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On (K_q, k) vertex stable graphs with minimum size

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Abstract

A graph G is a (K_q, k) vertex stable graph if it contains a K_q after deleting any subset of k vertices. We give a characterization of (K_q, k) vertex stable graphs with minimum size for $q = 3, 4, 5$.

Keywords: Stable graphs

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1. Introduction

For terms not defined here we refer to [1]. As usual, the *order* of a graph G is the number of its vertices (denoted by $|G|$) and the *size* of G is the number of its edges (denoted by $e(G)$). A complete subgraph of order q of G is called a q-*clique* of G. The complete graph of order q is denoted by K_q. When a graph G contains a q-clique as subgraph, we say "G contains a K_q". The union of p mutually disjoint copies of K_q is denoted by pK_q. When A is a set of vertices we denote by $G - A$ the subgraph induced by $V(G) - A$.

In [7, 8] Horvárth and G.Y Katona consider the notion of (H, k) *stable* graph: given a simple graph H, an integer k and a graph G containing H as subgraph, G is a *a* (H, k) *stable graph* whenever the deletion of any set of k edges does not lead to a H-free graph. These authors consider (P_n, k) stable graphs and prove a conjecture stated in [6] on the minimum size of a (P_4, k) stable graph. In [2], Dudek, Szymański and Zwonek are interested in a vertex version of this notion and introduce the (H, k) vertex stable graphs.

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Definition 1.1. [2] Let H be a graph. A graph is a (H,k) vertex stable graph if it contains a graph isomorphic to H after deleting any subset of k vertices. By $Q(H,k)$ we denote the minimum size of a (H,k) vertex stable graph. If G is (H,k) vertex stable of size $Q(H,k)$ we call it *minimum* (H,k) vertex stable.

In this paper, we are only interested by (H,k) vertex stable graphs and, since no confusion will be possible, a (H,k) vertex stable shall be simply called a (H,k) *stable graph.*

In [2], the authors give values of $Q(H,k)$ when H is isomorphic to C_3, C_4 or K_4 and provide upper bounds for some other cases while in [3, 4] the bipartite case is considered.

It must be pointed out that in some cases the value of $Q(H,k)$ can be obtained without the description of extremal graphs, that is (H,k) vertex stable graphs whose size is precisely $Q(H,k)$. In this paper we describe the extremal (H,k) stable graphs when H is isomorphic to K_q, for $q \in \{3,4,5\}$ while in [5] we describe the extremal (K_q,k) stable graphs when k is small with respect to q.

By considering (H,k) stable graph with minimum size, it must be clear that we can add some isolated vertices, the resulting graph remains to be a (H,k) stable graph with minimum size. From now on, the graphs considered have no isolated vertices.

Proposition 1.2. *[2] If G is a (H,k) stable graph with minimum size then every vertex as well as every edge is contained in a subgraph isomorphic to H.*

Remark 1.3. Proposition 1.2 implies, in particular, that when $H \equiv K_q$ then the minimum degree of a (H,k) stable graph with minimum size is at least $q-1$.

Lemma 1.4. *[2] Let $k \geq 1$. If G is (H,k) stable then for any vertex v, $G-\{v\}$ is $(H,k-1)$ stable.*

Definition 1.5. Let H be a non complete graph on $q+p+1$ ($p \geq 0$) vertices and u be one vertex. Let N be the neighbourhood of u and $R = V(H)-u-N$. We shall say that H is a *near complete graph (R,N,u) on $q+p+1$ vertices* (see Figure 1) when

- $H-\{u\}$ is complete,
- $d_H(u) = q+\epsilon$ ($\epsilon \in \{-1,0,1\}$).

Note that the set R is not empty since H is not complete. Hence, $|R| = p - \epsilon$, and since H is not complete we must have $p \geq 2$ when $d_H(u) = q + 1$ and $p \geq 1$ when $d_H(u) = q$.

