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ASYMPTOTIC LOWEST TWO-SIDED CELL

CÉDRIC BONNAFÉ AND JÉRÉMIE GUILHOT

ABSTRACT. To a Coxeter system (W,S) (with S finite) and a weight function $L : W \to \mathbb{N}$ is associated a partition of W into Kazhdan-Lusztig (left, right or two-sided) L-cells. Let $S^\circ = \{s \in S \mid L(s) = 0\}$, $S^+ = \{s \in S \mid L(s) > 0\}$ and let $\mathbf{c}$ be a Kazhdan-Lusztig (left, right or two-sided) L-cell. According to the semicontinuity conjecture of the first author, there should exist a positive natural number m such that, for any weight function $L' : W \to \mathbb{N}$ such that $L(s^+) = L'(s^+) > mL'(s^\circ)$ for all $s^+ \in S^+$ and $s^\circ \in S^\circ$, $\mathbf{c}$ is a union of Kazhdan-Lusztig (left, right or two-sided) L'-cells.

The aim of this paper is to prove this conjecture whenever (W,S) is an affine Weyl group and $\mathbf{c}$ is contained in the lowest two-sided L-cell.

1. Introduction

Let (W,S) be a Coxeter system (with S finite) and let Γ be a totally ordered abelian group. Let $L : W \to \Gamma$ be a weight function in the sense of Lusztig [14, §3.1]. To such a datum is associated a partition of W into Kazhdan-Lusztig left, right or two-sided L-cells [14, Chapter 8]. By virtue of [1, Corollary 2.5], the computation of these partitions can be reduced to the case where L has only non-negative values, which we assume here in this introduction. We then set

$$S^\circ = \{s \in S \mid L(s) = 0\} \qquad \text{and} \qquad S^+ = \{s \in S \mid L(s) > 0\}.$$

A particular case of the semicontinuity conjecture of the first author [1, Conjecture A(a)] can be stated as follows:

Semicontinuity Conjecture (asymptotic case). *There exists a positive integer m such that, for any Kazhdan-Lusztig (left, right or two-sided) L-cell $\mathbf{c}$ and for any weight function $L' : W \to \Gamma$ such that $L(s^+) = L'(s^+) > mL'(s^\circ)$ for all $s^+ \in S^+$ and $s^\circ \in S^\circ$, the subset $\mathbf{c}$ is a union of Kazhdan-Lusztig (left, right or two-sided) L'-cells.*

The computation of the partition into Kazhdan-Lusztig cells is in general a very tough problem and a general proof of the semicontinuity conjecture would be very helpful. Even whenever it is not proved, it gives a lot of speculative "upper bounds" for the cells (for the inclusion order): at least, it can be seen as a guide along the computations.

Note that the full semicontinuity conjecture [1, Conjecture A] (not only the asymptotic case) has been verified in different situations (see for instance the discussion in [1, §5]). Note also that it has been established by the second author

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whenever (W,S) is an affine Weyl group with $|S|=3$ (see [11]). Our aim here is to prove a result slightly different in spirit than the previous ones. Indeed, it works for all affine Weyl groups and non-negative weight function L but it focuses only on one particular two-sided cell, namely the lowest one (which we denote by $\mathbf{c}_{\min}^L$).

Theorem. *Assume that (W,S) is an affine Weyl group. There exists a positive integer m such that, for any Kazhdan-Lusztig (left, right or two-sided) L-cell $\mathbf{c}$ contained in $\mathbf{c}_{\min}^L$ and for any weight function $L' : W \to \Gamma$ such that $L(s^+) = L'(s^+) > mL'(s^\circ)$ for all $s^+ \in S^+$ and $s^\circ \in S^\circ$, the subset $\mathbf{c}$ is a union of Kazhdan-Lusztig (left, right or two-sided) L'-cells.*

The main ingredient of the proof of this result is the generalized induction of the second author [10] together with the particular geometric description of the lowest two-sided cell and its left subcells.

The paper is organized as follows. In the literature, the lowest two-sided cell is defined whenever L takes only positive values on S (i.e. $S = S^+$). The aim of the first four sections is to extend this description of the case where L is allowed to vanish on some elements of S and to relate it to the semidirect product decomposition of W associated to the partition $S = S^\circ \dot\cup S^+$ as in [2] (see also [1, §2.E]). It must be noticed that the proof of a key lemma (see Lemma 3.9) requires a case-by-case analysis: this lemma is of geometric nature and does not involve Kazhdan-Lusztig theory.

In Section 5, we introduce Kazhdan-Lusztig theory and, in Section 6, we recall a more sophisticated version of the semicontinuity conjecture and we state our main results. The proof of these results is then done in the last two sections.

2. AFFINE WEYL GROUPS AND GEOMETRIC REALIZATION

In this paper, we fix an euclidean $\mathbb{R}$-vector space V of dimension $r \geqslant 1$ and we denote by Φ an *irreducible* root system in V of rank r: the scalar product will be denoted by $(,) : V \times V \longrightarrow \mathbb{R}$. The dual of V will be denoted by V^* and $\langle,\rangle : V \times V^* \longrightarrow \mathbb{R}$ will denote the canonical pairing. If $\alpha \in \Phi$, we denote by $\check\alpha \in V^*$ the associated coroot (if $x \in V$, then $\langle x, \check\alpha\rangle = 2(x,\alpha)/(\alpha,\alpha)$) and by $\check\Phi$ the dual root system. We fix a positive system Φ^+ and for $\alpha \in \Phi^+$ we set

$$H_{\alpha,0} = \{x \in V \mid \langle x, \check\alpha\rangle = 0\}.$$

Then the Weyl group Ω_0 of Φ is generated by the orthogonal reflection with respect to the hyperplanes $H_{\alpha,0}$. It acts on the root lattice $\langle\Phi\rangle$ and the semidirect product $\Omega_0 \ltimes \langle\Phi\rangle$ is an affine Weyl group of type $\check\Phi$.

2.1. **Geometric realizations.** For $\alpha \in \Phi^+$ and $n \in \mathbb{Z}$, we set

$$H_{\alpha,n} = \{x \in V \mid \langle x, \check\alpha\rangle = n\}$$

Then $H_{\alpha,n}$ is an *affine* hyperplane in V. Let

$$\mathscr{F} = \{H_{\alpha,n} \mid \alpha \in \Phi^+ \text{ and } n \in \mathbb{Z}\}.$$

If $H \in \mathscr{F}$, we denote by σ_H the orthogonal reflection with respect to H. Let $\Omega = \langle \sigma_H \mid H \in \mathscr{F}\rangle$. Then Ω is isomorphic to $W_0 \ltimes \langle\Phi\rangle$. We shall regard Ω as acting on the right of V.

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An *alcove* is a connected component of the set

$$V - \bigcup_{H \in \mathscr{F}} H.$$

It is well-known that Ω acts simply transitively on the set of alcoves $\mathrm{Alc}(\mathscr{F})$. Recall also that, if A is an alcove, then its closure $\overline{A}$ is a fundamental domain for the action of Ω on V.

The group Ω acts on the set of faces (the codimension 1 facets) of alcoves. We denote by S the set of Ω-orbits in the set of faces. Note that if $A \in \mathrm{Alc}(\mathscr{F})$, then the faces of A is a set of representatives of S since $\overline{A}$ is a fundamental domain for the action of Ω. If a face f is contained in the orbit $s \in S$, we say that f is of type s. To each $s \in S$ we can associate an involution $A \to sA$ of $\mathrm{Alc}(\mathscr{F})$: the alcove sA is the unique alcove which shares with A a face of type s. Let W be the group generated by all such involutions. Then (W, S) is a Coxeter system and it is isomomorphic to the affine Weyl group $W_0 \ltimes \langle \Phi \rangle$ (hence we also have $\Omega \simeq W$). We shall regard W as acting on the left of $\mathrm{Alc}(\mathscr{F})$. The action of Ω commutes with the action of W.

We denote by A_0 the fundamental alcove associated to Φ:

$$A_0 = \{x \in V \mid 0 < \langle x, \check{\alpha} \rangle < 1 \text{ for all } \alpha \in \Phi^+\}.$$

Let $A \in \mathrm{Alc}(\mathscr{F})$. Then there exists a unique $w \in W$ such that $wA_0 = A$. We will freely identify W with the set of alcoves $\mathrm{Alc}(\mathscr{F})$

2.2. **Associated Coxeter system.** Let $\ell : W \to \mathbb{N}$ denote the length function (with respect to the Coxeter system (W, S)). We denote by $\mathscr{L}(W)$ the set of finite sequences $(w_1, \dots, w_n)$ of elements of W such that $\ell(w_1 \cdots w_n) = \ell(w_1) + \cdots + \ell(w_n)$. If $(w_1, \dots, w_n)$ is a finite sequence of elements of W then, in order to simplify notation, we shall write $w_1 \bullet w_2 \bullet \cdots \bullet w_n$ if $(w_1, \dots, w_n) \in \mathscr{L}(W)$. If I is a subset of S, we denote by W_I the subgroup of W generated by I. We denote by X_I the set of elements $w \in W$ which are of minimal length in wW_I: it is a set of representatives of W/W_I. It follows from the irreducibility of Φ that W_I is finite whenever I is a *proper* subset of S: in this case, the longest element of W_I will be denoted by w_I.

Example 2.1. Let $\lambda \in V$ be a 0-dimensional facet of an alcove. We denote by W_λ the stabilizer in W of the set of alcoves containing λ. It can be shown that W_λ is the standard parabolic subgroup of W generated by $S_\lambda = S \cap W_\lambda$ (in other words, with the previous notation, $W_\lambda = W_{S_\lambda}$). Note that W_λ is finite: the longest element of W_λ will be denoted by w_λ and we set $X_\lambda = X_{S_\lambda}$. ■

Let $H = H_{\alpha,n} \in \mathscr{F}$ with $\alpha \in \Phi^+$ and $n \in \mathbb{Z}$. Then H divides $V - H$ into two half-spaces

$$V_H^+ = \{\mu \in V \mid \langle x, \check{\alpha} \rangle > n\},$$
$$V_H^- = \{\mu \in V \mid \langle x, \check{\alpha} \rangle < n\}.$$

We say that an hyperplane H separates the alcoves A and B if $A \subset V_H^+$ and $B \subset V_H^-$ or $A \subset V_H^-$ and $B \subset V_H^+$. For $A, B \in \mathrm{Alc}(\mathscr{F})$, we set

$$H(A, B) = \{H \in \mathscr{F} | H \text{ separates } A \text{ and } B\}.$$

Proposition 2.2. *Let $x, y \in W$ and $A \in \mathrm{Alc}(\mathscr{F})$. We have*

(1) $\ell(x) = |H(A, xA)|$;
(2) $x \bullet y$ *if and only if* $H(A, yA) \cap H(yA, xyA) = \emptyset$.

2.3. **Weight functions.** Let $(\Gamma, +)$ be a totally ordered abelian group: the order on Γ will be denoted by $\leqslant$. Let $L : W \to \Gamma$ be a *weight function* on W, that is a function satisfying $L(ww') = L(w) + L(w')$ whenever $\ell(ww') = \ell(w) + \ell(w')$. Recall that this implies the following property:

(2.2) *If s, $t \in S$ are conjugate in W, then $L(s) = L(t)$.*

We denote by $\mathrm{Weight}(W, \Gamma)$ the set of weight function from W to Γ. Throughout this paper, we will always assume that L is *non-negative* that is $L(s) \geqslant 0$ for all $s \in S$. The weight function L is called *positive* if $L(s) > 0$ for all $s \in S$ (in other words, L is positive if and only if $L(w) > 0$ if $w \neq 1$). Note that a weight function on W is completely determined by its values on the generators $s \in S$: the element of the set $\{L(s) \mid s \in S\}$ are called the parameters.

Example 2.3. The map $W \to \Gamma$, $w \mapsto 0$ is a weight function (and will be denoted by 0): it is not positive (if $W \neq 1$). On the other hand, $\ell : W \to \mathbb{Z}$ is a positive weight function. ■

Here is a first consequence of the non-negativeness assumption:

Proposition 2.4. *Let x, $y \in W$. If $L(x) = 0$, then $L(xy) = L(yx) = L(y)$.*

Proof. Let $l = \ell(x)$ and let $s_1, \dots, s_l$ be elements of S such that $x = s_1 \cdots s_l$. Then $L(x) = L(s_1) + \cdots + L(s_l)$, so $L(s_i) = 0$ for all $i \in \{1, 2, \dots, l\}$, because L is non-negative. So, arguing by induction on the length of x, we may (and we will) assume that $\ell(x) = 1$, i.e. that $x = s_1$. Two cases may occur:

• If $xy > y$, then $\ell(xy) = \ell(x) + \ell(y)$, so $L(xy) = L(x) + L(y) = L(y)$, as desired.

• If $xy < y$, then $\ell(y) = \ell(x) + \ell(xy)$, so $L(y) = L(x) + L(xy) = L(xy)$, as desired. □

2.4. **L-special points.** Let $H \in \mathscr{F}$ and assume that H supports a face of type $s \in S$: we then set $L_H = L(s)$. Note that this is well defined since if H supports faces of type $s, t \in S$ then s and t are conjugate in W [4, Lemma 2.1]. Then $L_H = L_{H\sigma}$ for all $\sigma \in \Omega$. If λ is a 0-dimensional facet of an alcove, we set

$$L_\lambda = \sum_{\substack{H \in \mathscr{F} \\ \lambda \in H}} L_H = L(w_\lambda).$$

Note that $L_{\lambda\sigma} = L_\lambda$ for all $\sigma \in \Omega$. We set

$$\nu_L := \max_{\lambda \in V} L_\lambda.$$

We then say that λ is an *L-special point* if $L_\lambda = \nu_L$. We denote by $\mathrm{Spe}_L(W)$ the set of L-special points: it is stable under the action of Ω. Since $\overline{A_0}$ is a fundamental

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domain for the action of Ω, the set $\mathrm{Spe}_L(W)\cap\overline{A_0}$ is a set of representative of orbits of $\mathrm{Spe}_L(W)$ under the action of Ω.

Example 2.5. If $L=\ell$ is the usual length function then L_λ is just the number of hyperplanes which go through λ hence $\nu_\ell=|\Phi^+|$ and the set of ℓ-special points is equal to the weight lattice

$$P(\Phi)=\{v\in V \mid \forall\ \alpha\in\Phi,\ \langle v,\check{\alpha}\rangle\in\mathbb{Z}\}.$$

Hence we recover the original definition of special points by Lusztig in [13]. ■

Convention 2.6. *If W is not of type $\tilde{\mathrm{C}}_r$ $(r\geqslant 1)$ then any two parallel hyperplanes have same weight* [4, Lemma 2.2]*. In the case where W is of type $\tilde{\mathrm{C}}_r$ with generators t, s_1,..., s_{r-1}, t' such that $\langle t,s_1,\dots,s_{r-1}\rangle=W_0$ and $\langle s_1,\dots,s_{r-1}\rangle$ is of type A_{r-1}, by symmetry of the Dynkin diagram, we may (and we will) assume that $L(t)\geqslant L(t')$.*

Recall that the type $\tilde{\mathrm{C}}_1$ is also the type $\tilde{\mathrm{A}}_1$.

Remark 2.7. Note that with our Convention 2.6 for $\tilde{\mathrm{C}}_r$ $(r\geqslant 1)$, the point $0\in V$ is always an L-special point. ■

For $\alpha\in\Phi$ we set

$$L_\alpha := \max_{n\in\mathbb{Z}}\ L_{H_{\alpha,n}}.$$

Remark 2.8. Note that if W is not of type $\tilde{\mathrm{C}}$ then since any two parallel hyperplanes have same weight we have $L_\alpha = L_{H_{\alpha,n}}$ for all n. In general we will say that $H=H_{\alpha,n}\in\mathscr{F}$ is of *maximal weight* if $L_H=L_\alpha$. ■

We denote by Φ^L the subset of Φ which consists of all roots of positive weight. Note that Φ^L is a root system of rank r, not necessarily irreducible, and that $\Phi^L\cap\Phi^+$ is a positive system in Φ^L: see the proof of Lemma 3.9 where we classify the root systems Φ^L. We denote by Δ^L the unique simple system contained in $\Phi^L\cap\Phi^+$. We have

- If W is not of type $\tilde{\mathrm{C}}$ or if $L(t)=L(t')$ in type $\tilde{\mathrm{C}}$ then
$$\mathrm{Spe}_L(W)=\{v\in V\mid \forall\ \alpha\in\Phi^L,\ \langle v,\check{\alpha}\rangle\in\mathbb{Z}\}.$$
- If W is of type $\tilde{\mathrm{C}}$ and $L(t)>L(t')$ then
$$\mathrm{Spe}_L(W)=\langle\Phi^L\rangle\subsetneq\{v\in V\mid\forall\ \alpha\in\Phi^L,\ \langle v,\check{\alpha}\rangle\in\mathbb{Z}\}.$$

In other words, the L-special points are those points of V which lies in the intersection of $|\Phi^L\cap\Phi^+|$ hyperplanes of maximal weight.

Let $\mathscr{F}^L=\{H\in\mathscr{F}\mid L_H>0\}$. Let λ be a 0-dimensional facet of an alcove which is contained in an hyperplane of positive weight. An *L-quarter* with vertex λ is a connected component of

$$V-\bigcup_{\substack{H\in\mathscr{F}^L\\ \lambda\in H}} H.$$

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It is an open simplicial cone: it has r walls.

Let $\alpha \in \Phi^L$. A *maximal L-strip orthogonal to α* is a connected component of

$$V - \bigcup_{\substack{n \in \mathbb{Z} \\ L_{H_{\alpha,n}} = L_\alpha}} H_{\alpha,n}.$$

If A is an alcove, we denove by $\mathscr{U}_\alpha^L(A)$ the unique maximal L-strip orthogonal to α containing A. Finally, we set

$$\mathscr{U}^L(A_0) = \bigcup_{\substack{\alpha \in \Phi^+ \\ L_\alpha > 0}} \mathscr{U}_\alpha^L(A_0).$$

3. On the lowest two-sided cell

We keep the notation of the previous section. We fix a non-negative weight function $L \in \mathrm{Weight}(W, \Gamma)$.

3.1. **Definition and examples.** Recall that we have set $\nu_L := \max_{\lambda \in V} L_\lambda$. Since W_λ is a standard parabolic subgroup of W, one can easily see that

$$\nu_L = \max_{I \in \mathscr{P}_{\mathrm{fin}}(S)} L(w_I)$$

where $\mathscr{P}_{\mathrm{fin}}(S)$ denotes the set of subsets I of S such that W_I is finite. We set

$$\mathscr{W} = \bigcup_{I \in \mathscr{P}_{\mathrm{fin}}(S)} W_I.$$

Then we define the *lowest two-sided cell* of W by

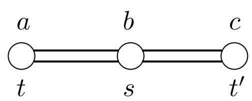
$$\mathbf{c}_{\min}^L(W) = \{xwy \mid w \in \mathscr{W},\ x{\bullet}w{\bullet}y \text{ and } L(w) = \nu_L\}.$$

We shall see later (see Section 5.2) the reason for this terminology. When the group W is clear from the context, we will write $\mathbf{c}_{\min}^L$ instead of $\mathbf{c}_{\min}^L(W)$. Note the following immediate property of $\mathbf{c}_{\min}^L$:

Lemma 3.1. *Let w, x, $y \in W$ be such that $w \in \mathbf{c}_{\min}^L$ and $x{\bullet}w{\bullet}y$. Then $xwy \in \mathbf{c}_{\min}^L$.*

The set $\mathbf{c}_{\min}^L$ can change quite dramatically when the parameters are varying as shown in the following example.

Example 3.2. Let W be of type $\tilde{\mathrm{C}}_2$ with diagram and weight function given by



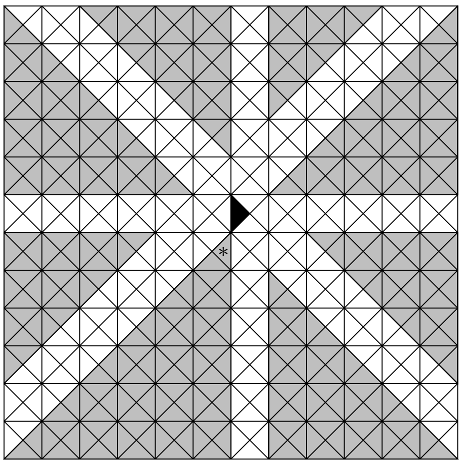
where $a, b, c \in \Gamma$ and, by convention, we assume that $a \geq c$. We start by describing the set

$$\mathscr{W}^{\max} = \{w \in \mathscr{W} \mid L(w) = \nu_L\}.$$

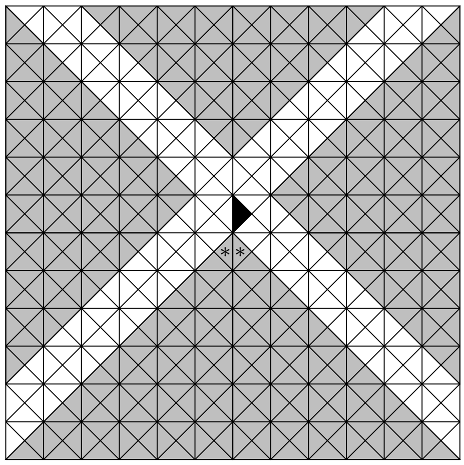
We have

$$\mathscr{W}^{\max} = \begin{cases} \{tsts\} & \text{if } a > c \text{ and } b > 0,\\ \{tsts, tst\} & \text{if } a > c \text{ and } b = 0,\\ \{tsts, t'st's\} & \text{if } a = c > 0 \text{ and } b > 0,\\ \{tsts, tst, t'st's, t'st', tt'\} & \text{if } a = c > 0 \text{ and } b = 0,\\ \{sts, stst, st's, st'st'\} & \text{if } a = c = 0 \text{ and } b > 0,\\ \mathscr{W} & \text{if } a, b, c = 0. \end{cases}$$

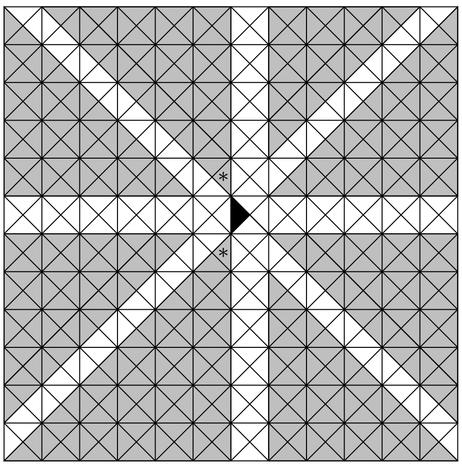
If $\mathscr{W}^{\max} = \mathscr{W}$ then we get $\mathbf{c}^L_{\min} = W$. Otherwise the corresponding sets $\mathbf{c}^L_{\min}$ are described in the following figures: the black alcove is the fundamental alcove A_0, the alcoves with a star correspond to the set $\mathscr{W}^{\max}$ and the set $\mathbf{c}^L_{\min}$ consists of all the alcoves lying in the gray area (via the identification $w \leftrightarrow wA_0$).



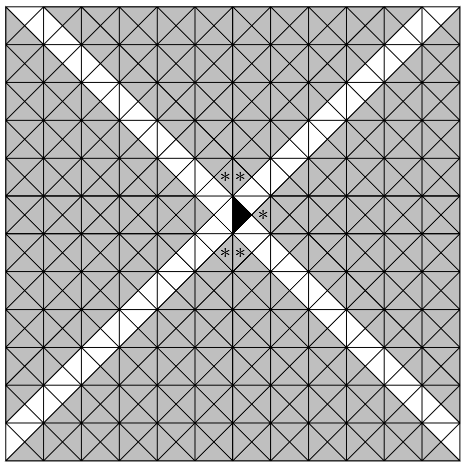
The set $\mathbf{c}^L_{\min}$ for $a > c$ and $b > 0$



The set $\mathbf{c}^L_{\min}$ for $a > c$ and $b = 0$



The set $\mathbf{c}^L_{\min}$ for $a = c > 0$ and $b > 0$



The set $\mathbf{c}^L_{\min}$ for $a = c > 0$ and $b = 0$

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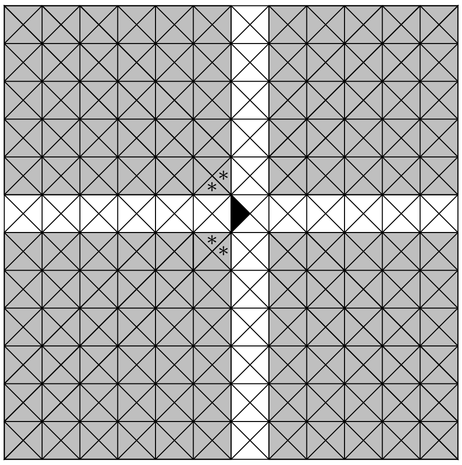

The set $\mathbf{c}_{\min}^L$ for $a = c = 0$ and $b > 0$

3.2. **Some alternative description of $\mathbf{c}_{\min}^L$.** Let $\mathscr{L}_L(W)$ be the set of finite sequences $(w_1, \ldots, w_n)$ of elements of W such that $L(w_1 \cdots w_n) = L(w_1) + \cdots + L(w_n)$.

