CREATING CHAOS FROM A GENERIC FAMILY OF VECTOR FIELDS ADMITTING A HOPF BIFURCATION

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Abstract. This text deals with the problem of creating a chaotic differential system from a generic one-parameter family of smooth vector fields on $\mathbb{R}^n$, that exhibits a Hopf bifurcation, by imposing a dynamics on the parameter. We prove that if all the non purely imaginary eigenvalues of the Jacobian at the bifurcation point have a strictly negative real part, we can create such a chaotic system by inducing a hysteresis dynamics on it. After having presented the proof of this result, we illustrate it by showing how to induce a chaotic behavior in a four-dimensional differential system modeling the behavior of a hypothetical regulatory system.

1. Introduction

Although far from experiment, this article is motivated by the growing interest in understanding and controlling biological regulatory systems, two main topics in systems and synthetic biology, where identification and characterization of regulatory units with prescribed dynamical features are essential (see [9] and [5]). Many metabolic and genetic intracellular regulations are homeostatic, that is they maintain constant some viability parameters (rate, concentration, level, etc.) by perpetually adapting the internal state of the cell to a changing environment. Temperature, pH, osmotic pressure are emblematic, but other examples are the control of internal concentration of metal ions which are essential as trace but become lethal at higher rate (as can be seen in [7]). In general, breaking of homeostasis can lead important damages, like Wilson's disease in the context of copper homeostasis ([7]).

The question addressed in this article is how to break homeostasis and induce complex and even chaotic dynamics in a given model of homeostatic regulatory unit. In the article [6], an analogous approach led to the following result: given a one-parameter family of smooth ordinary differential equations that all have a globally stable asymptotic state, it is possible, under very mild conditions, to construct a feedback on the parameter to get a chaotic dynamical system. Here we consider a generic one-parameter family of smooth vector fields on $\mathbb{R}^n$ admitting a local Hopf bifurcation: we prove that if the non purely imaginary eigenvalues of the Jacobian at the bifurcation point have a strictly negative real part, we can create under a generic hypothesis, a (fast) hysteresis dynamics on the parameter, so as to obtain a slow-fast chaotic dynamical system. Naturally, the expression "slow-fast" system refers to the singular perturbations theory. Indeed, we need to put two scales of time in our extended system, otherwise it would have too many possible behaviors. In our setting, a chaotic dynamical system is a system having a strictly positive topological entropy, which guarantees a sensitive dependence on initial conditions.

Date: 18 march 2011.

As usual, the transition from a time-continuous dynamical system to a discrete one relies on the construction of a Poincaré return map. And to prove the presence of a chaotic behavior, we will show that this return map has a Horseshoe-type dynamics. We refer the reader to [1], [2], [11] for more details on these standard techniques. Remark that concerning the proof of our result, our strategy differs from the one adopted by E.Pécou in [6] in which a homoclinic orbit is constructed, so as to meet or to reconstruct a well-known chaotic situation (namely those respectively described by the Shilnikov's theorem, and the geometric Lorenz attractors); here we directly construct a Horseshoe.

The paper is divided in three parts. In the first one (section 2), we state and prove the main result (Theorem (2.5)) that we have just presented above, in the case $n = 2$. The mechanism creating chaos relies on a linear model (see 2.3.b) with two spirals having close but distinct centers, for which we construct a one-dimensional Poincaré section which is covered twice. Then we prove that chaos is kept up when the situation is no more linear by constructing a Horseshoe from this one-dimensional section (see 2.3.c and 2.3.d). We finish this section by proving a corollary of Theorem (2.5).
Next, the general case $n \geq 2$ is proved in section 3 with the exactly same reasoning, because the assumption on the Jacobian at the bifurcation point forces the dynamics to locally restrict itself to a manifold of dimension two. Let us mention that the ideas developed here have been more or less already studied by many authors like R.Lozi ([4]), even though they did not consider the matching of an oscillating system (the one having a Hopf bifurcation) with a bistable one.
Lastly, in the third part (section 4) we illustrate our result by exhibiting a chaotic behavior in a system of four differential equations and nineteen parameters, called V-system. These equations have been introduced by J.J Tyson[1] and E.Pécou, and represent a hypothetical gene regulatory network with classical regulatory functions of Michaelis-Menten and Hill type. In order to be in the situation of the first part, we need to use three scales of time (subsection 4.2), which comes to imposing three different scales of parameters. In the last subsection 4.3, we numerically exhibits for this particular system a Poincaré map that covers twice the section on which it is defined, which guarantees the presence of chaos in this system.

Let us notice that our result is in coherence with Thomas' conjecture which states that a positive (the bistable switch) and a negative (the Hopf subsystem) feedbacks are necessary conditions for chaos ([8]).

2. The result in the case $n = 2$.

Let $(\mathrm{f}_\mu)_{\mu\in\mathbb{R}}$ be a generic family of smooth vector fields in $\mathbb{R}^n$, depending smoothly on the parameter μ, admitting a local Hopf bifurcation in a point x^0 of $\mathbb{R}^n$, for the value of parameter μ_0. For each real μ, the associated vector field has a flow governed by the equation:

$$\dot{x} = \mathrm{f}\left(x, \mu\right),$$

where f is the smooth function naturally defined on $\mathbb{R}^{n+1}$ by the equality $\mathrm{f}\left(x, \mu\right) = \mathrm{f}_\mu\left(x\right)$. Our goal is to prove that under an assumption of contractility of the eigenvalues of the Jacobian at x^0, and under a hypothesis on the curve of critical points

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of f, which is implicitely defined in a neighborhood of x^0, we can construct a smooth function g such that the extended singularly perturbed system:

$$\dot{x} = \mathrm{f}\,(x, \mu)$$
$$\epsilon\dot{\mu} = \mathrm{g}\,(x, \mu)\,,$$

where ϵ is a small positive number, is chaotic.
In the rest of the section we set $n = 2$.

2.1. **The hypotheses.** Here we precise the context given above.

2.1.a. *Assumptions on the function* f. **H1** There exists a value μ_0 and a critical point x^0 of the field f_{μ_0} such that the Jacobian $D_x\mathrm{f}_{\mu_0}\left(x^0\right)$ has a pair of pure imaginary complex eigenvalues $\pm \mathrm{i}\beta$ with $\beta > 0$.

By the implicit function theorem, the curve of zeros of f is, in a neighborhood of $\left(x^0, \mu_0\right)$, the graph of a smooth function in the variable μ, defined in a small open interval $\mathcal{U}$. We denote by $\phi = (\phi_1, \phi_2)$ this function.

H2 Let $\alpha\,(\mu) \pm \mathrm{i}\beta\,(\mu)$ be the eigenvalues of the Jacobian $D_x\mathrm{f}_\mu\,(\phi\,(\mu))$ that are equal to $\pm\mathrm{i}\beta$ in μ_0. We assume we have $\alpha'\,(\mu_0) \neq 0$, which means that the two eigenvalues cross the pure imaginary axis with a non-zero velocity.

Under these two assumptions, there exists a change of variables for which the Taylor expansion of degree three of f is of the form:

$$\dot{x_1} = \left(a_0\mu + b_0\left(x_1^2 + x_2^2\right)\right)x_1 - \left(\omega + a_1\mu + b_1\left(x_1^2 + x_2^2\right)\right)x_2 \tag{1}$$
$$\dot{x_2} = \left(\omega + a_1\mu + b_1\left(x_1^2 + x_2^2\right)\right)x_1 + \left(a_0\mu + b_0\left(x_1^2 + x_2^2\right)\right)x_2. \tag{2}$$

Generically, the coefficients a_0, b_0 are non null in the last equations: in this case, restricting the interval $\mathcal{U}$ if necessary, the Hopf bifurcation's theorem tells us that all the fixed points $\phi\,(\mu)$ with μ smaller than μ_0 are focuses encircled by limit cycles (which disappear in the bifurcation value μ_0), and all the fixed points $\phi\,(\mu)$ with μ greater than μ_0 are also focuses but with inverse stability. Recall that this Hopf bifurcation is supercritical if $\alpha'\,(\mu_0) < 0$, which means that the cycles are stable so that the focuses in the planes $\mu > \mu_0$ are stable ones, or subcritical ifnot (in which case the stability is reverse).

Remark 2.1. *It is well known that Assumptions* **H1**, **H2** *are not sufficient to get a Hopf bifurcation. This can be seen by looking at the system (written in complex form)* $\dot{z} = (\mu + \mathrm{i})\,z$. *Our result will also work for such a degenerate case.*

2.1.b. *Hysteresis curve.* We want to perturb the parameter μ by imposing a dynamics on it thanks to a curve called *hysteresis* (or sometimes *switch*):

Definition 2.2. *By hysteresis we mean a connected curve in* $\mathbb{R}^2$*, defined by some equations of the form* $\mathrm{g}\,(x, y) = 0$*, where* g *is a smooth function from* $\mathbb{R}^2$ *to* $\mathbb{R}$*, for which there exist two reals* $x^* < x^{**}$ *such that:*
(i)*For every* $x < x^*$ *or* $x > x^{**}$*, there is only one* z *in* $\mathbb{R}$ *verifying* $\mathrm{g}\,(x, z) = 0$*, and this unique critical point of the equation* $\dot{y} = \mathrm{g}\,(x, y)$ *is asymptotically stable.*
(ii)*For every* $x^* < x < x^{**}$*, there are exactly three zeros* $(x, z_i)_{i=1,2,3}$ *of* g*. Among them, two points say* z_1, z_3 *are asymptotically stable critical points of* $\dot{y} = \mathrm{g}\,(x, y)$ *(and the other one is repulsive).*

(iii)*For $x = x^*$ or $x = x^{**}$ there are exactly two zeros $(x, z_i)_{i=1,2}$ of* g*. One of the z_i's is asymptotically stable and the other one is degenerate.*

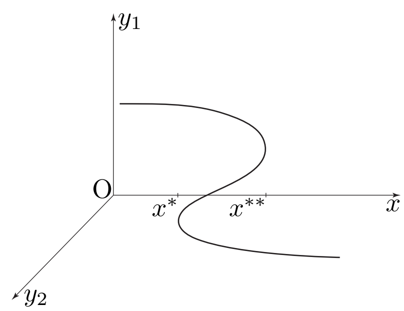

FIGURE 1. A hysteresis curve.

So a hysteresis can be decomposed in three parts: two stable curves that we call *the superior and inferior branches*, separated by the *unstable branch.* If y^*, y^{**} are the degenerate critical points of the vector fields g_{x^*} and $\mathrm{g}_{x^{**}}$ let us remark that locally in (x^*, y^*) appears a saddle-node bifurcation, since in a neighborhood of it we have a stable critical point and an unstable one that collapse and then disappear, as the parameter x varies. We call (x^*, y^*) the "first saddle-node bifurcation point". The same thing happens in the "second saddle-node bifurcation point" (x^{**}, y^{**}).