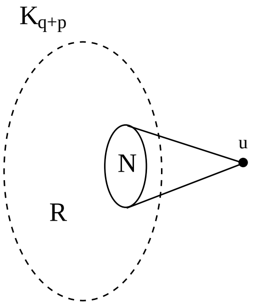

Figure 1: A near complete graph (R,N,u) on $q + p + 1$ vertices

2. Preliminary results

Proposition 2.1. *If G is a (K_q, k) stable graph with minimum size ($q \geq 3$) then G has no component isomorphic to a near complete graph (R,N,u) on $q + p + 1$ vertices.*

Proof. Suppose, by contradiction, that there exists such a component $H = (R, N, u)$ on $q+p+1$ vertices with $d_H(u) = q+\epsilon$ ($\epsilon \in \{-1, 0, 1\}$). Since G is a (K_q, k)-stable graph with minimum size, $G - \{u\}$ is not (K_q, k)-stable. There exists a set S with at most k vertices such that S intersects every K_q of $G - \{u\}$. There exists a K_q in $G - S$ and clearly such a K_q contains u. Since N is a $K_{q+\epsilon}$ and $N - S$ contains no K_q, $|S \cap N| \geq \epsilon + 1$ (trivial for $\epsilon = -1$). If $|S \cap N| \geq \epsilon + 2$ then $|N - S| \leq q - 2$, and hence S intersects every K_q containing u, a contradiction. Thus, $|S \cap N| = \epsilon + 1$ and $|N - S| = q - 1$. If there exists v in $R - S$ then $(N - S) + \{v\}$ is a K_q in $G - \{u\}$, a contradiction. Thus, $R \subset S$. Let $a \in R$ and $b \in N - S$, and set $S^{'} = S - \{a\} + \{b\}$. We have $| \ S^{'} \ | \leq k$ and $G - S^{'}$ contains no K_q, a contradiction.

□

Lemma 2.2. *Let $q \geq 3$ and $k \geq 1$ and let G be a minimum (K_q, k) stable graph.If u is a vertex of degree $q - 1$ then one of the following statements is true*

- $\forall v \in N(u) \quad d(v) \geq q+1$,
- $Q(K_q, k-1) + 3(q-2) \leq Q(K_q, k)$.

Proof. Since $d(u) = q-1$, $\{u\} + N(u)$ induces a complete graph on q vertices. Assume that some vertex $w \in N(u)$ has degree $q+a$ ($a = -1$ or $a = 0$) and let $v \in N(u)$ distinct from w. Then $G - v$ is a $(K_q, k-1)$ stable graph (Lemma 1.4). Since the degree of u in $G - \{v\}$ is $q-2$, no edge incident with u can be contained in a K_q. We can thus delete these $q-2$ edges and the resulting graph (say G') is still a $(K_q, k-1)$ stable graph. In G', the degree of w is now $q+a-2$. Hence, no edge incident with w in G' can be contained in a K_q. Deleting these $q+a-2$ edges from G' leads to a graph $G^{"}$ which remains to be a $(K_q, k-1)$ stable graph.

By deleting v, we have $e(G - \{v\}) \leq e(G) - (q-1)$ and hence

$$e(G^{'}) \leq e(G) - (q-1) - (q-2)$$

We get thus

$$Q(K_q, k-1) \leq e(G^{''}) \leq e(G) - (q-1) - (q-2) - (q+a-2)$$

Since $e(G) = Q(K_q, k)$, the result follows. □

Proposition 2.3. *If G is a minimum $(K_q, 1)$ stable graph ($q \geq 4$) then it is isomorphic to K_{q+1}.*

Proof. Let G be a minimum $(K_q, 1)$ stable graph. Since K_{q+1} is $(K_q, 1)$ stable, clearly $e(G) \leq \binom{q+1}{2}$. We can assume that G is connected. Otherwise, each component contains a K_q, but $\binom{q+1}{2} < 2\binom{q}{2}$ as soon as $q \geq 4$, a contradiction. Let u be a vertex of G and Q be a subgraph of $G - \{u\}$ isomorphic to K_q. Assume that there exists a vertex v outside Q and distinct from u. Note that v can be a neighbour of u. Since $d(u) \geq q-1$ and $d(v) \geq q-1$, $e(G) \geq e(Q) + 2(q-1) - 1 = \binom{q}{2} + 2q - 3 = \binom{q+1}{2} + q - 3$. Thus, $e(G) > e(K_{q+1})$, a contradiction. Hence, $V(G) = V(Q) \cup \{u\}$ with $d(u) \geq q-1$. Since for any edge e $K_{q+1} - \{e\}$ is not $(K_q, 1)$ stable, we see that $d(u) = q$, that is G is isomorphic to K_{q+1}. □

Remark 2.4. It is easy to see that the minimum $(K_3, 1)$ stable graphs are $2K_3$ and K_4.

