x \in \mathbb{Q}$ and p be any prime number, with the p -adic valuation of x we can construct a norm by writing

$$|x|_p = \begin{cases} p^{-\gamma} & \text{if } x \neq 0 \\ p^{-\infty} = 0 & \text{if } x = 0. \end{cases} \quad (6)$$

This expression satisfy the following properties

- a) $|x|_p \geq 0$, and $|x|_p = 0 \iff x = 0$;
- b) $|xy|_p = |x|_p |y|_p$;
- c) $|x + y|_p \leq \max\{|x|_p, |y|_p\}$, with equality when $|x|_p \neq |y|_p$.

When a norm satisfy *c*) it is called a non-Archimedean norm and an interesting fact is that over \mathbb{Q} all the possible norms are equivalent to $|\cdot|_p$ for some p : this is the so-called Ostrowski theorem, see [1] for a proof.

Definition 2.1 *Let p be a any prime number. We define the field of p -adic numbers \mathbb{Q}_p as the completion of \mathbb{Q} when using the norm $|\cdot|_p$.*

We present in the following lines the algebraic structure of the set \mathbb{Q}_p . Every p -adic number $x \neq 0$ can be represented in a unique manner by the formula

$$x = p^\gamma(x_0 + x_1p + x_2p^2 + \dots), \tag{7}$$

where $\gamma = \gamma(x)$ is the p -adic valuation of x and x_j are integers such that $x_0 > 0$ and $0 \leq x_j \leq p - 1$ for $j = 1, 2, \dots$. Remark that this canonical representation implies the identity $|x|_p = p^{-\gamma}$.

Let $x, y \in \mathbb{Q}_p$, using the formula (7) we define the sum of x and y by $x + y = p^{\gamma(x+y)}(c_0 + c_1p + c_2p^2 + \dots)$ with $0 \leq c_j \leq p - 1$ and $c_0 > 0$, where $\gamma(x + y)$ and c_j are the unique solution of the equation

$$p^{\gamma(x)}(x_0 + x_1p + x_2p^2 + \dots) + p^{\gamma(y)}(y_0 + y_1p + y_2p^2 + \dots) = p^{\gamma(x+y)}(c_0 + c_1p + c_2p^2 + \dots).$$

Furthermore, for $a, x \in \mathbb{Q}_p$, the equation $a + x = 0$ has a unique solution in \mathbb{Q}_p given by $x = -a$. In the same way, the equation $ax = 1$ has a unique solution in \mathbb{Q}_p : $x = 1/a$.

We take now a closer look at the topological structure of \mathbb{Q}_p . With the norm $|\cdot|_p$ we construct a distance over \mathbb{Q}_p by writing

$$d(x, y) = |x - y|_p \tag{8}$$

and we define the balls $B_\gamma(x) = \{y \in \mathbb{Q}_p : d(x, y) \leq p^{-\gamma}\}$ with $\gamma \in \mathbb{Z}$. Remark that, from the properties of the p -adic valuation, this distance has the *ultra-metric* property (*i.e.* $d(x, y) \leq \max\{d(x, z), d(z, y)\} \leq |x|_p + |y|_p$).

We gather with the next proposition some important facts concerning the balls in \mathbb{Q}_p .

Proposition 2.1 *Let γ be an integer, then we have*

- 1) *the ball $B_\gamma(x)$ is a open and a closed set for the distance (8).*
- 2) *every point of $B_\gamma(x)$ is its center.*
- 3) *\mathbb{Q}_p endowed with this distance is a complete Hausdorff metric space.*
- 4) *\mathbb{Q}_p is a locally compact set.*
- 5) *the p -adic group \mathbb{Q}_p is a totally discontinuous space.*

3 Functional spaces

In this article, we will work with the subset \mathbb{Z}_2 of \mathbb{Q}_2 which is defined by $\mathbb{Z}_2 = \{x \in \mathbb{Q}_2 : |x|_2 \leq 1\}$, and we will focus on real-valued functions over \mathbb{Z}_2 . Since \mathbb{Z}_2 is a locally compact commutative group, there exists a Haar measure dx which is translation invariant *i.e.*: $d(x + a) = dx$, furthermore we have the identity $d(ax) = |a|_2 dx$ for $a \in \mathbb{Z}_2^*$. We will normalize the measure dx by setting

$$\int_{\{|x|_2 \leq 1\}} dx = 1.$$

This measure is then unique and we will note $|E|$ the measure for any subset E of \mathbb{Z}_2 .

Lebesgue spaces $L^p(\mathbb{Z}_2)$ are thus defined in a natural way: $\|f\|_{L^p} = \left(\int_{\mathbb{Z}_2} |f(x)|^p dx \right)^{1/p}$ for $1 \leq p < +\infty$, with the usual modifications when $p = +\infty$.