Example 3.3. For instance, $\mathscr{L}_\ell(W) = \mathscr{L}(W)$. Note also that $\mathscr{L}(W) \subset \mathscr{L}_L(W)$, by definition of a weight function: the inclusion might be strict, as it is shown by the case where $L = 0$. Finally, if $L(s) > 0$ for all $s \in S$, then $\mathscr{L}_L(W) = \mathscr{L}(W)$. ■

Example 3.4. If $L = 0$, then $\nu_L = 0$, $\mathrm{Spe}_L(W)$ is the set of 0-dimensional facets, $\mathscr{L}_L(W)$ is the set of finite sequences of elements of W and $\mathbf{c}_{\min}^L = W$. ■

For $A, B \in \mathrm{Alc}(\Phi)$, we set

$$H^L(A, B) = \{H \in \mathscr{F}^L | H \text{ separates } A \text{ and } B\}.$$

Then we have (compare to Proposition 2.2):

Proposition 3.5. *Let $x, y \in W$ and $A \in \mathrm{Alc}(W)$. We have*

(1) $L(x) = \sum_{H \in H^L(A, xA)} L_H$*;*
(2) $(x, y) \in \mathscr{L}_L(W)$ *if and only if* $H^L(A, yA) \cap H^L(yA, xyA) = \emptyset$.

The set $\mathbf{c}_{\min}^L$ can be described as follows.

Proposition 3.6. *The following equalities hold:*

$$\begin{aligned} \mathbf{c}_{\min}^L &= \{xwy \mid w \in \mathscr{W}, (x, w, y) \in \mathscr{L}_L(W) \textit{ and } L(w) = \nu_L\} \\ &= \{xw_\lambda y \mid \lambda \in \mathrm{Spe}_L(W) \textit{ and } (x, w_\lambda, y) \in \mathscr{L}_L(W)\} \\ &= \{w \mid w(A_0) \not\subset \mathscr{U}^L(A_0)\}. \end{aligned}$$

Proof. Let

$$\begin{aligned} A &= \{xwy \mid w \in \mathscr{W}, (x, w, y) \in \mathscr{L}_L(W) \text{ and } L(w) = \nu_L\}, \\ B &= \{xw_\lambda y \mid \lambda \in \mathrm{Spe}_L(W) \text{ and } (x, w_\lambda, y) \in \mathscr{L}_L(W)\}, \\ C &= \{w \mid wA_0 \not\subset \mathscr{U}^L(A_0)\}. \end{aligned}$$

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It is clear that $\mathbf{c}_{\min}^L$, $B \subset A$. Now, let $z \in A$. Then there exists $w \in \mathscr{W}$ and x, $y \in W$ such that $z = xwy$, $L(z) = L(x) + L(w) + L(y)$ and $L(w) = \nu_L$.

Let us first prove that $z \in B$. There exists a 0-dimensional facet λ such that $w \in W_\lambda$. If $L(w_\lambda) < \nu_L$, then $L(w) \leqslant L(w_\lambda) < \nu_L$, which is impossible. Therefore, $L(w_\lambda) = \nu_L$, so $\lambda \in \mathrm{Spe}_L(W)$. Write $w = w_\lambda a$, with $a \in W_\lambda$. Then, since w_λ is the longest element of W_λ, we get that $\ell(w_\lambda) = \ell(w) + \ell(a)$, so $L(w) + L(a) = L(w_\lambda)$, so $L(a) = 0$. By Proposition 2.4, it follows that $L(ay) = L(y)$. Then $z = xw_\lambda ay$, and $L(z) = L(x) + L(w) + L(y) = L(x) + L(w_\lambda) + L(ay)$. This shows that $z \in B$. So $A = B$.

Let us now prove that $z \in \mathbf{c}_{\min}^L$. We shall argue by induction on $\ell(x)+\ell(y)$. The result being obvious if $\ell(x)+\ell(y) = 0$, we assume that $\ell(x)+\ell(y) > 0$. By symmetry, we may assume that $x = sx'$, with $s \in S$ and $sx' > x'$. Let $z' = sz$. Therefore, $L(z) = L(sx')+L(w)+L(y) = L(s)+L(x')+L(w)+L(y) \geqslant L(s)+L(x'wy) = L(s)+L(z') \geqslant L(z')$. Consequently, $L(z) = L(s)+L(z')$ and $L(z') = L(x')+L(w)+L(y)$. So $z' \in A$ and, by the induction hypothesis, $z' \in \mathbf{c}_{\min}^L$. Two cases may occur:

• If $sz' > z'$, then $z = sz' \in \mathbf{c}_{\min}^L$ by Lemma 3.1, as desired.

• If $sz' < z'$, then $L(z) = L(sz') \leqslant L(z')$. Since we have already proved that $L(z) \geqslant L(z')$, this forces $L(s) = 0$. Write $z' = aw'b$ with $w' \in \mathscr{W}$, $L(w') = \nu_L$ and $a \bullet w' \bullet b$. Since $z = sz' < z'$, this means that z is obtained from the expression $aw'b$ by removing a simple reflection s' conjugate to s from a reduced expression of a, b or w'. It it is removed from a or b, then $z = a'w'b'$ with $a' \bullet w' \bullet b'$, so $z \in \mathbf{c}_{\min}^L$. If it is removed from w', then $z = aw''b'$ with $L(w'') = L(w') = \nu_L$ and $a \bullet w'' \bullet b$, so $z \in \mathbf{c}_{\min}^L$.

Therefore, we have proved that $A = B = \mathbf{c}_{\min}^L$.

It remains to show that $C = \mathbf{c}_{\min}^L$. Let $z \in \mathbf{c}_{\min}^L = B$. Then there exist $x, y \in W$ and $\lambda \in \mathrm{Spe}_L(W)$ such that $z = xw_\lambda y$ and $(x, w_\lambda, y) \in \mathscr{L}_L(W)$. In particular we have $(w_\lambda, y) \in \mathscr{L}_L(W)$ hence, using Proposition 3.5, we get

$$(*) \qquad H^L(A_0, yA_0) \cap H^L(yA_0, w_\lambda yA_0) = \emptyset.$$

Let $\alpha \in \Phi^L$. Since λ is a special point there exists an hyperplane H_{α,n_λ} of weight L_α which contains λ. The hyperplane H_{α,n_λ} separates yA_0 and $w_\lambda A_0$ and is of maximal weight, therefore by $(*)$ it cannot lie in $H^L(A_0, yA_0)$ and it follows that it separates A_0 and $w_\lambda yA_0$. Therefore for all $\mu \in w_\lambda yA_0$ we have

$$\begin{array}{ll} \langle \mu, \check{\alpha} \rangle > n_\lambda & \text{if } n_\lambda \geq 1 \\ \langle \mu, \check{\alpha} \rangle < n_\lambda & \text{if } n_\lambda \leq 0 \end{array}$$

and $w_\lambda y \notin \mathscr{U}_\alpha^L(A_0)$. But this holds for all $\alpha \in \Phi^L$, thus $w_\lambda yA_0 \notin \mathscr{U}^L(A_0)$. Now we have $(x, w_\lambda, y) \in \mathscr{L}_L(W)$ therefore

$$H^L(A_0, w_\lambda yA_0) \cap H^L(w_\lambda yA_0, xw_\lambda yA_0) = \emptyset$$

from where we see that H_{α,n_λ} does not lie $H^L(w_\lambda yA_0, xw_\lambda yA_0)$. Hence $x_\lambda w_\lambda yA_0 \notin \mathscr{U}_\alpha^L(A_0)$ as required.

Let us now prove that $C \subset \mathbf{c}_{\min}^L$. Let $w \in C$. The idea is to follow the proof of [4, Proposition 5.5] using Φ^L instead of Φ. The alcove wA_0 lies in some connected component of

$$V - \bigcup_{\alpha \in \Phi^L} H_{\alpha,0}.$$

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The group $\Omega_0^L = \langle \sigma_{H_{\alpha,0}} \mid \alpha \in \Phi^L \rangle$ is easily seen to act simply transitively on this set of connected components, therefore there exists $\sigma \in \Omega_0^L$ such that

$$wA_0 \subset \mathscr{C}_\sigma := \{v \in V \mid \langle v, \check{\alpha} \rangle > 0 \text{ for } \alpha \in \sigma(\Delta^L)\}$$

This implies that there exist r linearly independent roots $\beta_1, \dots, \beta_r$ in $\Phi^L \cap \Phi^+$ such that

$$\mathscr{C}_\sigma := \{v \in V \mid \langle v, \check{\beta}_i \rangle < 0 \text{ for } 1 \leq i \leq k,\ \langle v, \check{\beta}_i \rangle > 0 \text{ for } k+1 \leq i \leq r \rangle\}.$$

Removing the alcoves which lies in $\mathscr{U}^L(A_0)$ we obtain the following L-quarter, which is a translate of $\mathscr{C}_\sigma$:

$$\mathscr{C}'_\sigma := \{v \in V \mid \langle v, \check{\beta}_i \rangle < b_i \text{ for } 1 \leq i \leq k,\ \langle v, \check{\beta}_i \rangle > b_i \text{ for } k+1 \leq i \leq r \rangle\}$$

where

$$b_i = \begin{cases} 0 & \text{if } 1 \leq i \leq k, \\ 1 & \text{if } k+1 \leq i \leq r \text{ and } L(H_{\beta_i,0}) = L(H_{\beta_i,1}), \\ 2 & \text{otherwise.} \end{cases}$$

Let λ_σ be the vertex of $\mathscr{C}'_\sigma$ that is the point of V which satisfies $\langle \lambda_\sigma, \check{\beta}_i \rangle = b_i$ for all $1 \leq i \leq r$. Then λ_σ is a special point: for $\alpha \in \Phi^L$ we set $n_\alpha^\lambda = \langle \lambda_\sigma, \check{\alpha} \rangle$. Note that for all $\alpha \in \Phi^L$ we have

(†) $$\mathscr{C}_\sigma \subset V^+_{H_{\alpha,n_\alpha^\lambda}} \text{ if } n_\alpha^\lambda > 0 \quad \text{ and } \mathscr{C}_\sigma \subset V^-_{H_{\alpha,n_\alpha^\lambda}} \text{ if } n_\alpha^\lambda \leq 0.$$

Now let $z \in W$ be such that $z(A_0) \subset \mathscr{C}'_\sigma$, $\lambda \in \overline{z_\sigma A_0}$. We get for $\alpha \in \Phi^L$ (using (†))

- if $H_{\alpha,n} \in H^L(wA_0, zA_0)$ then $|n| > |n_\alpha^\lambda|$;
- if $H_{\alpha,n} \in H^L(A_0, zA_0)$ then $|n| \leq |n_\alpha^\lambda|$;
- if $H_{\alpha,n} \in H^L(w_\lambda zA_0, zA_0)$ then $n = n_\alpha^\lambda$.

Finally putting all this together we get that

$$\begin{array}{rcl} H^L(A_0, w_\lambda zA_0) \cap H^L(w_\lambda zA_0, zA_0) & = & \emptyset \\ \text{and} \quad H^L(A_0, zA_0) \cap H^L(zA_0, wz^{-1}(zA_0)) & = & \emptyset. \end{array}$$

Hence $w = wz^{-1}w_\lambda w_\lambda z$ and $(wz^{-1}, w_\lambda, w_\lambda z) \in \mathscr{L}_L(W)$. □

Remark 3.7. By direct product, one can easily show that Proposition 3.6 still holds when W is not irreducible. ■

3.3. **The elements of $\mathbf{c}_{\min}^L$.** Keeping the notation of the proof of Proposition 3.6, every element $\sigma \in \Omega_0^L$ defines an L-quarter $\mathscr{C}_\sigma$ with vertex 0 and an L-quarter $\mathscr{C}'_\sigma$ (which is a translate of $\mathscr{C}_\sigma$) with vertex λ_σ. We get the following equality

$$\mathbf{c}_{\min}^L(W) = \bigcup_{\sigma \in \Omega_0^L} \{w \in W \mid wA_0 \subset \mathscr{C}'_\sigma\} = \bigcup_{\sigma \in \Omega_0^L} N_\sigma^L(W).$$

We will simply write N_σ^L if it is clear from the context what the group W is. Note that any two sets $\mathscr{C}_\sigma, \mathscr{C}_{\sigma'}$ are separated by at least one maximal strip, hence the above union is disjoint. In fact, the sets $\overline{\mathscr{C}_\sigma}$ are the connected components of the closure of $\{\mu \in V \mid \mu \in wA_0, w \in \mathbf{c}_{\min}^L\}$.

Let b_σ be the unique element such that $\lambda_\sigma \in \overline{b_\sigma A_0}$ and b_σ has minimal length in the coset $W_{\lambda_\sigma} b_\sigma$. For a 0-dimensional facet λ of an alcove, we set $S_\lambda^\circ := \{s \in S_\lambda \mid L(s) = 0\}$ and we denote by w_λ° the element of minimal length in $w_\lambda W_{S_\lambda^\circ}$

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Lemma 3.8. *Every element $w \in \mathbf{c}_{\min}^L$ can be uniquely written under the form $x_w a_w w^\circ_{\lambda_\sigma} b_\sigma$ where $\sigma \in \Omega_0^L$, $a_w \in W_{S^\circ_{\lambda_\sigma}}$ and $x_w \in X_{\lambda_\sigma}$.*

Proof. Let $w \in \mathbf{c}_{\min}^L$. We know by the previous proof that $w = wz^{-1}w_\lambda w_\lambda z$ and $(wz^{-1}, w_\lambda, w_\lambda z) \in \mathscr{L}_L(W)$ where z is such that $z(A_0) \subset \mathscr{C}'_\sigma$ and $\lambda_\sigma \in \overline{zA_0}$. Both $b_\sigma A_0$ and $w_\lambda z A_0$ contains λ_σ in their closure hence they lie in the same right coset with respect to W_{λ_σ}. Since b_σ has minimal length in $W_{\lambda_\sigma} b_\sigma$ there exists $y' \in W_{\lambda_\sigma}$ such that $w_\lambda z = y' b_\sigma$ and $(y', b_\sigma) \in \mathscr{L}(W)$. Next let x_w be the element of minimal length in $(wz^{-1})W_{\lambda_\sigma}$ and let x' be such that $wz^{-1} = x_w x'$. Assume for a moment that $x', y' \in W_{S^\circ_{\lambda_\sigma}}$. Then

$$x' w_{\lambda_\sigma} y' \in W_{S^\circ_{\lambda_\sigma}} w_{\lambda_\sigma} W_{S^\circ_{\lambda_\sigma}} = W_{S^\circ_{\lambda_\sigma}} w_{\lambda_\sigma}$$

and we write $x' w_{\lambda_\sigma} y' = a_w w^\circ_{\lambda_\sigma}$. Finally w can be written uniquely as $x_w a_w w^\circ_\lambda b_\sigma$.

Let us now prove that $x', y' \in W_{S^\circ_{\lambda_\sigma}}$ that is $L(x') = L(y') = 0$. Recall that b_σ has minimal length in $W_\lambda b_\sigma$. On the one hand we have

$$L(w_\lambda w_\lambda z) = L(w_\lambda) + L(w_\lambda z) = L(w_\lambda) + L(y' b_\sigma) = L(w_\lambda) + L(y') + L(b_\sigma).$$

On the other hand

$$L(w_\lambda w_\lambda z) = L(w_\lambda y' b_\sigma) = L(w_\lambda y') + L(b_\sigma) = L(w_\lambda) - L(y') + L(b_\sigma)$$

hence $L(y') = 0$. Similarly one can show that $L(x') = 0$. The proof is complete □

Later on, we will need the following result. We put it and prove it here because it uses the notation introduced in this section.

Lemma 3.9. *Let $w = x_w a_w w^\circ_{\lambda_\sigma} b_\sigma$ where $\sigma \in \Omega_0^L$, $a_w \in W_{S^\circ_{\lambda_\sigma}}$ and $x_w \in X_{\lambda_\sigma}$. Then we have*

1. *$x_w \bullet a_w \bullet w^\circ_{\lambda_\sigma} \bullet b_\sigma$;*
2. *if $w' < a_w w^\circ_\lambda b_\sigma$ and $w' \in \mathbf{c}_{\min}^L$ then either $w' = a_{w'} w^\circ_{\lambda_\sigma} b_\sigma$ where $a_{w'} < a_w$ or $w' = x_{w'} a_{w'} w^\circ_{\lambda_{\sigma'}} y_{\sigma'}$ where $y_{\sigma'} < b_\sigma$.*

Eventhough this result might look fairly natural, it is in fact quite long to prove and involved a case by case analysis.

Proof. We prove 1. The fact that $a_w \bullet w^\circ_{\lambda_\sigma} \bullet b_\sigma$ is clear by definition therefore we only need to show that $x_w \bullet (a_w w^\circ_{\lambda_\sigma} b_\sigma)$. To this end we show that

$$D := H(A_0, a_w w^\circ_{\lambda_\sigma} b_\sigma A_0) \cap H(a_w w^\circ_{\lambda_\sigma} b_\sigma A_0, x_w a_w w^\circ_{\lambda_\sigma} b_\sigma A_0) = \emptyset.$$

Claim 1. *If $\langle \lambda_\sigma, \check{\beta} \rangle \in \mathbb{Z}$ then $H_{\beta,n} \notin D$ for all $n \in \mathbb{Z}$.*

Proof. Let $n_\beta = \langle \lambda_\sigma, \check{\beta} \rangle$ and assume that $n_\beta > 0$, the case $n_\beta \leq 0$ is similar. Since $x_w \in X_{\lambda_\sigma}$, there are no hyperplane containing λ_σ which lies in

$$H(a_w w^\circ_{\lambda_\sigma} b_\sigma A_0, x_w a_w w^\circ_{\lambda_\sigma} b_\sigma A_0).$$

Hence $H_{\beta,n_\beta} \notin D$. Now let $n \neq n_\beta$. Then since $\lambda_\sigma \in \overline{a_w w^\circ_{\lambda_\sigma} b_\sigma A_0}$, we have

$$n < \langle \mu, \check{\beta} \rangle < n_\beta + 1 \text{ for all } \mu \in a_w w^\circ_{\lambda_\sigma} b_\sigma A_0.$$

If $n > n_\beta$ we have $H_{\beta,n} \notin H(A_0, a_w w^\circ_{\lambda_\sigma} b_\sigma A_0)$ and $H_{\beta,n} \notin D$. If $n < n_\beta$ then $H_{\beta,n} \notin H(a_w w^\circ_{\lambda_\sigma} b_\sigma A_0, x_w a_w w^\circ_{\lambda_\sigma} b_\sigma A_0)$ and $H_{\beta,n} \notin D$. □

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Claim 2. *Let $\beta \in \Phi^+$ be such that there exists a hyperplane of direction β in D. Then $H_{\beta,0} \cap \mathscr{C}_\sigma \neq \emptyset$*

Proof. Let $\beta \in \Phi^+$ be such that there exists a hyperplane of direction β in D. By the previous claim, we know that $\langle \lambda_\sigma, \check{\beta} \rangle \notin \mathbb{Z}$. Let $n \in \mathbb{Z}$ be such that $n < \langle \lambda_\sigma, \check{\beta} \rangle < n+1$. We will assume that $n > 0$. (The case $n \leq 0$ is similar.) Note that we must have $H_{\beta,n} \cap \mathscr{C}'_\sigma \neq 0$. Translating by $-\lambda_\sigma$ we get

$$t_{-\lambda_\sigma}(H_{\beta,n}) \cap \mathscr{C}_\sigma \neq \emptyset.$$

Let $x \in t_{-\lambda_\sigma}(H_{\beta,n}) \cap \mathscr{C}_\sigma$. Note that we have $\langle x, \check{\beta} \rangle = n - \langle \lambda_\sigma, \check{\beta} \rangle$. Let

$$\delta = \frac{\langle \lambda_y, \check{\beta} \rangle - n}{\langle \lambda_y, \check{\beta} \rangle} < 1.$$

Then $0 < \delta < 1$ and an easy calculation to show that $x + \delta\lambda_\sigma \in H_{\beta,0} \cap \mathscr{C}_\sigma$. □

Claim 3. *Let $\beta \in \Phi^+$ be such that $H_{\beta,0} \cap \mathscr{C}_\sigma \neq \emptyset$. Then we have either*

(1) $\langle \lambda_\sigma, \check{\beta} \rangle \in \mathbb{Z}$,
(2) $0 < \langle \lambda_\sigma, \check{\beta} \rangle < 1$.

We now prove that Claims (1)–(3) implies that $D = \emptyset$. By Claim 2, the only hyperplanes that can lie in D are those of the form $H_{\beta,n}$ where $H_{\beta,0} \cap \mathscr{C}_\sigma \neq \emptyset$. But then Claim 3 implies that we have either

(1) $\langle \lambda_\sigma, \beta \rangle \in \mathbb{Z}$,
(2) $0 < \langle \lambda_\sigma, \beta \rangle < 1$.

If we are in case (1), we have $H_{\beta,n} \notin D$ by Claim 1. If we are in case (2), then the alcove $a_w w^\circ_{\lambda_\sigma} b_\sigma A_0$, which contains λ_σ in its closure, must satisfies

$$a_w w^\circ_{\lambda_\sigma} b_\sigma A_0 \subset \{x \in V \mid 0 < \langle x, \check{\beta} \rangle < 1\}.$$

But so does A_0, therefore there are no hyperplane of direction β which lies on $H(A_0, a_w w^\circ_{\lambda_\sigma} b_\sigma A_0)$ and $H_{\beta,n} \notin D$. Thus $D = \emptyset$ as required.

It remains to prove Claim 3. We will proceed by a case by case analysis but first we want to express Claim 3 in a form which is easier to check. To do so, we need to introduce some more notation.

Any $\sigma \in \Omega_0^L$ defines a partition $\Delta_\sigma^+ \cup \Delta_\sigma^-$ of Δ^L where

$$\Delta_\sigma^+ = \{\alpha \in \Delta^L \mid \alpha\sigma \in \Phi^+\} \text{ and } \Delta_\sigma^- = \{\alpha \in \Delta^L \mid \alpha\sigma \in \Phi^-\}.$$

Remark 3.10. Note that we can obtain all partition of Δ^L in this way, but that two distinct σ might give rise to the same partition. ■

To such a partition, we associate $\lambda_{\Delta_\sigma^+,\Delta_\sigma^-} \in V$ defined by

$$\langle \lambda_{\Delta_\sigma^+,\Delta_\sigma^-}, \check{\alpha} \rangle = 0 \text{ if } \alpha \in \Delta_\sigma^- \text{ and } \langle \lambda_{\Delta_\sigma^+,\Delta_\sigma^-}, \check{\alpha} \rangle = b_\alpha \text{ if } \alpha \in \Delta_\sigma^+$$

where

$$b_\alpha = \begin{cases} 1 & \text{if } k+1 \leq i \leq r \text{ and } L(H_{\alpha,0}) = L(H_{\alpha,1}); \\ 2 & \text{otherwise.} \end{cases}$$

Then we have $\lambda_\sigma = (\lambda_{\Delta_\sigma^+,\Delta_\sigma^-})\sigma$. Claim 3 is then easily seen to be equivalent to the following statement, by applying σ^{-1}. (In the expression $\mathscr{C}_1$, the 1 denotes the identity of Ω_0^L.)

Claim 3'. *Let $\gamma \in \Phi^+$ such that $H_{\gamma,0} \cap \mathscr{C}_1$ and let $\sigma \in \Omega_0^L$. Then we have either*

(1) $\langle \lambda_{\tilde{\Delta}_\sigma^+,\tilde{\Delta}_\sigma^-}, \check{\gamma}\rangle \in \mathbb{Z}$
(2) $0 < \langle \lambda_{\tilde{\Delta}_\sigma^+,\tilde{\Delta}_\sigma^-}, \check{\gamma}\rangle < 1$ *if* $\gamma\sigma \in \Phi^+$
(3) $-1 < \langle \lambda_{\tilde{\Delta}_\sigma^+,\tilde{\Delta}_\sigma^-}, \check{\gamma}\rangle < 0$ *if* $\gamma\sigma \in \Phi^-$

Proof. As mentioned earlier, we proceed by a case by case analysis. Note that it is enough to prove the claim for $\gamma \notin \Phi^L$, since for all $\gamma \in \Phi^L$ we have $\langle \lambda_{\tilde{\Delta}_\sigma^+,\tilde{\Delta}_\sigma^-}, \check{\gamma}\rangle \in \mathbb{Z}$ as $\lambda_{\tilde{\Delta}_\sigma^+,\tilde{\Delta}_\sigma^-}$ is a L-special point.