Notation 2.3. *In the rest of the text we will only use the notations (x^*, y^*), (x^{**}, y^{**}) so as to mention the first and second saddle-node bifurcation points of a given hysteresis.*

Remark 2.4. *Notice that the inferior and superior branches of a hysteresis defined by a function* g *are no more invariant when we perturb the equation $\dot{y} = \mathrm{g}(x, y)$ by adding an equation of the form $\dot{x} = \mathrm{f}(x, y)$, as we are going to do in the following.*

We will denote by $\mathcal{G}_2$ the set of smooth real-valued functions g on $\mathbb{R}^2$ defining such curves, and by $\mathcal{G}_{c,2}$ the subset of $\mathcal{G}_2$ composed of hysteresis that can in addition be described as the graph of a (cubic-like) function in the variable y. (By a cubic function we mean a third degree polynomial function).
We will also use hysteresis having flat inferior and superior branches and a linear unstable one (see figure 2), and will refer to them as *hysteresis with flat branches.* They are defined by their points (x^*, y^*) and (x^{**}, y^{**}), that we'll also call *bifurcation points*, even though we cannot really say there is a bifurcation in this case. $\mathcal{G}_{p,2}$ will be the set of piecewise linear functions from $\mathbb{R}^2$ to $\mathbb{R}$, defining such curves. To construct an element h of $\mathcal{G}_{p,2}$, one can for instance take a negative real a and define the real function e by:

$$\forall x \in \mathbb{R},\ \mathrm{e}(x) = \begin{cases} y^{**} & \text{if}\, x \leq a, \\ (y^{**} - y^*)\, x/a + y^* & \text{if}\ x \in [a, 0], \\ y^* & \text{if}\, x \geq 0 \end{cases}$$

Then it suffices to bend the graph of e by considering the function $\mathrm{h}(x, y) = \mathrm{e}(x - by) - y$, for a convenient choice of the constant b, and to notice that the three constant branches of the hysteresis defined by h have the desired stability.

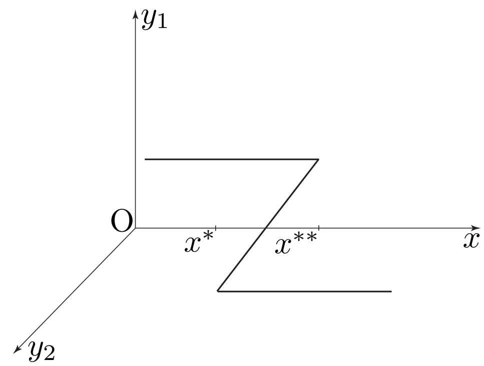

FIGURE 2. A hysteresis with flat branches.

2.2. **The theorem.** The two-dimensional version of our result is the following:

Theorem 2.5. *Let us assume Hypotheses* **H1**, **H2** *on* f *are satisfied. If the function* ϕ *verifies* $(\phi_1'(\mu_0), \phi_2'(\mu_0)) \neq (0,0)$*, then there exists a smooth function* h *in* $\mathcal{G}_2$*, and a non zero real number* a*, such that for every sufficiently small positive number* ϵ*, the singularly perturbed system:*

$$\left\{ \begin{array}{ll} \dot{x_1} & = \mathrm{f}_1(x_1, x_2, \mu) \\ \dot{x_2} & = \mathrm{f}_2(x_1, x_2, \mu) \\ \epsilon\dot{\mu} & = \mathrm{h}(x_1 + a x_2, \mu) \end{array} \right. , \tag{3}$$

taken in a neighborhood $\mathcal{V}$ *of* $\left(x^0, \mu_0\right)$ *enough small, is chaotic.*

Notice that the hypothesis on the curve of critical points of f is generic, and that this condition is the same as the one asked by E.Pécou, in her article [6].
If one wants to have a hysteresis that can be described as a cubic-like function in the variable μ, the analogous result is the following:

Corollary 2.6. *With the same assumptions as in Theorem* (2.5)*, there exists a smooth function* h *in* $\mathcal{G}_{c,2}$*, a non zero real* a*, and two small numbers* $0 < \epsilon_1 < \epsilon_2$*, such that for every* ϵ *in* $]\epsilon_1, \epsilon_2[$*, the system* (3)*, taken in a neighborhood* $\mathcal{V}$ *of* $\left(x^0, \mu_0\right)$ *enough small, is chaotic.*

The proof of this corollary is the same as the one of (2.5), except for its last step (see subsection 2.4).

2.3. **Proof of Theorem** (2.5)**.** *All along the proof, we assume that we have* $\alpha'(\mu_0) < 0$ *(in case of a Hopf bifurcation at* μ_0 *this means the cycles are stable ones). The case where* $\alpha'(\mu_0) > 0$ *is totally similar (see the remark at the end of the subsection).*

First we describe the hysteresis curves with flat branches that are conveniently placed comparing with the curve of critical points ϕ, in order to prove Theorem (2.5) in the (weaker) case when the function h is in $\mathcal{G}_{p,2}$. Later we will transform the convenient function h into a smooth one.
Let us fix a value μ^{**} in $\mathcal{U}$ greater than μ_0. Because of the generic assumption there exists a value μ^* for which we have $\phi(\mu^*) \neq \phi(\mu^{**})$. Without loss of generality, we suppose that the point $(\phi(\mu^*), \mu^*)$ is the origin O. We want to work with a function

g of which set of zeros is the cartesian product of the straight line $(\mathrm{O}\phi(\mu^{**}))$ with a hysteresis in the plane orthogonal to this line. To do this we begin by choosing two values x_1^*, x_1^{**} verifying $x_1^* < x_1^{**} < 0$ if $\phi_2(\mu^{**}) > 0$ (or else $x_1^* > x_1^{**} > 0$ if $\phi_2(\mu^{**}) < 0$) and such that the hysteresis with flat branches defined by the two bifurcation points $(x_1^*, 0)$ and (x_1^{**}, μ^{**}) does not intersect the graph of ϕ in another point than the origin. Besides we ask that the reals x_1^*, x_1^{**} are close enough to zero, so that the points $(x_1^*, 0)$ and $(x_1^{**}, 0)$ are inside the possible cycle C_0 belonging to the plane $\mu = 0$ (see the figure 3).

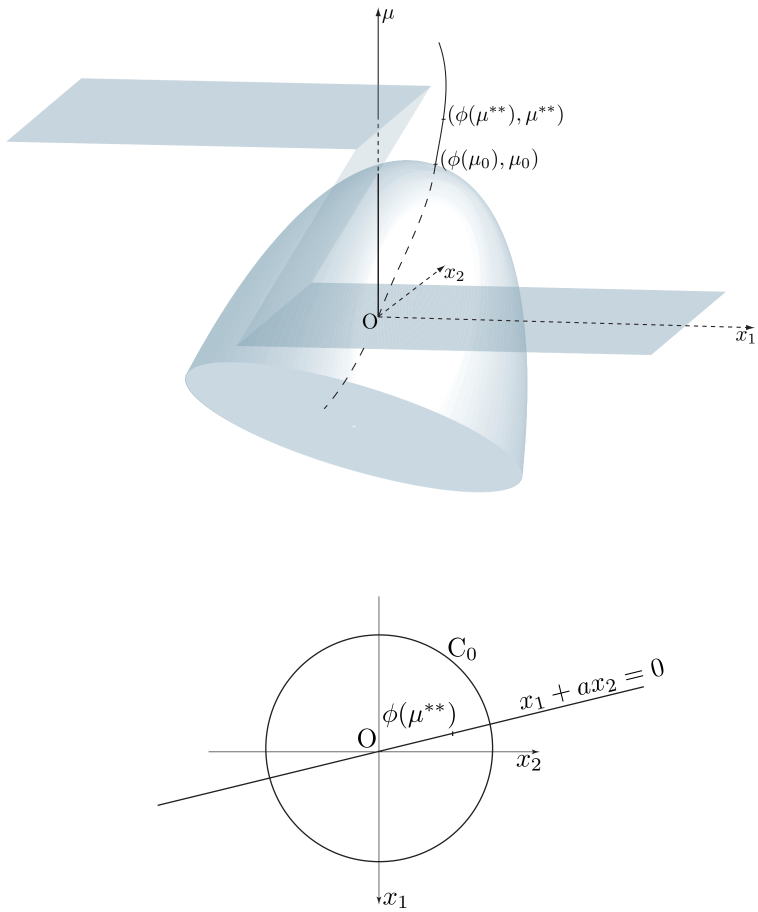

FIGURE 3. The Hopf bell-like surface and the surface defined by a piecewise-linear hysteresis.

In the rest of the text we will assume without loss of generality that $\phi_2(\mu^{**}) > 0$, and thus the values x_1^*, x_1^{**} will be taken negative. Now that we have defined our function h, we set $\mathrm{g}(x_1, x_2, \mu) = \mathrm{h}(x_1 + ax_2, \mu)$, where $x_1 + ax_2 = 0$ is an equation of the straight line $(\mathrm{O}\phi(\mu^{**}))$. Our first goal is to prove that for a certain choice of the values x_1^*, x_1^{**} and μ^{**}, such a function g verifies that the system:

(4)
$$\begin{cases} \dot{x_1} &= \mathrm{f}_1(x_1, x_2, \mu) \\ \dot{x_2} &= \mathrm{f}_2(x_1, x_2, \mu) \\ \epsilon\dot{\mu} &= \mathrm{g}(x_1, x_2, \mu) \end{cases},$$

is chaotic.

2.3.a. *Using the singular perturbations theory to describe the flow of* (4). Let's take a piecewise linear function g from $\mathbb{R}^3$ to $\mathbb{R}$ of the same kind as above. By construction, the system (4) admits the origin as unique critical point. Remark that since the function $x_1 \mapsto \mathrm{h}(x_1, 0)$ is constant, there exists locally in the origin, an invariant stable manifold of dimension one and an unstable one of dimension two. In fact, we even have that the inferior and superior half planes of the hysteresis, that are the cartesian products of the stable branches of h by the line $(\mathrm{O}\phi(\mu^{**}))$, are invariant except near the two straight lines Δ^* and Δ^{**} (that we call the two *fold lines* of the hysteresis), respectively defined by the equations $x_1 + ax_2 = x_1^*$ and $x_1 + ax_2 = x_1^{**}$. This is clear by the Cauchy-Lipschitz theorem, which can be applied here because the function h is Lipschitz continuous (indeed, the function e defined at the end of 2.1.b is Lipschitz continuous).
Moroever, these invariant half planes can be described as (constant) graphs in the variables x_1, x_2: namely the graphs $\xi_-(x_1, x_2) = 0$ for (x_1, x_2) above Δ^* and $\xi_+(x_1, x_2) = \mu^{**}$ for (x_1, x_2) below Δ^{**}.
This fact allows us to use singular perturbation theory:

Proposition 2.7. *As $\epsilon > 0$ tends to zero, the flow $\varphi_{2,\epsilon}$ of* (4) *is C^0-approached by a flow φ_2 of which trajectories are successions of continuous arcs, each of them being the union of a segment of the form $\left\{\left(x_1^0, x_2^0, \mu\right) : 0 \leq \mu \leq \mu^{**}\right\}$ with a solution of one of the equations $\dot{x} = \mathrm{f}(x, \xi_+(x))$ and $\dot{x} = \mathrm{f}(x, \xi_-(x))$. More precisely, we have:*

$$\forall M > 0,\, \forall p \in \mathcal{V},\, \forall t \in [0, M],\, \lim_{\epsilon \to 0} \varphi_{2,\epsilon}(p, t) = \varphi_2(p, t),$$

where $\mathcal{V}$ is a neighborhood of $\left(x^0, \mu_0\right)$ enough small.

