A Kinematic Model of the Nonholonomic n-bar System: Geometry and Flatness

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Abstract: We propose a kinematic model of a system moving in $\mathbb{R}^{m+1}$ and consisting of n rigid bars attached successively to each other and subject to the nonholonomic constraints that the velocity of the source point of each bar is parallel to that bar. We prove that the associated control system is controllable and feedback equivalent to the m-chained form around any regular configuration. Hence we deduce that the n-bar system is flat and the cartesian position of the source point of the last bar is a flat output. The n-bar system is a natural generalization of the n-trailer system and we provide a comparison of flatness properties of both systems.

Keywords: n-bar system, m-chained form, Cartan distribution, feedback equivalence, flatness

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1. INTRODUCTION

The well known n-trailer system was proposed by Laumond (1991) to model a unicycle-like robot towing trailers. This nonholonomic model has attracted a lot of attention and has been a source of inspiration to study its various properties: controllability (Laumond (1991)), structure (Jean (1998), Pasillas-Lépine and Respondek (2001c), Mormul (2000)), flatness (Fliess et al. (1995), Jakubczyk (1993)), motion planning and tracking (Laumond (1998), Murray and Sastry (1993), Pasillas-Lépine and Respondek (2001d)), optimal control (Laumond (1998)), etc. In this paper we propose its generalization, which we call the n-bar system, consisting of a "train" of n rigid bars subject to nonholonomic constraints (see a detailed description in Section 2 below). We study the geometry of the model of the n-bar system and prove that around any regular configuration (that is, none of the angles between two consecutive bars is $\pm\frac{\pi}{2}$), the associated control system is feedback equivalent to the m-chained form. This implies that the n-bar system is flat around any regular configuration and we show that the cartesian position of the source point of the last (from the top) bar is a flat output. We show also that all other minimal flat outputs are equivalent to that one. This is in contrast with the n-trailer system for which the position of the last trailer is also a flat output but there is a whole family of non equivalent flat outputs (parameterized by one function of three variables, see Li and Respondek (2010b)). As a by-product of our considerations we deduce the global controllability of the n-bar system since it is accessible at any (regular or not) configuration. We send the reader to Li (2010) and Li and Respondek (2010c) for proofs and a geometric analysis of the n-bar system and to Slayaman (2008) and Slayaman and Pelletier (2009) for another, although similar, model for the n-bar system (called there an articulated arm) and for a detailed analysis of singular configurations.

This paper is organized as follows. We define our model of the n-bar system in Section 2. We provide geometric notions and recall a characterization of Cartan distributions $\mathcal{CC}^n(\mathbb{R},\mathbb{R}^m)$ in Section 3. Then we give our main results: equivalence of the n-bar system in $\mathbb{R}^{m+1}$ to the m-chained form and global controllability in Section 4 and its flatness in Section 5.

2. N-BAR SYSTEM IN $\mathbb{R}^{M+1}$

In this section we will consider the n-bar system moving in $\mathbb{R}^{m+1}$, as shown on Figure 1, and derive a kinematic model for it. It is assumed that all n components of the n-bar system are attached in such a way that P_i is the source point of the $(i+1)$-th bar and simultaneously the endpoint of the i-th bar and that the instantaneous velocity of the point P_i is parallel to the vector $\overrightarrow{P_iP_{i+1}}$, for $0 \le i \le n-1$. Furthermore, each rigid bar is assumed to have length one. The coordinates of P_i in $\mathbb{R}^{m+1}$ are given by $P_i = (x_i^1, x_i^2, \dots, x_i^{m+1})$, $0 \le i \le n$. Clearly, the configuration of the n-bar system can be described completely by the $(n+1)(m+1)$ coordinates

$$x_0^1,\dots,x_0^{m+1},x_1^1,\dots,x_1^{m+1},\dots,x_n^1,\dots,x_n^{m+1}$$

in $X = \mathbb{R}^{(n+1)(m+1)}$. Due to the assumption $|\overrightarrow{P_iP_{i+1}}| = 1$, for $0 \le i \le n-1$, we have the holonomic constraints $\Psi(x) = 0$, where $\Psi = (\Psi_1,\dots,\Psi_n)^\top : X = \mathbb{R}^{(n+1)(m+1)} \to \mathbb{R}^n$ is given by

$$\begin{cases} \Psi_1(x) &= (x_1^1 - x_0^1)^2 + \cdots + (x_1^{m+1} - x_0^{m+1})^2 - 1 \\ \Psi_2(x) &= (x_2^1 - x_1^1)^2 + \cdots + (x_2^{m+1} - x_1^{m+1})^2 - 1 \\ &\vdots \\ \Psi_n(x) &= (x_n^1 - x_{n-1}^1)^2 + \cdots + (x_n^{m+1} - x_{n-1}^{m+1})^2 - 1. \end{cases} \tag{1}$$

Under these n holonomic constraints, the true configuration space of the n-bar system becomes the regular embedded submanifold $Q = \mathbb{R}^{m+1} \times (S^m)^n \subset X$ defined