Let us now introduce the Littlewood-Paley decomposition in \mathbb{Z}_2 . We note \mathcal{F}_j the Boole algebra formed by the equivalence classes $E \subset \mathbb{Z}_2$ modulo the sub-group $2^j\mathbb{Z}_2$. Then, for any function $f \in L^1(\mathbb{Z}_2)$, we call $S_j(f)$ the conditionnal expectation of f with respect to \mathcal{F}_j :

$$S_j(f)(x) = \frac{1}{|B_j(x)|} \int_{B_j(x)} f(y) dy.$$

The dyadic blocks are thus defined by the formula $\Delta_j(f) = S_{j+1}(f) - S_j(f)$ and the Littlewood-Paley decomposition of a function $f : \mathbb{Z}_2 \rightarrow \mathbb{R}$ is given by

$$f = S_0(f) + \sum_{j=0}^{+\infty} \Delta_j(f) \quad \text{where } S_0(f) = \int_{\mathbb{Z}_2} f(x) dx. \quad (9)$$

We will need in the sequel some very special sets noted $Q_{j,k}$. Here is the definition and some properties:

Proposition 3.1 *Let $j \in \mathbb{N}$ and $k = \{0, 1, \dots, 2^j - 1\}$. Define the subset $Q_{j,k}$ of \mathbb{Z}_2 by*

$$Q_{j,k} = \{k + 2^j\mathbb{Z}_2\}. \quad (10)$$

Then

- 1) We have the identity $\mathcal{F}_j = \bigcup_{0 \leq k < 2^j} Q_{j,k}$,
- 2) For $k = \{0, 1, \dots, 2^j - 1\}$ the sets $Q_{j,k}$ are mutually disjoint,
- 3) $|Q_{j,k}| = 2^{-j}$ for all k ,
- 4) the 2-adic valuation is constant over $Q_{j,k}$.

The verifications are easy and left to the reader.

With the Littlewood-Paley decomposition given in (9), we obtain the following equivalence for the Lebesgue spaces $L^p(\mathbb{Z}_2)$ with $1 < p < +\infty$:

$$\|f\|_{L^p} \simeq \|S_0(f)\|_{L^p} + \left\| \left(\sum_{j \in \mathbb{N}} |\Delta_j f|^2 \right)^{1/2} \right\|_{L^p}.$$

See the book [9], chapter IV, for a general proof.

For Besov spaces we will define them by the norm

$$\|f\|_{B_p^{s,q}} \simeq \|S_0 f\|_{L^p} + \left(\sum_{j \in \mathbb{N}} 2^{jsq} \|\Delta_j f\|_{L^p}^q \right)^{1/q} \quad (11)$$

where $s \in \mathbb{R}$, $1 \leq p, q < +\infty$ with the necessary modifications when $p, q = +\infty$.

Remark 1 For homogeneous functional spaces $\dot{B}_p^{s,q}$, we drop out the term $\|S_0 f\|_{L^p}$ in (11).

4 Proof of the theorem 1

To begin the construction of the counterexample we consider $0 < j_0 < j_1$ two integers and we fix $\alpha, \beta \in \mathbb{R}$ such that

$$2^{2j_0} \leq \frac{\beta}{\alpha}. \quad (12)$$

Take now a decreasing sequence $(\varepsilon_j)_{j \in \mathbb{N}} \in \ell^q(\mathbb{N})$ with $q > 2$ such that $\varepsilon_0 = 1$ and $(\varepsilon_j)_{j \in \mathbb{N}} \notin \ell^2(\mathbb{N})$.

Define N_j in the following form

$$N_j = \begin{cases} 2^j & \text{if } 0 < j < j_0, \\ 2^{-j} \frac{\beta}{\alpha} & \text{if } j_0 \leq j \leq j_1. \end{cases} \quad (13)$$

We construct a function $f : \mathbb{Z}_2 \rightarrow \mathbb{R}$ by considering his values over the dyadic blocs and we will use for this the sets $Q_{j,k}$ defined in (10):

$$\Delta_j f(x) = \begin{cases} \varepsilon_j \alpha 2^j & \text{over } Q_{j+1,0}, \\ -\varepsilon_j \alpha 2^j & \text{over } Q_{j+1,1}, \\ \varepsilon_j \alpha 2^j & \text{over } Q_{j+1,2}, \\ -\varepsilon_j \alpha 2^j & \text{over } Q_{j+1,3}, \\ \vdots & \\ \varepsilon_j \alpha 2^j & \text{over } Q_{j+1,2N_j-2}, \\ -\varepsilon_j \alpha 2^j & \text{over } Q_{j+1,2N_j-1}, \\ 0 & \text{elsewhere.} \end{cases}$$

Remark that, with this definition of $\Delta_j f(x)$ we have the identities

- $\|\Delta_j f\|_{L^\infty} = \varepsilon_j \alpha 2^j,$
- $\|\Delta_j f\|_{L^1} = \varepsilon_j \alpha N_j,$
- $\|\Delta_j f\|_{L^2}^2 = \varepsilon_j^2 \alpha^2 2^j N_j.$

From this quantities we construct the following norms

(a) for the Besov space $\dot{B}_\infty^{-1,\infty}$ we have

$$\|f\|_{\dot{B}_\infty^{-1,\infty}} = \sup_{j \in \mathbb{N}} 2^{-j} \|\Delta_j f\|_{L^\infty} = \alpha, \text{ since the sequence } (\varepsilon_j)_{j \in \mathbb{N}} \text{ is decreasing and } \varepsilon_0 = 1.$$