Proof. It suffices to apply the classical Tychonoff theorem on slow-fast systems, of which assumptions are very easy to verify here (see [3], [10]). Briefly, the idea is the following: for $\epsilon > 0$ enough small, any point which is not a zero of our function g will be carried vertically (that is to say along the μ-axis) by the flow $\varphi_{2,\epsilon}$ until it reaches a stable part of the hysteresis, in which case its motion will be defined by the flow *reduced* on this surface. The equation associated to such a reduced flow is called a *slow* equation.

To precise this idea, let φ_+ be the reduced flow associated to the slow equation $\dot{x} = \mathrm{f}(x, \xi_+(x))$. By construction of our hysteresis defined by g, its critical point $\phi(\mu^{**})$ (which is unique in a neighborhood of $\left(x^0, \mu_0\right)$ enough small) is stable. So, for initial conditions enough close to this point, the trajectories will hit the fold line Δ^{**} in a finite time, afterwards they will not exist anymore. Moroever, asking that the value x_1^{**} be closer to zero if necessary, we get that these trajectories are almost logarithmic spirals defined by a polar equation of the form:

$$\rho = \rho_0 e^{\dfrac{\alpha(\mu^{**})}{\beta(\mu^{**})}(\theta - \theta_0)},$$

in the sense that locally in $\phi(\mu^{**})$, the flow φ_+ is close to its linear part.

The same holds for the reduced flow φ_- of the other slow equation $\dot{x} = \mathrm{f}(x, \xi_-(x))$. In this case, the trajectories are nearby repulsive logarithmic spirals, having a polar equation defined by the positive coefficient $\alpha(0)/\beta(0)$ (see figure 4).

Now, let us denote by φ_2 the continuous flow, of which trajectories are those of the reduced flows φ_+, φ_- connected between them by vertical segments of the form $\{(x_1^*, x_2^0, \mu) : 0 \leq \mu \leq \mu^{**}\}$ or $\{(x_1^{**}, x_2^0, \mu) : 0 \leq \mu \leq \mu^{**}\}$. Tychonoff's theorem tells us that φ_2 is the limit of $\varphi_{2,\epsilon}$ as ϵ tends to zero, in the meaning of the proposition (2.7).

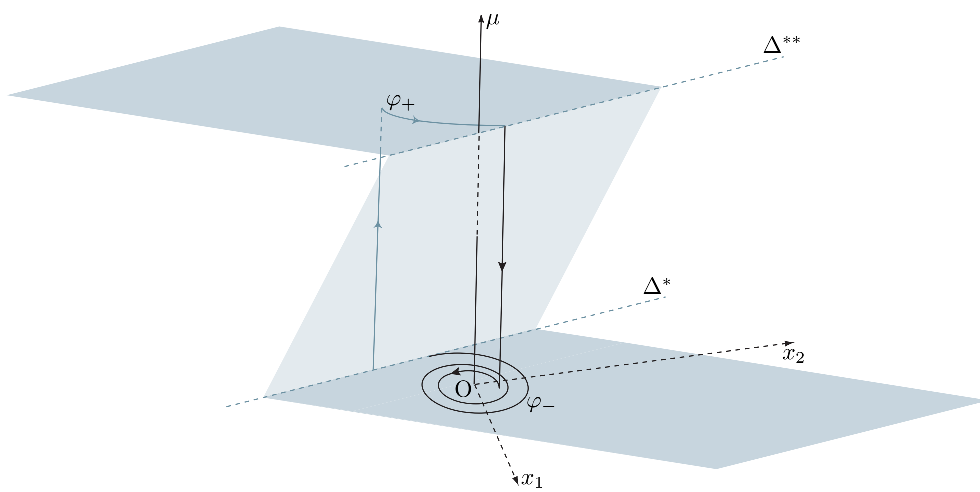

FIGURE 4. The dynamics of the flow φ_2.

□

Definition 2.8. *The flow φ_2 of the proposition* (2.7) *is called the limit flow associated to the system* (4).

2.3.b. *A linear model creating chaos.* Here we simplify again our problem by considering the case where the reduced flows φ_- and φ_+ (of which orbits are repulsive spirals centered at the origin, and attractive ones centered at $\phi(\mu^{**})$), are linear. The linear limit flow of the proposition (2.7) obtained by this way is denoted by φ_3. We claim we have:

Proposition 2.9. *There exists a choice of the values $x_1^* < x_1^{**} < 0$ and $\mu^{**} > 0$ such that the linear limit flow φ_3 associated to the system* (4) *covers at least twice a segment I belonging to the plane $\mu = 0$. More precisely, there exists a decomposition $I = I_1 \cup I_2$ in two sub-intervals and a Poincaré return map P (associated to φ_3) defined on I, such that the images $P^2(I_1)$, $P^2(I_2)$ strictly contain I.*

Proof. Let $\mathrm{F}_1, \mathrm{F}_2$ be the linear flows defined in the plane $\mu = 0$, of which orbits are respectively centered at the origin $(0,0)$ and at the point $\phi(\mu^{**})$, and have the

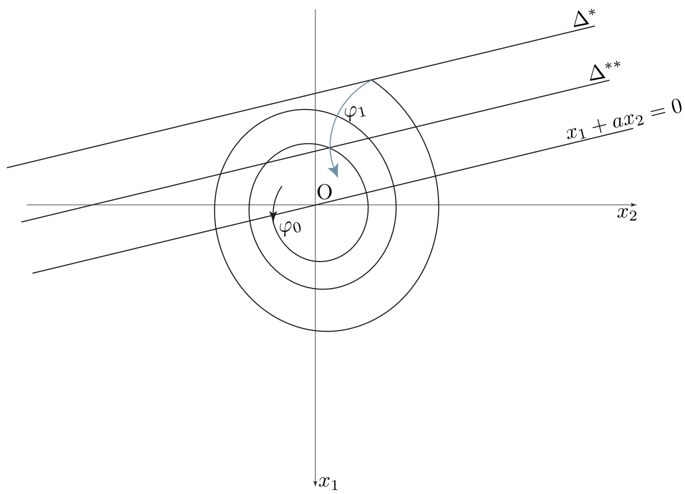

FIGURE 5. The flow φ_3.

polar equations:

$$\rho = \rho_0 e^{\dfrac{\alpha(0)}{\beta(0)}(\theta-\theta_0)} \text{ and } \rho = \rho_0 e^{\dfrac{\alpha(\mu^{**})}{\beta(\mu^{**})}(\theta-\theta_0)}.$$

We use these two flows so as to fix the position of the two fold lines Δ^*, Δ^{**} of our hysteresis h (i.e to fix the values x_1^*, x_1^{**}). Recall that their slopes is $-1/a$.
*(i)*We begin with the position of the fold Δ^*.
Given an initial condition $M_0 = (\rho_0, \theta_0)$ near the origin, any point $M = (\rho, \theta)$ belonging to the trajectory $(\mathrm{F}_1(M_0, t))_{t\in\mathbb{R}}$ satisfies:

$$M = \left(\rho_0 e^{\dfrac{\alpha(0)}{\beta(0)}(\theta-\theta_0)} \cos(\theta), \rho_0 e^{\dfrac{\alpha(0)}{\beta(0)}(\theta-\theta_0)} \sin(\theta) \right),$$

in cartesian coordinates. Derivating this expression in θ, we get that the locus of points at which the tangent of the flow F_1 has a slope equals to $-1/a$, is a straight line, having a constant angle with the half-axis $\mathrm{O}x_1$ equals to

$$\arctan\left(\frac{\dfrac{\alpha(0)}{\beta(0)\,a} + 1}{\dfrac{1}{a} - \dfrac{\alpha(0)}{\beta(0)}} \right),$$

or $\pi/2$ in the case where $a = \beta(0)/\alpha(0)$ (see figure 6).

Denote by A the intersection of this straight line with one having an equation of the form $x_1 + ax_2 = b$ where b is strictly negative. Call Δ this last line, and consider the first return of the point A in Δ, that is to say the point $B = \mathrm{F}_1(A, \tau(A))$, where $\tau(A)$ is the first strictly positive time necessary for A to hit this line. We obtain a straight line $(\mathrm{O}B)$ which is clearly the locus of first return of the points, in which

the slope of the tangent to F_1 is $-1/a$, in this tangent. To see this, it suffices to observe that the image of a logarithmic spiral under a homotethy is still a spiral of same nature, defined by the same coefficient (here $\alpha(0)/\beta(0)$). Remark that (OB) has a slope strictly smaller than a.
From the construction of these two lines (OA) and (OB), we adjust the position of Δ^* by taking the circle centered at $\phi(\mu^{**})$ of ray $O\phi(\mu^{**})$: it cuts (OB) in a point at which we draw Δ^* (figure 6).

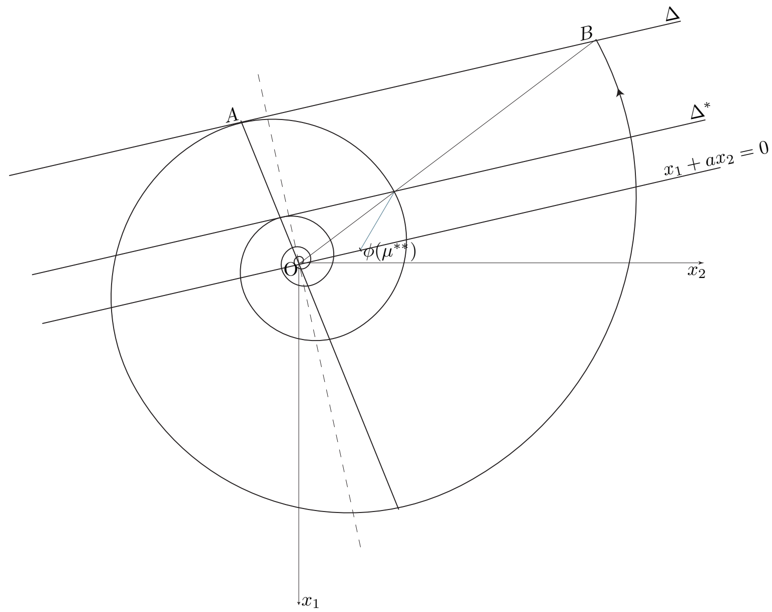

FIGURE 6. Construction of the fold line Δ^*.

(ii) Then, we consider the flow F_2, so as to position Δ^{**}.
Let us denote by $J = [A_1 B_1]$ the segment of which extremities are the intersections of Δ^* with our two straight lines (OA) and (OB). We know there is a unique trajectory S_0 of the flow F_1 which is tangent to Δ^*, and that the tangency point is the boundary A_1 of this segment.
The return of J in the line $(O\phi(\mu^{**}))$ by rotations centered at $\phi(\mu^{**})$ is a segment of the form $[OC]$ with C having a strictly non zero x_2-coordinate. As the value $\mu^{**} \geq \mu_0$ was chosen arbitrarily in the construction of our hysteresis h, we can take it closer to μ_0 if necessary, so that we can make the spiraling motion of the flow F_2 be very close to a rotation one. Thus denoting by Ψ_1 the application of first return in Δ^{**} (i.e defined by the equality $\Psi_1(M) = F_2(M, \tau(M))$, where $\tau(M)$ is the time necessary for the point M to reach Δ^{**}), we get that the image $J' = [A_1' B_1']$ of J under Ψ_1 is a segment of which extremity B_1' has a strictly positive x_2-coordinate. Moreover, fixing the value x_1^{**} so that the origin be enough close to Δ^{**}, we obtain that J' intersects the spiral S_0 in at least 3 points having a strictly negative x_2-coordinate, and such that the extremity $A_1' = \Psi_1(A_1)$ does not belong to the spiral S_1 (see figure 7).