Proposition 2.5. *If G is a minimum $(K_q, 2)$ stable graph ($q \geq 4$) then it is isomorphic to K_{q+2}.*

Proof. Since K_{q+2} is a $(K_q, 2)$ stable graph, we can suppose that G has at most $\binom{q+2}{2}$ edges. We can suppose, moreover, that G is not complete, otherwise G is obviously reduced to K_{q+2}. Let u be a vertex of minimum degree (recall that the minimum degree is at least $q-1$) and let v be one of its neighbours.

Assume that $d_G(u) = q-1$. $G - \{v\}$ is a $(K_q, 1)$ stable graph, but it is not minimum, since none of the remaining edge incident with u can be contained in a complete graph on q vertices. By deleting the $q-2$ edges incident with u, we get thus a $(K_q, 1)$ stable graph.

If $d(v) \geq q+1$, this graph has at most $\binom{q+2}{2} - (2q-1)$ edges. Since this number of edges must be greater than $\binom{q+1}{2}$ by Proposition 2.3, we have

$$(q+2)(q+1) - 4q + 2 \geq (q+1)q$$

That leads to $q \leq 2$, a contradiction. If $d(v) \leq q$, by Lemma 2.2, we have $Q(K_q, 1) + 3q - 6 \leq Q(K_q, 2)$ and hence

$$q(q+1) + 6q - 12 \leq (q+1)(q+2)$$

Which gives $q \leq 3$, a contradiction.

We can thus assume that the minimum degree of G is at least q. Let u and v be two non adjacent vertices of G. Since $G - \{u, v\}$ contains a K_q (say Q), let a and b be two distinct vertices of Q. Since $G - \{a, b\}$ must contain also a K_q, there is certainly a vertex w distinct from v and u, outside Q, inducing with $q-1$ other vertices of $G - \{a, b\}$ a K_q. Hence G contains three vertices (u, v and w) at least in $G - Q$ and we have:

$$\binom{q+2}{2} \geq e(G) \geq \binom{q}{2} + 3q - 2$$

Which gives $q < 3$, a contradiction. Hence G is complete and the proposition follows. □

Lemma 2.6. *Let G be a minimum $(K_q, 3)$ stable graph, $q \geq 5$. Let u be a vertex of minimum degree in G and suppose that $d_G(u) = q + l$, where $-1 \leq l \leq 1$. Then for every neighbour v of u we have $d_G(v) \geq q + l + 2$.*

Proof. Suppose, contrary to our claim, that $d_G(v) \leq q+l+1$ for a neighbour v of u. Since, by Proposition 1.2, the edge uv is contained in a clique of order q and $q \geq 5$, there is a set A of vertices of G such that $|A| = l+2$ and the vertices of the set $A \cup \{u, v\}$ are mutually adjacent. The graph $G' = G - A$ is $(K_q, 3-(l+2))$ stable. We have $d_{G'}(u) = q+l-(l+2) = q-2$, hence also $G'' = G' - \{u\}$ is $(K_q, 1-l)$ stable. But in G'' the degree of the vertex v is at most $q-2$ and therefore $G''' = G'' - \{v\}$ is $(K_q, 1-l)$ stable. Since every vertex of the set $A \cup \{u, v\}$ has at least $q-3$ neighbours outside this set, we have

$$\binom{q+1-l}{2} \leq e(G''') \leq \binom{q+3}{2} - (l+4)(q-3) - \binom{l+4}{2}$$

which contradicts $q \geq 5$. □

Proposition 2.7. *If G is a minimum $(K_q, 3)$ stable graph ($q \geq 5$) then it is isomorphic to K_{q+3}.*

Proof. Note first that to prove the proposition it is sufficient to prove that every vertex of G has the degree at least $q+2$.