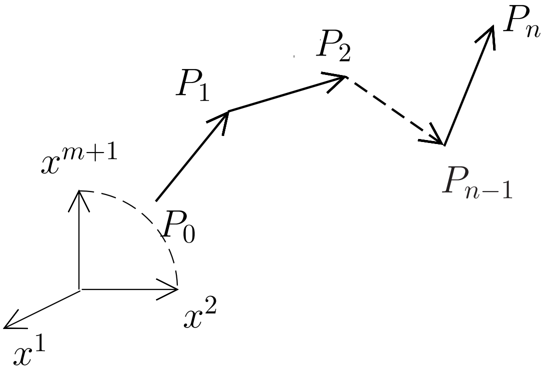

Fig. 1. n-bar system in $\mathbb{R}^{m+1}$

by $Q = \{x \in X : \Psi(x) = 0\}$. Moreover, the constraint $\Psi(x) = 0$ implies that for any $1 \leq i \leq n$, there always exists $1 \leq \sigma(i) \leq m+1$, such that $x_i^{\sigma(i)} - x_{i-1}^{\sigma(i)} \neq 0$. Now the assumption that the instantaneous velocity of the point P_i is parallel to the vector $\overrightarrow{P_iP_{i+1}}$, for $0 \leq i \leq n-1$, imposes the following nonholonomic constraints on the n-bar system: the velocity of the system along any trajectory is annihilated by the following differential 1-forms

$$\Omega_i^j = (x_i^j - x_{i-1}^j)\mathrm{d}x_{i-1}^{\sigma(i)} - (x_i^{\sigma(i)} - x_{i-1}^{\sigma(i)})\mathrm{d}x_{i-1}^j,$$

for $1 \leq i \leq n$, $1 \leq j \leq m+1$ and $j \neq \sigma(i)$. The distribution $\mathcal{E}$ annihilated by all forms Ω_i^j is given by

$$\mathcal{E} = \bigcap_{i,j} \ker \Omega_i^j = \operatorname{span}\{g_1, \ldots, g_{n+m+1}\},$$

where

$$\begin{aligned}
g_1 &= (x_1^1 - x_0^1)\frac{\partial}{\partial x_0^1} + \cdots + (x_1^{m+1} - x_0^{m+1})\frac{\partial}{\partial x_0^{m+1}} \\
g_2 &= (x_2^1 - x_1^1)\frac{\partial}{\partial x_1^1} + \cdots + (x_2^{m+1} - x_1^{m+1})\frac{\partial}{\partial x_1^{m+1}} \\
&\vdots \\
g_n &= (x_n^1 - x_{n-1}^1)\frac{\partial}{\partial x_{n-1}^1} + \cdots + (x_n^{m+1} - x_{n-1}^{m+1})\frac{\partial}{\partial x_{n-1}^{m+1}} \\
g_{n+i} &= \frac{\partial}{\partial x_n^i}, \quad 1 \leq i \leq m+1,
\end{aligned} \tag{2}$$

which defines the control-linear system on $X = \mathbb{R}^{(n+1)(m+1)}$

$$\Delta : \quad \dot{x} = \sum_{i=1}^{n+m+1} g_i(x)v_i, \quad x \in X. \tag{3}$$

To obtain a kinematic model of the n-bar system we have to constrain the system Δ to the regular submanifold $Q \subset X$. Consider the embedding $\Phi : Q \to X$ such that $\Phi(q) = q$, for any $q \in Q$. Let $\mathcal{J}$ be the codistribution spanned by all differential forms Ω_i^j, i.e.,

$$\mathcal{J} = \operatorname{span}\{\Omega_i^j,\ 1 \leq i \leq n,\ 1 \leq j \leq m+1,\ j \neq \sigma(i)\}. \tag{4}$$

Clearly, $\mathcal{J} = \mathcal{E}^\perp$ and the pull back Φ^* maps $\mathcal{J}$ into a codistribution $\mathcal{I} = \Phi^*\mathcal{J}$ on Q, i.e.,

$$\mathcal{I} = \operatorname{span}\{\omega_i^j,\ 1 \leq i \leq n,\ 1 \leq j \leq m+1,\ j \neq \sigma(i)\}, \tag{5}$$

where $\omega_i^j = \Phi^*\Omega_i^j$. Define a distribution $\mathcal{D}$ on Q as $\mathcal{D} = \mathcal{I}^\perp$. Notice that $\mathcal{D}$ is just the intersection $TQ \cap \mathcal{E}$ and is of constant rank equal to $m+1$ thus giving rise to a driftless (control-linear) system

$$\Gamma : \ \dot{q} = \sum_{i=0}^{m} f_i(q)u_i, \ q \in Q = \mathbb{R}^{m+1} \times (S^m)^n, \tag{6}$$

where locally $\mathcal{D} = \operatorname{span}\{f_0, \ldots, f_m\}$, which describes completely the n-bar system moving in $\mathbb{R}^{m+1}$. To summarize, the model Γ describing the n-bar moving in $\mathbb{R}^{m+1}$, for $m \geq 1$, is defined by the control-linear system $\triangle$, given by (2)-(3), together with the holonomic constraint $\Psi(x) = 0$. Notice that, we give explicit expression neither for the distribution $\mathcal{D}$ nor for the vector fields f_i, for $0 \leq i \leq m$. All their properties will be formulated and analyzed in terms of the distribution $\mathcal{E}$ and the holonomic constraint $\Psi(x) = 0$. Another, although similar, model for the n-bar system (called articulated arm) has been very recently introduced and studied in Slayaman (2008) and Slayaman and Pelletier (2009); in the former a detailed analysis of the singular locus (see also Section 4 below) has been performed.

The presented model of the n-bar system in $\mathbb{R}^{m+1}$ is a natural generalization of the well known n-trailer system on $\mathbb{R}^2$. The latter is a model for a unicycle-like mobile robot towing n-trailers such that the tow hook of each trailer is located at the center of its unique axle (with the assumption that the distances between any two consecutive trailers are equal). The n-trailer system is subject to nonholonomic constraints: it is assumed that the wheels of each individual trailer are aligned with the body and are not allowed to slip (Laumond (1991)). Clearly, the nonholonomic constraint that the wheel can not slip on the plane $\mathbb{R}^2$, can be equivalently rephrased that the instantaneous velocity of the middle point of the i-th trailer axle, say point P_i, is parallel to the vector $\overrightarrow{P_iP_{i+1}}$ joining the two consecutive axles. These are exactly our nonholonomic constraints imposed for the n-bar system, the only difference being to allow the vectors $\overrightarrow{P_iP_{i+1}}$ to move in $\mathbb{R}^{m+1}$ and not on the plane $\mathbb{R}^2$.