(iii) Now that we have fixed our hysteresis h, let us prove that φ_3 is chaotic.
By *(ii)* the segment J' contains three points M_1, M_2, M_3 of the spiral S_0 (all having a strictly negative x_2-coordinate) which are consecutive (i.e there exist two times $t_1, t_2 > 0$, such that $\varphi_3(t_1, M_3) = M_2$ and $\varphi_3(t_2, M_2) = M_1$). Then let us consider the first return map Ψ in the axis $(\mathrm{O}x_1)$ associated to the flow φ_3 (or equivalently F_1), and set:

$$\begin{cases} I_1 & = [\Psi(M_1)\,\Psi(M_2)] \\ I_2 & = [\Psi(M_2)\,\Psi(M_3)] \end{cases}.$$

By construction we have:

$$\begin{cases} \Psi(J') = \Psi \circ \Psi_1 \circ \Psi_0(I_1) \supsetneq I_1 \cup I_2 \\ \Psi \circ \Psi_1 \circ \Psi_0(I_1) = \Psi \circ \Psi_1 \circ \Psi_0(I_2) \end{cases},$$

which means that the map $\Psi \circ \Psi_1 \circ \Psi_0$ covers the interval $I = I_1 \cup I_2$ twice.

From all of this we conclude that the segment I is a Poincaré section for the linear limit flow φ_3. Let P be the return map of φ_3 associated to this section. This map does not cover I twice because we have $P(I_1) = I_2$, but the map P^2 does (because $P(I_2) = \Psi \circ \Psi_1 \circ \Psi_0(I_1)$ contains I_2), which finishes the proof. □

Our application P of the previous proposition is piecewise continuous, because the boundary points of the segments I_1, I_2 (denoted $\Psi(M_1), \Psi(M_2), \Psi(M_3)$ in the previous proof) are discontinuous ones. It is well known that any piecewise continuous application of an interval covering at least twice this interval has strictly positive topological entropy. Thus, we have proved:

Corollary 2.10. *There exists a choice of the three values $x_1^* < x_1^{**} < 0$ and $\mu^{**} > 0$ such that the linear limit flow φ_3 associated to the system* (4) *is chaotic.*

2.3.c. *Proof of the result when the hysteresis is piecewise linear with flat branches.* Now let us consider again the limit flow φ_2 of the system (4), but this time without assuming that φ_+ and φ_- are linear. In this case we still have a Poincaré return map $\mathcal{P}$ associated to φ_2 and defined on a segment $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$ belonging to the plane $\mu = 0$, such that the images $\mathcal{P}^2(\mathcal{I}_1)$ and $\mathcal{P}^2(\mathcal{I}_2)$ strictly contain $\mathcal{I}$.

Indeed, making a Taylor development of the function $x \mapsto \mathrm{f}(x, 0)$ at the origin $(0,0)$, we get by the implicit function theorem, that the locus of points at which the tangent of the flow φ_- is parallel to $(\mathrm{O}\phi(\mu^{**}))$ is (locally at $(0,0)$) a curve which is tangent to the straight line $(\mathrm{O}A)$ of the construction we made above. Similarly the points of first return in these tangents form now a curve tangent (at the origin $(0,0)$) to the straight line $(\mathrm{O}B)$ we had before. So the circle of ray $(\mathrm{O}\phi(\mu^{**}))$ still cuts this curve in a point at which we can draw the line Δ^*. Then we can go on the same construction as in the linear model.

Then, from this segment $\mathcal{I}$ constructed above, we can create a rectangular section transverse to the plane $\mu = 0$, admitting a Horseshoe:

Proposition 2.11. *For the same choice of the values $x_1^*, x_1^{**}, \mu^{**}$ taken in the proposition* (2.9)*, there exists a rectangle $\mathcal{R}$ transverse to the plane $\mu = 0$, such that for any $\epsilon > 0$ enough small, the flow $\varphi_{2,\epsilon}$ of the system* (4) *covers at least twice this section.*
More precisely, there exists a decomposition $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ in two rectangles such that

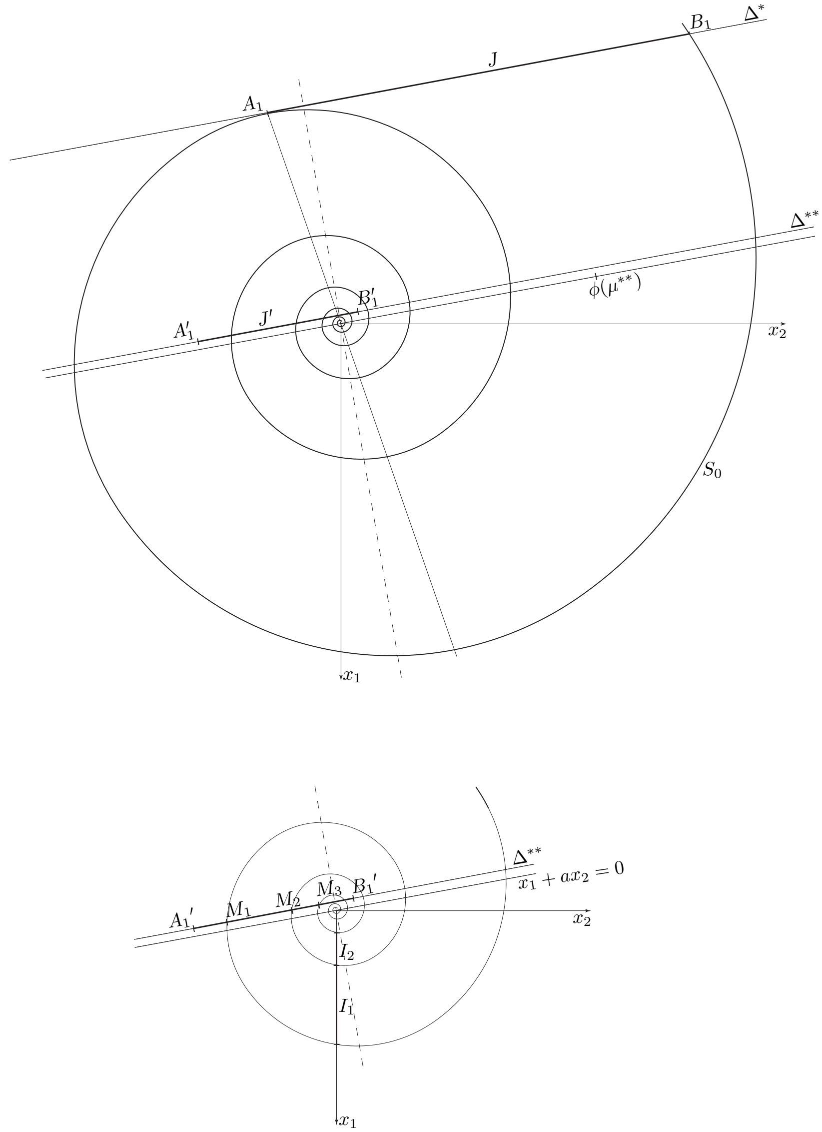

FIGURE 7. The one-dimensional section I covered twice by the flow φ_3.

for any $\epsilon > 0$ enough small, the flow $\varphi_{2,\epsilon}$ admits a Poincaré return map $\mathcal{P}_\epsilon$ defined on $\mathcal{R}$, such that $\mathcal{P}_\epsilon^2(\mathcal{R}_1), \mathcal{P}_\epsilon^2(\mathcal{R}_2)$ are disjoint and contain both one rectangle that intersects $\mathcal{R}$ along all its length (see figure 8).

Proof. Let us fix a positive number $s > 0$. We consider the rectangle $\mathcal{R}_1(s)$ which is above $\mathcal{I}_1$, that is to say defined by:

$$\mathcal{R}_1(s) = \{(x_1, x_2, \mu) : (x_1, x_2) \in \mathcal{I}_1 \text{ and } \mu \in [0, s]\},$$

and set $\mathcal{R}(s) = \mathcal{R}_1(s) \cup \mathcal{R}_2(s)$, where $\mathcal{R}_2(s)$ is the rectangle of height s, associated to the segment $\mathcal{I}_2$.
As previously, for any ϵ enough small, there exists a Poincaré return map $\mathcal{P}_\epsilon$ associated to the flow $\varphi_{2,\epsilon}$ which is defined on the rectangle $\mathcal{R}(s)$. Each of the images $\mathcal{P}_\epsilon^2(\mathcal{R}_1(s))$ and $\mathcal{P}_\epsilon^2(\mathcal{R}_2(s))$ is a rectangle that crosses $\mathcal{R}(s)$ by covering all its lenght. Therefore, to obtain a horseshoe, it suffices to verify that these two images are disjoint.

We first remark that the images under $\mathcal{P}_\epsilon^2$ of the interiors $\overset{\circ}{\mathcal{R}_1}(s), \overset{\circ}{\mathcal{R}_2}(s)$ of the two rectangles, must be disjoint: indeed there is no point z_1, z_2 in $\overset{\circ}{\mathcal{R}_1}(s) \times \overset{\circ}{\mathcal{R}_2}(s)$ such that $\mathcal{P}_\epsilon(z_1) = \mathcal{P}_\epsilon(z_2)$, otherwise we would have

$$\varphi_{2,\epsilon}(\tau(z_1) - \tau(z_2), z_1) = z_2,$$

and thus either z_1 or z_2 would return in $\mathcal{R}(s)$ in a time strictly smaller than the first return time in this section. Then, let $\Gamma(s)$ be the edge common to our two rectangles, that is the segment above the point $\Psi(M_2)$ belonging to both $\mathcal{I}_1$ and $\mathcal{I}_2$. By construction of the return map $\mathcal{P}$, the distance between $\mathcal{P}^2(\Psi(M_2))$ and the boundary of $\mathcal{I}$ is strictly positive (see the proof of the proposition (2.9)). Thus, as this distance does not depend on ϵ and by continuity of each flow $\varphi_{2,\epsilon}$, there exists a value s_0 such that for any $\epsilon > 0$ enough small, the image $\mathcal{P}_\epsilon^2(\Gamma(s_0))$ does not intersect the section $\mathcal{R}(s_0)$. The assertion is proved. ☐

The conclusion of all we have done in subsection 2.3, is that there exists a choice of the values $x_1^*, x_1^{**}, \mu^{**}$ for which the associated function g constructed at the beginning of subsection 2.3 verifies that (for every $\epsilon > 0$ enough small) the system (4) is chaotic, which is the piecewise linear version of Theorem (2.5) .

2.3.d. *End of the proof of the result.* Finally, we can transform the piecewise linear hysteresis h into a smooth one, by smoothing the rough edges near the points (x_1^*, μ^*) and (x_1^{**}, μ^{**}): this does not affect the dynamics of our system since the images (under the flow $(\phi_t)_{t\in\mathbb{R}}$) of the invariant Cantor set included in $\mathcal{R}(s_0)$ are all at a distance strictly positive of the flat branches, in particular of the rough edges of the hysteresis. The result is proved in the case $\alpha'(\mu_0) < 0$.