Type $\tilde{\mathrm{G}}_2$. It is a straightforward verification.

Type $\tilde{\mathrm{F}}_4$. Let $V = \mathbb{R}^4$ with orthonormal basis $(\varepsilon_i)_{1\leq i\leq 4}$. The root sytem Φ of type F_4 consists of 24 long roots and 24 short roots:

$$\pm\varepsilon_i \pm \varepsilon_j \text{ and } \pm\varepsilon_i, \frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4).$$

Assume that $S^\circ = \{s_1, s_2\}$. We get that Φ^L is of type D_4 and consists of the roots $\pm\varepsilon_i \pm \varepsilon_j$. We choose the following simple system:

$$\Delta^L = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_4, \varepsilon_3 + \varepsilon_4\}.$$

We have $\mathscr{C}_1 = \{x \in V | \langle x, \check{\alpha}\rangle > 0, \text{for all } \alpha \in \Delta^L\}$. In other words

$$\begin{cases} x_1 - x_2 & > 0 \\ x_2 - x_3 & > 0 \\ x_3 - x_4 & > 0 \\ x_3 + x_4 & > 0 \end{cases}$$

for all $x_1, x_2, x_3, x_4 \in \mathscr{C}_1$. In particular, we have $x_1 > x_2 > x_3 > |x_4|$. The first step is to determine the set $\mathfrak{B}$ of roots $\gamma \in \Phi\backslash\Phi^L$ which satisfies

$$H_{\gamma,0} \cap \mathscr{C}_1 \neq \emptyset.$$

We find

$$\mathfrak{B} = \{\pm\frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4), \pm\frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)\}.$$

The set of points $\{\lambda_{\Delta_\sigma^+,\Delta_\sigma^-} \mid \sigma \in \Omega_0^L\}$ is the set of points $(x_1, x_2, x_3, x_4) \in V$ which are solutions to the systems

$$\begin{cases} x_1 - x_2 & = \delta_1 \\ x_2 - x_3 & = \delta_2 \\ x_3 - x_4 & = \delta_3 \\ x_3 + x_4 & = \delta_4 \end{cases}$$

where $\delta_i = 0$ or 1. Claim 3' then follows by a straightforward computations. We find that $\langle \lambda_{\Delta_\sigma^+,\Delta_\sigma^-}, \check{\gamma}\rangle \in \mathbb{Z}$ in all cases.

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Assume that $S^\circ = \{t_1, t_2, t_3\}$. We get that Φ^L is of type D_4 and consists of the roots $\pm\varepsilon_i, \frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)$. We choose the following simple system:

$$\Delta^L = \{\frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4), \varepsilon_2, \varepsilon_3, \varepsilon_4\}.$$

We have $\mathscr{C}_1 = \{x \in V | \langle x, \check{\alpha}\rangle > 0, \text{for all } \alpha \in \Delta^L\}$. The set $\mathfrak{B}$ of roots $\gamma \in \Phi \backslash \Phi^L$ which satisfies $H_{\gamma,0} \cap \mathscr{C}_0^L \neq \emptyset$ is

$$\mathfrak{B} = \{\pm(\varepsilon_2 - \varepsilon_3), \pm(\varepsilon_3 - \varepsilon_4), \pm(\varepsilon_2 - \varepsilon_4)\}.$$

The set of points $\lambda_{\Delta_\sigma^+, \Delta_\sigma^-}$ is the set of points $(x_1, x_2, x_3, x_4) \in V$ which are solutions to the systems

$$\begin{cases} x_1 - x_2 - x_3 - x_4 & = \delta_1 \\ x_2 & = \delta_2 \\ x_3 & = \delta_3 \\ x_4 & = \delta_4 \end{cases}$$

where $\delta_i = 0$ or $\frac{1}{2}$ (since for all $\alpha \in \Delta^L$ we have $\check{\alpha} = 2\alpha$). Let $\gamma = \varepsilon_i - \varepsilon_j$ where $j > i > 1$ and let $\sigma \in \Omega_L^0$. Then σ defines a partition of Δ^L, which in turns defines the δ's. There are 3 cases to consider:

- Suppose that $\delta_i = \delta_j$. Then $\langle \lambda_{\Delta_\sigma^+, \Delta_\sigma^-}, \check{\gamma}\rangle = x_i - x_j = 0 \in \mathbb{Z}$ as required;
- Suppose that $\frac{1}{2} = \delta_i > \delta_j = 0$. Then σ sends ε_i to a positive root and ε_j to a negative one. Hence it sends γ to a positive root. We get

  $$\langle \lambda_{\Delta_\sigma^+, \Delta_\sigma^-}, \check{\gamma}\rangle = x_i - x_j = \frac{1}{2}$$

  as required.
- Suppose that $\frac{1}{2} = \delta_j > \delta_i = 0$. Then σ sends ε_i to a negative root and ε_j to a positive one. Hence it sends γ to a negative root. We get

  $$\langle \lambda_{\Delta_\sigma^+, \Delta_\sigma^-}, \check{\gamma}\rangle = x_i - x_j = -\frac{1}{2}$$

  as required.

Type $\tilde{\mathrm{B}}_n$. Let $V = \mathbb{R}^n$ with basis $(\varepsilon_i)_{1 \leqslant i \leqslant n}$. The root sytem Φ of type B_n consists of $2n$ short roots $\pm\varepsilon_i$ and $2n(n-1)$ long roots roots $\pm\varepsilon_i \pm \varepsilon_j$.

Assume that $S^\circ = \{t\}$. We get that Φ^L of type D_n and consists of the roots $\pm\varepsilon_i \pm \varepsilon_j$. We choose the following simple system:

$$\Delta^L = \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_{n-1} + \varepsilon_n\}.$$

The set $\mathfrak{B}$ of roots $\gamma \in \Phi \backslash \Phi^L$ which satisfies $H_{\gamma,0} \cap \mathscr{C}_1 \neq \emptyset$ is

$$\mathfrak{B} = \{\pm\varepsilon_n\}.$$

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The set of points $\lambda_{\Delta^+_\sigma,\Delta^-_\sigma}$ is the set of points $(x_1,x_2,x_3,x_4)\in V$ which are solutions to the systems

$$\begin{cases} x_1-x_2 &= \delta_1\\ x_2-x_3 &= \delta_2\\ \quad\vdots & \\ x_{n-1}-x_n &= \delta_{n-1}\\ x_{n-1}+x_n &= \delta_n\end{cases}$$

where $\delta_i=0$ or 1. This implies that $x_n=-1/2,0$ or $1/2$. Therefore we get

$$\langle\lambda_{\Delta^+_\sigma,\Delta^-_\sigma},\check{\varepsilon}_n\rangle = 2(\lambda_{\Delta^+_\sigma,\Delta^-_\sigma},\varepsilon_n) = -1,0 \text{ or } 1$$

as required.

Assume that $I_0=\{s_1,\dots,s_n\}$. We get that Φ^L of type $(\mathrm{A}_1)^n$ and consists of the roots $\pm\varepsilon_i$. We choose the following simple system:

$$\Delta^L=\{\varepsilon_i\}.$$

The set $\mathfrak{B}$ of roots $\gamma\in\Phi\backslash\Phi^L$ which satisfies $H_{\gamma,0}\cap\mathscr{C}_1\neq\emptyset$ is

$$\mathfrak{B}=\{\varepsilon_i-\varepsilon_j | i<j\}$$

The set of points $\lambda_{\Delta^+_\sigma,\Delta^-_\sigma}$ is the set of points $(x_1,x_2,x_3,x_4)\in V$ which are solutions to the systems

$$\begin{cases} x_1 &= \delta_1\\ x_2 &= \delta_2\\ \quad\vdots & \\ x_n &= \delta_n\end{cases}$$

where $\delta_i=0$ or $\frac{1}{2}$. Let $\gamma=\varepsilon_i-\varepsilon_j$ where $j>i$ and let $\sigma\in\Omega^0_L$. Then σ defines a partition of Δ^L, which in turns defines the δ's. There are 3 cases to consider:

- Suppose that $\delta_i=\delta_j$. Then $\langle\lambda_{\Delta^+_\sigma,\Delta^-_\sigma},\check{\gamma}\rangle = x_i-x_j=0\in\mathbb{Z}$ as required;
- Suppose that $\frac{1}{2}=\delta_i>\delta_j=0$. Then σ sends ε_i to a positive root and ε_j to a negative one. Hence it sends γ to a positive root. We get

  $$\langle\lambda_{\Delta^+_\sigma,\Delta^-_\sigma},\check{\gamma}\rangle = x_i-x_j=\frac{1}{2}$$

  as required.
- Suppose that $\frac{1}{2}=\delta_j>\delta_i=0$. Then σ sends ε_i to a negative root and ε_j to a positive one. Hence it sends γ to a negative root. We get

  $$\langle\lambda_{\Delta^+_\sigma,\Delta^-_\sigma},\check{\gamma}\rangle = x_i-x_j=-\frac{1}{2}$$

  as required.

Type $\tilde{\mathrm{C}}_n$. Let $V=\mathbb{R}^n$ with orthonormal basis $(\varepsilon_i)_{1\leqslant i\leqslant n}$. The root sytem Φ of type C_n consists of $2n$ long roots $\pm 2\varepsilon_i$ and $2n(n-1)$ short roots roots $\pm\varepsilon_i\pm\varepsilon_j$.

Assume that $I_0=\{t'\}$. Then $\Phi^L=\Phi$ and the statement is trivial since $\Phi\backslash\Phi^L=\emptyset$.

Assume that $I_0 = \{s_1, \dots, s_{n-1}\}$. We get that Φ^L is of type $(\mathrm{A}_1)^n$ and consists of the roots $\pm 2\varepsilon_i$. We choose the following simple system:

$$\Delta^L = \{2\varepsilon_i\}.$$

The set $\mathfrak{B}$ of roots $\gamma \in \Phi \backslash \Phi^L$ which satisfies $H_{\gamma,0} \cap \mathscr{C}_1 \neq \emptyset$ is

$$\mathfrak{B} = \{\varepsilon_i - \varepsilon_j | i < j\}$$

The set of points $\lambda_{\Delta_\sigma^+, \Delta_\sigma^-}$ is the set of points $(x_1, \dots, x_n) \in V$ which are solutions to the systems

$$\begin{cases} x_1 & = \delta_1 \\ x_2 & = \delta_2 \\ & \vdots \\ x_n & = \delta_n \end{cases}$$

where $\delta_i = 0$ or 1 if $L(t) = L(t')$ or where $\delta_i = 0$ or 2 if $L(t) > L(t')$. We find that $\langle \lambda_{\Delta_\sigma^+, \Delta_\sigma^-}, \check{\gamma} \rangle \in \mathbb{Z}$ in all cases.

Assume that $I_0 = \{s_1, \dots, s_{n-1}, t'\}$. It is the same thing as the previous case, except that the δ's only take values 0 or 2.

Assume that $I_0 = \{t, t'\}$. We get that Φ^L is of type D_n and consists of the roots $\pm\varepsilon_i \pm \varepsilon_j$. We choose the following simple system:

$$\Delta^L = \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_{n-1} + \varepsilon_n\}.$$

The set $\mathfrak{B}$ of roots $\gamma \in \Phi \backslash \Phi^L$ which satisfies $H_{\gamma,0} \cap \mathscr{C}_1 \neq \emptyset$ is

$$\mathfrak{B} = \{\pm 2\varepsilon_n\}.$$

The set of points $\lambda_{\Delta_\sigma^+, \Delta_\sigma^-}$ is the set of points $(x_1, \dots, x_n) \in V$ which are solutions to the systems

$$\begin{cases} x_1 - x_2 & = \delta_1 \\ x_2 - x_3 & = \delta_2 \\ & \vdots \\ x_{n-1} - x_n & = \delta_{n-1} \\ x_{n-1} + x_n & = \delta_n \end{cases}$$

where $\delta_i = 0$ or 1. Let $\gamma = 2\varepsilon_n$ and let $\sigma \in \Omega_L^0$. Then σ defines a partition of Δ^L, which in turns defines the δ's. There are 3 cases to consider:

- Suppose that $\delta_{n-1} = \delta_n$. Then $\langle \lambda_{\Delta_\sigma^+, \Delta_\sigma^-}, \check{\gamma} \rangle = x_n = 0 \in \mathbb{Z}$ as required;
- Suppose that $\frac{1}{2} = \delta_{n-1} > \delta_n = 0$. Then σ sends $2\varepsilon_n = \gamma$ to a negative root and we have
$$\langle \lambda_{\Delta_\sigma^+, \Delta_\sigma^-}, \check{\gamma} \rangle = x_n = -\frac{1}{2}$$
as required.
- Suppose that $\frac{1}{2} = \delta_n > \delta_{n-1} = 0$. Then σ sends $2\varepsilon_n = \gamma$ to a positive root and we have
$$\langle \lambda_{\Delta_\sigma^+, \Delta_\sigma^-}, \check{\gamma} \rangle = x_n = \frac{1}{2}$$
as required.

The proof of Claim 3' (hence of Statement (1)) is complete. □

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We now prove (2). Let $w' < a_w w^{\circ}_{\lambda_\sigma} b_\sigma$ be such that $w' \in \mathbf{c}^L_{\min}$ and write

$$w' = x_{w'} a_{w'} w^{\circ}_{\lambda_{\sigma'}} b_{\sigma'}.$$

Assume that $W_{\lambda_{\sigma'}} = W_{\lambda_\sigma}$. Then since

$$a_{w'} w^{\circ}_{\lambda_{\sigma'}} b_{\sigma'} < z_{w'} a_{w'} w^{\circ}_{\lambda_{\sigma'}} b_{\sigma'} < a_w w^{\circ}_{\lambda_\sigma} b_\sigma$$

and $a_{w'} w^{\circ}_{\lambda_{\sigma'}}, a_w w^{\circ}_{\lambda_{\sigma'}} \in W_{\lambda_\sigma}$, we get (see [14, Proof of Lemma 9. 10]) that either $b_{\sigma'} < b_\sigma$ or $b_{\sigma'} = b_\sigma$ and $a_{w'} w^{\circ}_{\lambda'} < a_w w^{\circ}_{\lambda'}$ which implies that $a_{w'} < a_w$ as required.

Assume that $W_{\lambda_{\sigma'}} \neq W_{\lambda_\sigma}$. Write $w' = w_\sigma z'$ where $w_\sigma \in W_{\lambda_\sigma}$ and z' has minimal length in the coset $W_{\lambda_\sigma} w'$.

Note that since $w' = w_\sigma z' < a_w w^{\circ}_{\lambda_\sigma} b_\sigma$ we get that $z' \leq b_\sigma$. But we must have $z' < b_\sigma$ otherwise we would have $w_\sigma < a_w w^{\circ}_{\lambda_\sigma}$, which together with the condition $w' \in \mathbf{c}^L_{\min}$ would imply that $w_\sigma = u' w^{\circ}_{\lambda_\sigma}$ for some $u' \in W^{\circ}_{\lambda_\sigma}$ and $\lambda_\sigma = \lambda_{\sigma'}$.

If we show that

$$D' = H(b_{\sigma'} A_0, z' A_0) \cap H(A_0, b_{\sigma'} A_0) = \emptyset$$

then the result will follow. Indeed if $D' = \emptyset$ then there exists an $x \in W$ such that $z' = x b_{\sigma'}$ and $x \bullet b_{\sigma'}$ and we get $b_{\sigma'} < b_\sigma$ since $b_{\sigma'} \leq z < b_\sigma$.

Let λ' be the unique L-special point which contains $z' A_0$ and $w^{\circ}_{\lambda_\sigma} z' A_0$. A hyperplane H which lies in D' cannot contain $\lambda_{\sigma'}$ (otherwise $H \notin H(A_0, b_{\sigma'} A_0)$) nor λ' (otherwise $H \notin H(b_{\sigma'} A_0, z' A_0)$) but it has to separate these two points. Hence it also separates any alcoves which contains $\lambda_{\sigma'}$ and any alcoves which contains λ'. In particular it separates $a_{w'} w^{\circ}_{\lambda_{\sigma'}} b_{\sigma'} A_0$ and $w' A_0 = x_{w'} a_{w'} w^{\circ}_{\lambda_{\sigma'}} b_{\sigma'} A_0$ but there are no such hyperplanes as we have shown in the proof of Statement (1). □

Example 3.11 (positive weight functions). In this example, and only in this example, we assume that L is ***positive***. Let $\mathscr{P}_L(S)$ be the set of proper subsets I of S such that $L(w_I) = \nu_L$. If $I \in \mathscr{P}_L(S)$, then $W_I \simeq W_0$ (because L is positive). Note also that, since L is positive,

$$\mathscr{P}_L(S) = \{S_\lambda \mid \lambda \in \operatorname{Spe}_L(W)\}$$

and $\quad \{w \in \mathscr{W} \mid L(w) = \nu_L\} = \{w_I \mid I \in \mathscr{P}_L(S)\} = \{w_\lambda \mid \lambda \in \operatorname{Spe}_L(W)\}$.

Recall also that $\mathscr{L}_L(W) = \mathscr{L}(W)$. Therefore

$$\begin{aligned} \mathbf{c}^L_{\min} &= \{x w_I y \in W \mid x, y \in W,\ x \bullet w_I \bullet y \text{ and } I \in \mathscr{P}_L(S)\} \\ &= \{x w_\lambda y \mid x, y \in W,\ x \bullet w_\lambda \bullet y \text{ and } \lambda \in \operatorname{Spe}_L(W)\}. \end{aligned}$$

It we set $M^L_\lambda = \{z \in W \mid w_\lambda \bullet z \text{ and } s w_\lambda z \notin \mathbf{c}^L_{\min} \text{ for all } s \in S_\lambda\}$ as in [4, Proof of Theorem 5.4], then we obtain the following decomposition of $\mathbf{c}^L_{\min}$

$$\mathbf{c}^L_{\min} = \dot{\bigcup_{\substack{\lambda \in \operatorname{Spe}^0_L(W) \\ z \in M^L_\lambda}}} N_{\lambda, z} \quad \text{(disjoint union)}$$

where $\operatorname{Spe}^0_L(W) = \operatorname{Spe}_L(W) \cap \overline{A}_0$ is a set of representatives for the Ω-orbits on $\operatorname{Spe}_L(W)$ and

$$N_{\lambda, z} = \{x w_\lambda z \mid x \in X_\lambda\}.$$

It is easily seen that the set M^L_λ consists of our elements b_σ and that the sets $N_{\lambda,z}$ correspond to N^L_σ. ■

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4. Semidirect product decomposition

We fix a non-negative weight function $L : W \to \Gamma$ where Γ is a totally abelian group. The aim of this section is to express the lowest two-sided cell $\mathbf{c}_{\min}^L$ in relation to the decomposition of W as semidirect product of two Coxeter groups as in [2].

4.1. **Coxeter groups.** If I is a subset of S, we set

$$I^\circ = \{s \in I \mid L(s) = 0\} \quad \text{and} \quad I^+ = \{s \in I \mid L(s) > 0\},$$

so that $I = I^\circ \dot\cup I^+$. We also set

$$\tilde{I} = \{wsw^{-1} \mid w \in W_{I^\circ} \text{ and } s \in I^+\}$$

and we denote by $\tilde{W}_{\tilde{I}}$ the subgroup of W generated by $\tilde{I}$. For simplification, we set $W^\circ = W_{S^\circ}$ and $\tilde{W} = \tilde{W}_{\tilde{S}}$.

Note that, if $s \in I^\circ$ and $t \in I^+$, then $L(s) \neq L(t)$, so s and t are not conjugate in W. It then follows from [2] that

$$W_I = W_{I^\circ} \ltimes \tilde{W}_{\tilde{I}} \quad \textit{and} \quad (\tilde{W}_{\tilde{I}}, \tilde{I}) \textit{ is a Coxeter group.} \tag{4.1}$$

If $I = S$, we get that

$$W = W^\circ \ltimes \tilde{W} \quad \text{and} \quad (\tilde{W}, \tilde{S}) \text{ is a Coxeter group.}$$

We will assume that W° is finite. Note however, that this assumption is not very restrictive when dealing with an affine Weyl group. Indeed, by direct products, we can assume that W is irreducible. In this case, either $L = 0$ (and then $\mathbf{c}_{\min}^L = W$ and the problem is uninteresting) or S° is a proper subset of S (and then W° is finite because W is irreducible).

4.2. **The group $\tilde{\Omega}$.** We keep the notation of Section 2.1. Let

$$S_\Omega = \{\sigma_H \mid H \text{ is a wall of } A_0\}.$$

Then (Ω, S_Ω) is a Coxeter system. Let

$$S_\Omega^\circ = \{\sigma_H \mid H \text{ is a wall of } A_0 \text{ and } L_H = 0\}$$

and

$$S_\Omega^+ = \{\sigma_H \mid H \text{ is a wall of } A_0 \text{ and } L_H > 0\}.$$

Then we have $\Omega = \Omega^\circ \ltimes \tilde{\Omega}$ where Ω° is generated by S_Ω° and $\tilde{\Omega}$ is generated by

$$\tilde{S}_\Omega := \{\rho\sigma_H\rho^{-1} \mid \rho \in \Omega^\circ \text{ and } \sigma_H \in S_\Omega^+\} = \{\sigma_{H\rho} \mid \rho \in \Omega^\circ \text{ and } \sigma_H \in S_\Omega^+\}.$$

We set

$$\tilde{\mathscr{F}} = \{H \in \mathscr{F} \mid \sigma_H \in \tilde{\Omega}\}.$$

It is clear by definition that $\tilde{\Omega}$ is generated by $\{\sigma_H \mid H \in \tilde{\mathscr{F}}\}$. Further, the following conditions are satisfied

(D1) $\tilde{\Omega}$ stabilizes $\tilde{\mathscr{F}}$.

(D2) The group $\tilde{\Omega}$, endowed with the discrete topology, acts properly on V.

We prove (D1). Let $\tilde{\sigma} \in \tilde{\Omega}$ and $H \in \tilde{\mathscr{F}}$, that is $\sigma_H \in \tilde{\Omega}$. We have $\tilde{\sigma}\sigma_H\tilde{\sigma}^{-1} = \sigma_{H\tilde{\sigma}} \in \tilde{\Omega}$ and, therefore, $H\tilde{\sigma} \in \tilde{\mathscr{F}}$. Condition (D2) follows easily form the fact that Ω, endowed with the discrete topology, acts properly on V.

We denote by $\mathrm{Alc}(\tilde{\mathscr{F}})$ the set of alcoves with respect to $\tilde{\mathscr{F}}$, that is the connected components of

$$V - \bigcup_{H \in \tilde{\mathscr{F}}} H.$$

Let $\tilde{A}_0$ be the unique alcove (with respect to $\tilde{\mathscr{F}}$) which contains A_0. Then we have (see [3, Chapter 5, §3] and [2, §4])

(1) The group $\tilde{\Omega}$ is generated by the orthogonal reflections with respect to the wall of $\tilde{A}_0$.
(2) $\overline{\tilde{A}_0}$ is a fundamental domain for the action of $\tilde{\Omega}$.
(3) $\overline{\tilde{A}_0} = \bigcup_{\rho \in \Omega^\circ} \overline{A_0}\rho$.
(4) Any element $\sigma_H \in \tilde{\Omega}$ is conjugate in $\tilde{\Omega}$ to an orthogonal reflection with respect to a wall of $\tilde{A}_0$.

It follows that $\tilde{\Omega}$ is an affine Weyl group (see [2, §4]). Note that $\tilde{\Omega}$ is not necessarily irreducible. In fact, as we expect, the group $\tilde{\Omega}$ is nothing else that the group generated by the reflections with respect to the hyperlanes in $\mathscr{F}^L = \{H \in \mathscr{F} \mid L_H > 0\}$.

Lemma 4.2. *We have $\tilde{\mathscr{F}} = \mathscr{F}^L$.*

Proof. Let $H \in \mathscr{F}^L$. There exists $\sigma \in \Omega$ and a wall H' of A_0 of positive weight such that $H'\sigma = H$. Write $\sigma = \rho\tilde{\sigma}$ where $\rho \in \Omega^\circ$ and $\tilde{\sigma} \in \tilde{\Omega}$. Then $\sigma_{H'\rho} \in \tilde{\Omega}$ and we have

$$\sigma_H = \tilde{\sigma}\sigma_{H'\rho}\tilde{\sigma}^{-1}$$

therefore $\sigma_H \in \tilde{\Omega}$.