Let u be a vertex of the minimum degree in G and suppose, contrary to our claim, that $d_G(u) \leq q+l$, where $-1 \leq l \leq 1$.

Let $v_1, v_2, ..., v_{l+2}$ be such vertices of G that the set $\{u, v_1, v_2, ..., v_{l+2}\}$ induce a clique in G (such vertices exist since u is contained in a clique of order q by Proposition 1.2 and $q \geq 5$). By Lemma 2.6, we have $d_G(v_i) \geq q+l+2$ for $i = 1, 2, ..., l+2$. Set $G' = G - \{v_1, v_2, ..., v_{l+2}\}$. The graph G' is clearly $(K_q, 1-l)$ stable. Moreover, since $d_{G'}(u) = q-2$, the graph $G'' = G' - \{u\}$ is also $(K_q, 1-l)$ stable and we have

$$\binom{q+1-l}{2} \leq e(G'') \leq \binom{q+3}{2} - (l+2)(q-1) - (q-2) - \binom{l+3}{2}$$

which contradicts $q \geq 5$.

□

3. A characterization of (K_3, k) stable graph with minimum size

Dudek, Szymański and Zwonek in [2] have shown that $Q(K_3, k) = 3k + 3$ for every nonnegative integer k. In this section we characterize all that (K_3, k) stable graphs with minimum size.

Clearly, K_3 is the unique minimum $(K_3, 0)$ stable graph.

The following theorem characterize all graphs which are (K_3, k) stable with minimum size.

Theorem 3.1. *Let* $G = (V, E)$ *be a* (K_3, k) *stable graph with minimum size.* G *is isomorphic to* $pK_4 + qK_3$*, where* p *and* q *are such nonnegative integers that* $2p + q = k + 1$.

Proof. By Remark 2.4, K_3 is the unique minimum $(K_3, 0)$ stable graph, and the minimum $(K_3, 1)$ stable graphs are $2K_3$ and K_4. Clearly, the graph $(k + 1)K_3$ is a (K_3, k) stable graph and has $3k + 3$ edges. Let $k_0 \geq 1$ and suppose that for every $k < k_0$ every minimum (K_3, k) stable graph is a union of p copies of K_4 and q copies K_3 with $2p + q = k + 1$.

Let G be a (K_3, k_0) stable graph of minimum size. Since $G - \{v\}$ is $(K_3, k_0 - 1)$ stable for every vertex v, we have $3k_0 \leq e(G - \{v\}) \leq e(G) - d_G(v) \leq 3k_0 + 3 - d_G(v)$, that is $d_G(v) \leq 3$. If every vertex of G has degree equal to 2, then G is a union of $k_0 + 1$ copies of K_3, and the theorem is proved. So we may suppose that there is a vertex v_0 of degree 3. But then $G - \{v_0\}$ is $(K_3, k_0 - 1)$ stable and $e(G - \{v_0\}) = 3k_o$, that is $G - \{v_0\}$ is minimum $(K_3, k_0 - 1)$ stable. By the induction hypothesis, $G - \{v_0\}$ is isomorphic to $p'K_4 + q'K_3$, where $2p' + q' = k_0$. It is clear that all the neighbours of v_0 are in the same component of G, (otherwise one of the edges incident with v_0 is not contained in any triangle, contrary to Proposition 1.2). Now it is easy to see that G is isomorphic to $(p' + 1)K_4 + (q' - 1)K_3$ and $2(p' + 1) + (q' - 1) = k_0 + 1$ (otherwise there is a set A of cardinality k_0 which is transversal of all cliques of order 3 in G). □

4. A characterization of (K_4, k) stable graph with minimum size

In [2] the minimum number of edges of a (K_4, k) stable graph is given.

Theorem 4.1. *[2] If G is a (K_4, k) stable graph with minimum size ($k \geq 1$) then*

- $Q(K_4, 0) = 6$,
- $Q(K_4, k) = 5k + 5$ *when* $k \geq 1$.