3. CHARACTERIZATION OF CARTAN DISTRIBUTION $\mathcal{CC}^N(\mathbb{R}, \mathbb{R}^M)$

Consider an arbitrary distribution $\mathcal{D}$. The *derived flag* of $\mathcal{D}$ is the sequence of modules of vector fields $\mathcal{D}^{(0)} \subset \mathcal{D}^{(1)} \subset \cdots$ defined inductively by

$$\mathcal{D}^{(0)} = \mathcal{D} \ \text{ and } \ \mathcal{D}^{(i+1)} = \mathcal{D}^{(i)} + [\mathcal{D}^{(i)}, \mathcal{D}^{(i)}], \ \text{for } i \geq 0.$$

The *Lie flag* of $\mathcal{D}$ is the sequence of modules of vector fields $\mathcal{D}_0 \subset \mathcal{D}_1 \subset \cdots$ defined inductively by

$$\mathcal{D}_0 = \mathcal{D} \ \text{ and } \mathcal{D}_{i+1} = \mathcal{D}_i + [\mathcal{D}_0, \mathcal{D}_i], \ \text{ for } i \geq 0.$$

In general, the derived and Lie flags are different; though for any point p in the underlying manifold the inclusion $\mathcal{D}_i(p) \subset \mathcal{D}^{(i)}(p)$ holds, for $i \geq 0$. Two distributions D and $\tilde{\mathcal{D}}$ defined on two manifolds M and $\tilde{M}$, respectively, are *equivalent* if there exists a smooth diffeomorphism φ between M and $\tilde{M}$ such that $(\varphi_*\mathcal{D})(\tilde{p}) = \tilde{\mathcal{D}}(\tilde{p})$, for each point $\tilde{p}$ in $\tilde{M}$.

An alternative description of the above notions can also be given using the dual language of differential forms. A *codistribution* $\mathcal{I}$ of rank s on a smooth manifold M (or a *Pfaffian system*) is a map that assigns smoothly to each point p in M a linear subspace $\mathcal{I}(p) \subset T_p^*M$ of dimension s. Such a field of cotangent s-planes is spanned locally by s pointwise linearly independent smooth differential 1-forms $\omega_1, \ldots, \omega_s$ on M, which will be denoted by $\mathcal{I} = \text{span}\,\{\omega_1, \ldots, \omega_s\}$. Two codistributions (Pfaffian systems) $\mathcal{I}$ and $\tilde{\mathcal{I}}$ defined on two manifolds M and $\tilde{M}$, respectively, are equivalent if there exists a smooth diffeomorphism φ between M and $\tilde{M}$ such that $\mathcal{I}(p) = (\varphi^*\tilde{\mathcal{I}})(p)$ for each point p in M. For a codistribution $\mathcal{I}$, its *derived flag* $\mathcal{I}^{(0)} \supset \mathcal{I}^{(1)} \supset \cdots$ can be defined by

$$\mathcal{I}^{(0)} = \mathcal{I},\ \ \mathcal{I}^{(i+1)} = \{\omega \in \mathcal{I}^{(i)} : \mathrm{d}\omega \equiv 0 \bmod \mathcal{I}^{(i)}\}, \text{ for } i \geq 0,$$

provided that each element $\mathcal{I}^{(i)}$ of this sequence has constant rank. In this case, it is immediate to see that the derived flag of the distribution $\mathcal{D} = \mathcal{I}^{\perp}$ coincides with the sequence of distributions that annihilate the elements of the derived flag of $\mathcal{I}$, that is $\mathcal{D}^{(i)} = (\mathcal{I}^{(i)})^{\perp}$, for $i \geq 0$. The *Engel rank*, at a point p, of a codistribution $\mathcal{I} = \text{span}\,\{\omega_1, \cdots, \omega_s\}$ is the largest integer ρ such that there exists a 1-form α in $\mathcal{I}$ for which $((d\alpha)^{\rho} \wedge \omega_1 \wedge \cdots \wedge \omega_s)(p) \neq 0$. A characteristic distribution of $\mathcal{D}$ is $\mathcal{C}(\mathcal{D}) = \{f \in \mathcal{D} \ : \ [f, \mathcal{D}] \in \mathcal{D}\}$.

Consider $J^n(\mathbb{R}, \mathbb{R}^m)$, the space of n-jets of smooth maps from $\mathbb{R}$ into $\mathbb{R}^m$ and denote its canonical coordinates by $x_0^0, x_1^0, \ldots, x_m^0, x_1^1, \ldots, x_m^1, \ldots, x_1^n, \ldots, x_m^n$, where x_0^0 represents the independent variable and x_i^0 for $1 \leq i \leq m$, represent the dependent variables, and x_i^j, for $1 \leq i \leq m$ and $1 \leq j \leq n$, correspond to the ordinary derivatives $\dfrac{\mathrm{d}^j x_i^0}{\mathrm{d}(x_0^0)^j}$. The *Cartan distribution* on $J^n(\mathbb{R}, \mathbb{R}^m)$, which we will denote by $\mathcal{CC}^n(\mathbb{R}, \mathbb{R}^m)$, is the completely nonholonomic distribution spanned by the following family of vector fields

$$\frac{\partial}{\partial x_0^0} + \sum_{j=0}^{n-1}\sum_{i=1}^{m} x_i^{j+1}\frac{\partial}{\partial x_i^j},\ \frac{\partial}{\partial x_1^n},\ \ldots,\ \frac{\partial}{\partial x_m^n}$$

or, equivalently, annihilated by the following family of differential 1-forms $\mathrm{d}x_i^j - x_i^{j+1}\mathrm{d}x_0^0,\ \ 0 \leq j \leq n-1,\ 1 \leq i \leq m$.