Conversely, let $H \in \tilde{\mathscr{F}}$ that is $\sigma_H \in \tilde{\Omega}$. By (4), σ_H is conjugate (in $\tilde{\Omega}$) to $\sigma_{H'}$ where H' is a wall of $\tilde{A}_0$. By (3), we know that the walls of $\tilde{A}_0$ are of the form $H\rho$ where H is a wall of A_0 of positive weight. In particular, H' has positive weight. It follows that H has positive weight and $H \in \mathscr{F}^L$ as required. □

Finally we want to define a root system associated to $\tilde{\Omega}$. Let

$$\tilde{\Phi} := \{b_\alpha \alpha \mid \alpha \in \Phi^L\}$$

where b_α is defined to be the smallest integer such that H_{α,b_α} has positive weight. We also fix a set of positive roots

$$\tilde{\Phi}^+ = \{b_\alpha \alpha \mid \alpha \in \Phi^L \cap \Phi^+\}.$$

Remark 4.3. If Ω is not of type $\tilde{\mathrm{C}}$ then we simply have $\tilde{\Phi} = \Phi^L$. Indeed in this case, any two parallele hyperplane have same weights, hence $b_\alpha = 1$ for all $\alpha \in \Phi^L$. If Ω is of type $\tilde{\mathrm{C}}$, then we may have $b_\alpha = 2$ for some choices of parameters, namely when $L(t) > L(t') = 0$ (see Convention 2.6). ■

Remark 4.4. We have $\Omega_0^L = \tilde{\Omega}_0$ where $\tilde{\Omega}_0 = \langle \sigma_{H_{\tilde{\alpha},0}} \mid \tilde{\alpha} \in \tilde{\Phi}\rangle$. ■

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Lemma 4.5. *The group $\tilde{\Omega}$ is the affine Weyl group associated to $\tilde{\Phi}$. Further the alcove $\tilde{A}_0$ is the fundamental alcove associated to $\tilde{\Phi}$, that is*

$$\tilde{A}_0 = \{x \in V \mid 0 < \langle x, \check{\alpha}\rangle < 1 \text{ for all } \alpha \in \tilde{\Phi}^+\}$$

Proof. The first statement is clear since we have

$$\mathscr{F}^L = \{H_{b_\alpha \alpha, n} \mid \alpha \in \Phi^L \cap \Phi^+, n \in \mathbb{Z}\}.$$

The second statement follows easily from the above equality and the fact that $A_0 \subset \tilde{A}_0$. □

Doing as in Section 2.1, we obtain another geometric realization of $\tilde{\Omega}$, namely as a group generated by involutions on the set $\mathrm{Alc}(\tilde{\mathscr{F}})$. Indeed, $\tilde{\Omega}$ acts transitively on the set of faces of alcoves in $\mathrm{Alc}(\tilde{\mathscr{F}})$: we denote by $\{\tilde{t}_1, \ldots, \tilde{t}_m\}$ the set of $\tilde{\Omega}$-orbits in the set of faces. Note that the set of faces of $\tilde{A}_0$ is a set of representatives of the set of orbits. To each $\tilde{t}_i$ we can associate an involution $\tilde{A} \mapsto \tilde{t}_i\tilde{A}$ of $\mathrm{Alc}(\tilde{\mathscr{F}})$ where $\tilde{t}_i\tilde{A}$ is the unique alcove of $\mathrm{Alc}(\tilde{\mathscr{F}})$ which shares with $\tilde{A}$ a face of type $\tilde{t}_i$. The group generated by all the $\tilde{t}_i$ is an affine Weyl group isomorphic to $\tilde{\Omega}$. We would like to use the notation $\tilde{W}$ and $\tilde{S}$ for this group, and eventually we will, but before one needs to be careful since $\tilde{W}$ also denotes the group appearing in the semidirect product decomposition of W (where W is the group generated by involutions on $\mathrm{Alc}(\mathscr{F})$).

4.3. **Alcoves of $\tilde{W}$.** Recall the definition of (W, S) in Section 2.1 and that

$$S^\circ = \{s' \in S \mid L(s) = 0\} \text{ and } S^+ = \{s \in S \mid L(s) > 0\}.$$

Then we have $W = W^\circ \ltimes \tilde{W}$ where W° is generated by S° and $\tilde{W}$ is generated by

$$\tilde{S} = \{wtw^{-1} \mid t \in S^+ \text{ and } w \in W^\circ\}.$$

Lemma 4.6. *Let $\tilde{t} \in \tilde{S} = \{wtw^{-1} \mid t \in S^+$ and $w \in W^\circ\}$. Then there exists a unique wall H of $\tilde{A}_0$ such that*

$$\tilde{t}A_0 = A_0\sigma_H.$$

Proof. Let $w \in W^\circ$ and $t \in S^+$ be such that $\tilde{t} = wtw^{-1}$. Let $\rho \in \Omega^\circ$ be such that $wA_0 = A_0\rho$ and let H' be the unique hyperplane which contains the face of type t of A_0. Then we have

$$wtw^{-1}A_0 = wtA_0\rho^{-1} = wA_0\sigma_{H'}\rho^{-1} = A_0\rho\sigma_{H'}\rho^{-1} = A_0\sigma_{H'\rho}$$

and the result follows. □

Therefore there is a natural bijection between the set $\tilde{S}$ and the set of faces of $\tilde{A}_0$ and therefore between $\tilde{S}$ and the set of orbits $\{\tilde{t}_1, \ldots, \tilde{t}_m\}$: we will freely identify those two sets. Note that an element $\tilde{t} \in \tilde{S}$ can be viewed as acting on the set of alcoves $\mathrm{Alc}(\mathscr{F})$ when it is considered as an element of $\tilde{W} \subset W$ but it can also be viewed as acting on $\mathrm{Alc}(\tilde{\mathscr{F}})$ if $\tilde{t}$ is considered as acting on $\mathrm{Alc}(\tilde{\mathscr{F}})$ via the action defined at the end of the previous section. In the following lemma, we show that these two actions behaves well with one another.

Lemma 4.7. *If $\tilde{w} \in \tilde{W}$, then*

$$\tilde{w}A_0 \subset \tilde{w}\tilde{A}_0.$$

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From where it follows that

$$\overline{\bigcup_{w^\circ \in W^\circ} w^\circ \tilde{w} A_0} = \overline{\tilde{w}\tilde{A}_0}.$$

Proof. Let $\tilde{t} \in \tilde{S}$. Then $\tilde{t}\tilde{A}_0$ is the unique alcove in $\mathrm{Alc}(\tilde{\mathscr{F}})$ which shares with $\tilde{A}_0$ a face of type $\tilde{t}$, hence we have

$$\tilde{t}\tilde{A}_0 = \tilde{A}_0 \sigma_H$$

where H is the hyperplane which supports the face of $\tilde{A}_0$ of type $\tilde{t}$. By the previous lemma we see that

$$\tilde{t}A_0 = A_0 \sigma_H.$$

Hence since $A_0 \subset \tilde{A}_0$, the first assertion follows. The second assertion follows from $\overline{\tilde{A}_0} = \bigcup_{\rho \in \Omega^\circ} \overline{A_0}\rho = \bigcup_{w \in W^\circ} \overline{wA_0}$. □

4.4. **The lowest two-sided cell of $\tilde{W}$.** Let $\tilde{L}$ denote the restriction of L to $\tilde{W}$. By [2, Corollary 1.4], it is a *positive* weight function. Note that we have $\tilde{L}(wtw^{-1}) = L(t)$ for all $w \in W^\circ$.

Theorem 4.8. *We have*

$$\mathbf{c}^L_{\min}(W) = W^\circ \cdot \mathbf{c}^{\tilde{L}}_{\min}(\tilde{W}) \quad \textit{and} \quad N^L_\sigma(W) = W^\circ \cdot N^{\tilde{L}}_\sigma(\tilde{W})$$

for all $\sigma \in \Omega^L_0$.

Proof. First, since $\tilde{\mathscr{F}} = \mathscr{F}^L$ and $A_0 \subset \tilde{A}_0$ we see that

$$\mathscr{U}^{\tilde{L}}(\tilde{A}_0) = \mathscr{U}^L(A_0).$$

Then applying the results of the previous section we get

$$\mathbf{c}^{\tilde{L}}_{\min}(\tilde{W}) = \{\tilde{w} \in \tilde{W} \mid \tilde{w}(\tilde{A}_0) \not\subset \mathscr{U}^{\tilde{L}}(\tilde{A}_0)\}$$
$$\mathbf{c}^L_{\min}(W) = \{w \in W \mid w(A_0) \not\subset \mathscr{U}^L(A_0)\}.$$

Let $w \in \mathbf{c}^L_{\min}$ and write $w = w^\circ \tilde{w}$ where $w^\circ \in W^\circ$ and $\tilde{w} \in \tilde{W}$. We have $w^\circ \tilde{w} A_0 \notin \mathscr{U}^L(A_0)$ that is $w^\circ \tilde{w} A_0 \notin \mathscr{U}^{\tilde{L}}(\tilde{A}_0)$. Since the only hyperplane separating $\tilde{w}A_0$ and $w^\circ \tilde{w} A_0$ are hyperplanes of weight 0, this implies that $\tilde{w}A_0 \notin \mathscr{U}^{\tilde{L}}(\tilde{A}_0)$. Hence, by Lemma 4.7, we get that $\tilde{w}\tilde{A}_0 \notin \mathscr{U}^{\tilde{L}}(\tilde{A}_0)$ and $\tilde{w} \in \mathbf{c}^{\tilde{L}}_{\min}(\tilde{W})$ as required.

Conversely let $w^\circ \tilde{w} \in W^\circ \cdot \tilde{\mathbf{c}}^{\tilde{L}}_{\min}$. Since $\tilde{w} \in \tilde{\mathbf{c}}^{\tilde{L}}_{\min}$ we have $\tilde{w}\tilde{A}_0 \notin \mathscr{U}^{\tilde{L}}(\tilde{A}_0)$. By Lemma 4.7, it follows that $\tilde{w}A_0 \notin \mathscr{U}^L(A_0)$ and $w^\circ \tilde{w} A_0 \notin \mathscr{U}^L(A_0)$ as required.

The second equality in the theorem follows easily from the fact that $\tilde{\Omega}_0 = \Omega^L_0$, Lemma 4.7 and

$$N^L_\sigma(W) = \{w \in W \mid wA_0 \subset \mathscr{C}'_\sigma\} \quad \text{and} \quad N^{\tilde{L}}_\sigma(\tilde{W}) = \{\tilde{w} \in \tilde{W} \mid \tilde{w}\tilde{A}_0 \subset \mathscr{C}'_\sigma\}.$$

□

Remark 4.9. Since $\mathbf{c}^L_{\min}(W)$, $\mathbf{c}^{\tilde{L}}_{\min}(\tilde{W})$ and W° are stable by taking the inverse, we get that

$$\mathbf{c}^L_{\min}(W) = W^\circ \cdot \mathbf{c}^{\tilde{L}}_{\min}(\tilde{W}) = \mathbf{c}^{\tilde{L}}_{\min}(\tilde{W}) \cdot W^\circ = W^\circ \cdot \mathbf{c}^{\tilde{L}}_{\min}(\tilde{W}) \cdot W^\circ. \blacksquare$$

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5. Kazhdan-Lusztig cells

5.1. **Iwahori-Hecke algebras.** Recall that Γ is a totally ordered abelian group, whose law is denoted by $+$ and whose order relation is denoted by $\leqslant$. Let $\mathscr{A}$ be the group algebra of Γ over $\mathbb{Z}$. We shall use the following notation for $\mathscr{A}$

$$\mathscr{A} = \bigoplus_{\gamma\in\Gamma} \mathbb{Z}v^\gamma \text{ where } v^\gamma.v^{\gamma'} = v^{\gamma+\gamma'}.$$

Let $L : W \to \Gamma$ be a weight function. For $s \in S$ we set $v_s = v^{L(s)}$.

We denote by $\mathscr{H} = \mathscr{H}(W, S, L)$ the corresponding generic Iwahori-Hecke algebra, that is the free associative $\mathscr{A}$-algebra with $\mathscr{A}$-basis $\{T_w \mid w \in W\}$ and multiplication given by

(a) $\quad T_wT_{w'} = T_{ww'} \quad$ if $\ell(ww') = \ell(w) + \ell(w')$
(b) $\quad T_s^2 = (v_s - v_s^{-1})T_s + 1 \quad$ if $s \in S$.

Let $^-$ be the involution of $\mathscr{A}$ which takes v^γ to $v^{-\gamma}$. It is well known that this map can be extended to a ring involution on $\mathscr{H}$ (we will also denote it by $^-$) via the formula:

$$\overline{\sum_{w\in W} a_wT_w} = \sum_{w\in W} \overline{a_w}T_{w^{-1}}^{-1}.$$

For all $w \in W$, by [14, Theorem 5.2], there exists a unique element $C_w \in \mathscr{H}$ such that

- $\overline{C_w} = C_w$,
- $C_w \in T_w + (\bigoplus_{y<w} \mathscr{A}_{<0}T_y)$ where $\mathscr{A}_{<0} = \bigoplus_{\gamma<0} \mathbb{Z}v^\gamma$.

From the second condition, it is clear that the set $\{C_w, w \in W\}$ forms an $\mathscr{A}$-basis of $\mathscr{H}$, known as the Kazhdan-Lusztig basis.

We write

$$C_w = \sum_{y\in W} P_{y,w}T_w \text{ where } P_{y,w} \in \mathscr{A}.$$

The elements $P_{y,w}$ are called the Kazhdan-Lusztig polynomials and they satisfy the following properties ([14, §5.3])

(1) $P_{y,y} = 1$
(2) $P_{y,w} = 0$ if $y \not\leqslant w$,
(3) $P_{y,w} \in \mathscr{A}_{<0}$ if $y < w$,
(4) $P_{y,w} = v_s^{-1}P_{sy,w}$ if $sy > y$ and $sw < w$.

Following [14, §6], we now describe the multiplication rule for the C_w's. For each $y, w \in W$ and $s \in S$ such that $sy < y < w < sw$ we define $M^s_{y,w} \in \mathscr{A}$ by the inductive condition

$$M^s_{y,w} - \sum_{\substack{y<z<w\\ sz<z}} P_{y,z}M^s_{z,w} - v_sP_{y,w} \in \mathscr{A}_{<0}$$

and the symmetry condition

$$\overline{M^s_{y,w}} = M^s_{y,w}.$$

For $w \in W$ and $s \in S$, we obtain the following multiplication formula for the Kazhdan-Lusztig basis

$$C_sC_w = \begin{cases} C_{sw} + \sum_{z; sz<z<w} M^s_{z,w} C_z, & \text{if } w < sw, \\ (v_s + v_s^{-1})C_w, & \text{if } sw < w. \end{cases}$$

Since $C_s = T_s + v_s^{-1}T_1$ for all $s \in S$, one can see that

$$T_sC_w = \begin{cases} C_{sw} - v_s^{-1}C_w + \sum_{z; sz<z<w} M^s_{z,w} C_z, & \text{if } w < sw, \\ v_sC_w, & \text{if } sw < w. \end{cases}$$

We will also need the following relation for Kazhdan-Lusztig polynomials. Let $y < w \in W$ and $s \in S$ such that $sw < w$. We have

(1) $$P_{y,w} = v_sP_{y,sw} + P_{sy,sw} - \sum_{\substack{y \leq z < sw \\ sz < z}} P_{y,z}M_{z,sw} \qquad \text{if } sy < y$$

(2) $$P_{y,w} = v_s^{-1}P_{sy,w} \qquad \text{if } sy > y$$

Finally we define the preorders $\leq_{\mathscr{L}}, \leq_{\mathscr{R}}, \leq_{\mathscr{LR}}$ as in [14]. For instance $\leq_{\mathscr{L}}$ is the transitive closure of the relation:

$$y \leftarrow_{\mathscr{L}} w \Longleftrightarrow \text{there exists } s \in S \text{ such that } M^s_{y,w} \neq 0.$$

Each of these preorders give rise to an equivalence relation $\sim_{\mathscr{L}}, \sim_{\mathscr{R}}$ and $\sim_{\mathscr{LR}}$. The equivalence classes associated to $\sim_{\mathscr{L}}$, $\sim_{\mathscr{R}}$ and $\sim_{\mathscr{LR}}$ are called left, right and two-sided cells, respectively. The partition of W in cells *depends on the choice of the weight function*. The preorders $\leq_{\mathscr{L}}, \leq_{\mathscr{R}}, \leq_{\mathscr{LR}}$ induce partial orders on the left, right and two-sided cells, respectively.

Remark 5.1. We have $x \sim_{\mathscr{L}} y$ if and only if $x^{-1} \sim_{\mathscr{R}} y^{-1}$ [14, §8]. It follows easily that a union of left cells which is stable by taking the inverse is a also a union of two-sided cells. ∎

Remark 5.2. All the above can also be defined for weight functions which take negative values. It is shown in [1] that the partition into cells with respect to a weight function L^- is the same as the partition into cells with respect to L where L is defined by

$$L(s) = \begin{cases} L(s) & \text{if } L^-(s) \geq 0, \\ -L(s) & \text{if } L^-(s) < 0. \end{cases}$$

Note that L is a non-negative weight function. Hence the computation of Kazhdan-Lusztig cells can be reduced to the non-negative case. ∎

5.2. **Kazhdan-Lusztig lowest two-sided cell.** In the case where L is a positive weight function, it is a well known fact that there is a lowest (Kazhdan-Lusztig) two sided cell with respect to the partial order $\leq_{\mathscr{LR}}$. This two-sided cell has been thoroughly studied [15, 16, 17, 4, 9] and it is equal to

$$\mathbf{c}^L_{\min} = \{xwy \mid w \in \mathscr{W},\ x.w.y \text{ and } L(w) = \nu_L\}$$

(Hence the name for the set $\mathbf{c}^L_{\min}$.) The aim of this section is to show that this presentation also holds for non-negative weight function.

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Let L be a non-negative weight function. Then, following Section 4, we have $W = W^\circ \ltimes \tilde{W}$. Let $\tilde{L}$ be the restriction of L to $\tilde{W}$. Then $\tilde{L}$ is a positive weight function on $\tilde{W}$ and $\tilde{L}(wtw^{-1}) = L(t)$ for all $w \in W^\circ$ and $t \in S^+$. We denote by $\tilde{\mathscr{H}} = \mathscr{H}(\tilde{W}, \tilde{S}, \tilde{L})$ the corresponding Hecke algebra. The group W° acts on $\tilde{W}$ and stabilizes $\tilde{S}$ and $\tilde{L}$, therefore it naturally acts on $\tilde{\mathscr{H}}$ and we can define the the semidirect product of algebras

$$W^\circ \ltimes \tilde{\mathscr{H}}.$$

It has an $\mathscr{A}$-basis $(x \cdot T_{\tilde{w}})_{x \in W^\circ, \tilde{w} \in \tilde{W}}$ and the map

$$x \cdot T_{\tilde{w}} \longmapsto T_{x\tilde{w}}$$

defines an isomorphism of $\mathscr{A}$-algebras from $\tilde{\mathscr{H}}$ to $\mathscr{H}(W, S, L)$. The cells of (W, S, L) can then be described in the following way.

Theorem 5.3. *(*[1, Corollary 2.13]*) The left cells (respectively the two-sided cells) of (W, S, L) are of the form $W^\circ.C$ (respectively $W^\circ.C.W^\circ$) where C is a left cell (respectively a two-sided cell) of $(\tilde{W}, \tilde{S}, \tilde{L})$.*

Finally we are ready to prove one of the main result of this paper which gives a general presentation of the lowest two-sided cell, including the case when the weight function L vanishes on some generators. Note that this theorem is already known when the parameters are positive: see [15, 16] for the equal parameter case and [4, §5], [17, Chapter 3] and [9] for the unequal parameters.

Theorem 5.4. *Let (W, S, L) be an irreducible affine Weyl group and let L be a non-negative weight function on W. Set*

$$\nu_L = \max_{I \subset S} w_I \quad \text{and} \quad \mathscr{W} = \bigcup_{I \subset S} W_I$$

where I runs over the subset of S such that W_I is finite. Then the lowest two-sided cell of W is

$$\mathbf{c}_{\min}^L = \{xwy \mid w \in \mathscr{W},\ x \bullet w \bullet y \text{ and } L(w) = \nu_L\}.$$

Further, the decomposition of $\mathbf{c}_{\min}^L$ into left cells is

$$\mathbf{c}_{\min}^L = \bigcup_{\sigma \in \Omega_0^L} N_\sigma^L.$$

Proof. As mentioned previously this result is already known when L is a positive weight function. On the one hand, by Theorem 5.3, the lowest two-sided Kazhdan-Lusztig cell of W with respect to $\leq_{\mathscr{LR}}$ and the weight function L is

$$W^\circ \cdot \mathbf{c} \cdot W^\circ$$

where $\mathbf{c}$ is the lowest Kazhdan-Lusztig cell of $(\tilde{W}, \tilde{L})$. But in this case we know that

$$\mathbf{c} = \mathbf{c}_{\min}^{\tilde{L}}(\tilde{W})$$

since $\tilde{L}$ is a positive weight function. Then the result follows from Theorem 4.8, where we proved that

$$W^\circ \cdot \mathbf{c}_{\min}^{\tilde{L}}(\tilde{W}) \cdot W^\circ = \mathbf{c}_{\min}^L(W).$$

The left cells lying in $\mathbf{c}_{\min}^L$ are of the form $W^\circ \cdot N_\sigma^{\tilde{L}}(\tilde{W})$ where $N_\sigma^{\tilde{L}}(\tilde{W})$ is a left cell of $\mathbf{c}_{\min}^{\tilde{L}}(\tilde{W})$. Once again, by Theorem 4.8, we know that $W^\circ \cdot N_\sigma^{\tilde{L}}(\tilde{W}) = N_\sigma^L$ as required. □

6. On the asymptotic semicontinuity of the lowest two-sided cell

In this section, we fix a totally ordered abelian group Γ.

6.1. **Semicontinuity conjecture.** Let $\bar{S} = \{\omega_1, ..., \omega_m\}$ be the set of conjugacy classes in S. Let $\mathbb{Z}[\bar{S}]$ be the free $\mathbb{Z}$-module with basis $\bar{S}$ and let $V' = \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}[\bar{S}]$. We shall view the elements of $\mathbb{Z}[\bar{S}]$ as embedded in V'. We denote by $\omega_1^*, \dots, \omega_m^*$ the dual basis of $\omega_1, \dots, \omega_m$. For $(n_1, ..., n_m) \in \mathbb{Q}^r - \{0\}$ we set

$$H_{n_1\omega_1+...+n_m\omega_m} := \ker(\sum n_i\omega_i^*).$$

Such an hyperplane is called a rational hyperplane.

Since Γ is torsion-free, the natural map $\Gamma \to \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma$ is injective, so we shall view Γ as embedded in the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma$: in particular, if $r \in \mathbb{Q}$ and $\gamma \in \Gamma$ then $r\gamma$ is well-defined. Moreover, the order on Γ extends uniquely to a total order on $\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma$ that we still denote by $\leq$.

Following [3] we now introduce the notion of facets and chambers associated to a finite set of rational hyperplanes. Let $H = H_{n_1\omega_1+...+n_m\omega_m}$ where $n_i \in \mathbb{Q}$. We say that a weight function $L \in \text{Weight}(W, \Gamma)$ lies on H if we have

$$\sum_{i=1}^{m} n_i L(\omega_i) = 0.$$

We say that two weight functions L, L' lie on the same side of $H_{n_1\omega_1+...+n_m\omega_m}$ if we have

$$\sum_{i=1}^{m} n_i L(\omega_i) > 0 \text{ and } \sum_{i=1}^{m} n_i L'(\omega_i) > 0$$

or

$$\sum_{i=1}^{m} n_i L(\omega_i) < 0 \text{ and } \sum_{i=1}^{m} n_i L'(\omega_i) < 0.$$

Let $\mathfrak{H}$ be a finite set of rational hyperplanes. We define an equivalence relation on $\text{Weight}(W, \Gamma)$: we write $L \sim_{\mathfrak{H}} L'$ if for all $H \in \mathfrak{H}$ we have either

(1) $L, L' \in H$;
(2) L, L' lie on the same side of H.

The equivalence classes associated to this relation will be called $\mathfrak{H}$-facets. A $\mathfrak{H}$-chamber is a $\mathfrak{H}$-facet $\mathscr{F}$ such that no weight function in $\mathscr{F}$ lies on a hyperplane $H \in \mathfrak{H}$.