Proposition 4.2. *If G is a (K_4, k) stable graph with minimum size ($k \geq 1$) then it has no connected component isomorphic to K_4.*

Proof. Let us consider $k \geq 2$. Assume that some component H of G is isomorphic to a K_4 with the vertices of H being a, b, c, d. Then $G - H$ has $5k - 1$ edges. Since $G - H$ is not a $(K_4, k - 1)$ stable graph, there is a set S with at most $k - 1$ vertices intersecting each K_4 of $G - H$. Then $S + \{a\}$ intersects each K_4 of G while S has at most $k - 1$ vertices, a contradiction.

When $k = 1$, G must have 10 edges by Theorem 4.1. Since for each vertex v the graph $G - v$ contains a K_4, v is joined to this K_4 by 4 edges. Hence G is a K_5 and the result holds. □

Proposition 4.3. *If G is a (K_4, k) stable graph with minimum size ($k \geq 1$) then every vertex of G has degree $3, 4$ or 5.*

Proof. By Proposition 1.2 every vertex is contained in a K_4, hence its degree is at least 3. Assume that G has a vertex v with $d(v) \geq 6$. Then, by Lemma 1.4, $G - v$ is a $(K_4, k - 1)$ stable graph and therefore has at least $5k$ edges, which is impossible since G has exactly $5k + 5$ edges, by Theorem 4.1. □

Proposition 4.4. *Let $G = (V, E)$ be a (K_4, k) stable graph with minimum size ($k \geq 1$). If H is a component containing no vertex of degree 5, then each vertex of H has degree 4.*

Proof. By Proposition 4.3 the vertices of G have degree 3 or 4. Assume to the contrary that H contains some vertex v with degree 3. Let $N(v) = \{u_1, u_2, u_3\}$ be its neighbourhood. By Proposition 1.2, $N(v)$ is complete. Since H is not isomorphic to K_4 by Proposition 4.2, assume that, without loss of generality, u_1 is joined to some new vertex w. Since u_1w must be contained in a K_4 by Proposition 1.2, the vertices of this K_4 are in $\{u_1, u_2, u_3, w, v\}$, thus w must be adjacent to u_2 and u_3. By Proposition 2.1, H is not isomorphic to a K_5 minus one edge, hence there must exist some new vertex w' adjacent to w. Since each vertex in $\{u_1, u_2, u_3, w\}$ has degree 4, we cannot find a K_4 using the edge ww', a contradiction with Proposition 1.2. $\square$

Theorem 4.5. *If G is (K_4, k) stable ($k \geq 1$) with minimum size then it is isomorphic to $pK_5 + qK_6$, where p and q are nonnegative integers such that $2p + 3q = k + 1$.*

Proof. The proof is by induction on k. By Proposition 2.3, the only minimum $(K_4, 1)$ stable graph with minimum size is K_5. Let $k_0 \geq 2$ and suppose that for every integer k, such that $1 \leq k < k_0$ every (K_4, k) stable graph with minimum size is isomorphic to $pK_5 + qK_6$, where p and q are nonnegative integers such that $2p + 3q = k + 1$.

Let G be a (K_4, k_0) stable graph with minimum size. By Theorem 4.1 we have $e(G) = 5k_0 + 5$. Note that it is sufficient to prove that every component of G is isomorphic either to K_5 or to K_6.

By Proposition 4.3, we have $3 \leq d_G(v) \leq 5$ for every vertex v of G. Since by Proposition 1.2, every edge of G is contained in a K_4, all the neighbours of a vertex v are in the same component of $G - \{v\}$.

Suppose first that there is a vertex v in G such that $d_G(v) = 5$. Then $G - \{v\}$ is $(K_4, k_0 - 1)$ stable and moreover, since $e(G - v) = 5k_0$, $G - \{v\}$ is minimum $(K_4, k_0 - 1)$ stable. Hence every component of $G - \{v\}$ is either isomorphic to K_5 or to K_6. If v is connected in G to a K_6, then the component of G which contains v is a near complete graph, contradicting Proposition 2.1. So v is connected to a K_5 and G is a union of graphs isomorphic to K_5 or K_6, as desired.