Remark 2.12. *In the case where $\alpha'(\mu_0) > 0$ (which corresponds to a subcritical Hopf bifurcation), it suffices to take the symmetric of the hysteresis we considered above with regard to the axis $\mathrm{O}x_1$. The exactly same reasoning applies in this case. The proof is achieved.*

2.4. **Proof of Corollary** (2.6)**.** Let us consider again the set $\mathcal{R}(s_0)$ defined above. There exist two small numbers $0 < \epsilon_1 < \epsilon_2$ such that for any ϵ in $]\epsilon_1, \epsilon_2[$, both $\mathcal{P}_\epsilon^2(\mathcal{R}_1(s_0))$ and $\mathcal{P}_\epsilon^2(\mathcal{R}_2(s_0))$ will cover this set while being at a bounded distance of the plane $\mu = 0$.
We can therefore bend the inferior stable branch of the piecewise linear hysteresis h without modifying the dynamics established above (see Proposition (2.14) at the end of the remark 2.13 below), and also smooth the rough edge near the point (x_1^*, μ^*). We obtain an inferior branch that can be described as the graph of a smooth function in the variable μ. Applying the same for the superior branch of h, we get the result.

Remark 2.13. *To conclude section 2, a natural question that remains is to know whether the same construction of our horseshoe-type dynamics still works if we take*

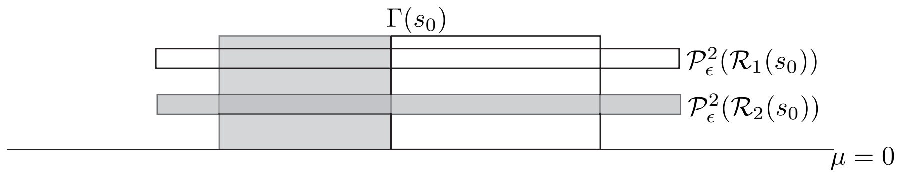

FIGURE 8. The covering of the rectangle $\mathcal{R}(s_0)$ by the map $\mathcal{P}_\epsilon^2$ (for a fixed ϵ in $]\epsilon_1, \epsilon_2[$).

a cubic-like hysteresis in $\mathcal{G}_{c,2}$ instead of smoothing a flat branches one as we did. In fact, the answer is no. Indeed, let us take again the vector field

$$\mathrm{F}(x,\mu) = (\mathrm{f}(x,\mu), \mathrm{h}(x_1 + a x_2, \mu)/\epsilon)$$

of our problem, where this time h *is a function in* $\mathcal{G}_{c,2}$*. Then, the Jacobian at the origin (which is the only fixed point of* F*) is:*

$$D\mathrm{F}(\mathrm{O}) = \begin{bmatrix} & & \frac{\partial \mathrm{f}_1}{\partial \mu}(\mathrm{O}) \\ & D\mathrm{f}(\mathrm{O}) & \\ & & \frac{\partial \mathrm{f}_2}{\partial \mu}(\mathrm{O}) \\ \frac{1}{\epsilon}\frac{\partial \mathrm{h}(0,0)}{\partial x_1} & \frac{a}{\epsilon}\frac{\partial \mathrm{h}(0,0)}{\partial x_1} & \frac{1}{\epsilon}\frac{\partial \mathrm{h}(0,0)}{\partial \mu} \end{bmatrix}.$$

Since the coefficients at the bottom left of this matrix are no more equal to zero, then when ϵ vary in $]0,1[$ all the possible cases can happen concerning its spectrum: it can, for instance, be only composed of real eigenvalues, in which case the spiraling motion would not exist, and thus our construction could not be applied.

What we only have is the following proposition (of which proof is just an application of the implicit function theorem), which tells us that given a very small ϵ, if the stable branches of the hysteresis have a slope of order ϵ, then the dynamics is the same as in the flat case. In other words, our choice of the hysteresis in our results is optimum.

Proposition 2.14. *Let B be a square matrix of size two, of which spectrum is $(\alpha \pm \mathrm{i}\beta)$, with $\alpha, \beta > 0$, and let $A(\epsilon)$ be the matrix of size three:*

$$A(\epsilon) = \begin{bmatrix} \multicolumn{2}{c}{\multirow{2}{*}{B}} & a_{1,3} \\ & & a_{2,3} \\ a_{3,1} & a a_{3,1} & -\frac{\gamma}{\epsilon} \end{bmatrix},$$

where γ is a strictly positive number. Then for any ϵ enough small, the spectrum $\mathfrak{S}(A(\epsilon))$ of $A(\epsilon)$ has the form:

$$\mathfrak{S}(A(\epsilon)) = \{\alpha + \epsilon z_1(\epsilon) \pm \mathrm{i}(\beta + \epsilon z_2(\epsilon)), -\gamma/\epsilon + \epsilon z_3(\epsilon)\},$$

where the $z_i(\epsilon)$ are bounded functions of ϵ.

3. The result in the general case $n \geq 2$

Theorem (2.5) can be extended to the case where $(\mathrm{f}_\mu)_{\mu\in\mathbb{R}}$ is a family of vector fields in $\mathbb{R}^n$, with $n \geq 2$, provided we make an additional hypothesis of contractility on the other eigenvalues of the Jacobian at the bifurcation point.

3.1. **The hypotheses.** **H1′** There exists a value μ_0 and a critical point x^0 of the field f_{μ_0} such that the Jacobian $D_x\mathrm{f}_{\mu_0}\left(x^0\right)$ has a pair of pure imaginary complex eigenvalues $\pm \mathrm{i}\beta$ with $\beta > 0$, and the other eigenvalues $\lambda_3, \cdots, \lambda_n$ have a strictly negative real part.
Here again, we denote by $\phi = (\phi_1, \cdots, \phi_n)$ the smooth function defined on a neighborhood $\mathcal{U}$ of the bifurcation value μ_0, that locally defines the graph of fixed points associated to the function f.

Hypothesis **H2** on the cross of the pure imaginary axis with a non-zero velocity, has not changed.
Remark that restricting $\mathcal{U}$ if necessary, we have that the eigenvalues $\lambda_2(\mu), \cdots, \lambda_n(\mu)$ (which are equal to $\lambda_2, \cdots, \lambda_n$ in $\mu = \mu_0$) have also a strictly negative real part.
For each μ in $\mathcal{U}$, we denote by Π_μ the plane span by the real and imaginary parts of the eigenvectors associated to $\alpha(\mu) \pm \mathrm{i}\beta(\mu)$. Note that all those planes are almost parallel because these eigenvectors vary smoothly with the parameter μ.

Under these two assumptions, the center manifold theorem gives us the existence, for every μ in $\mathcal{U}$, of a smooth manifold of dimension two $\mathcal{W}_\mu$, tangent at the point $(\phi(\mu), \mu)$ to the plane Π_μ. This manifold is attracting for the flow of the equation $\dot{x} = \mathrm{f}(x, \mu)$. Moroever, for every μ in $\mathcal{U}$, the dynamics of this system restricted to $\mathcal{W}_\mu$ is given by equations of the form:

$$(5) \qquad \left\{ \begin{array}{ll} \dot{u_1} & = \left(a_0\mu + b_0\left(u_1^2+u_2^2\right)\right)u_1 - \left(\omega + a_1\mu + b_1\left(u_1^2+u_2^2\right)\right)u_2 \\ \dot{u_2} & = \left(\omega + a_1\mu + b_0\left(u_1^2+u_2^2\right)\right)u_1 + \left(a_0\mu + b_0\left(u_1^2+u_2^2\right)\right)u_2 \end{array} \right. .$$

3.2. **The theorem and its proof.** The n-dimensional version of our result is the following:

Theorem 3.1. *Let us suppose that Hypotheses* **H1′**, **H2** *on the function* f *are satisfied. Assume moroever that the projection of the graph $x = (\phi_1(\mu), \cdots, \phi_n(\mu))$ on the plane Π_{μ_0} is not reduced to the point $(\phi(\mu_0), \mu_0)$. Then, there exist two distinct integers i, j in $[1, n]$, two non zero real numbers a_i, a_j and a smooth function* h *in $\mathcal{G}_2$ such that for every $\epsilon > 0$ enough small, the singularly perturbed system:*

$$(6) \qquad \left\{ \begin{array}{ll} \dot{x} & = \mathrm{f}(x, \mu) \\ \epsilon\dot{\mu} & = \mathrm{h}(a_i x_i + a_j x_j, \mu) \end{array} \right. ,$$

taken in a neighborhood of $\left(x^0, \mu_0\right)$ enough small, is chaotic.

Proof. The exactly same reasoning as in the two-dimensional case can be applied. *(i)*Indeed, let us first fix a value μ^{**} in $\mathcal{U}$, strictly greater than μ_0. By assumption on the graph $x = \phi(\mu)$, there exists a value of the parameter (say 0) which is strictly smaller than μ_0 and such that the projection X_0 of the point $(\phi(\mu^{**}), 0)$ on the plane Π_0 is not equal to the fixed point $(\phi(0), 0)$. Without loss of generality, we assume that this last point is the origin $\mathrm{O} = (0, \cdots, 0)$ of $\mathbb{R}^{n+1}$.
Then we consider two planes H^*, H^{**} which are parallel to the plane containing

the origin and the two points $(\phi(\mu^{**}), 0), X_0$. There exist two distinct integers i, j in $[1, n]$, such that the equations of these planes are of the form:

$$a_i x_i + a_j x_j = x^*$$
$$a_i x_i + a_j x_j = x^{**},$$

where a_i, a_j and x^*, x^{**} are four non zero numbers. The intersections $H^* \cap \Pi_0$ and $H^{**} \cap \Pi_0$ are two lines that will be our two fold lines Δ^*, Δ^{**} of subsection 2.3. To finish this first step we consider the element h in $\mathcal{G}_{p,2}$ defining the hysteresis with flat branches of which bifurcation points are $(x^*, 0)$ and (x^{**}, μ^{**}), in order to prove the piecewise linear version of the result before smoothing this convenient hysteresis.
(ii) Without loss of generality, we can assume that the origin is very close to the two fold lines that intersect by the way a small neighborhood of the origin in $\mathbb{R}^{n+1}$, in which the flow of the system (6) is very close to its linear part. Thanks to the hypothesis **H1′**, we can therefore construct a rectangular section $\mathcal{I}$ in the hyperplane $\mu = 0$, that transversely cross the plane Π_0, and which is covered twice by the limit flow associated to (6). It remains to extend vertically this section $\mathcal{I}$, that is to say to consider the sets

$$\mathcal{R}(s) = \{(x, \mu) \in \mathcal{I} \times [0, s]\},$$

for some small strictly positive s. There exist a value s_0 such that $\mathcal{R}(s_0)$ is a Horseshoe for the flow of the system (6). We conclude by taking off this rectangle from the hyperplane $\mu = 0$, and by smoothing the hysteresis as in the two-dimensional case. □

To finish section 3, we also have the similar corollary as in section 2:

Corollary 3.2. *With the same assumptions as in Theorem* (3.1)*, there exist two integers* i, j *in* $[1, n]$*, two non zero real numbers* a_i, a_j*, a smooth function* h *in* $\mathcal{G}_{c,2}$ *and two small numbers* $0 < \epsilon_1 < \epsilon_2$*, such that for every* ϵ *in* $]\epsilon_1, \epsilon_2[$*, the system* (6)*, taken in a neighborhood of* (x^0, μ_0) *enough small, is chaotic.*

4. Application to a four-dimensional differential system modeling the behavior of a hypothetical gene regulatory network.