The problem of characterizing distributions that are locally equivalent to the Cartan distribution $\mathcal{CC}^n(\mathbb{R}, \mathbb{R}^m)$ has been studied and solved in the following way by Pasillas-Lépine and Respondek (2001b) (see also Yamaguchi (1982), Pasillas-Lépine and Respondek (2001a), Mormul (2004)):

Theorem 1. A distribution $\mathcal{D}$ of rank $m+1$, with $m \geq 2$, on a manifold M of dimension $(n+1)m+1$ is equivalent, in a small enough neighborhood of a point p in M, to the Cartan distribution $\mathcal{CC}^n(\mathbb{R}, \mathbb{R}^m)$ if and only if the following conditions hold:

(i) $\mathcal{D}^{(n)}(p) = T_pM$;
(ii) $\mathcal{D}^{(n-1)}$ is of constant rank $nm+1$ and contains an involutive subdistribution $\mathcal{L}_{n-1}$ that has constant corank one in $\mathcal{D}^{(n-1)}$;
(iii) $\mathcal{D}(p)$ is not contained in $\mathcal{L}_{n-1}(p)$.

Moreover, if $m \geq 3$, $\mathcal{L}_{n-1}$ exists if and only if the Engel rank of $(\mathcal{D}^{(n-1)})^{\perp}$ equals 1 and $\text{rank}\,\mathcal{C}(\mathcal{D}^{(n-1)}) = (n-1)m$ and is given as $\mathcal{L}_{n-1} = \mathcal{F}_1 + \cdots + \mathcal{F}_m$, where $\mathcal{F}_i = \{f \in \mathcal{D}^{(n-1)} : f \lrcorner \mathrm{d}\omega_i \in (\mathcal{D}^{(n-1)})^{\perp}\}$ and ω_i's are any differential 1-forms such that $\mathcal{I}^{(n-1)} = (\mathcal{D}^{(n-1)})^{\perp} = \text{span}\{\omega_1, \ldots, \omega_m\}$.

Remarks 1. The involutive subdistribution $\mathcal{L}_{n-1}$, whose existence is claimed by (ii), is unique (if it exists) and will be called the canonical involutive subdistribution in $\mathcal{D}^{(n-1)}$. The uniqueness, involutivity, and the explicit form of $\mathcal{L}_{n-1} = \mathcal{F}_1 + \cdots + \mathcal{F}_m$ follow from a result of Bryant (1979) and have been shown in Pasillas-Lépine and Respondek (2001b).

Remark 2. Item (i) and (ii) describe the essential geometric property of distributions equivalent to the Cartan distribution $\mathcal{CC}^n(\mathbb{R}, \mathbb{R}^m)$ while the condition (iii) distinguishes regular points p at which $D(p) \not\subset \mathcal{L}_{n-1}(p)$ from singular points, where this last condition is violated.

The case $m = 1$ is excluded from Theorem 1 because if an involutive subdistribution of corank one $\mathcal{L}_{n-1} \subset \mathcal{D}^{(n-1)}$ exists it cannot be unique and therefore there is not a canonical one. However, a "non-canonical" version of Theorem 1 holds for $m = 1$ as well, as proved in Pasillas-Lépine and Respondek (2001b): a rank-two distribution is equivalent to $\mathcal{CC}^n(\mathbb{R}, \mathbb{R})$, called also the Goursat normal form or chained form, if and only if there exists a distribution $\mathcal{L}_{n-1}$ satisfying the conditions (i), (ii) and (iii) of Theorem 1.

Let a distribution $\mathcal{D}$ of rank $m+1$, with $m \geq 1$, satisfy the items (i) and (ii) of Theorem 1. The *regular locus* of $\mathcal{D}$, denoted by $Reg(\mathcal{D})$, is the subset of M consisting of points at which $\mathcal{D}$ is equivalent to the Cartan distribution $\mathcal{CC}^n(\mathbb{R}, \mathbb{R}^m)$ at $0 \in \mathbb{R}^{(n+1)(m+1)}$. It can be proved that $Reg(\mathcal{D})$ is an open and dense subset of M. In the case $m \geq 2$, since $\mathcal{L}_{n-1}$ is unique we clearly have $Reg(\mathcal{D}) = \{p \in M : \mathcal{D}(p) \not\subset \mathcal{L}_{n-1}(p)\}$.

4. FIRST MAIN RESULT: EQUIVALENCE OF THE N-BAR SYSTEM TO THE M-CHAINED FORM

Consider two driftless control systems

$$\Sigma: \quad \dot{x} = \sum_{i=0}^{m} f_i(x)u_i = f(x)u, \quad x \in M,$$

$$\text{and } \ \tilde{\Sigma}: \quad \dot{\tilde{x}} = \sum_{i=0}^{m} \tilde{f}_i(\tilde{x})\tilde{u}_i = \tilde{f}(\tilde{x})\tilde{u}, \quad \tilde{x} \in \tilde{M},$$

where $u = (u_0, \ldots, u_m)^{\top} \in \mathbb{R}^{m+1}$, $\tilde{u} = (\tilde{u}_0, \ldots, \tilde{u}_m)^{\top} \in \mathbb{R}^{m+1}$ and the rows $f = (f_0, \ldots, f_m)$ and $\tilde{f} = (\tilde{f}_0, \ldots, \tilde{f}_m)$ are formed by C^{∞}-smooth vector fields f_i and $\tilde{f}_i$, $0 \leq i \leq m$, on M and $\tilde{M}$, respectively. We say that Σ and $\tilde{\Sigma}$ are feedback equivalent if there exists a diffeomorphism $\varphi : M \to \tilde{M}, \tilde{x} = \varphi(x)$ and a feedback transformation $u = \beta(x)\tilde{u}$, where $\beta(x)$ is an invertible C^{∞}-smooth $(m+1) \times (m+1)$-matrix such that $\mathrm{D}\varphi(x) \cdot f(x)\beta(x) = \tilde{f}(\varphi(x))$.