Remark 6.1. In [3] the equivalence relation $\sim_{\mathfrak{H}}$ is defined on V' in the following way: $\lambda \sim_{\mathfrak{H}} \mu \in V'$ if for all for all $H \in \mathfrak{H}$ we have either

(1) $\lambda, \mu \in H$;
(2) λ, μ lie on the same side of H.

There is a one to one correspondance between the equivalence classes of this relation in V' and the sets of facets in $\text{Weight}(W, \Gamma)$. We will freely identify those two sets. ■

For an $\mathfrak{H}$-facet $\mathscr{F}$ we denote by $W_{\mathscr{F}}$ the parabolic subgroup generated by

$$\{s \in S | L(s) = 0 \text{ for all } L \in \mathscr{F}\}.$$

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We say that a subset X of W is stable by translation by W_I ($I \subset S$) on the left (respectively on both sides) if for all $w \in X$ we have $zw \in X$ (respectively $zwz' \in X$) for all $z \in W_I$ (respectively for all $z, z' \in W_I$). Finally we denote by $\mathscr{C}_{\mathscr{L}}(L)$ (respectively $\mathscr{C}_{\mathscr{L}\mathscr{R}}(L)$) the partition of W into left (respectively two-sided) cells with respect to the weight function L.

We can now state the first author's conjecture for the partition of W into cells. It is enough to state it for left and two-sided cells (see Remark 5.1).

Conjecture 6.2. *There exists a finite set of rational hyperplanes $\mathfrak{H}$ of V' satisfying the following properties*

(1) *If L_1, L_2 are two weight functions belonging to the same $\mathfrak{H}$-facet $\mathscr{F}$ then $\mathscr{C}_{\mathscr{L}}(L_1)$ (respectively $\mathscr{C}_{\mathscr{L}\mathscr{R}}(L_1)$) and $\mathscr{C}_{\mathscr{L}}(L_2)$ (respectively $\mathscr{C}_{\mathscr{L}\mathscr{R}}(L_2)$) coincide (we denote these partitions by $\mathscr{C}_{\mathscr{L}}(\mathscr{F})$ and $\mathscr{C}_{\mathscr{L}\mathscr{R}}(\mathscr{F})$).*
(2) *Let $\mathscr{F}$ be an $\mathfrak{H}$-facet. Then the cells of $\mathscr{C}_{\mathscr{L}}(\mathscr{F})$ (respectively $\mathscr{C}_{\mathscr{L}\mathscr{R}}(\mathscr{F})$) are the smallest subsets of W which are at the same time unions of cells of $\mathscr{C}_{\mathscr{L}}(C)$ (respectively $\mathscr{C}_{\mathscr{L}\mathscr{R}}(C)$) for all chamber C such that $\mathscr{F} \subset \bar{C}$ and stable by translation on the left (respectively on both sides) by $W_{\mathscr{F}}$.*

Remark 6.3. There are no restriction, such as non-negativity, on the weight functions in this conjecture. However, changing the sign of some values of the weight function L has no effect on the partition of W into cells (see Remark 5.2). Therefore, to prove the conjecture, it is enough to find a finite set of rational hyperplanes $\mathfrak{H}$ such that Statements (1) and (2) hold for all non-negative weight functions. Indeed, the conjecture will then hold for the minimal finite set of hyperplane which contain $\mathfrak{H}$ and which is stable under the action of the linear maps $\tau_i : V' \to V'$ defined by $\tau_i(\omega_i) = -\omega_i$ and $\tau_i(\omega_k) = \omega_k$ if $k \neq i$.

When only looking at the lowest two-sided cell, Statement (1) in the above conjecture is a direct consequence of Theorem 5.4 (see below). We denote by $\mathrm{Left}(\mathbf{c}_{\min}^L)$ the set of left cells of (W, S, L) lying in $\mathbf{c}_{\min}^L$.

Corollary 6.4 (of Theorem 5.4)**.** *Let W be an irreducible affine Weyl group. There exists a finite set of rational hyperplanes $\mathfrak{H}$ such that*

(1) *If L_1, L_2 are two weight functions belonging to the same $\mathfrak{H}$-facet $\mathscr{F}$ then $\mathbf{c}_{\min}^{L_1} = \mathbf{c}_{\min}^{L_2}$ (we denote this set $\mathbf{c}_{\min}^{\mathscr{F}}$) and $\mathrm{Left}(\mathbf{c}_{\min}^{L_1})$ and $\mathrm{Left}(\mathbf{c}_{\min}^{L_2})$ coincide (we denote this partition by $\mathrm{Left}(\mathbf{c}_{\min}^{\mathscr{F}})$).*

Proof. By Theorem 5.4, $\mathbf{c}_{\min}^L$ only depends on the values of L on the elements of the set $\mathscr{W}$ (see Section 3.1). But $\mathscr{W}$ is finite, hence it is easy to find a finite set of rational hyperplanes such that (1) holds. □

In the remaining of this paper, we will prove the following theorem which is concerned with the asymptotic behaviour of the lowest two-sided cell, hence providing new evidences for the semicontinuity conjecture.

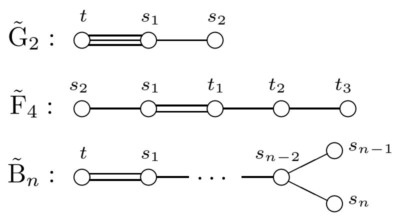
Theorem 6.5. *Let W be an irreducible affine Weyl group. There exists a finite set of rational hyperplanes $\mathfrak{H}$ satisfying property (1) in Corollary 6.4 and satisfying the following property: if $\mathscr{F}$ is an $\mathfrak{H}$-facet which is contained in H_{ω_i} for some i, then $\mathbf{c}_{\min}^{\mathscr{F}}$ is a union of two-sided cells of $\mathscr{C}_{\mathscr{L}\mathscr{R}}(C)$ and the left cells in $\mathrm{Left}(\mathbf{c}_{\min}^{\mathscr{F}})$ are union of left cells of $\mathscr{C}_{\mathscr{L}}(C)$ for all $\mathfrak{H}$-facets C such that $\mathscr{F} \subset \bar{C}$.*

Remark 6.6. (a) At the end of this Theorem, we really mean *for all* $\mathfrak{H}$*-facets* C *such that* $\mathscr{F} \subset \bar{C}$ and not *for all* $\mathfrak{H}$*-chambers* C *such that* $\mathscr{F} \subset \bar{C}$. It is clear that if it is true for all $\mathfrak{H}$-facets such that $\mathscr{F} \subset \bar{C}$ then it will also be true for all $\mathfrak{H}$-chambers C' such that $\mathscr{F} \subset \bar{C}'$. But the converse is true only if the semicontinuity conjecture holds!

(b) Arguying as in Remark 6.3, to prove the theorem, it is enough to find a finite set of rational hyperplanes such that (1) and (2) holds for non-negative weight functions and then take its closure under the action of the τ_i's.

(c) To prove the theorem, it is "enough" to show that the left cells in Left$(\mathbf{c}_{\min}^{\mathscr{F}})$ are union of left cells of $\mathscr{C}_{\mathscr{L}}(C)$ for all $\mathfrak{H}$-facets C such that $\mathscr{F} \subset \bar{C}$. Indeed $\mathbf{c}_{\min}^{\mathscr{F}}$ is stable by taking the inverse, hence, by Remark 5.1, if it is a union of left cells of $\mathscr{C}_{\mathscr{L}}(C)$, it is also a union of two-sided cells of $\mathscr{C}_{\mathscr{L}\mathscr{R}}(C)$. ■

6.2. **Irreducible affine Weyl groups of type** $\tilde{\mathrm{B}}_n$ **,** $\tilde{\mathrm{F}}_4$ **or** $\tilde{\mathrm{G}}_2$**.** Let (W, S) be an irreducible affine Weyl group of one of the following types



Then $|\bar{S}| = 2$. We set $\bar{S} = \{\mathbf{s}, \mathbf{t}\}$ where $\mathbf{s}$ (respectively $\mathbf{t}$) is the subset of S which consists of all the generators named with the letter s (respectively t). In this case, we will identify $\mathbb{Z}[\bar{S}]$ with $\mathbb{Z}^2$ through $(i, j) \longrightarrow i\mathbf{s} + j\mathbf{t}$.

Let $m_1, m_2 \in \mathbb{Q}_{>0}$. We define the following finite set of rational hyperplanes of V'

$$\mathfrak{H}(m_1, m_2) := \{H_{\mathbf{s}+m_1\mathbf{t}}, H_{\mathbf{s}-m_1\mathbf{t}}, H_{\mathbf{s}+m_2\mathbf{t}}, H_{\mathbf{s}-m_2\mathbf{t}}, H_{\mathbf{s}}, H_{\mathbf{t}}\}.$$

Note that $\mathfrak{H}(m, M)$ is stable under the actions of the τ_i (see Remark 6.3). In Figure 1, we draw the finite set of hyperplanes $\mathfrak{H}(m, M)$ for some choice of constants $M, m \in \mathbb{Q}_{>0}$.

The set of weight functions corresponding to the $\mathfrak{H}$-facet $\mathscr{C}_1$ of V' is

$$\{L \in \mathrm{Weight}(W, \Gamma) \mid L(\mathbf{s}) > m_1 \cdot L(\mathbf{t}) \text{ and } L(\mathbf{s}), L(\mathbf{t}) > 0\}.$$

Theorem 6.7. *There exists* $m_1, m_2 \in \mathbb{Q}_{>0}$ *such that Theorem 6.5 holds for* $\mathfrak{H}(m_1, m_2)$.

The proof of this theorem will be given in Section 8.2.

Remark 6.8. Note that this theorem is equivalent to Theorem 1. Let L be a non-negative weight function on W which vanishes on a proper non-empty subset S° of S. Then we have either $L \in H_{\mathbf{s}}$ or $H_{\mathbf{t}}$. Assume that $L \in H_{\mathbf{t}}$, that is $L(t) = 0$ for all $t \in \mathbf{t}$. Let $\mathbf{c}$ be an L-cell contained in $\mathbf{c}_{\min}^{L}$ (note that we have either $\mathbf{c} = \mathbf{c}_{\min}^{L}$ or $\mathbf{c} = N_\sigma^L$ for some $\sigma \in \Omega_0^L$). Then Theorem 1 implies that there exists an integer m such that for all weight functions L' such that $L'(\mathbf{s}) > m \cdot L'(\mathbf{t})$, the set $\mathbf{c}$ is a union of L'-cells. In other words, $\mathbf{c}$ is a union of L'-cell for all weight function L' in $\mathscr{C}_1$ (with $m_1 = m$). Conversely, if Theorem 6.9 holds, then Theorem 1 holds for all for any integer m greater than m_1. The case $L \in H_{\mathbf{s}}$ is similar using $m = 1/m_2$.

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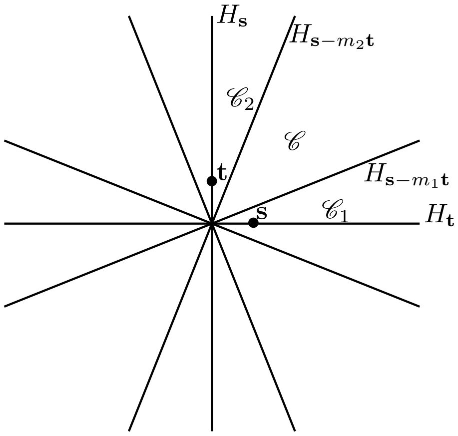
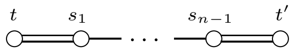
FIGURE 1. Set of hyperplanes $\bar{\mathfrak{H}}(m, M)$



6.3. **Irreducible affine Weyl group of type $\tilde{\mathrm{C}}$.** Let W be an irreducible affine Weyl group of type $\tilde{\mathrm{C}}$ with diagram as follows



Then $|\bar{S}| = 3$. We set $\bar{S} = \{\mathbf{t}, \mathbf{s}, \mathbf{t'}\}$ where $\mathbf{t} = \{t\}$, $\mathbf{s} = \{s_1, \dots, s_{n-1}\}$ and $\mathbf{t'} = \{t'\}$. In this case, we will identify $\mathbb{Z}[\bar{S}]$ with $\mathbb{Z}^3$ through $(i, j, k) \longrightarrow i\mathbf{t} + j\mathbf{s} + k\mathbf{t'}$.

Let $\mathbf{m} = (m_1, \dots, m_6) \in \mathbb{Q}^6_{>0}$. We define the following finite set of rational hyperplanes of V'

$$\mathfrak{H}(\mathbf{m}) := \{H_{\mathbf{s}}, H_{\mathbf{t}}, H_{\mathbf{t'}}, H_{\mathbf{t}-\mathbf{t'}}, H_{\mathbf{t}-m_1\mathbf{s}}, H_{\mathbf{t'}-m_2\mathbf{s}}, H_{\mathbf{t}-m_3(\mathbf{s}+\mathbf{t'})},$$
$$H_{\mathbf{t'}-m_4(\mathbf{s}+\mathbf{t})}, H_{(\mathbf{t}-\mathbf{t'})\pm m_5\mathbf{s}}, H_{(\mathbf{t}+\mathbf{t'})-m_6\mathbf{s}}\}.$$

We then set $\bar{\mathfrak{H}}(\mathbf{m})$ to be the closure of $\mathfrak{H}(\mathbf{m})$ under the actions of the τ_i (see Remark 6.3). In Figure 2, we draw the finite set of hyperplanes $\mathfrak{H}(\mathbf{m})$ for some choice of constants $\mathbf{m} \in \mathbb{Q}^6_{>0}$. We intersect on the affine hyperplane with equation $\mathbf{s}^*(\mu) = 1$. We put an arrow in a chamber $\mathscr{C}$ pointing towards $\mathscr{F} \subset V'$ to indicate that $\overline{\mathscr{C}} \cap \mathscr{F} \neq \emptyset$.
The chamber $\mathscr{C}_1$ corresponds to the weight functions

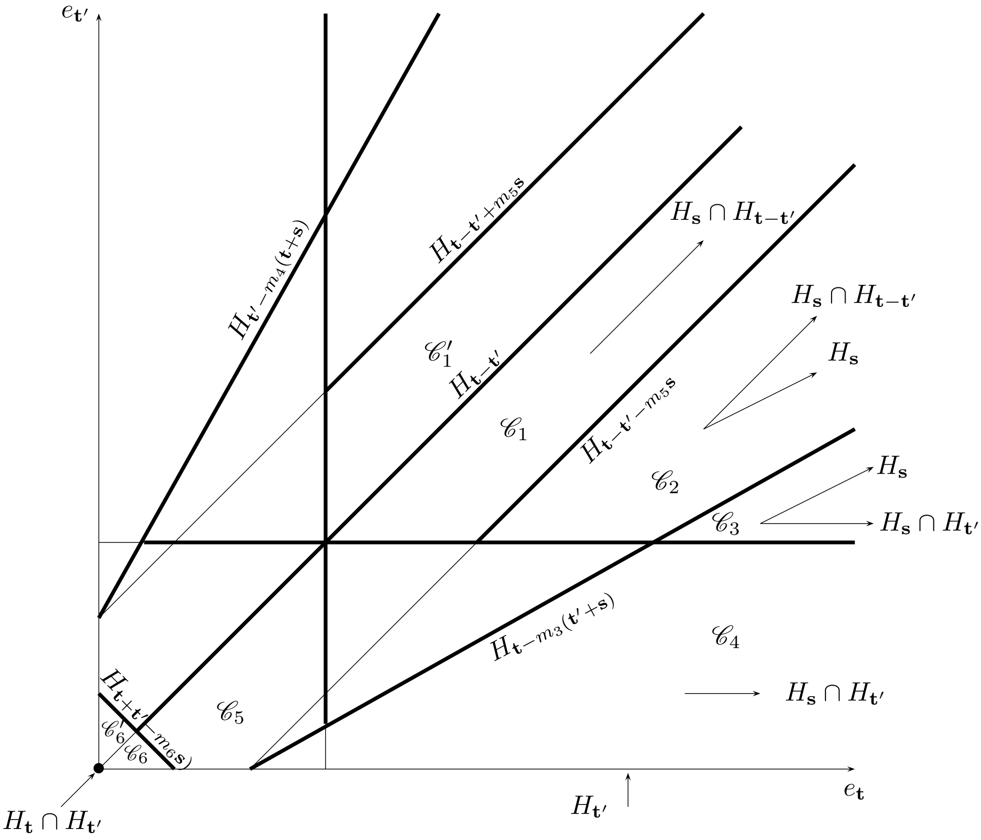
$$\{L \mid L(\mathbf{t}) > L(\mathbf{t'}), L(\mathbf{t'}) > m_2 \cdot L(\mathbf{s}), L(\mathbf{t}) - L(\mathbf{t'}) < m_5 \cdot L(\mathbf{s})\}.$$

Theorem 6.9. *There exist $\mathbf{m} \in \mathbb{Q}^6_{>0}$ such that Theorem 6.5 holds for $\bar{\mathfrak{H}}(\mathbf{m})$.*

Remark 6.10. Theorem 6.9 is stronger than Theorem 1. Indeed, let L be a non-negative weight function on W which vanishes on a proper non-empty subset S° of S. Then, Theorem 1 only gives us informations on weight functions L' satisfying $L'(s^+) = L(s^+)$ for all $s^+ \in S^+$. However, in Theorem 6.9, we may have $L'(s^+) \neq L(s^+)$ for some $s^+ \in S^+$. For instance, if $L(\mathbf{t}) = L(\mathbf{t'}) > 0$ and $L(\mathbf{s}) = 0$, then Theorem 6.9 implies that for all weight functions L' such that

$$\{L' \mid L'(\mathbf{t}) \geq L'(\mathbf{t'}), L'(\mathbf{t'}) > m_2 \cdot L'(\mathbf{s}), L'(\mathbf{t}) - L'(\mathbf{t'}) < m_5 \cdot L'(\mathbf{s})\}$$

FIGURE 2. Set of hyperplanes $\mathfrak{H}(\mathbf{m})$



any L-cell contained in $\mathbf{c}_{\min}^L$ is a union of L'-cells. But Theorem 1 does not tell us anything in this case as we do not have $L'(\mathbf{t}) = L(\mathbf{t}) = L(\mathbf{t}') = L'(\mathbf{t}')$. We now show in more details that Theorem 6.9 implies Theorem 1.

(i) Assume that $S^\circ = \mathbf{s}$ and $L(\mathbf{t}) = L(\mathbf{t}')$. Let $m \geq \max\{m_1, m_2\}$. Then if L' satisfies $L'(\mathbf{t}) = L'(\mathbf{t}') > mL'(\mathbf{s})$ we must have $L' \in \bar{\mathscr{C}}_1 \cap \bar{\mathscr{C}}_1'$. But $\bar{\mathscr{C}}_1 \cap \bar{\mathscr{C}}_1'$ contains L in its closure, therefore Theorem 6.9 tells us that any L-cells included in $\mathbf{c}_{\min}^L$ is a union of L' cells (see also Claim 8.10).

(ii) Assume that $S^\circ = \mathbf{s}$ and $L(\mathbf{t}) > L(\mathbf{t}')$. Let m be such that

$$\frac{L(\mathbf{t}) - L(\mathbf{t}')}{m_5} m > L(\mathbf{t}) \text{ and } m > m_2.$$

Then if L' satisfies $L'(\mathbf{t}) = L(\mathbf{t}) > mL'(\mathbf{s})$ and $L'(\mathbf{t}') = L(\mathbf{t}') > mL'(\mathbf{s})$ we must have

$$L'(\mathbf{t}') - L'(\mathbf{t}) > m_5 L'(\mathbf{s}) \text{ and } L'(\mathbf{t}') > m_2 L'(\mathbf{s})$$

that is $L' \in \mathscr{C}_2 \cup \mathscr{C}_3 \cup (\overline{\mathscr{C}_2} \cap \overline{\mathscr{C}_3})$. Then Theorem 1 then follows from Theorem 6.9 (see also Claim 8.8).

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(iii) Assume that $S^\circ = \mathbf{s} \cup \mathbf{t}'$. Let m be such that $m > 2m_3$. Then if L' satisfies $L'(\mathbf{t}) = L(\mathbf{t}) > mL'(\mathbf{s})$ and $L'(\mathbf{t}) = L(\mathbf{t}) > mL'(\mathbf{t}')$ we must have

$$2L'(\mathbf{t}) > 2m_3(L'(\mathbf{t}') + L'(\mathbf{s}))$$

that is $L' \in \mathscr{C}_3 \cup \mathscr{C}_4 \cup (\overline{\mathscr{C}_3} \cap \overline{\mathscr{C}_4})$. Then Theorem 1 then follows from Theorem 6.9 (see also Claim 8.4).

(iv) Assume that $S^\circ = \mathbf{t}' \cup \mathbf{t}$ that is $L \in H_{\mathbf{t}'} \cap H_{\mathbf{t}}$. Let m be such that $m > \frac{2}{m_6}$. Then if L' satisfies $L'(\mathbf{s}) = L(\mathbf{s}) > mL'(\mathbf{t})$ and $L'(\mathbf{s}) = L(\mathbf{s}) > mL'(\mathbf{t}')$ we must have $m_6 L'(\mathbf{s}) > L'(\mathbf{t}) + L'(\mathbf{t}')$. In other words $L' \in \mathscr{C}_6$ and Theorem 1 follows from Theorem 6.9 (see also Claim 8.6).

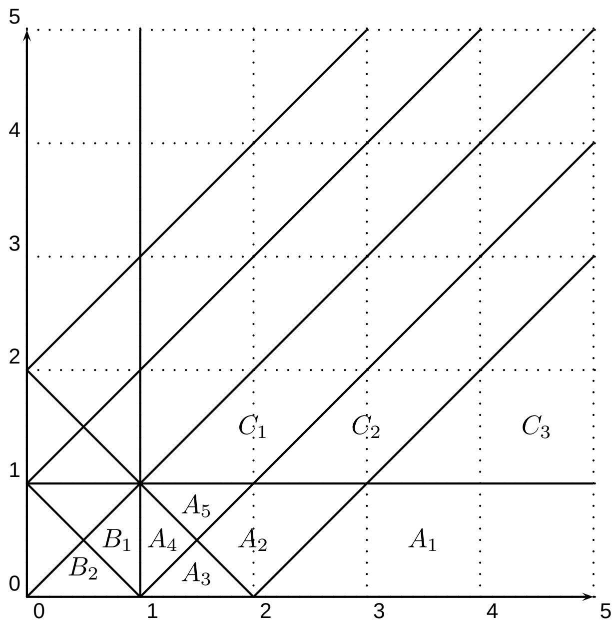
(v) Assume that $S^\circ = \mathbf{t}'$. Then the result is trivial since $\mathbf{c}^L_{\min} = \mathbf{c}^{L'}_{\min}$ for all L, L' such that
- $L(\mathbf{t}), L(\mathbf{t}') > L(\mathbf{t}') = 0$
- $L'(\mathbf{s}) = L(\mathbf{s}), L'(\mathbf{t}) = L(\mathbf{t}) > L'(\mathbf{t}')$ and $L'(\mathbf{t}') > 0$.

Remark 6.11. In this remark, we explain why we do need an hyperplane of the form $H_{(\mathbf{t}-\mathbf{t}')-m_5\mathbf{s}}$ in our finite set of hyperplanes in Theorem 6.9, eventhough the lowest two-sided cell is the same whether the weight function lies in $\mathscr{C}_1$ or $\mathscr{C}_2$. Assume that W is of type $\tilde{\mathrm{C}}_2$. It is shown in [11, 12] that Conjecture 6.2 holds for the following set of hyperplanes

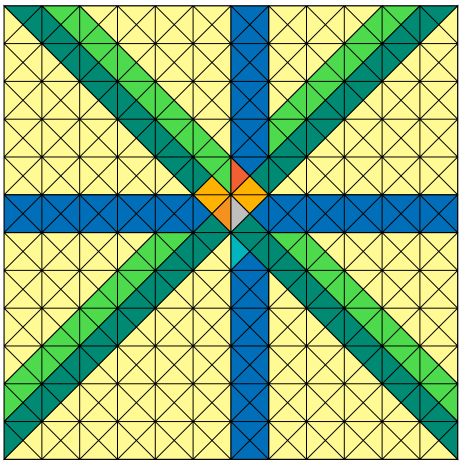
$$\mathfrak{H} := \{\mathscr{H}_{(1,0,0)}, \mathscr{H}_{(0,1,0)}, \mathscr{H}_{(0,0,1)}, \mathscr{H}_{(\varepsilon,\varepsilon,0)}, \mathscr{H}_{(0,\varepsilon,\varepsilon)}, \mathscr{H}_{(\varepsilon,0,\varepsilon)}, \mathscr{H}_{(\varepsilon,\varepsilon,\varepsilon)}, \mathscr{H}_{(\varepsilon,2\varepsilon,\varepsilon)}\}.$$

We describe this set of hyperplanes in Figure 3, projecting on the affine hyperplane with equation $\mathbf{s}^*(\mu) = 1$.