Here we use the reasoning made in the first part in order to induce chaos in the following differential system:

$$\begin{cases} \dot{A} = \dfrac{k_{sa} + k_{saa}\left(\dfrac{A}{j_{aa}}\right)^2 + k_{sac}\left(\dfrac{C}{j_{ac}}\right)^2}{1 + \left(\dfrac{A}{j_{aa}}\right)^2 + \left(\dfrac{B}{j_{ab}}\right)^2 + \left(\dfrac{C}{j_{ac}}\right)^2} - k_{da}A \\ \dot{B} = \dfrac{k_{sb} + k_{sba}\left(\dfrac{A}{j_{ba}}\right)^2}{1 + \left(\dfrac{A}{j_{ba}}\right)^2} - k_{db}B \\ \dot{C} = \dfrac{k_{sc}}{1 + \left(\dfrac{D}{j_{cd}}\right)^2} - k_{dc}C \\ \dot{D} = \dfrac{k_{sd} + k_{sda}\left(\dfrac{A}{j_{da}}\right)^2}{1 + \left(\dfrac{A}{j_{da}}\right)^2 + \left(\dfrac{C}{j_{dc}}\right)^2} - k_{dd}D \end{cases},$$

and to explain how this chaotic motion is created. As said in the introduction, this system has been provided by J.J Tyson[1]. We will call it V-system in recognition of its origin, and will denote it by $\mathscr{V}$.

This is an autonomous system with nineteen parameters and four equations. The variables A, B, C, D represent concentrations of four constituants evolving in a cell: thus they will be taken only positive. The equations describe the chemical reactions between them and come from the Michaelis-Menten laws. The other quantities $k_{sa}, k_{saa}, j_{aa}, k_{sac}$... are also concentrations but assumed to be constant. The aim is to find a set of values of these parameters that we'll denote $\mathscr{P}_1$, for which the associated V-system $\mathscr{V}^1$ is chaotic, with the only rule that these values must be positive.

Numerical tests realized with the software Xdim lead us to consider the set of values $\mathscr{P}_0$:

$$\begin{cases} k_{sa} = 0.05,\ k_{saa} = 5,\ k_{sac} = 2.4,\ k_{da} = 0.1 \\ j_{aa} = 2.5,\ j_{ab} = 0.5,\ j_{ac} = 2 \\ k_{sb} = 0,\ k_{sba} = 0.3,\ j_{ba} = 17.5,\ k_{db} = 0.03 \\ k_{sc} = 48,\ k_{dc} = 4,\ j_{cd} = 1.5 \\ k_{sd} = 16,\ k_{sda} = 16,\ j_{da} = 3,\ j_{dc} = 1,\ k_{dd} = 4 \end{cases},$$

because the system $\mathscr{V}^0$ seemed to present an unpredictable behavior. Here again, these numerical values have been provided J.J Tyson[1].

4.1. **The V-system as the coupling of two sub-systems of $\mathbb{R}^2$.** The strategy adopted is to divide the V-system in two sub-systems of two equations (denoted by $\mathscr{V}_{A,B}$ for the one in the variables A, B and $\mathscr{V}_{C,D}$ for the one in the variables C, D) and to recognize in $\mathscr{V}_{A,B}$ a negative feedback circuit (by showing that this

sub-system admits a local Hopf bifurcation) and in $\mathscr{V}_{C,D}$ a positive feedback circuit (caracterized by a hysteresis). By a circuit in $\mathscr{V}_{A,B}$, we mean that the constituants A, B mutually influence their becoming; the fact that it is negative signifies that one of them activates the production of the other while this last one inhibits the production of the first. The same holds for $\mathscr{V}_{C,D}$.
In the rest of the text, $\mathscr{V}^0_{A,B}, \mathscr{V}^0_{C,D}$ will be the sub-systems of the model $\mathscr{V}^0$, and similarly for $\mathscr{V}^1, \mathscr{V}^1_{A,B}, \mathscr{V}^1_{C,D}$.

4.1.a. *A local Hopf bifurcation in the sub-system* $\mathscr{V}_{A,B}$. The sub-system $\mathscr{V}_{A,B}$ is the one defined by the equations:

$$\begin{cases} \dot{A} = \dfrac{k_{sa} + k_{saa}\left(\dfrac{A}{j_{aa}}\right)^2 + k_{sac}\left(\dfrac{C}{j_{ac}}\right)^2}{1 + \left(\dfrac{A}{j_{aa}}\right)^2 + \left(\dfrac{B}{j_{ab}}\right)^2 + \left(\dfrac{C}{j_{ac}}\right)^2} - k_{da}A \\ \dot{B} = \dfrac{k_{sb} + k_{sba}\left(\dfrac{A}{j_{ba}}\right)^2}{1 + \left(\dfrac{A}{j_{ba}}\right)^2} - k_{db}B \end{cases},$$

where C is considered here as a parameter.
For every C, let $\mathrm{F}_C(A, B)$ be the vector field in $\mathbb{R}^2$ associated to $\mathscr{V}_{A,B}$. Making $\dot{B} = 0$ and replacing the expression of B in the first equation, we can express the nullcline of this system, that is to say the set $\{(A, B, C) : \mathrm{F}_C(A, B) = (0, 0)\}$, as the graph of a function in the variable A of the form:

$$\left\{\left(A, \frac{k_{sb} + k_{sba}\left(\frac{A}{j_{ba}}\right)^2}{k_{db}\left(1 + \left(\frac{A}{j_{ba}}\right)^2\right)}, \mathrm{f}\left(A, \frac{k_{sb} + k_{sba}\left(\frac{A}{j_{ba}}\right)^2}{k_{db}\left(1 + \left(\frac{A}{j_{ba}}\right)^2\right)}\right)\right), A \geq 0\right\},$$

where f is a complicated function that we do not need to write.

The Jacobian matrix $D\mathrm{F}_C(A, B)$ of our vector field is:

$$\begin{bmatrix} \dfrac{2A(k_{saa} - k_{da}A)}{j_{aa}^2\left(1 + \left(\frac{A}{j_{aa}}\right)^2 + \left(\frac{B}{j_{ab}}\right)^2 + \left(\frac{C}{j_{ac}}\right)^2\right)} - k_{da} & \dfrac{-2k_{da}AB}{j_{ab}^2\left(1 + \left(\frac{A}{j_{aa}}\right)^2 + \left(\frac{B}{j_{ab}}\right)^2 + \left(\frac{C}{j_{ac}}\right)^2\right)} \\ \dfrac{2j_{ba}^2 A(k_{sba} - k_{sb})}{(j_{ba}^2 + A^2)^2} & -k_{db} \end{bmatrix}.$$

Naturally we are interested in the eigenvalues of this matrix for points (A, B, C) belonging to the nullcline. The calculations are inextricable, but using Mathematica, we find that for the values $(A, B, C) = (16.5139, 4.71033, 3.30896)$, the jacobian matrix of $\mathscr{V}^0_{A,B}$ has two pure imaginary eigenvalues, and so this system admits a local Hopf bifurcation. Remark that since the function f above is not constant, the assumption of Theorem (2.5) (which asks that the curve of critical points is not a vertical axis) is satisfied.

4.1.b. *A hysteresis in the sub-system $\mathscr{V}_{C,D}$.* Now we study the second sub-system $\mathscr{V}_{C,D}$ defined by the two equations:

$$\begin{cases} \dot{C} = \dfrac{k_{sc}}{1+\left(\dfrac{D}{j_{cd}}\right)^2} - k_{dc}C \\ \dot{D} = \dfrac{k_{sd}+k_{sda}\left(\dfrac{A}{j_{da}}\right)^2}{1+\left(\dfrac{A}{j_{da}}\right)^2+\left(\dfrac{C}{j_{dc}}\right)^2} - k_{dd}D \end{cases}.$$

Making $\dot{D}=0$ and replacing the new expression of D in the third equation, we get, provided that the values k_{sd} and k_{sda} are the same, that the nullcline associated to this subsystem is:

$$\left\{\left(A, C, \frac{1}{k_{dd}}\left(\frac{k_{sd}+k_{sda}\left(\dfrac{A}{j_{da}}\right)^2}{1+\left(\dfrac{A}{j_{da}}\right)^2+\left(\dfrac{C}{j_{dc}}\right)^2}\right)\right) : (A,C)\in\mathscr{C}_{A,C}\right\},$$

where we have set:

$$\mathscr{C}_{A,C} = \left\{(A,C)\in\mathbb{R}_+^2 : k_{dc}C = \frac{k_{sc}\left(1+\left(\dfrac{A}{j_{da}}\right)^2+\left(\dfrac{C}{j_{dc}}\right)^2\right)^2}{\left(1+\left(\dfrac{A}{j_{da}}\right)^2+\left(\dfrac{C}{j_{dc}}\right)^2\right)^2+\dfrac{k_{sd}{}^2}{k_{dd}^2 j_{cd}^2}\left(1+\left(\dfrac{A}{j_{da}}\right)^2\right)^2}\right\}.$$

We claim this last set is, under certain conditions on the parameters (which are satisfied by $\mathscr{P}_0$) a hysteresis. To do this we prove the following property:

Property 4.1. *Assuming we have* $\dfrac{k_{sd}}{k_{dd}j_{cd}} > 2$ *and* $k_{sd} = k_{sda}$*, there exist two numbers* $0 < A^* < A^{**}$ *such that:*

- $\forall A < A^*$ *or* $A > A^{**}$*, there is exactly one point in* $\mathscr{C}_{A,C}$
- $\forall A \in (A^*, A^{**})$*, there are exactly three points in* $\mathscr{C}_{A,C}$
- *If* $A = A^*$ *or* $A = A^{**}$*, there are exactly two points in* $\mathscr{C}_{A,C}$*.*

Proof. To reduce the expressions appeared above, we introduce a new variable:

$$X = 1+\left(\frac{A}{j_{da}}\right)^2.$$

As we are only interested in the positive values of our variables, we can use X instead of using A in our calculations. Let $\mathrm{f}_X(C)$ be the right-hand side of the equation defining the set $\mathscr{C}_{A,C}$. A simple calculus shows that f'_X is strictly positive on $\mathbb{R}_+^*$, tends to zero as C tends to infinity, and that we have $\mathrm{f}'_X(0) = 0$. Thus the graph of f_X has a form given by the figure (9):

Now the idea is to study the intersection of the previous graph with the line $k_{dc}C$ when k_{dc} varies. Let search on positive values C_0 for which the tangent at the point $(C_0, \mathrm{f}_X(C_0))$ passes through $(0,0)$. Such a point C_0 verifies the equality

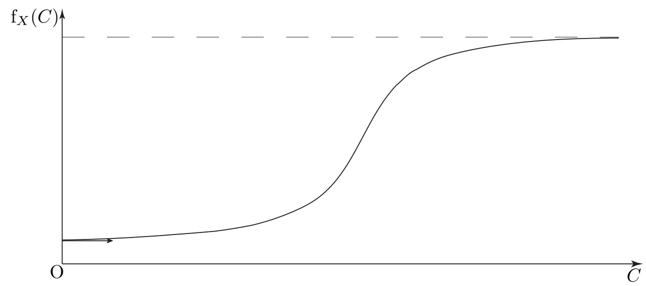

FIGURE 9. Graph of the function f_X.