Definition 2. An $(m+1)$-input driftless control system $\Sigma : \ \dot{x} = \sum_{i=0}^{m} u_i f_i(x)$, defined on $\mathbb{R}^{(n+1)m+1}$, is said to be in the *m-chained form* if it is represented by

$$\begin{array}{ccccccccc}
\dot{x}_0^0 = u_0 & \dot{x}_1^0 & = & x_1^1 u_0 & \cdots & \dot{x}_m^0 & = & x_m^1 u_0 \\
 & & \cdots & & \cdots & & \cdots & \\
 & \dot{x}_1^{n-1} & = & x_1^n u_0 & \cdots & \dot{x}_m^{n-1} & = & x_m^n u_0 \\
 & \dot{x}_1^n & = & u_1 & \cdots & \dot{x}_m^n & = & u_m.
\end{array}$$

A system in the m-chained form is also called the *canonical contact system* on $J^n(\mathbb{R}, \mathbb{R}^m)$. In fact, the vector fields

$f_0, \ldots, f_m$ of the m-chained form coincide with those generating the Cartan distribution $\mathcal{CC}^n(\mathbb{R}, \mathbb{R}^m)$ given in Section 3. To any control-linear system Σ, we associate the distribution spanned by all its vector fields $\mathcal{D}_\Sigma = \operatorname{span}\{f_0, \ldots, f_m\}$. The (local) feedback equivalence of Σ and $\tilde{\Sigma}$ coincides with the (local) equivalence of the associated distributions $\mathcal{D}_\Sigma$ and $\mathcal{D}_{\tilde{\Sigma}}$. Therefore the statement that a control system Σ is locally feedback equivalent to the m-chained form (equivalently, to the canonical contact system on $J^n(\mathbb{R}, \mathbb{R}^m)$) will always mean that the associated distribution $\mathcal{D}_\Sigma$ is locally equivalent to the Cartan distribution $\mathcal{CC}^n(\mathbb{R}, \mathbb{R}^m)$.

In this section we will formulate our first main result. See Slayaman (2008) and Slayaman and Pelletier (2009) for another approach to the problem of equivalence of the n-bar system (called there the articulated arm system) to the Cartan distribution (and, more generally, to the multi-flag system).

Theorem 3. The n-bar system Γ moving in $\mathbb{R}^{m+1}$, for $m \geq 1$, defined by (6), is locally feedback equivalent to the m-chained form at any point $x \in X = \mathbb{R}^{(n+1)(m+1)}$ satisfying $\Psi(x) = 0$ (that is, at x corresponding to a point $q \in Q$) such that

(R1) $\sum_{j=1}^{m+1}(x_i^j - x_{i-1}^j)(x_{i+1}^j - x_i^j) \neq 0$, for $1 \leq i \leq n-1$, if $m \geq 2$;
(R2) $\sum_{j=1}^{m+1}(x_i^j - x_{i-1}^j)(x_{i+1}^j - x_i^j) \neq 0$, for $2 \leq i \leq n-1$, if $m = 1$.

Moreover, at any point $q \in Q$ (equivalently, at any point $x \in X = \mathbb{R}^{(n+1)(m+1)}$ satisfying $\Psi(x) = 0$), the n-bar system satisfies the condition (i) and (ii) of Theorem 1.

Remark 1. Let $\mathcal{D}_\Gamma$ be the distribution associated to the n-bar system Γ. Define the regular locus $Reg(\Gamma)$ of Γ as $Reg(\Gamma) = Reg(\mathcal{D}_\Gamma)$. Then Theorem 3 implies that the regular loci of Γ are different for the case $m \geq 2$ and $m = 1$ which are defined by (R1) and (R2), respectively, together with the condition $\Psi(x) = 0$. It is obvious that $Reg(\Gamma)$ is open and dense in the configuration space Q for both $m \geq 2$ and $m = 1$.

Remark 2. The regularity condition $\sum_{j=1}^{m+1}(x_i^j - x_{i-1}^j)(x_{i+1}^j - x_i^j) \neq 0$ has a clear interpretation for the n-bar system. Let θ_i, for $1 \leq i \leq n-1$, denote the angles of the $(i+1)$-th bar with respect to the i-th bar, i.e., the angle between the vectors $\overrightarrow{P_{i-1}P_i}$ and $\overrightarrow{P_iP_{i+1}}$. Then clearly the regularity conditions mean that θ_i are different from $\pm\frac{\pi}{2}$, in other words, the i-th bar is not perpendicular to the $(i+1)$-th one. Using the angles θ_i, the regular locus can be rewritten as

$$Reg(\Gamma)_{m\geq 2} = \{q \in Q : \theta_i \neq \pm\frac{\pi}{2}, \quad 1 \leq i \leq n-1\}$$
$$Reg(\Gamma)_{m=1} = \{q \in Q : \theta_i \neq \pm\frac{\pi}{2}, \quad 2 \leq i \leq n-1\}.$$

It is interesting to observe the difference between the planar ($m = 1$) and all other cases ($m \geq 2$). Namely, the angle $\pm\frac{\pi}{2}$ between the bars $\overrightarrow{P_0P_1}$ and $\overrightarrow{P_1P_2}$ (the two most far from the controlled one) is a singularity for $m \geq 2$ but is not for the planar case. The latter implies, in particular, that the 2-bar system in $\mathbb{R}^2$ is transformable into the chained form even if the bars are perpendicular. This is not true any longer if we consider the 2-bar system in the space $\mathbb{R}^{m+1}$, $m \geq 2$ (in $\mathbb{R}^3$, for instance). Of course, the 2-bar system in $\mathbb{R}^2$ is just the 1-trailer system (a unicycle-like mobile robot towing one trailer or, equivalently, a nonholonomic car) and it is well known that the system can be brought into the chained form even if the axles are perpendicular. In other words, the rank 2 distributions on 4-dimensional manifolds with the growth vector $(2, 3, 4)$ have no singularities, a result that goes back to Engel (1890).