Assume now that no component has a vertex of degree 5. Then, by Proposition 4.4, each component is a 4-regular subgraph.

Let v be any vertex and let $N(v) = \{u_1, u_2, u_3, u_4\}$ be its neighbourhood. Since v is contained in a K_4 by Proposition 1.2, we can suppose, without restriction of generality, that u_1u_2, u_1u_3 and u_2u_3 are edges of G. Since vu_4 must be contained in a K_4 by Proposition 1.2, u_4 must be adjacent to at least 2 vertices of N (say, without loss of generality, u_2 and u_3).

case 1: $u_1u_4 \in E(G)$. Then the component containing v is a K_5.

case 2: $u_1u_4 \notin E(G)$. Let w be a new vertex adjacent to u_1 (this new vertex must exist since the component of v is 4-regular). Then u_1w cannot be contained in a K_4, a contradiction.

□

5. A characterization of (K_5, k) stable graph with minimum size

In this section we provide the value of $Q(K_5, k)$ for $k \geq 5$, as well as a description of the corresponding minimum stable graphs.

Lemma 5.1. *Let G be a (K_5, k) stable graph containing a component isomorphic to K_p with $p \geq 9$. Then the graph G' obtained from G by deleting two vertices v and v' in this K_p and adding a disjoint K_6 is a (K_5, k) stable graph such that*

- *if $p \geq 10$ then $e(G') < e(G)$,*
- *if $p = 9$ then $e(G') = e(G)$.*

Proof. Let A be the set of vertices created by the adjunction of the new K_6. Let S be a set of vertices with $|S| \leq k$ in G'. If $|S \cap A| \leq 1$, $G - S$ obviously contains a K_5. If $|S \cap A| \geq 2$ then $S^{'} = S - A + \{v, v'\}$ is a subset of G with at most k vertices. Hence $G - S'$ contains a K_5 which still exists in $G^{'} - S$.

If $p \geq 10$ then at least 17 edges are deleted and 15 edges are created, thus $e(G') < e(G)$. If $p = 9$, 15 edges are deleted while 15 edges are created so $e(G) = e(G')$. □

Lemma 5.2. *Let G be a (K_5, k) stable graph with minimum size. Then G does not contain 2 components isomorphic to a K_p with $5 \leq p \leq 6$.*

Proof. If we have two components (say K and L) isomorphic to a complete graph with 5 vertices then the graph G' obtained from G by deleting these two components and adding a complete graph on 6 vertices is still a (K_5, k) stable graph. Indeed, let S' be any subset of $V(G')$ with $|S'| \leq k$. If $G^{'} - S'$ does not contain any K_5 then S' must contain at least 2 vertices v and w of the new K_6. Let $S = S' - \{v, w\} + \{a, b\}$, where $a \in K$ and $b \in L$, then $G - S$ does not contain any K_5, a contradiction.

When we have a K_5 and a K_6, we get the same kind of contradiction when replacing these two complete graphs with a K_7 as well as when we have two K_6s replaced by a K_8. $\square$

Lemma 5.3. *Let $k \geq 5$ and let G be a (K_5, k) stable graph with minimum size which is the vertex disjoint union of complete graphs. Then each component is a K_7 or a K_8.*

Proof. By Lemma 5.1, we can consider that each component is a K_p with $5 \leq p \leq 9$. By Lemma 5.2, at most one component is a K_5 or a K_6. If some component is isomorphic to a K_9 then let us replace this component by a K_6 and a K_7. By Lemma 5.1 the resulting graph is still a (K_5, k) stable graph with minimum size. It is clear that no component is isomorphic to a K_9 now. Indeed, applying once more the operation described above leads to a (K_5, k) stable graph with minimum size having two K_6s, a contradiction with Lemma 5.2.

Therefore, we have to consider only the case when G is the vertex disjoint union of complete graphs isomorphic to K_7 or K_8 and at most one K_5 or one K_6. Replacing a K_5 and a K_7 by one K_8 leads to a (K_5, k) stable graph with a number of edges less than the number of edges of G, a contradiction. Replacing a K_6 and a K_8 by two K_7 leads to a (K_5, k) stable graph with a number of edges less than the number of edges of G, a contradiction.