$\mathrm{f}'_X(C_0)\, C_0 = \mathrm{f}_X(C_0)$, that is to say:

$$4\frac{{k_{sd}}^2}{k_{dd}^2 j_{cd}^2}\frac{\left(\frac{C_0}{j_{dc}}\right)^2 X^2}{\left(X+\left(\frac{C_0}{j_{dc}}\right)^2\right)^2+\frac{{k_{sd}}^2}{k_{dd}^2 j_{cd}^2}X^2} = X + \left(\frac{C_0}{jdc}\right)^2,$$

which is equivalent to:

$$\mathscr{C}_0{}^3 + 3X^2\mathscr{C}_0\left(1-\frac{{k_{sd}}^2}{k_{dd}^2 j_{cd}^2}\right) + 3X\mathscr{C}_0{}^2 + X^3\left(1+\frac{{k_{sd}}^2}{k_{dd}^2 j_{cd}^2}\right) = 0,$$

where $\mathscr{C}_0$ is equal to $\frac{C_0^2}{j_{dc}^2}$. And since we have $\frac{k_{sd}}{k_{dd} j_{cd}} > 2$, there exist two strictly positive solutions (depending on the variable A) $0 < \mathscr{C}_1 < \mathscr{C}_2$ of this equation, and so two values $0 < C_1 < C_2$ for which the tangent at $(C_i, \mathrm{f}_X(C_i))$ passes through the origin.

So for every positive A, there exist two thresholds $0 < k_1(A) < k_2(A)$ (which are the two slopes of the tangents) such that:

- if $k_{dc} < k_1(A)$ or $k_{dd} > k_2(A)$, there is only one point in $\mathscr{C}_{A,C}$,
- if $k_{dc} = k_1(A)$ or $k_{dd} = k_2(A)$ there are two points in this curve,
- if $k_1(A) < k_{dc} < k_2(A)$ there are three points in it.

Considering again k_{dc} as a fixed parameter, it remains to justify why we can express this last result in terms of the variable A. To do this, we look at the monotony of k_i. We have:

$$k_i(A) = \frac{4k_{sc}{k_{sd}}^2}{j_{dc}k_{dd}^2 j_{cd}^2}\frac{X^2\sqrt{\mathscr{C}_i}}{(X+\mathscr{C}_i)^3\left(1+\frac{{k_{sd}}^2}{k_{dd}^2 j_{cd}^2}\left(\frac{X}{X+\mathscr{C}_i}\right)^2\right)^2},$$

and because $\mathscr{C}_i$ is a root of the equation written above, so:

$$(\mathscr{C}_i + X)^3 = \frac{{k_{sd}}^2}{k_{dd}^2 j_{cd}^2}(3\mathscr{C}_i - X)\, X^2.$$

From this last equality, we get two informations: the first one is that the quotient $\mathscr{C}_i/X$ is a constant and thus does not depend on the variable A, the second one is

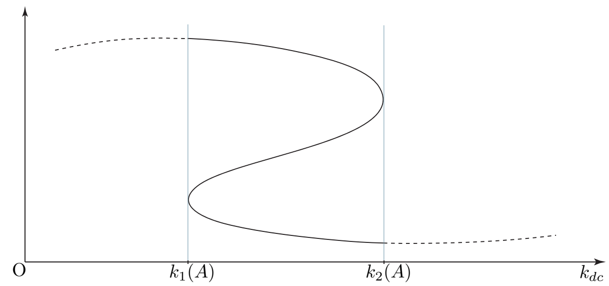

FIGURE 10. The curve $\mathscr{C}_{A,C}$ for a fixed A.

the following expression of the square root $\sqrt{\mathscr{C}_i}$:

$$\sqrt{\mathscr{C}_i} = \frac{k_{dd} j_{cd}}{k_{sd}\sqrt{3}} \sqrt{\mathscr{C}_i + X} \sqrt{\frac{(\mathscr{C}_i + X)^2}{X^2} + \frac{{k_{sd}}^2}{k_{dd}^2 j_{cd}^2} \frac{X}{\mathscr{C}_i + X}}.$$

Substituting in the expression of $k_i(A)$ we obtain the existence of two strictly positive constants $(\gamma_i)_{i=1,2}$, such that for every positive A, we have:

$$k_i(A) = \frac{\gamma_i}{\sqrt{1 + \left(\frac{A}{j_{da}}\right)^2}},$$

which implies the strict monotony of the functions k_i. Thus the numbers $k_1^{-1}(k_{dc})$ and $k_2^{-1}(k_{dc})$ are well defined and distinct because $k_1(A) \neq k_2(A)$ for any positive A. Setting $A^* = k_1^{-1}(k_{dc})$, and $A^{**} = k_2^{-1}(k_{dc})$ we get the result. □

Property 4.2. *Assuming the parameters satisfy the conditions* $\frac{k_{sd}}{k_{dd} j_{cd}} > 2$ *and* $k_{sd} = k_{sda}$*, then* $\mathscr{C}_{A,C}$ *is a hysteresis defined by a function in* $\mathcal{G}_{c,2}$*. In particular, this is the case of the set* $\mathscr{C}^0_{A,C}$ *associated to the* V*-system* $\mathscr{V}^0$*.*

Proof. Since the set $\mathscr{C}_{A,C}$ can be expressed as the graph of a function in the variable C, and thanks to the property (4.1), we are sure it has the form of a hysteresis defined by an element of the set $\mathcal{G}_{c,2}$. To see that the stability conditions required in our definition of a hysteresis are satisfied, it suffices to consider the sign of the derivatives $\frac{\partial \mathrm{g}}{\partial C}(A, C)$ where $\mathrm{g}(A, C) = k_{dc}C - \mathrm{f}_X(C)$.

Concerning the set of parameters $\mathscr{P}_0$, we have $\frac{k_{sd}}{k_{dd} j_{cd}} = 32$ and $k_{sd} = k_{sda}$ (= 16), and the function g belongs to $\mathcal{G}_{c,2}$. □

Lastly, the isocline of $\mathscr{V}_{C,D}$ appears as the intersection of the cartesian product of $\mathscr{C}_{A,C}$ by $\mathbb{R}_+$ with the set:

$$\left\{ (A, C, D) : \; D = \frac{1}{k_{dd}} \left(\frac{k_{sd} + k_{sda}\left(\frac{A}{j_{da}}\right)^2}{1 + \left(\frac{A}{j_{da}}\right)^2 + \left(\frac{C}{j_{dc}}\right)^2} \right) \right\}.$$

This intersection is a smooth curve in the three-dimensional space (A, C, D) (see the figure 11).

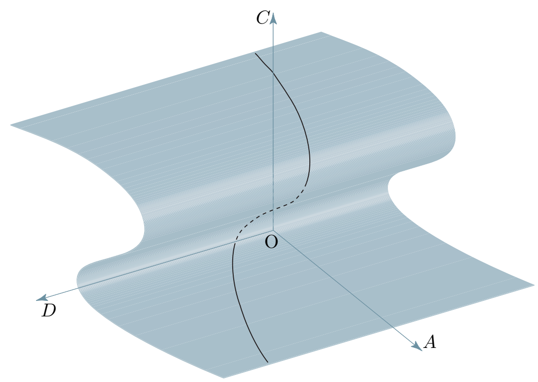

FIGURE 11. The isocline of the sub-system $\mathscr{V}_{C,D}$

4.2. **Using three scales of time to be in the situation of Theorem** (2.5)**.** In order to apply the theorem (2.5), we force the parameters k_{sd}, k_{sda}, k_{dd} to be much more greater than the other ones, by considering the following system, which is still of V-system type:

$$\left\{\begin{aligned}
\dot{A} &= \frac{k_{sa} + k_{saa}\left(\dfrac{A}{j_{aa}}\right)^2 + k_{sac}\left(\dfrac{C}{j_{ac}}\right)^2}{1 + \left(\dfrac{A}{j_{aa}}\right)^2 + \left(\dfrac{B}{j_{ab}}\right)^2 + \left(\dfrac{C}{j_{ac}}\right)^2} - k_{da}A \\
\dot{B} &= \frac{k_{sb} + k_{sba}\left(\dfrac{A}{j_{ba}}\right)^2}{1 + \left(\dfrac{A}{j_{ba}}\right)^2} - k_{db}B \\
\dot{C} &= \frac{k_{sc}}{1 + \left(\dfrac{D}{j_{cd}}\right)^2} - k_{dc}C \\
\dot{D} &= \frac{1}{\epsilon}\left(\frac{k_{sd} + k_{sda}\left(\dfrac{A}{j_{da}}\right)^2}{1 + \left(\dfrac{A}{j_{da}}\right)^2 + \left(\dfrac{C}{j_{dc}}\right)^2} - k_{dd}D\right)
\end{aligned}\right. ,$$

where ϵ is a very small positive number.

Indeed, by Tychonoff's theorem that we used before, the dynamics of this last system is asymptotically (and approximately) the same as the following slow system:

$$\begin{cases} A' = \dfrac{k_{sa} + k_{saa}\left(\dfrac{A}{j_{aa}}\right)^2 + k_{sac}\left(\dfrac{C}{j_{ac}}\right)^2}{1 + \left(\dfrac{A}{j_{aa}}\right)^2 + \left(\dfrac{B}{j_{ab}}\right)^2 + \left(\dfrac{C}{j_{ac}}\right)^2} - k_{da}A \\ B' = \dfrac{k_{sb} + k_{sba}\left(\dfrac{A}{j_{ba}}\right)^2}{1 + \left(\dfrac{A}{j_{ba}}\right)^2} - k_{db}B \\ C' = \dfrac{k_{sc}\left(1 + \left(\dfrac{A}{j_{da}}\right)^2 + \left(\dfrac{C}{j_{dc}}\right)^2\right)^2}{\left(1 + \left(\dfrac{A}{j_{da}}\right)^2 + \left(\dfrac{C}{j_{dc}}\right)^2\right)^2 + \dfrac{{k_{sd}}^2}{k_{dd}^2 j_{cd}^2}\left(1 + \left(\dfrac{A}{j_{da}}\right)^2\right)^2} - k_{dc}C \end{cases},$$

where $'$ stands for the derivative according to the fast time. The assumptions required to get this asymptotic behavior are those of 2.3.a (and are again very easy to verify), of which is added the fact that the reduced problem admits an asymptotically stable critical point. For this reason, the approximation stated by Tychonoff's theorem remains true for an infinite time (see [3] for further explanations).