The property of controllability of the n-bar system can also be obtained from Theorem 3.

Corollary 4. The n-bar system Γ is globally controllable on $\mathbb{R}^{m+1} \times (S^m)^n$.

5. SECOND MAIN RESULT: FLATNESS OF THE N-BAR SYSTEM IN $\mathbb{R}^{M+1}$

Consider a smooth nonlinear control system $\Xi : \dot{x} = f(x, u)$, where $x \in X$, an n-dimensional manifold and $u \in U$, an m-dimensional manifold. Given any integer l, we associate to Ξ its l-prolongation Ξ^l given by

$$\Xi^l : \quad \begin{array}{rl} \dot{x} & = f(x, u^0) \\ \dot{u}^0 & = u^1 \\ & \vdots \\ \dot{u}^l & = u^{l+1} \end{array}$$

which can be considered as a control system on $X^l = X \times U \times \mathbb{R}^{ml}$, whose state variables are $(x, u^0, u^1, \ldots, u^l)$ and whose m controls are the m components of u^{l+1}. Denote $\bar{u}^l = (u^0, u^1, \ldots, u^l)$.

Definition 5. The system Ξ is called *flat* at a point $(x_0, \bar{u}_0^l) \in X^l = X \times U \times \mathbb{R}^{ml}$, for some $l \geq 0$, if there exist a neighborhood $\mathcal{O}^l$ of $(x_0, \bar{u}_0^l)$ and m smooth functions

$$h_i = h_i(x, u^0, u^1, \ldots, u^l), \quad 1 \leq i \leq m,$$

called *flat outputs*, defined in $\mathcal{O}^l$, having the following property: there exist an integer s and smooth functions γ_i, $1 \leq i \leq n$, and δ_i, $1 \leq i \leq m$, such that we have

$$x_i = \gamma_i(h, \dot{h}, \ldots, h^{(s)}), \quad 1 \leq i \leq n$$
$$u_i = \delta_i(h, \dot{h}, \ldots, h^{(s)}), \quad 1 \leq i \leq m,$$

where $h = (h_1, \ldots, h_m)^\top$, along any trajectory $x(t)$ given by a control $u(t)$ that satisfies $(x(t), u(t), \dot{u}(t), \ldots, u^{(l)}(t)) \in \mathcal{O}^l$.

The compositions $\gamma_i(h, \dot{h}, \ldots, h^{(s)})$ and $\delta_i(h, \dot{h}, \ldots, h^{(s)})$ are, a priori, defined in an open set $\mathcal{O}^{s+l} \subset X^{s+l} = X \times U \times \mathbb{R}^{m(s+l)}$. The above definition requires that $\pi(\mathcal{O}^{s+l}) \supset \mathcal{O}^l$, where $\pi(x, \bar{u}^{s+l}) = (x, \bar{u}^l)$, and that for all such $(x, \bar{u}^{s+l})$, the compositions yield, respectively, x_i and u_i. If $h_i = h_i(x, u^0, u^1, \ldots, u^r)$, $r \leq l$, we will say that the system is $(x, u, \ldots, u^r)$-flat and, in particular, x-*flat* if $h_i = h_i(x)$. In the case $h_i = h_i(x, u^0, u^1, \ldots, u^r)$, we will assume that they are defined on $\mathcal{O}^r \subset X^r = X \times U \times \mathbb{R}^{mr}$, where $\pi^{-1}(\mathcal{O}^r) \supset \mathcal{O}^l$ and π stands for the projection $\pi(x, u^0, \ldots, u^r, \ldots, u^l) = (x, u^0, \ldots, u^r)$.

Let $h_1, \ldots, h_m$ be flat outputs of the system Ξ. It has been observed in Respondek (2003) that there exist integers $k_1, \ldots, k_m$ such that $\operatorname{span}\{\mathrm{d}x_1, \ldots, \mathrm{d}x_n, \mathrm{d}u_1, \ldots, \mathrm{d}u_m\} \subset \operatorname{span}\{\mathrm{d}h_i^{(j)}, 1 \leq i \leq m,\ 0 \leq j \leq k_i\}$, and if at the same time $\operatorname{span}\{\mathrm{d}x_1, \ldots, \mathrm{d}x_n, \mathrm{d}u_1, \ldots, \mathrm{d}u_m\} \subset \operatorname{span}\{\mathrm{d}h_i^{(j)}, 1 \leq$

$i \le m,\ 0 \le j \le \mu_i\}$, then $k_i \le \mu_i$, for $1 \le i \le m$. The m-tuple $(k_1, \ldots, k_m)$ will be called the *differential m-weight* of $h = (h_1, \ldots, h_m)$ and $k = \sum_{i=1}^{m} k_i$ will be called the *differential weight* of h.

Definition 6. Flat outputs of Ξ at $(x_0, \bar{u}_0^l)$ are called minimal if their differential weight is the lowest among all flat outputs of Ξ at $(x_0, \bar{u}_0^l)$.

Let $U_{\text{sing}}(x)$ be the m-dimensional subspace of $\mathbb{R}^{m+1}$ such that for any control $(u_0(x), \ldots, u_m(x))^\top = u(x) \in U_{\text{sing}}(x)$ we have $\sum_{i=0}^{m} f_i(x)u_i(x) \in \mathcal{C}_1(x)$, where $\mathcal{C}_1 \subset \mathcal{D} = \text{span}\{f_0, \ldots, f_m\}$ is the characteristic subdistribution of $\mathcal{D}^{(1)}$. Any control $u(t) \in U_{\text{sing}}(x(t))$ will be called *singular* and the trajectories of the system governed by a singular control remain *tangent* to the characteristic distribution $\mathcal{C}_1$. The following theorem, given in Respondek (2003), characterizes the minimal flat outputs for systems that are feedback equivalent to the m-chained form (i.e., the canonical contact system on $J^n(\mathbb{R}, \mathbb{R}^m)$), with $m \ge 2$.