It remains to consider the case where the components are all isomorphic to a K_7 with the exception of one K_6 or all isomorphic to a K_8 with the exception of one K_5. When we have at least two K_7 and a K_6, these three complete graphs can be replaced by two K_8, the resulting graph is still a (K_5, k) stable graph, but the number of edges is less than the number of edges of G, a contradiction. When we have at least two K_8 and a K_5, these three complete graphs can be replaced by three K_7, the resulting graph is still a (K_5, k) stable graph, but the number of edges is less than the number of edges of G, a contradiction.

When G is reduced to a K_8 and a K_5 or to a K_7 and a K_6 , we must have $k \leq 4$, which is impossible.

$\square$

Lemma 5.4. *Let G be a (K_5, k) stable graph with minimum size and maximum degree 6. Assume that some component contains a K_6. Then either the component is equal to this K_6 or to K_7.*

Proof. Let $A = \{v_1 \dots v_6\}$ be the set of vertices of the K_6. If $d(v_i) = 5$ for each vertex in A the proof is complete. Assume that the vertex v_1 has degree 6 and let w be its neighbour outside A. Since $v_1 w$ must be contained in a K_5 by Proposition 1.2, w must be adjacent to 3 other vertices in V (say v_2, v_3 and v_4). In the same way, if v_5 or v_6 has a neighbour outside A, this vertex must be adjacent to 4 vertices of A, which is impossible if this vertex is distinct from w.

Let $w' \notin A$ be a neighbour of w (if any). Since ww' must be contained in a K_5 by Proposition 1.2, w' must have at least 3 neighbours in A, which is impossible. Hence the connected component containing the K_6 contains at most one vertex more (the vertex w). If w is not adjacent to at least one of v_5 or v_6 (say v_5) then this component is a near complete graph (R,N,u) on 7 vertices (with $u = w$, $N = A$ or $N = A + \{v_5\}$, $R = \{v_5, v_6\}$ or $R = \{v_5\}$ respectively), which is impossible by Proposition 2.1. If w is adjacent to v_4 and v_5, the component containing the K_6 is a K_7 as claimed. $\square$

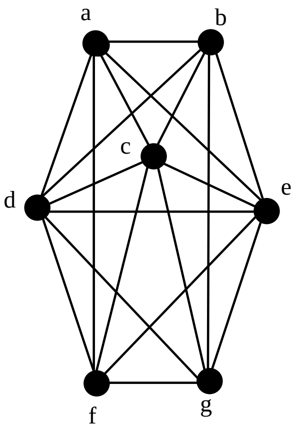

Figure 2: Forbidden component of a (K_5, k) stable graph with minimum size.

Lemma 5.5. *Let G be a (K_5, k) stable graph with minimum size. Then no component of G is isomorphic to the subgraph depicted in Figure 2.*

Proof. Since $G - \{a\}$ is not a (K_5, k) stable graph, there exists a set S with $|S| \leq k$ which intersects each K_5 in $G - \{a\}$. If S contains one of the the vertices in $\{c, d, e\}$, then S intersects each K_5 in G, which is impossible. Since $\{c, d, e, f, g\}$ induces a K_5, S contains at least one vertex in $\{f, g\}$. When $g \in S$, S intersects each K_5 in G, which is impossible. Assume that $f \in S$ then $S' = S - \{f\} + \{c\}$ intersects each K_5 in G, a contradiction since $|S'| \leq k$. □

Lemma 5.6. *Let G be a (K_5, k) stable graph with minimum size. Assume that some component contains vertices with degree* 5 *or* 6 *only. Then this component is a complete graph with at least* 5 *vertices.*

Proof. Let H be a connected component containing vertices of degree 5 or 6 only. By Proposition 1.2, every edge is contained in a K_5. Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be a set of vertices inducing a K_5 in H.

case 1 : $\exists \quad i \quad 1 \leq i \leq 5 \quad d_H(u_i) = 6$.