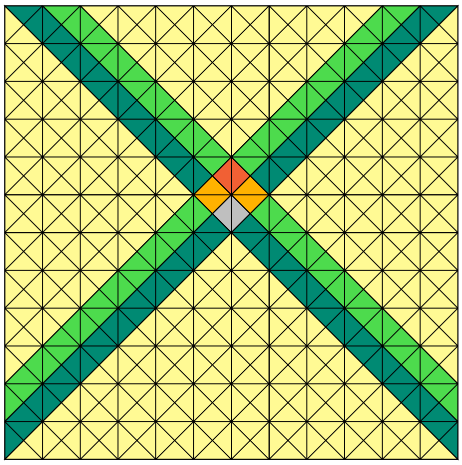
FIGURE 3. Hyperplanes in $\mathfrak{H}$.



In the figure below, we show the partition of W into cells for a weight function in C_1 and for a weight function L' such that $L'(\mathbf{s}) = 0$ and $L'(\mathbf{t}) > L'(\mathbf{t}') > 0$. The set $\mathbf{c}_{\min}^{L'}$ consists of the yellow alcoves. We see that $\mathbf{c}_{\min}^{L'}$ is NOT a union of cells of (W, S, L). Hence, we need the hyperplane $H_{(\mathbf{t}-\mathbf{t}')-m_5\mathbf{s}}$ so that there are no weight function L' such that $L'(\mathbf{s}) = 0$ and $L'(\mathbf{t}) > L'(\mathbf{t}') > 0$ and which lies in the closure of $\mathscr{C}_1$ (see Figure 2).



Partition of W into cells for $L \in C_1$.



Partition of W into cells for L'.

7. Proof of Theorem 6.5 in the generic setting

7.1. **Hypothesis and notation.** Let (W, S) be an irreducible affine Weyl group generated by S. Let $S = S^\circ \cup S^+$ be a partition of S such that no element of S° is conjugate to an element of S^+ and $S^\circ, S^+ \neq S$. For a subset I of S we set $I^\circ = I \cap S^\circ$ and $I^+ = I \cap S^+$. We denote by $\bar{S}$ the set of conjugacy classes in S in W and we set

$$\bar{S}^+ = \{\omega \in \bar{S} | \omega \subset S^+\} \text{ and } \bar{S}^\circ = \{\omega \in \bar{S} | \omega \subset S^\circ\}$$

As in the previous section, $\mathbb{Z}[\bar{S}]$ denotes the free $\mathbb{Z}$-module with basis $\bar{S}$ and $V' = \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}[\bar{S}]$. We identify $\mathbb{Z}[\bar{S}]$ with $\mathbb{Z}^{|S|}$.

A subset X of $\mathbb{Z}[\bar{S}]$ is called positive if the following three conditions hold

(1) $\mathbb{Z}[\bar{S}] = X \cup (-X)$;
(2) $X + X \subset X$;
(3) $X \cap (-X)$ is a subgroup of $\mathbb{Z}[\bar{S}]$.

Any positive subset X defines a total order $\leq_X$ on $\mathbf{\Gamma} := \mathbb{Z}[\bar{S}]/(X \cap (-X))$ simply by setting

$$\gamma \geq_X 0 \Longleftrightarrow \text{all the representatives of } \gamma \text{ belong to } X.$$

We briefly explain how to classify all the positive subsets of $\mathbb{Z}[\bar{S}]$. Let $\mathscr{P}(\mathbb{Z}[\bar{S}])$ be the set of all sequences $(\varphi_1, \ldots, \varphi_d)$ such that φ_i is a non-zero linear form defined on $\ker(\varphi_{i-1}) \subset V'$, with the convention that $\varphi_0 = 0$. Then we can associate to $\Phi = (\varphi_1, \ldots, \varphi_d) \in \mathscr{P}(\mathbb{Z}[\bar{S}])$ a positive subset $\mathrm{Pos}(\Phi)$ of $\mathbb{Z}[\bar{S}]$ by setting

$$\mathrm{Pos}(\Phi) = \{\gamma \in \mathbb{Z}[\bar{S}] \mid \exists\, 0 \leq k \leq d-1; \gamma \in \ker \varphi_k \text{ and } \varphi_{k+1}(\gamma) > 0\} \cup \ker \varphi_d.$$

It can be shown that all positive subsets can be obtained this way. We denote by $\mathscr{P}_+(\mathbb{Z}[\bar{S}])$ the subset of $\mathscr{P}(\mathbb{Z}[\bar{S}])$ which consists of all sequences $\Phi = (\varphi_1, \ldots, \varphi_d)$

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such that

$$\varphi_1 = \sum_{\omega \in \bar{S}^+} \omega^* \quad \text{and} \quad \varphi_k(\omega) > 0 \text{ for all } \omega \in \bar{S}.$$

Hypothesis. *From now on and until the end of this section, we fix a positive subset $X = Pos(\Phi)$ such that $\Phi \in \mathscr{P}_+(\mathbb{Z}[\bar{S}])$. In type $\tilde{\mathrm{C}}$ we assume that $\mathbf{t} \geq \mathbf{t}'$.*

To simplify the notation, we will denote by $\leq$ the total order on $\mathbf{\Gamma}$ instead of $\leq_X$.

Example 7.1. Assume that W is of type $\tilde{\mathrm{B}}_r$, $\tilde{\mathrm{F}}_4$ or $\tilde{\mathrm{G}}_2$ and let $\bar{S}^+ = \{\mathbf{s}\}$ and $\bar{S}^\circ = \{\mathbf{t}\}$. Let $X \in \mathscr{P}_+(\mathbb{Z}[\bar{S}])$. Then we have $\varphi_1 = \mathbf{s}^*$ and $\ker(\varphi_1) = \mathbb{R}\mathbf{t}$. Since we assumed that $\varphi_2(\mathbf{t}) > 0$ we must have $\varphi_2 = \kappa \mathbf{t}^*$ where $\kappa > 0$. It follows that $\mathbf{\Gamma} = \mathbb{Z}[\bar{S}]$ and that the order on $\mathbf{\Gamma}$ is simply the lexicographic order:

$$(i,j) < (i',j') \Longleftrightarrow i < i' \text{ or } (i = i' \text{ and } j < j').$$

Assume that W is of type $\tilde{\mathrm{C}}_r$ and that $S^+ = \mathbf{t}$. Let $\varphi_1 = \mathbf{t}^*$ and φ_2 be defined by $\varphi_2(0,j,k) = bj + ck$ for $b, c \in \mathbb{N}$. Then we have $\ker(\varphi_2) = \langle (0,-c,b) \rangle$. Finally we define φ_3 by $\varphi_3(0,-c,b) = 1$ and we extend it by linearity. Then the order associated to $(\varphi_1, \varphi_2, \varphi_3)$ can be describe as follows:

$$\{(i,j,k) \mid (i,j,k) > 0\} = \{(i,j,k) \mid i > 0\} \cup \{(0,j,k) \mid bj + ck > 0\} \cup \{(0,-kc,kb) \mid k > 0\} \blacksquare$$

Let $\mathbf{L} : W \longrightarrow \mathbf{\Gamma}$ be the weight function defined by $\mathbf{L}(s) = \omega_i$ if $s \in \omega_i$. Let $\mathbf{A}$ be the group algebra of $\mathbf{\Gamma}$ over $\mathbb{Z}$. Recall that we use the exponential notation for $\mathbf{A}$

$$\mathbf{A} = \bigoplus_{\gamma \in \mathbf{\Gamma}} \mathbb{Z} v^\gamma \text{ where } v^\gamma . v^{\gamma'} = v^{\gamma + \gamma'}.$$

Let $\mathbf{H} = \mathscr{H}(W, S, \mathbf{L})$ be the associated Hecke algebra. We will denote by $\mathbf{T}_x$ the element of the standard bases of $\mathbf{H}$, by $\mathbf{C}_x$ the elements of the Kazhdan-Lusztig basis of $\mathbf{H}$ and by $\mathbf{P}_{x,y}, \mathbf{M}_{x,y}$ the polynomials in $\mathbf{A}$ defined in Section 5.1. We set

$$\mathbf{A}_{<0} = \bigoplus_{\gamma < 0} \mathbb{Z} v^\gamma \quad , \quad \mathbf{A}_{\geq 0} = \bigoplus_{\gamma \geq 0} \mathbb{Z} v^\gamma$$

and

$$\mathbf{H}_{<0} = \bigoplus_{w \in W} \mathbf{A}_{<0} \mathbf{T}_w.$$

We denote by $^+ : \mathbf{\Gamma} \to \mathbf{\Gamma}$ (respectively $^\circ$) the map induced by the projection of $\mathbb{Z}[\bar{S}]$ onto $\oplus_{\omega \in \bar{S}^+} \mathbb{Z}\omega$ (respectively $\oplus_{\omega \in \bar{S}^\circ} \mathbb{Z}\omega$). For $a = \sum_{\gamma \in \mathbf{\Gamma}} a_\gamma v^\gamma \in \mathbf{A}$ we define

$$\deg(a) = \max\{\gamma \in \mathbf{\Gamma} \mid a_\gamma \neq 0\}.$$

We will write $\deg^+(a)$ instead of $\deg(a)^+$.

Remark 7.2. Note that if $a = \sum a_\gamma v^\gamma$ satisfies $\varphi_1(\deg^+(a)) < 0$, then $a \in \mathbf{A}_{<0}$. Indeed, if $a_\gamma \neq 0$, then $\gamma = \gamma^+ + \gamma^\circ$ where $\gamma^+ \leq \deg^+(a)$. Applying φ_1 yields $\varphi_1(\gamma) = \varphi_1(\gamma^+) + 0 \leq \varphi_1(\deg^+(a)) < 0$ that is $\gamma < 0$ as required. $\blacksquare$

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In this section we will have to distinguish the following cases. (We keep the notation of Section 6.2 and 6.3.)

Case 1. W is of type $\tilde{\mathrm{B}}_r$, $\tilde{\mathrm{F}}_4$ or $\tilde{\mathrm{G}}_2$.

Case 2. W is of type $\tilde{\mathrm{C}}_r$, $\bar{S}^+ \neq \{\mathbf{t}, \mathbf{t}'\}$.

Case 3. W is of type $\tilde{\mathrm{C}}_r$, $\bar{S}^+ = \{\mathbf{t}, \mathbf{t}'\}$ and $(-1, k, 1) < 0$ for all $k > 0$.

Case 4. W is of type $\tilde{\mathrm{C}}_r$, $\bar{S}^+ = \{\mathbf{t}, \mathbf{t}'\}$ and $(-1, k, 1) > 0$ for some $k > 0$.

If we are in Case (1)–(3), we define the weight function $\mathbf{L}^+$ by

$$\mathbf{L}^+(\omega) = \mathbf{L}(\omega) \text{ if } \omega \in \bar{S}^+ \text{ and } \mathbf{L}^+(\omega) = 0 \text{ if } \omega \in \bar{S}^\circ.$$

Hence we have $\mathbf{L}^+(w) = (\mathbf{L}(w))^+$.

If we are in Case (4), we define the weight function $\mathbf{L}^+$ by

$$\mathbf{L}^+(t) = \mathbf{L}^+(t') = \mathbf{t} \text{ and } \mathbf{L}^+(s) = 0 \text{ if } s \in \mathbf{s}.$$

Note that in this case, we have $\mathbf{L}^+(w) \neq (\mathbf{L}(w))^+$.

Recall that, for a weight function L, we say that a hyperplane H of direction α is of maximal L-weight if $L_H = L_\alpha$ where $L_\alpha = \max_{n\in\mathbb{Z}} H_{\alpha,n}$. Let $H, H' \in \mathscr{F}$. Then we have either [4, Lemma 2.2] (a) $L_H = L_{H'}$ or (b) W is of type $\tilde{\mathrm{C}}_r$, H contains a face of type t_1 and H' a face of type t_2 and $\{t_1, t_2\} = \{t, t'\}$ and $L(t) \neq L(t')$.

Lemma 7.3. *Let λ be a $\mathbf{L}^+$-special point and let H be an hyperplane orthogonal to α which contains λ and such that $\mathbf{L}^+_H > 0$. Then in Case 1–3, $\mathbf{L}_H$ is of maximal $\mathbf{L}$-weight. In Case 4, we may have $\mathbf{L}_H = \mathbf{t}' < \mathbf{t} = \mathbf{L}_\alpha$.*

Proof. Let λ be a $\mathbf{L}^+$-special point and let H be an hyperplane which contains λ and such that $\mathbf{L}^+_H > 0$. In Case 1, the result is clear since any hyperplane is of maximal weight. In Case 2, since $\mathbf{t} > \mathbf{t}'$ and $\bar{S}^+ \neq \{\mathbf{t}, \mathbf{t}'\}$, we must have $\mathbf{t}' \subset \bar{S}^\circ$. If $\mathbf{t} \in \bar{S}^\circ$, then since $\mathbf{L}^+_H > 0$ we have $L_H = \mathbf{s}$ and H is of maximal weight. If $\mathbf{t}' \in \bar{S}^+$, then since λ is a $\mathbf{L}^+$-special we have $S_\lambda = \{t, s_1, \ldots, s_{n-1}\}$ and we have $L_H = \mathbf{t}$ or $L_H = \mathbf{s}$ and the result follows. In Case 3, since $\mathbf{t} > \mathbf{t}'$, we must have $S_\lambda = \{t, s_1, \ldots, s_{n-1}\}$ and the result follows as above. Finally in Case 4, the result is clear. □

Remark 7.4. In Case 3, if $\deg^+(a) \leq (-1, 0, 1)$ then $a \in \mathbf{A}_{<0}$. Indeed, we have $\deg(a) = \deg^+(a) + \gamma$ where $\gamma \in \ker(\varphi_1)$. But then $\gamma = (0, k, 0)$ for some $k \in \mathbb{Z}$. Therefore $\deg(a) = (-1, k, 1) < 0$. ■

7.2. **Generalized induction of Kazhdan-Lusztig cells.** Recall that every element $w \in \mathbf{c}^{\mathbf{L}^+}_{\min}$ can be uniquely written under the form $x_w a_w w^\circ_{\lambda_\sigma} b_\sigma$ where $\sigma \in \Omega^{\mathbf{L}^+}_0$, $a_w \in W_{S^\circ_{\lambda_\sigma}}$ and $x_w \in X_{\lambda_\sigma}$. For $\sigma \in \Omega^{\mathbf{L}^+}_0$, we set

$$N^{\mathbf{L}^+}_\sigma := \{w \in W \mid w = x a w_{\lambda_\sigma} b_\sigma, x \in X_{\lambda_\sigma}, a \in W_{S^\circ_{\lambda_\sigma}}\},$$
$$U_\sigma := \{w \in W \mid w = a w_{\lambda_\sigma} b_\sigma, a \in W_{S^\circ_{\lambda_\sigma}}\}$$

and

$$N_\sigma^{\leq} := \bigcup_{\sigma' \in \Omega_0^L, b_{\sigma'} \leq b_\sigma} N_{\sigma'}^{\mathbf{L}^+},$$

$$U_\sigma^{\leq} := \bigcup_{\sigma' \in \Omega_0^L, b_{\sigma'} \leq b_\sigma} U_{\sigma'}.$$

For $u = aw_{\lambda_\sigma}^\circ y_\sigma \in U_\sigma$, we set $X_u := X_{\lambda_\sigma}$.

Theorem 7.5. *For all $\sigma \in \Omega_0^{\mathbf{L}^+}$, the set $U_\sigma^{\leq}$ together with the collection of subsets $\{X_u \mid u \in U_\sigma^{\leq}\}$ satisfy the following condition*

I1. *for all $u \in U_\sigma^{\leq}$, we have $e \in X_u$,*
I2. *for all $u \in U_\sigma^{\leq}$ and $x \in X_u$ we have $\ell(xu) = \ell(x) + \ell(u)$,*
I3. *for all $u, v \in U_\sigma^{\leq}$ such that $u \neq v$ we have $X_u u \cap X_v v = \emptyset$,*
I4. *the submodule $\mathscr{M} := \langle \mathbf{T}_x \mathbf{C}_u \mid u \in U_\sigma^{\leq},\ x \in X_u \rangle_{\mathscr{A}} \subseteq \mathscr{H}$ is a left ideal.*
I5. *for all $v \in U_\sigma^{\leq}$ and $y \in X_v$ we have*

$$\mathbf{T}_y \mathbf{C}_v \equiv \mathbf{T}_{yv} + \sum_{\substack{\ell(xu) < \ell(yv) \\ u \in U_\sigma^{\leq}}} a_{xu,yv} \mathbf{T}_{xu} \bmod \mathbf{H}_{<0}$$

Assuming that this theorem holds, we would get, using the Generalised Induction Theorem [11, Theorem 6.3], that the set

$$N_\sigma^{\leq} = \{xu \mid u \in U_\sigma^{\leq}, x \in X_u\}$$

is a left ideal of $(W, \mathbf{L})$ (i.e. $y \leq_{\mathscr{L}} w \in N_\sigma^{\leq}$ implies $y \in N_\sigma^{\leq}$) for all $\sigma \in \Omega_0^{\mathbf{L}^+}$. In particular it would be a union of left cells. Then by an easy induction on the length of $b_\sigma \in W$, we would get that each N_σ is a union of left cells. In turn, since $\mathbf{c}_{\min}^{\mathbf{L}^+}$ is stable by taking the inverse, this would implie that $\mathbf{c}_{\min}^{\mathbf{L}^+}$ is indeed a union of two-sided cells of $(W, \mathbf{L})$.

Remark 7.6. Condition **I5** is stated slightly differently in [11, §6]: *for all $v \in U$, $y \in X_v$ we have*

$$T_y C_v \equiv T_{yv} + \sum_{xu \sqsubset yv} a_{xu,yv} T_x C_u \bmod \mathscr{H}_{<0}$$

where $\sqsubset$ denotes a preorder such that $xu \sqsubset yv$ implies $\ell(xu) < \ell(yv)$. It is a straightforward induction on $\ell(xu)$ to show that those two conditions are equivalent. ■

7.3. **Kazhdan-Lusztig Polynomials and M-polynomials.** Let $x \in W$ and let $I \subset S$ be such that W_I is finite. There exist unique $x' \in W_I$ and $d_x \in X_I^{-1}$ such that $x = x' d_x$. Next $x' \in W_I$ can be uniquely written as $x' = au$ where $a \in W_{I^\circ}$ and u has minimal length in the coset $W_{I^\circ} x'$ of W_I. We will write $x = a_x^I u_x^I d_x^I$ for this decomposition or simply $x = a_x u_x d_x$ if it is clear from the context what the subset I should be. We denote by w_I° (not to confuse with w_{I°) the element of minimal length in the coset $W_{I^\circ} w_I$.

Remark 7.7. Note that, for all $a \in W_{I^\circ}$ and all $t \in I^+$, we must have $taw_I^\circ < aw_I^\circ$ since the number of elements of I^+ appearing in any reduced expression of aw_I° is maximal. ■

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Lemma 7.8. *Let $I \subset S$ be such that W_I is finite and let $y \in W$ be such that $y = aw_I^\circ z$ where $a \in W_{I^\circ}$ and $z \in X_I^{-1}$. Let $x = a_x u_x d_x < y$. Then*

$$\deg^+(\mathbf{P}_{x,y}) \leq \mathbf{L}(u_x)^+ - \mathbf{L}(w_I^\circ)^+$$

Furthermore, for all $s \in S^\circ$ such that $sx < x < y < sy$ we have

$$\mathbf{M}^s_{x,y} \neq 0 \Longrightarrow u_x = w_I^\circ.$$

Proof. We prove the result by induction. To this end, to any element $x, y \in W$ satisfying the hypothesis of the lemma, we associate a pair

$$\mathscr{P}(x,y) := (\ell(y) - \ell(x), \ell_0 - \ell(a_x))$$

where $\ell_0 = \ell(w_{I^\circ})$ and $x = a_x u_x d_x$. We order such pairs by the usual lexicographic order. Let $x < y$. If $\mathbf{L}(u_x)^+ = \mathbf{L}(w_I^\circ)^+$ then the result is clear. Thus we may assume that $\mathbf{L}(u_x)^+ < \mathbf{L}(w_I^\circ)^+$.

First assume that there exists $t \in I^+$ such that $tx > x$. Then, since $ty < y$, we have

$$\mathbf{P}_{x,y} = v^{-\mathbf{L}(t)}\mathbf{P}_{tx,y}$$

and the result follows by induction.

Next assume that $tx < x$ for all $t \in I^+$. Since we supposed that $\mathbf{L}(u_x)^+ < \mathbf{L}(w_I^\circ)^+$, there exists $s \in I_0$ such that $sx > x$. If $sy < y$ then

$$\mathbf{P}_{x,y} = v^{-\mathbf{L}(s)}\mathbf{P}_{sx,y}$$

and the result follows by induction since $\ell_0 - \ell(sa_x) = \ell_0 - \ell(a_x) - 1$.
If $sy > y$ then we have

$$(\dagger) \qquad \mathbf{P}_{sx,sy} = v^{\mathbf{L}(s)}\mathbf{P}_{sx,y} + \mathbf{P}_{x,y} - \sum_{\substack{sx \leq z < y \\ sz < z}} \mathbf{P}_{sx,z}\mathbf{M}^s_{z,y}.$$

By induction we know that

$$\deg^+(v^{\mathbf{L}(s)}\mathbf{P}_{sx,y}) \leq \mathbf{L}(u_x)^+ - \mathbf{L}(w_I^\circ)^+.$$

Further if $\mathbf{M}^s_{z,y} \neq 0$ then $\mathbf{L}(u_z)^+ = \mathbf{L}(w_I^\circ)^+$ where $z = a_z u_z d_z$. We know that $\deg(\mathbf{M}^s_{x,y}) < \mathbf{L}(s)$ (see [14, §6.3]). Thus if $\mathbf{M}^s_{z,y} \neq 0$, we get using the induction hypothesis

$$\deg_+(\mathbf{P}_{sx,z}\mathbf{M}^s_{z,y}) \leq \mathbf{L}(u_x)^+ - \mathbf{L}(w_I^\circ)^+.$$

Now we have

$$\mathscr{P}(sx,sy) = (\ell(sy) - \ell(sx), \ell_0 - \ell(sx_0)) = (\ell(y) - \ell(x), \ell_0 - \ell(a_x) - 1) < \mathscr{P}_{x,y}.$$

Hence by induction

$$\deg^+(\mathbf{P}_{sx,sy}) \leq \mathbf{L}(u_{sx})^+ - \mathbf{L}(w_I^\circ)^+ = \mathbf{L}(u_x)^+ - \mathbf{L}(w_I^\circ)^+.$$

The result follows using (†). □

Remark 7.9. The same proof can easily be generalised to any Coxeter group (W, S). Assume that W is finite and let w_S be the longest element of W. Using the previous lemma, we can show that

$$W_{S^\circ} \quad \text{and} \quad W_{S^\circ}w_0$$

are union of cells of $(W, S, \mathbf{L})$. Indeed, let $y \in W_{S^\circ}w_S$ and let $w \in W_{S^\circ}$ be such that $y = ww_S^\circ$ and $w{\bullet}w_S^\circ$. Let $x \in W$ be such that $x \leq_{\mathscr{L}} y$. We may assume that there exists $s \in S$ such that $sx < x < y < sy$ and $\mathbf{M}^s_{x,y} \neq 0$. Note that $y < sy$ we implies

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that $s \in S^\circ$. Write $x = a_x^S u_x^S d_x^S$. Since $\mathbf{M}^s_{x,y} \neq 0$ the previous lemma implies that $\mathbf{L}(u_x^S) = \mathbf{L}(w_S^\circ)$ where which in turn implies that $x \in W_{S^\circ} w_S$. Thus we have shown that $W_{S^\circ} w_S$ is a left ideal of W and thus a union of (left, right and two-sided) cells of $(W, \mathbf{L})$. Multiplying by the longest element sends (left, right and two-sided) cells to (left, right and two-sided) cells thus we get that W_{S° is also a union of (left, right, two-sided) cells of $(W, \mathbf{L})$. This argument provides an alternative proof of Theorem 1.1 in [10] when W is finite. ■

7.4. **Multiplication of the standard basis.** We set

$$\mathbf{T}_x \mathbf{T}_y = \sum_w \mathbf{f}_{x,y,w} \mathbf{T}_w \text{ where } \mathbf{f}_{x,y,w} \in \mathbf{A}.$$

Following [9, §2.3], we want to study the degree of the polynomials $\mathbf{f}_{x,y,w}$. We will need more precise result than in [9], but the method of the proof is similar.