Theorem 7. Consider the driftless control-linear system $\Sigma : \dot{x} = \sum_{i=0}^{m} f_i(x)u_i$, defined on a manifold X and let $\mathcal{D} = \text{span}\{f_0, \ldots, f_m\}$ be the distribution associated to Σ. If Σ is locally feedback equivalent, at $x_0 \in X$, to the m-chained form, with $m \ge 2$, then the following conditions are equivalent:

(i) $(\mathcal{L}_{n-1})^\perp = \text{span}\{\mathrm{d}h_0, \ldots, \mathrm{d}h_m\}$ around x_0, where $\mathcal{L}_{n-1}$ denotes the subdistribution that is involutive and of corank one in $\mathcal{D}^{(n-1)}$;

(ii) $h_0, \ldots, h_m$ are minimal flat outputs of Σ at (x_0, u^0), where $u^0 \notin U_{\text{sing}}(x_0)$.

The following theorem describes the flatness property of the n-bar system Γ moving in $\mathbb{R}^{m+1}$.

Theorem 8. (Flatness of the n-bar system) For the n-bar system Γ moving in $\mathbb{R}^{m+1}$, where $m \ge 2$, we have

(i) Γ is x-flat at any $(q_0, u^0) \in Q \times \mathbb{R}^{m+1}$ satisfying
  (a) $\Psi(x) = 0$ and $\sum_{j=1}^{m+1}(x_i^j - x_{i-1}^j)(x_{i+1}^j - x_i^j) \neq 0$, where $q_0 \in Q$ is identified with a point $x \in \mathbb{R}^{(n+1)(m+1)}$ satisfying $\Psi(x) = 0$;
  (b) u^0 is such that the instantaneous velocity $\dot{P}_0$ of the point P_0 is nonzero (and thus the instantaneous velocities $\dot{P}_i$, $0 \le i \le n-1$, are nonzero).

(ii) The coordinates $P_0 = (x_0^1, x_0^2, \ldots, x_0^{m+1})$ are minimal x-flat outputs of Γ at any (q_0, u^0) as above.

(iii) If $h_0, \ldots, h_m$ are any minimal flat outputs at (q_0, u^0), then locally around q_0 we have
$$\text{span}\{\mathrm{d}h_0, \ldots, \mathrm{d}h_m\} = \text{span}\{\mathrm{d}x_0^1, \mathrm{d}x_0^2, \ldots, \mathrm{d}x_0^{m+1}\}.$$

The item (iii) is in contrast with the planar case $m = 1$, where minimal flat outputs are not unique and their totality is actually parameterized by an arbitrary function of three variables (See a detailed analysis in Li (2010), Li and Respondek (2010a) and Li and Respondek (2010b)).

Proof: The items (i), (ii) and (iii) are natural consequences of Theorem 3 and Theorem 7. Theorem 3 assures that for $m \ge 2$, the n-bar system Γ is locally feedback equivalent to the m-chained form at any point q_0 that corresponds to $x \in \mathbb{R}^{(n+1)(m+1)}$ satisfying $\Psi(x) = 0$ and $\sum_{j=1}^{m+1}(x_i^j - x_{i-1}^j)(x_{i+1}^j - x_i^j) \neq 0$, for $1 \le i \le n-1$. Moreover, it can be proved that around q_0, the subdistribution $\mathcal{L}_{n-1}$, which is involutive and of corank one in $\mathcal{D}^{(n-1)}$, is given by
$$(\mathcal{L}_{n-1})^\perp = \text{span}\{\mathrm{d}\Phi^* x_0^1, \ldots, \mathrm{d}\Phi^* x_0^{m+1}\}.$$
Notice that on the configuration space Q, we have always that $\Phi^* x_0^j = x_0^j$, for $1 \le j \le m+1$. Thus according to Theorem 7, the coordinates of $P_0 = (x_0^1, x_0^2, \ldots, x_0^{m+1})$ are minimal x-flat outputs of Γ around q_0 which implies immediately that Γ is x-flat at (q_0, u^0) for some control u^0. Before we characterize the control u^0, notice that Theorem 7 implies that for control systems that are feedback equivalent to the m-chained form, for $n \ge 2, m \ge 2$, the minimal flat outputs are equivalent in the sense that for any two families of minimal flat outputs $(h_0, \ldots, h_m)$ and $(\tilde{h}_0, \ldots, \tilde{h}_m)$, we have $\text{span}\{\mathrm{d}h_0, \ldots, \mathrm{d}h_m\} = \text{span}\{\mathrm{d}\tilde{h}_0, \ldots, \mathrm{d}\tilde{h}_m\}$. In view of this and the item (ii) of Theorem 8, any minimal flat outputs $(h_0, \ldots, h_m)$ of the n-bar system in $\mathbb{R}^{m+1}$ for $n \ge 2$, $m \ge 2$ satisfy $\text{span}\{\mathrm{d}h_0, \ldots, \mathrm{d}h_m\} = \text{span}\{\mathrm{d}x_0^1, \mathrm{d}x_0^2, \ldots, \mathrm{d}x_0^{m+1}\}$. This proves (iii).