Now in this slow system we recognize the hysteresis defined by the set $\mathscr{C}_{A,C}$. Thus it suffices to choose another positive number ϵ' very small but greater than ϵ and to consider the V-system:

$$\begin{cases} \dot{A} = \dfrac{k_{sa} + k_{saa}\left(\dfrac{A}{j_{aa}}\right)^2 + k_{sac}\left(\dfrac{C}{j_{ac}}\right)^2}{1 + \left(\dfrac{A}{j_{aa}}\right)^2 + \left(\dfrac{B}{j_{ab}}\right)^2 + \left(\dfrac{C}{j_{ac}}\right)^2} - k_{da}A \\ \dot{B} = \dfrac{k_{sb} + k_{sba}\left(\dfrac{A}{j_{ba}}\right)^2}{1 + \left(\dfrac{A}{j_{ba}}\right)^2} - k_{db}B \\ \dot{C} = \dfrac{1}{\epsilon'}\left(\dfrac{k_{sc}}{1 + \left(\dfrac{D}{j_{cd}}\right)^2} - k_{dc}C\right) \\ \dot{D} = \dfrac{1}{\epsilon}\left(\dfrac{k_{sd} + k_{sda}\left(\dfrac{A}{j_{da}}\right)^2}{1 + \left(\dfrac{A}{j_{da}}\right)^2 + \left(\dfrac{C}{j_{dc}}\right)^2} - k_{dd}D\right) \end{cases}.$$

The dynamics of this system is in a first time captured in a three-dimensional space in which it is approximately described by the following slow system $\mathscr{S}$:

$$\begin{cases} A' = \dfrac{k_{sa} + k_{saa}\left(\dfrac{A}{j_{aa}}\right)^2 + k_{sac}\left(\dfrac{C}{j_{ac}}\right)^2}{1 + \left(\dfrac{A}{j_{aa}}\right)^2 + \left(\dfrac{B}{j_{ab}}\right)^2 + \left(\dfrac{C}{j_{ac}}\right)^2} - k_{da}A \\ B' = \dfrac{k_{sb} + k_{sba}\left(\dfrac{A}{j_{ba}}\right)^2}{1 + \left(\dfrac{A}{j_{ba}}\right)^2} - k_{db}B \\ C' = \dfrac{1}{\epsilon'}\left(\dfrac{k_{sc}\left(1 + \left(\dfrac{A}{j_{da}}\right)^2 + \left(\dfrac{C}{j_{dc}}\right)^2\right)^2}{\left(1 + \left(\dfrac{A}{j_{da}}\right)^2 + \left(\dfrac{C}{j_{dc}}\right)^2\right)^2 + \dfrac{{k_{sd}}^2}{k_{dd}^2 j_{cd}^2}\left(1 + \left(\dfrac{A}{j_{da}}\right)^2\right)^2} - k_{dc}C\right) \end{cases}.$$

Now $\mathscr{S}$ is of the same type as the system studied in the first, namely of the kind

$$\begin{cases} \dot{x} = \mathrm{f}(x, \mu) \\ \epsilon\dot{\mu} = \mathrm{g}(x, \mu) \end{cases},$$

where f admits a local Hopf bifurcation and g defines a hysteresis curve. According to the first part this system is chaotic if the bifurcation surfaces are well-placed.

4.3. **Numerical evidence of a chaotic motion.** Obviously the position of the bifurcations surfaces and the shape of the hysteresis are changed when we modify the values of the parameters. We consider the set $\mathscr{P}_1$:

$$\begin{cases} k_{sa} = 0.05,\ k_{saa} = 5,\ k_{sac} = 2.4,\ k_{da} = 0.1 \\ j_{aa} = 2.5,\ j_{ab} = 0.5,\ j_{ac} = 2 \\ k_{sb} = 0,\ k_{sba} = 0.3,\ j_{ba} = 17.5,\ k_{db} = 0.03 \\ k_{sc} = 330,\ k_{dc} = 32,\ j_{cd} = 1.5 \\ k_{sd} = 3.87,\ k_{sda} = 3.87,\ j_{da} = 2.9119,\ j_{dc} = 0.75,\ k_{dd} = 1.001 \end{cases},$$

in which we have imposed the different scales of parameters explained above. We cannot really increase the values because of numerical instabilities appeared in the software Xdim for such high values. Remark that the parameters of the system $\mathscr{V}^1_{A,B}$ and $\mathscr{V}^0_{A,B}$ are the same, thus $\mathscr{V}^1_{A,B}$ still admits a Hopf bifurcation. Moroever we have $\dfrac{k_{sd}}{k_{dd}j_{cd}} = 2.57742$ and $k_{sd} = k_{sda}$ thus by the proposition 4.2 the set $\mathscr{C}^1_{A,C}$ is a hysteresis. The figure (12) shows the bifurcation surfaces associated to the slow system $\mathscr{S}^1$.

With Xdim we found a Poincaré map that covers twice the section on which it is defined, in an exactly similar way as planned (see figures (13), (14) and (15)). This convinces us that $\mathscr{S}^1$ is chaotic. Such a section can be found for the V-system $\mathcal{V}^1$, by multiplying the parameters k_{sd}, k_{sda}, k_{dd} by a very great number.

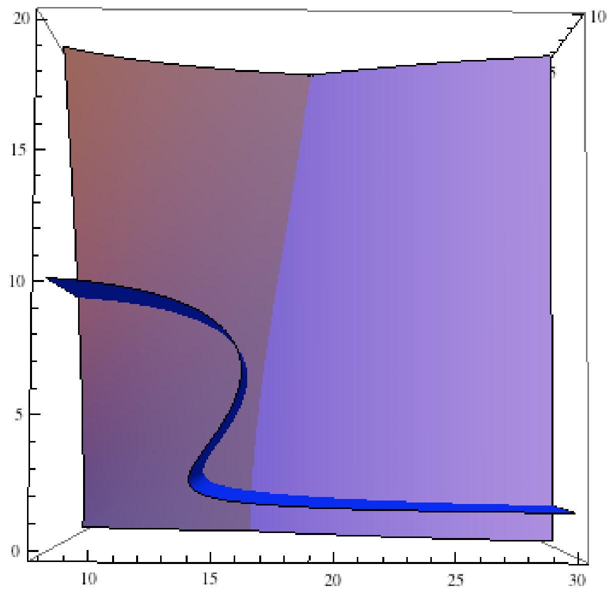

Figure 12. The hysteresis and the curve of critical points associated to the Hopf bifurcation of the system $\mathscr{S}^1$

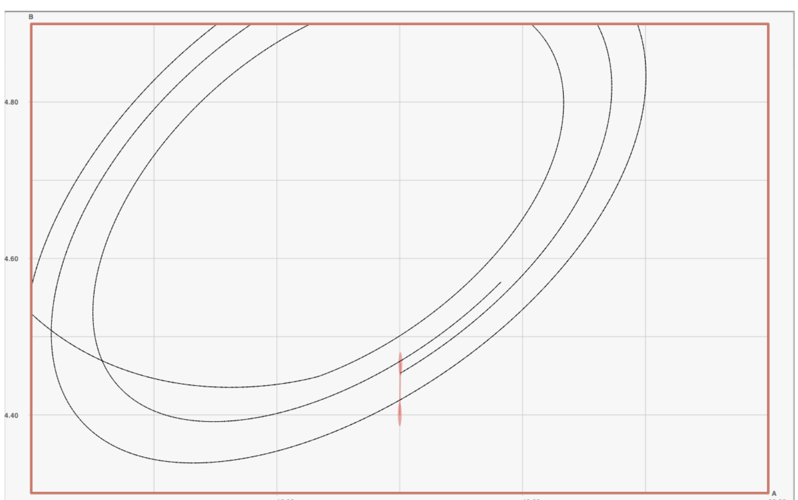

Figure 13. A trajectory of the system $\mathscr{S}^1$.

5. Conclusion

We have investigated how to create a chaotic behavior from a family of vector fields admitting (generically) a Hopf bifurcation. This study permitted us to see this problem in two different ways: the first one is the perturbation of the parameter of this family of vector fields, the second, of which interest comes from biological

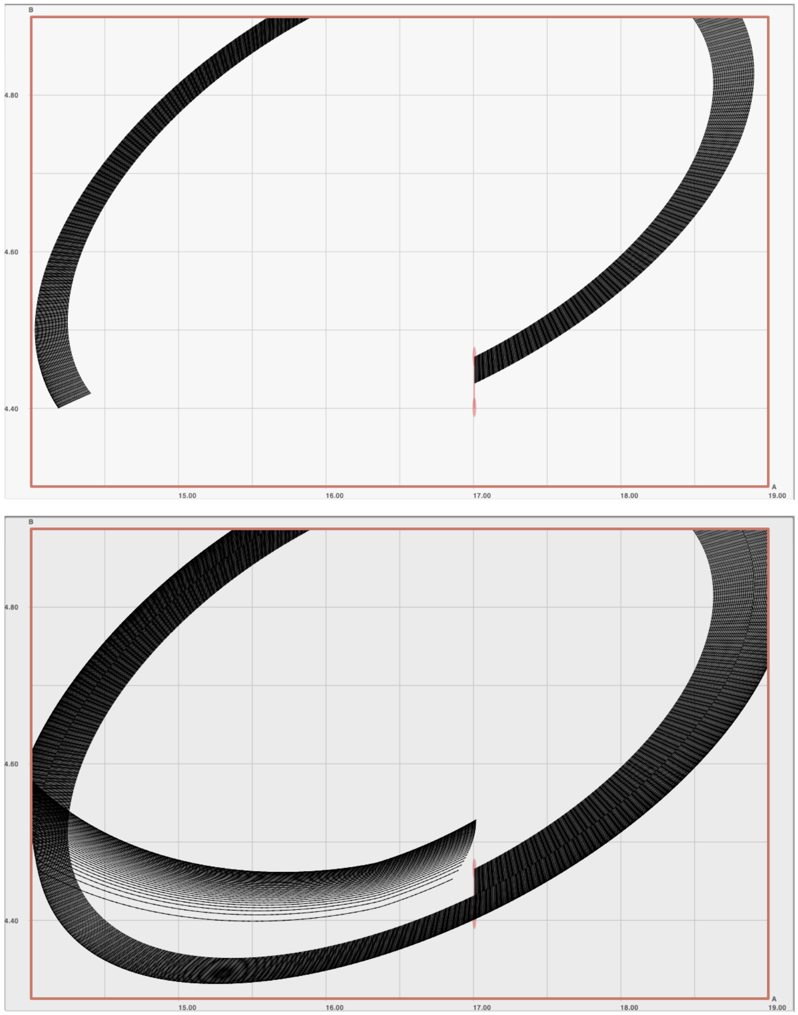

FIGURE 14. A Poincaré map for the flow of $\mathscr{S}^1$ that covers twice the segment on which it is defined.

regulatory systems, is the coupling of the differential system associated to this bifurcation, with a bistable one.

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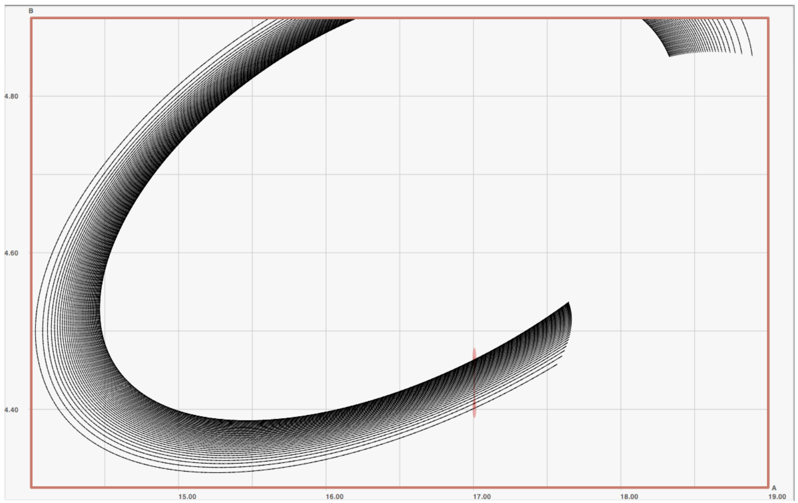

Figure 15

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