(b) for the Besov space $\dot{B}_1^{1,q}$ we write

$$\|f\|_{\dot{B}_1^{1,q}}^q = \sum_{j=0}^{j_1} (2^j \|\Delta_j f\|_{L^1})^q = \sum_{j=0}^{j_1} 2^{jq} \varepsilon_j^q \alpha^q N_j^q = \alpha^q \left(\sum_{j=0}^{j_0} 2^{jq} \varepsilon_j^q N_j^q + \sum_{j>j_0}^{j_1} 2^{jq} \varepsilon_j^q N_j^q \right)$$

We use now the values of N_j given in (13) and the relationship (12) to obtain

$$= \alpha^q \left(\sum_{j=0}^{j_0} 2^{2jq} \varepsilon_j^q + \sum_{j>j_0}^{j_1} \varepsilon_j^q \frac{\beta^q}{\alpha^q} \right) = \beta^q \left(\sum_{j=0}^{j_0} 2^{2jq} \frac{\alpha^q}{\beta^q} \varepsilon_j^q + \sum_{j>j_0}^{j_1} \varepsilon_j^q \right) \simeq \beta^q \left(\sum_{j=0}^{j_0} 2^{q(2j-2j_0)} \varepsilon_j^q + \sum_{j>j_0}^{j_1} \varepsilon_j^q \right).$$

$$\text{Then we have } \|f\|_{\dot{B}_1^{1,q}} \simeq \beta \left(C_1 + \sum_{j>j_0}^{j_1} \varepsilon_j^q \right)^{1/q}.$$

(c) For the Lebesgue space L^2 we use the same arguments above to obtain

$$\|f\|_{L^2}^2 = \sum_{j=0}^{j_1} \varepsilon_j^2 \alpha^2 2^j N_j = \alpha^2 \left(\sum_{j=0}^{j_0} 2^{2j} \varepsilon_j^2 + \sum_{j>j_0}^{j_1} \varepsilon_j^2 \frac{\beta}{\alpha} \right) \simeq \alpha\beta \left(C_2 + \sum_{j>j_0}^{j_1} \varepsilon_j^2 \right).$$

Once these norms are computed, we go back to the inequality

$$\|f\|_{L^2}^2 \leq C \|f\|_{\dot{B}_1^{1,q}} \|f\|_{\dot{B}_\infty^{-1,\infty}}$$

and we have

$$\alpha\beta \left(C_2 + \sum_{j>j_0}^{j_1} \varepsilon_j^2 \right) \leq C \times \alpha \times \beta \left(C_1 + \sum_{j>j_0}^{j_1} \varepsilon_j^q \right)^{1/q}.$$

But, by hypothesis, we have $(\varepsilon_j)_{j \in \mathbb{N}} \notin \ell^2(\mathbb{N})$ and $(\varepsilon_j)_{j \in \mathbb{N}} \in \ell^q(\mathbb{N})$, thus, for j_1 big enough it is impossible to find an universal constant C such that the above inequality is true. ■

References

- [1] Yvette AMICE. *Les nombres p-adiques*. PUF, 1975.
- [2] J. BERGH & J. LÖFSTRÖM. *Interpolation Spaces*. Grundlehren der mathematischen Wissenschaften, 223. Springer Verlag (1976).
- [3] D. CHAMORRO. *A counterexample for Improved Sobolev Inequalities over the 2-adic group*, HAL : hal-00531555, (2011).
- [4] A. COHEN, W. DAHMEN, I. DAUBECHIES & R. DE VORE. *Harmonic Analysis of the space BV*. Rev. Mat. Iberoamericana 19, n°1, 235-263 (2003).
- [5] P. GÉRARD, Y. MEYER & F. ORU. *Inégalités de Sobolev Précisées*. Equations aux Dérivées Partielles, Séminaire de l'Ecole Polytechnique, exposé n° IV (1996-1997).
- [6] K. IKEDA, T. KIM, TK. SHIRATANI *On p-adic bounded functions*. Mem. Fac. Sci. Kyushu Univ. Ser. A 46, no 2, 341-349 (1992).
- [7] L.C. JANG, T. KIM, J.-W. SON, S.-H. RIM *On p-adic bounded functions II*. J.Math. Anal. Appl. 264, no 1, 21-31 (2001).
- [8] N. KOBLITZ, *p-adic Numbers, p-adic Analysis and Zeta-functions*. GTM 58. Springer Verlag, 1977.
- [9] E. M. STEIN. *Topics in Harmonic analysis*. Annals of mathematics studies, 63. Princeton University Press (1970).
- [10] V. S. VLADIMIROV, I. V. VOLOVICH and E. I. ZELENOV. *p-Adic Analysis and Mathematical Physics*, World Scientific, Singapore (1994).

Diego CHAMORRO

Laboratoire d'Analyse et de Probabilités
Université d'Evry Val d'Essonne & ENSIIE

1 square de la résistance,
91025 Evry Cedex

diego.chamorro@m4x.org