Now we are going to characterize the control u^0. According to the definition of the flat output, the entire state and all the controls should be functions of the coordinates $x_0^1, x_0^2, \ldots, x_0^{m+1}$ and their time-derivatives. Recall the system Δ given be (2) and (3) and consider the system of equation for the x_0^j-variables
$$\begin{cases} \dot{x}_0^1 &= v_1(x_1^1 - x_0^1) \\ &\vdots \\ \dot{x}_0^{m+1} &= v_1(x_1^{m+1} - x_0^{m+1}) \\ \Psi_1(x) &= \displaystyle\sum_{j=1}^{m+1}(x_1^j - x_0^j)^2 - 1 = 0. \end{cases} \tag{7}$$
A direct computation shows that
$$v_1 = \left((\dot{x}_0^1)^2 + \cdots + (\dot{x}_0^{m+1})^2\right)^{\frac{1}{2}} = \eta_1(P_0, \dot{P}_0), \tag{8}$$
for some function η_1. Substituting (8) into (7), we get
$$\begin{aligned} x_1^1 &= x_0^1 + \frac{\dot{x}_0^1}{v_1} &&= \varphi_1^1(P_0, \dot{P}_0) \\ &\vdots &&\vdots \\ x_1^{m+1} &= x_0^{m+1} + \frac{\dot{x}_0^{m+1}}{v_1} &&= \varphi_1^{m+1}(P_0, \dot{P}_0), \end{aligned}$$
for some functions φ_1^i, for $1 \le i \le m+1$. Put $\varphi_1 = (\varphi_1^1, \ldots, \varphi_1^{m+1})$, we thus have
$$P_1 = (x_1^1, \ldots, x_1^{m+1}) = (\varphi_1^1, \ldots, \varphi_1^{m+1})(P_0, \dot{P}_0) = \varphi_1(P_0, \dot{P}_0).$$
In the same way, we obtain that, for $2 \le i \le n$,
$$v_i = \eta_i(P_{i-1}, \dot{P}_{i-1}) = \tilde{\eta}_i(P_0, \dot{P}_0, P_0^{(2)}, \ldots, P_0^{(i)})$$
$$\text{and} \quad P_i = \varphi_i(P_{i-1}, \dot{P}_{i-1}) = \tilde{\varphi}_i(P_0, \dot{P}_0, P_0^{(2)}, \ldots, P_0^{(i)}),$$
for some functions $\tilde{\eta}_i$ and $\tilde{\varphi}_i$. Finally, the controls v_{n+j}, for $1 \le j \le m+1$, can be expressed by
$$\begin{aligned} v_{n+j} = \dot{x}_n^j &= \frac{\mathrm{d}}{\mathrm{d}t}(\tilde{\varphi}_n^j(P_0, \dot{P}_0, P_0^{(2)}, \ldots, P_0^{(n)})) \\ &= \tau_j(P_0, \dot{P}_0, P_0^{(2)}, \ldots, P_0^{(n+1)}), \end{aligned}$$
for some functions τ_j. In this way, the entire state and all controls v_i, for $1 \le i \le n+m+1$, are expressed as functions of the coordinates of $P_0 = (x_0^1, x_0^2, \ldots, x_0^{m+1})$ and their derivatives up to order $n+1$. The n-bar system Γ has $m+1$ controls while the system Δ has $n+m+1$ controls.

So there must be n relations between controls of Δ when restricted to $Q = \{x \in X : \Psi(x) = 0\}$. We will see this below and at the same time we will clarify the problem of singularities. Clearly, in order that the above computations hold, all the controls v_i, for $1 \leq i \leq n$, cannot vanish. It is sufficient, however, to assume that the control v_1 is nonzero since around any point x satisfying $\Psi(x) = 0$ and the regularity condition $\sum_{j=1}^{m+1}(x_i^j - x_{i-1}^j)(x_{i+1}^j - x_i^j) \neq 0$, the condition $v_1 \neq 0$ implies that $v_i \neq 0$, for $2 \leq i \leq n$. In fact, differentiating the constraint
$\Psi_1(x) = (x_1^1 - x_0^1)^2 + (x_1^2 - x_0^2)^2 + \cdots + (x_1^{m+1} - x_0^{m+1})^2 - 1 = 0$,
we get $\sum_{j=1}^{m+1} 2\left((x_1^j - x_0^j)\dot{x}_1^j - (x_1^j - x_0^j)\dot{x}_0^j\right) = 0$.

Substituting $\dot{x}_0^j = v_1(x_1^j - x_0^j)$ and $\dot{x}_1^j = v_2(x_2^j - x_1^j)$, for $1 \leq j \leq m+1$, into the above equation, by a simple calculation we get $v_1 = v_2 \sum_{j=1}^{m+1}(x_1^j - x_0^j)(x_2^j - x_1^j)$. Recall that around any regular point q_0, we have always that $\sum_{j=1}^{m+1}(x_1^j - x_0^j)(x_2^j - x_1^j) \neq 0$. Therefore, the condition $v_1 \neq 0$ implies that $v_2 \neq 0$ and similarly, it can be shown that

$$v_i = v_{i+1} \sum_{j=1}^{m+1}(x_i^j - x_{i-1}^j)(x_{i+1}^j - x_i^j),$$

for $1 \leq i \leq n-1$ and $v_n = \sum_{j=1}^{m+1}(x_n^j - x_{n-1}^j)v_{n+j}$.

The above equations show, first, that $v_1 \neq 0$ is equivalent to $v_i \neq 0$, $1 \leq i \leq n$. Secondly, they imply that there exist n relations between the controls v_i, $1 \leq i \leq n+m+1$, of Δ implying that the n-bar system possesses, indeed, $m+1$ controls. Moreover, the condition $v_1 \neq 0$ (which yields $v_i \neq 0$, $1 \leq i \leq n$) implies that the instantaneous velocity $\dot{P}_0$ of the point P_0 is nonzero (and, consequently, the instantaneous velocities $\dot{P}_i$ of the points P_i, $0 \leq i \leq n-1$, are nonzero). Therefore the condition (b) holds.

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