We introduce some notation. For $\alpha \in \Phi^+$, we set $\mathscr{F}_\alpha = \{H_{\alpha,n} \mid n \in \mathbb{Z}\}$. For $x, y \in W$ we set

$$H_{x,y} = \{H \in \mathscr{F} \mid H \in H(A_0, yA_0) \cap H(yA_0, xyA_0)\},$$
$$I_{x,y} = \{\alpha \in \Phi^+ \mid H_{x,y} \cap \mathscr{F}_\alpha \neq \emptyset\}.$$

For $\alpha \in I_{x,y}$ we set

$$c^{\mathbf{L}}_{x,y}(\alpha) = \max_{H \in H_{x,y} \cap \mathscr{F}_\alpha} \mathbf{L}_H.$$

Let

$$c^{\mathbf{L}}_{x,y} = \sum_{\alpha \in I_{x,y}} c^{\mathbf{L}}_{x,y}(\alpha).$$

The following two lemmas can be found in [9].

Lemma 7.10. *Let $x, y \in W$ and $s \in S$ be such that $xs > x$. Then*

$$I_{x,sy} \subseteq I_{xs,y}.$$

Lemma 7.11. *Let $x, y \in W$ and $s \in S$ be such that $xs > x$ and $sy < y$. Let $\alpha \in \Phi^+$ and $n \in \mathbb{Z}$ be such that $H_{\alpha,n}$ is the unique hyperplane separating yA_0 and syA_0. There is an injective map φ from $I_{x,y}$ to $I_{xs,y} - \{\alpha\}$. Furthermore if $\beta \in I_{x,y}$ we have either $\varphi(\beta) = \beta$ or $\varphi(\beta) = \pm\sigma_{H_{\alpha,0}}(\beta)$.*

Using these two lemmas, one can obtain the following bound on the degree of $f_{x,y,z}$ in terms of x and y.

Theorem 7.12. *We have* $\deg(\mathbf{f}_{x,y,z}) \leq c^{\mathbf{L}}_{x,y}$ *for all $z \in W$.*

Note that this implies that $\deg^+(\mathbf{f}_{x,y,z}) \leq (c^{\mathbf{L}}_{x,y})^+$.

Let $x = s_N \dots s_1$ be a reduced expression of x. We denote $\mathfrak{I}_{x,y}$ the collection of all subsets $I = \{i_1, \dots, i_p\}$ such that $1 \leq i_1 < \dots < i_p \leq k$ and

$$s_{i_t} \dots \hat{s}_{i_{t-1}} \dots \hat{s}_{i_1} \dots s_1 y < \hat{s}_{i_t} \dots \hat{s}_{i_{t-1}} \dots \hat{s}_{i_1} \dots s_1 y.$$

For all $I = \{i_1, \dots, i_p\} \in \mathfrak{I}_{x,y}$ and all $1 \leq k \leq p$, we set

$$x_k = s_N \dots s_{i_p} \dots \hat{s}_{i_k} \text{ and } y_k = \hat{s}_{i_k} \dots \hat{s}_{i_{k-1}} \dots \hat{s}_{i_1} \dots s_1 y$$

and

$$z_I = s_N \dots \hat{s}_{i_p} \dots \hat{s}_{i_1} \dots s_1 y.$$

Then we have [4, Proof of Proposition 5.1]

$$\mathbf{T}_x\mathbf{T}_y = \sum_{I\in\mathfrak{I}_{x,y}} \left(\prod_{k=1}^{p}(v^{\mathbf{L}(s_{i_k})} - v^{-\mathbf{L}(s_{i_k})})\right) T_{z_I}.$$

Hence

$$\deg(\mathbf{f}_{x,y,z_I}) = \sum_{k=1}^{p} \mathbf{L}(s_{i_k}).$$

Fix k such that $2 \leq k \leq p$. Using the previous lemmas, there exists an injective map φ_k such that

$$\begin{aligned} I_{x_k,y_k} = I_{s_N\ldots s_{i_p}\ldots\hat{s}_{i_k},\hat{s}_{i_k}\ldots\hat{s}_{i_{k-1}}\ldots\hat{s}_{i_1}\ldots s_1 y} &\xrightarrow{\varphi_k} I_{s_N\ldots s_{i_p}\ldots s_{i_k},\hat{s}_{i_k}\ldots\hat{s}_{i_{k-1}}\ldots\hat{s}_{i_1}\ldots s_1 y} \\ &\subseteq I_{s_N\ldots s_{i_p}\ldots s_{i_k}(s_{i_k-1}\ldots s_{i_{k-1}+1}),\hat{s}_{i_{k-1}}\ldots\hat{s}_{i_1}\ldots s_1 y} = I_{x_{k-1},y_{k-1}} \end{aligned}$$

Thus we have a sequence

$$I_{x_p,y_p} \xrightarrow{\varphi_p} I_{x_p s_{i_p},y_p} \subseteq I_{x_{p-1},y_{p-1}} \xrightarrow{\varphi_{p-1}} I_{x_{p-1}s_{i_{p-1}},y_{p-1}} \ldots I_{x_1,y_1} \xrightarrow{\varphi_1} I_{x_1 s_{i_1},y_1} \subseteq I_{x,y}.$$

If we denote by α_{i_k} the positive root such that the only hyperplane separating $y_k A_k$ and $s_{i_k} y_k A_0$ lies in $\mathscr{F}_{\alpha_{i_k}}$ then we have

$$\{\alpha_{i_1}, \varphi_1(\alpha_{i_2}), \ldots, (\varphi_1 \ldots \varphi_{p-1})(\alpha_{i_p})\} \subseteq I_{x,y}.$$

Theorem 7.13. *Let $w = a_w w^\circ_{\lambda_\sigma} b_\sigma \in \mathbf{c}^{\mathbf{L}^+}_{\min}$. Let $x \in X_{\lambda_\sigma}$. We have*

$$T_x C_w \equiv T_{xw} + \sum_{z\in N_{\sigma'}, b_{\sigma'}<b_\sigma} a_z T_z \mod \mathbf{H}_{<0}.$$

Proof. We have

$$\begin{aligned} T_x C_w &= T_x T_w + T_x\left(\sum_{y<w} P_{y,w} T_y\right) \\ &= T_{xw} + \sum_{y\in\mathbf{c}^{\mathbf{L}^+}_{\min}} P_{y,w} T_x T_y + \sum_{y\notin\mathbf{c}^{\mathbf{L}^+}_{\min}} P_{y,w} T_x T_y \\ &= T_{xw} + \sum_{a'_w<a_w} P_{y,w} T_x T_{a'_w w^\circ_{\lambda_\sigma} d_\sigma} + \sum_{\substack{y\in N_{\sigma'}\\ b_{\sigma'}<b_\sigma}} P_{y,w} T_x T_y + \sum_{y\notin\mathbf{c}^{\mathbf{L}^+}_{\min}} P_{y,w} T_x T_y \\ &= T_{xw} + \sum_{a'_w<a_w} P_{y,w} T_{x a'_w w^\circ_{\lambda_\sigma} d_\sigma} + \sum_{\substack{y\in N_{\sigma'}\\ b_{\sigma'}<b_\sigma}} P_{y,w} T_x T_y + \sum_{y\notin\mathbf{c}^{\mathbf{L}^+}_{\min}} P_{y,w} T_x T_y \\ &\equiv T_{xw} + \sum_{\substack{y\in N_{\sigma'}\\ b_{\sigma'}<b_\sigma}} P_{y,w} T_x T_y + \sum_{y\notin\mathbf{c}^{\mathbf{L}^+}_{\min}} P_{y,w} T_x T_y \mod \mathbf{H}_{<0} \end{aligned}$$

Fix a y in the on of the sum above. Let λ be the unique $\mathbf{L}^+$-special point which is contained in the closure of yA_0 and which lies in the same orbit as λ_σ (i.e. $W_\lambda = W_{\lambda_\sigma}$). Write $y = a_y u_\lambda d_y$ where $d_y \in X_\lambda^{-1}$, u_λ has minimal length in the coset $W_{S^\circ_\lambda} y d_y^{-1}$ of W_λ and $a_y \in W_{S^\circ_\lambda}$.

Let $a_y u_\lambda = s_k \ldots s_1$ be a reduced expression and let $v = s_n \ldots s_{k+1}$ be such that $\mathbf{L}^+(s_n \ldots s_1) = \nu_{\mathbf{L}^+}$ and $\ell(s_n \ldots s_1) = n$. Let H_i be the unique hyperplane which separates $s_i \ldots s_1 d_y A_0$ and $s_{i+1} \ldots s_1 d_y A_0$ and let $\alpha_i \in \Phi^+$ be such that $H_i \in \mathscr{F}_{\alpha_i}$.

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Let $1 \leq i \leq k$ and assume that $H_i = H_{\alpha_i,n}$ where $n > 0$ (the case $n \leq 0$ is similar). We have $H_i \in H(A_0, yA_0)$ and since $\lambda \in H_i \cap \overline{yA_0}$ we see that

$$n < \langle \mu, \check{\alpha_i} \rangle < n+1 \text{ for all } \mu \in yA_0.$$

It follows that $H_{\alpha_i,m} \notin H(A_0, yA_0)$ for all $m \geq n+1$. Next, since $x \in X_{\lambda_\sigma} = X_\lambda$, we see that $H_i \notin H(yA_0, xyA_0)$ and it follows that $H_{\alpha_i,m} \notin H(yA_0, xyA_0)$ for all $m \leq n$. Finally, since λ is a $\mathbf{L}^+$-special point, we must have

$$\{\alpha_i \mid 1 \leq i \leq n\} = \Phi^+ \cap \Phi^{\mathbf{L}^+}.$$

It follows that

$$I_{x,y} \cap \Phi^{\mathbf{L}^+} \subseteq \{\alpha_{k+1}, \dots, \alpha_n\}.$$

By Lemma 7.8, we know that

$$\deg^+(\mathbf{P}_{y,w}) \leq \mathbf{L}(u_\lambda)^+ - \mathbf{L}(w_\lambda^\circ)^+ = -\mathbf{L}(v)^+.$$

Therefore, we have

$$(*) \qquad \deg^+(\mathbf{P}_{x,y}\mathbf{f}_{x,y,z}) \leq \deg^+(\mathbf{f}_{x,y,z}) - \mathbf{L}(v)^+ \leq (c^{\mathbf{L}}_{x,y})^+ - \mathbf{L}(v)^+.$$

If we are in Case 1–3, all the hyperplane which contains λ must be of maximal weight, hence we have $\mathbf{L}(s_i) = \mathbf{L}_{\alpha_i}$ for all i. Hence

$$(1) \qquad (c^{\mathbf{L}}_{x,y})^+ \leq \sum_{i=k+1}^{n} \mathbf{L}(s_i) = \mathbf{L}(v)^+.$$

In Case 4, we may have $\mathbf{L}^+(s_i) = \mathbf{t}' < \mathbf{t}$. Let $i, k, i', k' \in \mathbb{N}$ be such that

$$(2) \qquad \mathbf{L}(v)^+ = i\mathbf{t} + k\mathbf{t}' \text{ and } (c^{\mathbf{L}}_{x,y})^+ = i'\mathbf{t} + k'\mathbf{t}'.$$

Then by the work above we know that $i' + k' \leq i + k$.

Claim 7.14. If $y \in N^{\mathbf{L}^+}_{\sigma'}$ then $\mathbf{P}_{x,y}\mathbf{f}_{x,y,z} \in \mathbf{A}_{<0}$ whenever $z \notin N^{\mathbf{L}^+}_{\sigma'}$.

Proof. Let $x = s_N \dots s_1$ be a reduced expression of x. There exists $I = \{i_1, \dots, i_p\} \in \mathfrak{I}_{x,y}$ such that $z_I = z$. Assume that $z_I \notin N^{\mathbf{L}^+}_{\sigma}$. Then there exist $\alpha \in \Phi^{\mathbf{L}^+}$ and $i_{k+1} - 1 > M > i_k$ such that

$$s_M \dots \hat{s}_{i_k} \dots \hat{s}_{i_1} \dots s_1 yA_0 \notin U^{\mathbf{L}^+}_{\alpha}(A_0) \text{ and } s_{M+1} \dots \hat{s}_{i_k} \dots \hat{s}_{i_1} \dots s_1 yA_0 \in U^{\mathbf{L}^+}_{\alpha}(A_0)$$

and the unique hyperplane which separates these two alcoves is a wall of $N^{\mathbf{L}^+}_{\sigma'}$. That means that in the sequence

$$I_{x_p,y_p} \xrightarrow{\varphi_p} I_{x_p s_{i_k}, y_p} \subseteq I_{x_{p-1},y_{p-1}} \xrightarrow{\varphi_{p-1}} I_{x_{p-1}s_{i_{p-1}},y_{p-1}} \dots I_{x_1,y_1} \xrightarrow{\varphi_1} I_{x_1 s_{i_1},y_1} \subseteq I_{x,y}$$

we have $H \in I_{x_{k+1}s_{i_{k+1}},y_{k+1}}$ but $H \notin I_{x_k,y_k}$. Further, if there is an hyperplane of direction α in I_{x_k,y_k} then it can't be of maximal weight. Hence we have

$$c^{\mathbf{L}}_{x_k,y_k} \geq c^{\mathbf{L}}_{x_{k+1}s_{i_{k+1}},y_{k+1}} + \mathbf{C}_\alpha$$

where

$$\mathbf{C}_\alpha = \begin{cases} \mathbf{L}_\alpha & \text{if we are in Case 1.} \\ \mathbf{L}_\alpha \text{ or } \mathbf{t} - \mathbf{t}' & \text{if we are in Case 2.} \\ \mathbf{t} \text{ or } \mathbf{t} - \mathbf{t}' & \text{if we are in Case 3.} \\ \mathbf{t} \text{ or } \mathbf{t}' & \text{if we are in Case 4.} \end{cases}$$

Next, by repetitive use of Lemmas 7.10 and 7.11 we get the following inequalities

$$\begin{aligned}
c^{\mathbf{L}}_{x,y} &\geq c^{\mathbf{L}}_{x_1 s_{i_1}, y_1} \geq c^{\mathbf{L}}_{x_1,y_1} + \mathbf{L}(s_{i_1})\\
&\geq c^{\mathbf{L}}_{x_2 s_{i_2}, y_2} + \mathbf{L}(s_{i_1}) \geq c^{\mathbf{L}}_{x_2,y_2} + \sum_{\ell=1}^{2} \mathbf{L}(s_{i_\ell})\\
&\dots\\
&\geq c^{\mathbf{L}}_{x_k s_{i_k}, y_k} + \sum_{\ell=1}^{k-1} \mathbf{L}(s_{i_\ell}) \geq c^{\mathbf{L}}_{x_k,y_k} + \sum_{\ell=1}^{k} \mathbf{L}(s_{i_\ell})\\
&\geq c^{\mathbf{L}}_{x_{k+1} s_{i_{k+1}}, y_{k+1}} + \sum_{\ell=1}^{k} \mathbf{L}(s_{i_\ell}) + \mathbf{C}_\alpha\\
&\geq c^{\mathbf{L}}_{x_{k+1},y_{k+1}} + \sum_{\ell=1}^{k+1} \mathbf{L}(s_{i_\ell}) + \mathbf{C}_\alpha\\
&\geq \dots \geq c^{\mathbf{L}}_{x_p,y_p} + \sum_{\ell=1}^{p} \mathbf{L}(s_{i_p}) + \mathbf{C}_\alpha.
\end{aligned}$$

Hence, we get $(\mathbf{c}^{\mathbf{L}}_{x,y})^+ \geq \deg^+(\mathbf{f}_{x,y,z}) + \mathbf{C}^+_\alpha$.

If we are in Case 1 or in Case 2 with $\mathbf{C}_\alpha = \mathbf{L}_\alpha$ then we have, using (1)

$$\deg^+(\mathbf{f}_{x,y,z}) - \mathbf{L}(v)^+ \leq (c^{\mathbf{L}}_{x,y})^+ - \mathbf{L}(v)^+ - \mathbf{C}^+_\alpha \leq -\mathbf{L}_\alpha.$$

But we know that $\varphi_1(\mathbf{L}_\alpha) < 0$ hence $\varphi_1(\deg^+(\mathbf{P}_{x,y}\mathbf{f}_{x,y,z})) < 0$ and $\mathbf{P}_{x,y}\mathbf{f}_{x,y,z}$ lies in $\mathbf{A}_{<0}$; see Remark 7.2.

If we are in Case 2 and $C_\alpha = \mathbf{t} - \mathbf{t}'$, then, we must have $\mathbf{t} \in \bar{S}^+$ and $\mathbf{t}' \notin \bar{S}^\circ$. Therefore we have $C^+_\alpha = \mathbf{t}$ and we can conclude as above.

If we are in Case 3 then $C_\alpha \geq \mathbf{t} - \mathbf{t}'$ and we get

$$\deg^+(\mathbf{f}_{x,y,z}) - \mathbf{L}(v)^+ \leq (c^{\mathbf{L}}_{x,y})^+ - \mathbf{L}(v)^+ - \mathbf{C}^+_\alpha \leq (-1, 0, 1).$$

But since we are in Case 3, this implies that $\mathbf{P}_{x,y}\mathbf{f}_{x,y,z} \in \mathbf{A}_{<0}$; see Remark 7.4.

Finally, if we are in Case 4 then $\mathbf{C}_\alpha \geq \mathbf{t}'$. Using (2), we get

$$\begin{aligned}
\deg^+(\mathbf{f}_{x,y,z}) - \mathbf{L}(v)^+ &\leq (c^{\mathbf{L}}_{x,y})^+ - \mathbf{L}(v)^+ - \mathbf{t}'\\
&\leq (i' + k')\mathbf{t} - (i+k)\mathbf{t}' - \mathbf{t}'\\
&= (i' + k', 0, -(i+k) - 1)
\end{aligned}$$

and since $(i+k) \geq (i'+k') \geq 0$ we have $\varphi_1((i'+k', 0, -(i+k)-1)) < 0$. The result follows; see Remark 7.2. $\square$

Claim 7.15. If $y \notin \mathbf{c}^{\mathbf{L}^+}_{\min}$ then $\mathbf{P}_{x,y}\mathbf{f}_{x,y,z} \in \mathbf{A}_{<0}$.

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Proof. Since $y \notin \mathbf{c}_{\min}^{\mathbf{L}^+}$ we have $y \in U_{\alpha_m}^{\mathbf{L}^+}(A_0)$ for some $k+1 \leq m \leq n$. Then

$$c_{x,y}^{\mathbf{L}}(\alpha_m) = \begin{cases} 0 & \text{if we are in Case 1.} \\ 0 \text{ or } \mathbf{t}' & \text{if we are in Case 2.} \\ 0 \text{ or } \mathbf{t}' & \text{if we are in Case 3.} \\ 0 & \text{if we are in Case 4.} \end{cases}$$

We have

$$(c_{x,y}^{\mathbf{L}})^+ \leq \sum_{i=m+1}^{n} \mathbf{L}_{\alpha_i} + c_{x,y}^{\mathbf{L}}(\alpha_m)^+.$$

If we are in Case (1) or in Case 2 with $c_{x,y}^{\mathbf{L}}(\alpha_m) = 0$, then $\mathbf{L}(s_i) = \mathbf{L}_{\alpha_i}$ for all $1 \leq i \leq n$ (see Lemma 7.3). Hence

$$(c_{x,y}^{\mathbf{L}})^+ - \mathbf{L}(v)^+ \leq -\mathbf{L}_{\alpha_m}$$

and the result follows using $(*)$.

If we are in Case (2) with $c_{x,y}^{\mathbf{L}}(\alpha_m) = \mathbf{t}'$. Then

$$(c_{x,y}^{\mathbf{L}})^+ - \mathbf{L}(v)^+ \leq -\mathbf{L}_{\alpha_m} + (c_{x,y}^{\mathbf{L}})^+ = -\mathbf{t}.$$

and the result follows using $(*)$ and Remark 7.2.

If we are in Case (3), then $c_{x,y}^{\mathbf{L}}(\alpha_m) \leq \mathbf{t}'$. Then $\mathbf{L}(s_i) = \mathbf{L}_{\alpha_i}$ for all $1 \leq i \leq n$ and

$$(c_{x,y}^{\mathbf{L}})^+ - \mathbf{L}(v)^+ \leq -\mathbf{L}_{\alpha_m} + (c_{x,y}^{\mathbf{L}})^+ \leq -\mathbf{t} + \mathbf{t}'.$$

and the result follows; ; see Remark 7.4.

Finally, if we are in Case 4, then using (2) we must have $i + k > i' + k'$ since $y \in U_{\alpha_m}^{\mathbf{L}^+}(A_0)$. The result follows, arguying in a similar fashion as at the end of the previous Claim. □

The theorem follows easily from the two claims and the expression:

$$T_x C_w = T_{xw} + \sum_{\substack{y \in N_{\sigma'} \\ b_{\sigma'} < b_\sigma}} P_{y,w} T_x T_y + \sum_{y \notin \mathbf{c}_{\min}^{\mathbf{L}^+}} P_{y,w} T_x T_y \mod \mathbf{H}_{<0}$$

□

7.5. **Proof of Conditions I1–I5.** Condition **I1** it is clear. Condition **I2** is a direct consequence of Lemma 3.9. Condition **I3** follows from the fact that $\mathbf{c}_{\min}^{\mathbf{L}^+}$ is a disjoint union of the sets $N_\sigma^{\mathbf{L}^+}$. Condition **I5** is Theorem 7.13. Hence we only to prove **I4**, that is, we need to show that the **A**-module

$$\mathscr{M} := \langle \mathbf{T}_x \mathbf{C}_u | \ u \in U_\sigma^{\leq}, \ x \in X_u \rangle_{\mathscr{A}} \subseteq \mathscr{H}$$

is a left ideal of $\mathscr{H}$. Let $u = a_u w_{\lambda_\sigma}^\circ b_\sigma \in U_{\lambda,z}$ and $x \in X_{\lambda_\sigma}$. It is enough to show that $T_s T_x C_u \in \mathscr{M}$ for all $s \in S$. There is 3 cases to consider:

(1) $sx > x$ and $sx \in X_{\lambda_\sigma}$;
(2) $sx < x$ and $sx \in X_{\lambda_\sigma}$;
(3) $sx > x$ and $sx \notin X_{\lambda_\sigma}$.

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The result is clear in the first two cases since we have respectively

$$\mathbf{T}_s\mathbf{T}_x\mathbf{C}_u = \mathbf{T}_{sx}\mathbf{C}_u \in \mathscr{M}$$

and

$$\mathbf{T}_s\mathbf{T}_x C_u = (\mathbf{T}_{sx} + (v^{\mathbf{L}(s)} - v^{-\mathbf{L}(s)})\mathbf{T}_x)\mathbf{C}_u \in \mathscr{M}$$

Assume that we are in the third case. Then by Deodhar's lemma (see [7, Lemma 2.1.2]), there exists $s' \in S_{\lambda_\sigma}$ such that $sx = xs'$. If $s'u < u$ we get

$$\mathbf{T}_s\mathbf{T}_x\mathbf{C}_u = \mathbf{T}_{sx}\mathbf{C}_u = \mathbf{T}_x\mathbf{T}_{s'}\mathbf{C}_u = v^{\mathbf{L}(s')}\mathbf{T}_x\mathbf{C}_u \in \mathscr{M}$$

as required. Assume $s'u > u$. Note that this implies that $s' \in S^\circ$ since for all $s' \in S^+$ we have $s'u < u$. Then

$$\mathbf{T}_s\mathbf{T}_x\mathbf{C}_u = \mathbf{T}_x\mathbf{T}_{s'}\mathbf{C}_u = \mathbf{T}_x\big(\mathbf{C}_{s'u} + \sum_{z<w}\mathbf{M}^{s'}_{z,u}\mathbf{C}_z\big).$$

In the above sum, we know that the term $\mathbf{T}_x\mathbf{C}_{s'u} \in \mathscr{M}$. If $\mathbf{M}^{s'}_{z,u} \neq 0$ then by Lemma 7.8, we must have $z \in \mathbf{c}^{\mathbf{L}^+}_{\min}$ which in turn implies that by Lemma 3.9, that either $z = a_z w^\circ_{\lambda_\sigma} b_\sigma$ with $a_z < a_u$ (in which case $\mathbf{T}_z\mathbf{C}_u \in \mathscr{M}$) or $z = a_z w^\circ_{\lambda_{\sigma'}} b_{\sigma'}$ with $b_{\sigma'} < b_\sigma$. From there, the result follows by an easy induction on the length of b_σ.

8. Proof of Theorem 6.5

Let Γ be a totally ordered group abelian group. In this section we study the relation between $\mathbf{H} = (W, S, \mathbf{L})$ as define in the previous section and $\mathscr{H} = \mathscr{H}(W, S, L)$ where $L \in \mathrm{Weight}(W, \Gamma)$. The element of $\mathbf{H}$ and $\mathbf{A}$ will be written with a bold symbols.

8.1. **Specialisation.** Let $L : W \to \Gamma$ be weight function. Then the map $\theta^{\mathbf{L}}_{\Gamma} : \mathbf{\Gamma} \to \Gamma$ which sends $\mathbf{L}(s)$ to $L(s)$ is a group homomorphism. Further, this homomorphism induces a morphism of $\mathbb{Z}$-algebras $\theta^L_{\mathbf{A}} : \mathbf{A} \to \mathscr{A}$ which sends $v^{\mathbf{L}(s)}$ to $v^{L(s)}$. If $\mathscr{H}$ is viewed as a $\mathbf{A}$-algebra through $\theta^L_{\mathbf{A}}$, then there is a unique morphism of $\mathbf{A}$-algebras $\theta^L_{\mathbf{H}} : \mathbf{H} \to \mathscr{H}$ such that $\theta(\mathbf{T}_x) = T_x$ for all $x \in W$.

We recall the some result of [8, 6]. Let $N \in \mathbb{N}$ and let $X_N = \{z \in W \mid \ell(z) \leq N\}$. We now define three subsets $\mathbf{\Gamma}^1_+(N)$, $\mathbf{\Gamma}^2_+(N)$, $\mathbf{\Gamma}^3_+(N) \subset \mathbf{\Gamma}$. First, let $\mathbf{\Gamma}^1_+(N)$ be the set of all elements $\gamma > 0 \in \mathbf{\Gamma}$ such that $v^{-\gamma}$ occurs with a non-zero coefficient in a polynomial $\mathbf{P}_{z_1,z_2}$ for some $z_1 < z_2 \in X_N$. Next for any z_1, z_2 in X_N such that $\mathbf{M}^s_{z_1,z_2} \neq 0$ for some s, we write $\mathbf{M}^s_{z_1,z_2} = n_1 v^{\gamma_1} + \ldots + n_\ell v^{\gamma_\ell}$ where $0 \neq n_i \in \mathbb{Z}$, $\gamma_i \in \mathbf{\Gamma}$ and $\gamma_i - \gamma_{i-1} > 0$ for $2 \leq i \leq \ell$. Let $\mathbf{\Gamma}^2_+(N)$ be the set of all elements $\gamma_i - \gamma_{i-1} > 0$ arising in this way, for any $z_1, z_2 \in X_N$ and $s \in S$. Finally let $\mathbf{\Gamma}^3_+(N)$ be the set of all elements $\gamma > 0 \in \mathbf{\Gamma}$ such that $v^{-\gamma}$ occurs with a non-zero coefficient in a polynomial of the form

$$\sum_{z;z_1\leq z<z_2;sz<z} \mathbf{P}_{z_1,z}\mathbf{M}^s_{z,z_2} - v_s\mathbf{P}_{z_1,z_2}$$

for some $z_1, z_2 \in X_N$ and $s \in S$. We set $\mathbf{\Gamma}_+(N) = \mathbf{\Gamma}^1_+(N) \cup \mathbf{\Gamma}^2_+(N) \cup \mathbf{\Gamma}^3_+(N)$.

Proposition 8.1. *(see* [8, Proposition 3.3]*) Let $L : W \to \Gamma$ be a weight function such that the ring homomorphism θ_L satisfies the condition*

$$(*) \qquad \theta^L_{\mathbf{\Gamma}}(\mathbf{\Gamma}_+(N)) \subseteq \{\gamma \mid \gamma > 0\}.$$

Then, for all $x, y \in X_N$, we have $\theta^L_{\mathbf{A}}(\mathbf{P}_{x,y}) = P_{x,y}$ and $\theta^L_{\mathbf{A}}(\mathbf{M}^s_{x,y}) = M^s_{x,y}$. In particular $\theta^L_{\mathbf{H}}(\mathbf{C}_x) = C_x$.

The following Lemma is a straightforward generalisation of [6, Lemma 3.4].

Lemma 8.2. *We have*

$$\mathbf{\Gamma}_+(N) \subset \{(\gamma_1, \ldots, \gamma_{|S|}) \in \mathbb{Z}^{|S|} \mid -N \leq \gamma_i \leq N\}$$

We now give an outline of the proof of Theorem 6.5. First, note that the proof of Theorem 7.5 in the previous section only involved elements of bounded length, say by $N_0 \in \mathbb{N}$. Let (W, S) be an irreducible affine Weyl group and assume that $S = S^+ \cup S^\circ$ is such that no element of S^+ is conjugate to an element of S°. Let $X \in \mathscr{P}_+(\mathbb{Z}[\bar{S}])$ as in Section 7.1 and $\mathbf{\Gamma} = \mathbb{Z}[\bar{S}]/(X \cap (-X))$. Then, using N_0 and the previous lemma, we will determine a set of weight functions $\mathscr{C}$ (which will correspond at the end to a union of chambers) such that for all $L \in \mathscr{C}$ we have

$$\theta^L_{\mathbf{\Gamma}}(\mathbf{\Gamma}_+(N_0)) \subseteq \{\gamma \mid \gamma > 0\}.$$

For all such weight functions, we can apply the same proof as before to (W, S, L) by Proposition 8.1 and we get that $N^{\mathbf{L}^+}_\sigma$, for all $\sigma \in \Omega^{\mathbf{L}^+}_0$ is a union of cells of (W, S, L). But then, it will be easy to see that $N^{\mathbf{L}^+}_\sigma = N^{L'}_\sigma$ for all weight function L' which takes some zero values and which lies in $\overline{\mathscr{C}}$. Finally by changing the sets S^+ and S° (and also the subset X in type $\tilde{\mathrm{C}}$), we will eventually cover all the cases and determine the constants such that Theorem 6.7 and Theorem 6.9 holds. The constants we are determining are nowhere near the best we can find as one can see by looking at the essential hyperplanes in type $\tilde{\mathrm{C}}_2$ in Remark 6.11.

8.2. **Affine Weyl group of type** $\tilde{\mathrm{B}}_r$**,** $\tilde{\mathrm{F}}_4$ **or** $\tilde{\mathrm{G}}_2$**.** We keep the notation of Section 6.2. Let W be an irreducible affine Weyl group of type $\tilde{\mathrm{B}}_r$, $\tilde{\mathrm{F}}_4$ or $\tilde{\mathrm{G}}_2$.

First let $\bar{S}^+ = \{\mathbf{s}\}$ and $\bar{S}^\circ = \{\mathbf{t}\}$. Let $\Phi \in \mathscr{P}_+(\mathbb{Z}[\bar{S}])$. Then $\mathbf{\Gamma} = \mathbb{Z}[\bar{S}]$ and the order is the lexicographic order (see Example 7.1).

Claim 8.3. Let $L \in \mathrm{Weight}(W, \Gamma)$ be such that $L(\mathbf{s}) > N_0 \cdot L(\mathbf{t})$. Then

$$\theta^L_{\mathbf{\Gamma}}(\mathbf{\Gamma}_+(N_0)) \subset \{v^\gamma \mid \gamma > 0\}$$

Proof. Since the order on $\mathbf{\Gamma}$ is the lexicographic order, we must have

$$\mathbf{\Gamma}_+(N_0) \subset \{(i,j) \mid i > 0, -N_0 \leq i, j, k \leq N_0\} \cup \{(0,j) \mid j > 0\}.$$

Let $i > 0$ and $-N_0 \leq j \leq N_0$. Then

$$i \cdot \mathbf{L}(s) + j \cdot \mathbf{L}(t) > iN_0 \cdot L(\mathbf{t}) - N_0 \cdot L(\mathbf{t}) \geq 0.$$

The result follows. □

Thus Condition $(*)$ in Proposition 8.1 holds for all weight functions satisfying the hypothesis of the lemma. Therefore we get that $N^{\mathbf{L}^+}_\sigma$ is a union of cells of (W, S, L). But one can easily see that for all weight functions $L^+ : W \to \Gamma$ such that $L^+(\mathbf{s}) > 0$ and $L^+(\mathbf{t}) = 0$ we have $\Omega^{\mathbf{L}^+}_0 = \Omega^{L^+}_0$ and $N^{L^+}_\sigma = N^{\mathbf{L}^+}_\sigma$.

Next let $\bar{S}^+ = \{\mathbf{t}\}$ and $\bar{S}^\circ = \{\mathbf{s}\}$. Then we get that the order on $\mathbf{\Gamma}$ is as follows

$$(i,j) < (i',j') \Longleftrightarrow j < j' \text{ or } (j = j' \text{ and } i < i').$$

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Arguying as above, we get that for all weight functions $L \in \mathrm{Weight}(W,\Gamma)$ such that $L(\mathbf{t}) > N_0 \cdot L(\mathbf{s})$ and all weight functions $L^+ : W \to \Gamma$ such that $L^+(\mathbf{t}) > 0$ and $L^+(\mathbf{s}) = 0$, the sets $N_\sigma^{L^+}$ are union of left cells of (W, S, L).

Finally, putting all this together, we get Theorem 6.5 holds for the finite set of rational hyperplanes $\mathfrak{H}(N_0, 1/N_0)$.

8.3. **Affine Weyl group of type $\tilde{\mathrm{C}}$.** We keep the notation of Section 6.3. We have $\bar{S} = \{\mathbf{t}, \mathbf{s}, \mathbf{t}'\}$.

Claim 8.4. Let $\bar{S}^+ = \{\mathbf{t}\}$ and $\bar{S}^\circ = \{\mathbf{s}, \mathbf{t}'\}$ and let $L \in \mathrm{Weight}(W,\Gamma)$ be such that $L(\mathbf{t}) > N_0 \cdot L(\mathbf{s}) + N_0 \cdot L(\mathbf{t}')$. Then, there exists $\Phi = (\varphi_1, \dots, \varphi_d) \in \mathscr{P}_+(\mathbb{Z}[\bar{S}])$ such that

$$\theta^L_{\mathbf{\Gamma}}(\mathbf{\Gamma}_+(N_0)) \subset \{v^\gamma \mid \gamma > 0\}$$

where $\mathbf{\Gamma}$ is the totally ordered abelian group associated to $\mathrm{Pos}(\Phi)$.

Proof. Since $\bar{S}^+ = \{\mathbf{t}\}$ we set $\varphi_1 = \mathbf{t}^*$. Hence

$$\ker(\varphi_1) = \{(0, j, k) \mid j, k \in \mathbb{R}\}.$$

Now we want to find a linear map $\varphi_2 : \ker(\varphi_1) \to \mathbb{R}$ defined by $\varphi_2((0, j, k)) = bj + ck$ (where $b, c \geq 0$) such that the following property holds for all $-N_0 \leq i, j \leq N_0$:

$$(\dagger) \qquad bj + ck > 0 \text{ then } L(\mathbf{s})j + L(\mathbf{t})k > 0$$

To do so we proceed as in [6, §3]. Set $\mathscr{E} := \{x \in \mathbb{Q} \mid x = \pm\frac{k}{j}$ where $j, k \neq 0$ and $-N_0 \leq j, k \leq N_0\}$ and write $\mathscr{E} = \{x_1, \dots, x_n\}$ where $x_1 < x_2 < \dots < x_n$. We set $x_0 = 0$ and $x_{n+1} = +\infty$. Let $b, c \geq 0$ be integers such that

$$x_k = \frac{b}{c} = \max\{r \in \mathscr{E} \mid L(\mathbf{s}) \geq rL(\mathbf{t}')\}.$$

Note that we must have $x_{k+1}L(\mathbf{t}') > L(\mathbf{s})$. Then we claim that property (†) holds. Let $-N_0 \leq j, k \leq N_0$ be such that $bj + ck > 0$. If $j > 0$ and $k < 0$ then $b/c > -k/j$ and we have

$$L(\mathbf{s}) > -\frac{k}{j} L(\mathbf{t}') \text{ that is } jL(\mathbf{s}) + kL(\mathbf{t}') > 0$$

as required. If $j < 0$ and $k > 0$ (in this case we have $x_{k+1} < \infty$) then $b/c < -k/j$ but this forces $x_{k+1} \leq -k/j$. Hence

$$-\frac{k}{j} L(\mathbf{t}') > L(\mathbf{s}) \text{ that is } jL(\mathbf{s}) + kL(\mathbf{t}') > 0$$

as required.
Finally we set $\varphi_2 : \ker(\varphi_1) \to \mathbb{R}$ by $\varphi_2((0, j, k)) = bj + ck$ where $b, c \geq 0$ are chosen as above. Then $\mathrm{Pos}(\varphi_1, \varphi_2) \in \mathscr{P}_+(\mathbb{Z}[\bar{S}])$ and we have

$$\mathbf{\Gamma}_+(N_0) \subset \{(i, j, k) \mid i > 0, -N_0 \leq i, j, k \leq N_0\} \cup \{(0, j, k) \mid L(\mathbf{s})j + L(\mathbf{t}')k > 0\}.$$

The result follows easily from our assumptions on L. □

Let $L \in \mathrm{Weight}(W,\Gamma)$ and Φ as in the previous claim. By Proposition 8.1, we know that $N_\sigma^{\mathbf{L}^+}$ (for all $\sigma \in \Omega_0^{\mathbf{L}^+}$) is a union of left cells in (W, S, L). But one can see that $N_\sigma^{\mathbf{L}^+} = N_\sigma^{L^+}$ for all $L^+ \in \mathrm{Weight}(W,\Gamma)$ such that $L^+(\mathbf{t}) > 0$ and $L^+(\mathbf{s}) = L^+(\mathbf{t}') = 0$ and all $\sigma \in \Omega_0^{\mathbf{L}^+} = \Omega_0^{L^+}$.

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Remark 8.5. If we set $m_3 = N_0$, the above claim implies that Theorem 6.9 holds for all positive weight functions lying in $\mathscr{C}_3 \cup \mathscr{C}_4 \cup (\bar{\mathscr{C}}_3 \cap \bar{\mathscr{C}}_4)$ and all non-negative weight functions lying in $H_{\mathbf{s}} \cap H_{\mathbf{t}'}$.

Claim 8.6. Let $\bar{S}^+ = \{\mathbf{s}\}$ and $\bar{S}^\circ = \{\mathbf{t}, \mathbf{t}'\}$ and let $L \in \mathrm{Weight}(W, \Gamma)$ be such that $L(\mathbf{s}) > N_0 \cdot L(\mathbf{t}) + N_0 \cdot L(\mathbf{t}')$. Then, there exists $\Phi = (\varphi_1, \dots, \varphi_d) \in \mathscr{P}_+(\mathbb{Z}[\bar{S}])$ such that

$$\theta^L_{\mathbf{\Gamma}}(\mathbf{\Gamma}_+(N_0)) \subset \{v^\gamma \mid \gamma > 0\}$$

where $\mathbf{\Gamma}$ is the totally ordered abelian group associated to $\mathrm{Pos}(\Phi)$.

Proof. Since $\bar{S}^+ = \{\mathbf{s}\}$ and we set $\varphi_1 = \mathbf{t}^*$. Hence

$$\ker(\varphi_1) = \{(i, 0, k) \mid i, k \in \mathbb{R}\}.$$

Arguing as in the proof of the previous claim, there exist integers $a, c \geq 0$ such that if we define $\varphi_2 : \ker(\varphi_1) \to \mathbb{R}$ by $\varphi_2((i, 0, k)) = ai + ck$ then $\varphi_2((i, 0, k)) > 0$ implies $L(\mathbf{t})i + L(\mathbf{t}')k > 0$. For such φ_2 we have $\mathrm{Pos}(\varphi_1, \varphi_2) \in \mathscr{P}_+(\mathbb{Z}[\bar{S}])$ and

$$\mathbf{\Gamma}_+(N_0) \subset \{(i, j, k) \mid j > 0, -N_0 \leq i, j, k \leq N_0\} \cup \{(i, 0, k) \mid \mathbf{L}(\mathbf{t})j + \mathbf{L}(\mathbf{t}')k > 0\}.$$

The result follows easily from our assumptions on L. □

Let $L \in \mathrm{Weight}(W, \Gamma)$ and Φ as in the previous claim. Arguying as above,we get that $N^{L^+}_\sigma$ is a union of left cells of (W, S, L) for all $L^+ \in \mathrm{Weight}(W, \Gamma)$ such that $L^+(\mathbf{s}) > 0$ and $L^+(\mathbf{t}) = L^+(\mathbf{t}') = 0$.

Remark 8.7. If we set $m_6 = 1/N_0$, the above claim implies that Theorem 6.9 holds for all positive weight functions lying in $\mathscr{C}_6 \cup (\bar{\mathscr{C}}_6 \cap \bar{\mathscr{C}}'_6)$ and all non-negative weight functions lying in $H_{\mathbf{t}} \cap H_{\mathbf{t}'}$.

Claim 8.8. Let $\bar{S}^+ = \{\mathbf{t}, \mathbf{t}'\}$ and $\bar{S}^\circ = \{\mathbf{s}\}$ and let $L \in \mathrm{Weight}(W, \Gamma)$ be such that $L(\mathbf{t}) > N_0^2 \cdot L(\mathbf{s})$, $L(\mathbf{t}') > N_0^2 \cdot L(\mathbf{s})$ and $L(\mathbf{t}) - L(\mathbf{t}') > N_0 \cdot L(\mathbf{s})$. Then, there exists $\Phi = (\varphi_1, \dots, \varphi_d) \in \mathscr{P}_+(\mathbb{Z}[\bar{S}])$ such that

$$\theta^L_{\mathbf{\Gamma}}(\mathbf{\Gamma}_+(N_0)) \subset \{v^\gamma \mid \gamma > 0\}.$$

where $\mathbf{\Gamma}$ is the totally ordered abelian group associated to $\mathrm{Pos}(\Phi)$.

Proof. Since $\bar{S}^+ = \{\mathbf{t}, \mathbf{t}'\}$ we set $\varphi_1 = \mathbf{t}^* + \mathbf{t}'^*$. Hence

$$\ker(\varphi_1) = \{(i, j, -i) \mid i, j \in \mathbb{R}\}.$$

We define $\varphi_2 := \mathbf{t}^*$ and $\varphi_3 = \mathbf{s}^*$. Then $\mathrm{Pos}(\varphi_1, \varphi_2, \varphi_3) \in \mathscr{P}_+(\mathbb{Z}[\bar{S}])$ and we have

$$\begin{aligned} \mathbf{\Gamma}_+(N_0) \subset &\{(i, j, k) \mid i + k > 0, -N_0 \leq i, j, k \leq N_0\} \\ &\cup \{(i, j, -i) \mid i > 0, -N_0 \leq i, j \leq N_0\} \\ &\cup \{0, j, 0 \mid j > 0\}. \end{aligned}$$

The result follows from our assumptions on L. Note that the N_0^2 in the hypothesis of the lemma comes from the fact that we need to have

$$\theta^L_{\mathbf{\Gamma}}(-(N_0 - 1), -N_0, N_0) > 0.$$

That is

$$L(\mathbf{t}') - (N_0 - 1)(L(\mathbf{t}) - L(\mathbf{t}')) - N_0 L(\mathbf{s}) > 0.$$

The last inequality will hold whenever $L(\mathbf{t}') > N_0^2 L(\mathbf{s})$ and $L(\mathbf{t}) - L(\mathbf{t}') > N_0 \cdot L(\mathbf{s})$. □

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Let $L \in \text{Weight}(W, \Gamma)$ and Φ as in the previous claim. Arguying as above,we get that $N_\sigma^{L^+}$ is a union of left cells of (W, S, L) for all $L^+ \in \text{Weight}(W, \Gamma)$ such that $L^+(\mathbf{t}) > L^+(\mathbf{t}') > 0$ and $L^+(\mathbf{s}) = 0$.

Remark 8.9. If we set $m_1 = m_2 = N_0^2$ and $m_5 = N_0$, the above claim implies that Theorem 6.9 holds for all positive weight functions lying in $\mathscr{C}_2 \cup \mathscr{C}_3 \cup (\bar{\mathscr{C}}_2 \cap \bar{\mathscr{C}}_3)$ and all non-negative weight functions in $H_{\mathbf{s}} \cap H_{\mathbf{t}'}$.

Claim 8.10. Let $\bar{S}^+ = \{\mathbf{t}, \mathbf{t}'\}$ and $\bar{S}^\circ = \{\mathbf{s}\}$ and let $L \in \text{Weight}^+(W, \Gamma)$ be such that $L(\mathbf{t}) > N_0^2 \cdot L(\mathbf{s})$, $L(\mathbf{t}') > N_0^2 \cdot L(\mathbf{s})$. Then, there exists $\Phi = (\varphi_1, \ldots, \varphi_d) \in \mathscr{P}_+(\mathbb{Z}[\bar{S}])$ such that $(1, j_0, -1) > 0$ for some $j_0 \in \mathbb{N}$ and

$$\theta_{\mathbf{\Gamma}}^L(\mathbf{\Gamma}_+(N_0)) \subset \{v^\gamma \mid \gamma > 0\}$$

where $\mathbf{\Gamma}$ is the totally ordered abelian group associated to $\text{Pos}(\Phi)$.

Proof. Since $\bar{S}^+ = \{\mathbf{t}, \mathbf{t}'\}$ we set $\varphi_1 = \mathbf{t}^* + \mathbf{t}'^*$. Hence

$$\ker(\varphi_1) = \{(i, j, -i) \mid i, j \in \mathbb{R}\}.$$

Arguing as in the proof of Claim 8.4 there exist integers $d, b \geq 0$ (with $b > 0$ so that $(1, j_0, -1) > 0$ for some j_0) such that if we define $\varphi_2 : \ker(\varphi_1) \to \mathbb{R}$ by $\varphi_2((i, j, -i)) = di + bj$ then $\varphi_2((i, j, -i)) > 0$ implies $(L(\mathbf{t}) - L(\mathbf{t}'))i + L(\mathbf{s})j > 0$. For such φ_2 we have $\text{Pos}(\varphi_1, \varphi_2) \in \mathscr{P}_+(\mathbb{Z}[\bar{S}])$ and

$$\begin{aligned}\mathbf{\Gamma}_+(N_0) \subset &\{(i, j, k) \mid i + k > 0, -N_0 \leq i, j, k \leq N_0\} \\ &\cup \{(i, j, -i) \mid (L(\mathbf{t}) - L(\mathbf{t}')) \cdot i + L(\mathbf{s}) \cdot j > 0\}.\end{aligned}$$

The result follows from our assumptions on L. □

Let $L \in \text{Weight}(W, \Gamma)$ and Φ as in the previous claim (so that we are in Case 4). Arguying as above,we get that $N_\sigma^{L^+}$ is a union of left cells of (W, S, L) for all $L^+ \in \text{Weight}(W, \Gamma)$ such that $L^+(\mathbf{t}) = L^+(\mathbf{t}') > 0$ and $L^+(\mathbf{s}) = 0$.

Remark 8.11. If we set $m_1 = m_2 = N_0^2$, then above claim implies that Theorem 6.9 holds for all positive weight functions lying in $\mathscr{C}_1 \cup (\bar{\mathscr{C}}_1 \cap \bar{\mathscr{C}}_1')$ and all non-negative weight functions in $H_{\mathbf{s}} \cap H_{\mathbf{t}-\mathbf{t}'}$.

Finally we still have to consider the case $\bar{S}^+ = \{\mathbf{t}, \mathbf{s}\}$ and $\bar{S}^\circ = \{\mathbf{t}'\}$. However in this case, there is nothing to prove since for all $L, L^+ \in \text{Weight}(W, \Gamma)$ such that

- $L(\mathbf{s}), L(\mathbf{t}) > L(\mathbf{t}')$
- $L^+(\mathbf{t}) = L(\mathbf{t})$, $L^+(\mathbf{s}) = L(\mathbf{s})$ and $L^+(\mathbf{t}') = 0$

we have $\mathbf{c}_{\min}^L = \mathbf{c}_{\min}^{L^+}$. Indeed, in both cases, we have $\mathscr{W}^{\max} = \{w_0\}$ where w_0 is the longest element of the group generated by $t, s_1, \ldots, s_{n-1}$.

The proof of Theorem 6.9 for type C is now complete.

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C. Bonnafé, Institut de Mathématiques et de Modélisation de Montpellier (CNRS: UMR 5149), Université Montpellier 2, Case Courrier 051, Place Eugène Bataillon, 34095 MONTPELLIER Cedex, FRANCE

E-mail address: `cedric.bonnafe@math.univ-montp2.fr`

J. Guilhot, School of Mathematics, University of East Anglia, Norwich NR4 7TJ, UK

E-mail address: `j.guilhot@uea.ac.uk`