Without loss of generality we may assume that $i = 1$. Let w and w' the two neighbours of u_1 outside U. Since u_1w must be contained in a K_5 and since this K_5 contains 4 neighbours of u_1, w must be adjacent to at least two vertices in $U - \{u_1\}$. Without loss of generality, assume that $wu_2 \in E(G)$ and $wu_3 \in E(G)$. Let us remark that w is not joined to the two vertices u_4 and u_5, otherwise, H contains a K_6 and H is thus isomorphic to a complete graph by Lemma 5.4. For the same reason, w' is not joined to all the vertices in U.

subcase 1.1 : If w or w' has no other neighbour in U, say w, we must have $ww' \in E(G)$, $w'u_2 \in E(G)$ and $w'u_5 \in E(G)$. One of u_4 or u_5, say u_4, is not adjacent to w', and there must be a vertex w'' adjacent to u_4 ($d_H(u_4) \geq 5$), but the edge u_4w'' cannot be on any K_5, which is impossible.

subcase 1.2 : If w has an other neighbour (say u_5) in U. When w' is not adjacent to w, w' must be adjacent to precisely 3 vertices in $\{u_2, u_3, u_4, u_5\}$. If u_4w' is an edge, there must be an edge incident with w' ($d_H(w') \geq 5$), but this edge cannot be contained in any K_5, a contradiction. If u_4w' is not an edge, there must be an edge incident with u_4 and this edge cannot be contained in any K_5, which is impossible. Thus, w and w' are adjacent and there must be 2 vertices in $\{u_2, u_3, u_5\}$ adjacent to w', say u_2 and u_3. But now, there is an additionnal edge incident with u_4 and this edge is u_4w' otherwise it is not contained in any K_5. It is a routine matter to check that there is no additionnal vertex nor edge in H. Hence H is isomorphic to the graph depicted in Figure 2, a contradiction with Lemma 5.5.

case 2 : $\forall \quad i \quad 1 \leq i \leq 5 \quad d_H(u_i) = 5$.

Let w be the last neighbour of u_1 outside U. Since wu_1 must be contained in a K_5, w must be adjacent to u_2, u_3 and u_4, without loss of generality. Hence, $wu_5 \notin E(G)$ or H is complete. Since $d_H(u_5) = 5$, let $w' \neq w$ be the last neighbour of u_5 outside U. Then u_5w' is not contained in a K_5, which is impossible.

$\square$

Lemma 5.7. $Q(K_5, 4) = 36$.

Proof. Since K_9 and $K_6 + K_7$ are $(K_5, 4)$ stable graphs, we certainly have $Q(K_5, 4) \leq 36$.

Assume that some graph G with $e(G) \leq 35$ is a $(K_5, 4)$ stable graph with minimum size. Let v be a vertex with maximum degree. If $d(v) \geq 8$ then $G - v$ is a $(K_5, 3)$ stable graph with at most 27 edges, a contradiction with Proposition 2.7. If $d(v) = 7$ then $G - \{v\}$ is a $(K_5, 3)$ stable graph with at most 28 edges. Hence we must have $e(G - \{v\}) = 28$ and G is a $(K_5, 3)$ stable graph with minimum size. By Proposition 2.7, $G - \{v\}$ is a K_8 and G is a K_9 minus one edge, a contradiction with Proposition 2.1.

We can thus assume that the maximum degree of G is at most 6. If some vertex u has degree 4, let v be one of its neighbours. We know, by Lemma 2.6 that $d(v) = 6$. By deleting v, we get a graph $G - v$ which is a $(K_5, 3)$ stable graph. In that graph, the edges incident with u are not contained in a K_5 since the degree of u is now 3. We can thus delete these edges and we obtain a $(K_5, 3)$ stable graph with at most 27 edges, a contradiction with Proposition 2.7.

Hence every vertex must have degree 5 or 6. By Lemma 5.6, the components of G are complete graphs. It can be easily checked that the only convenient graphs are K_9

and $K_6 + K_7$, a contradiction with $e(G) \leq 35$. □

Lemma 5.8. *$K_6 + K_7$ and K_9 are the unique $(K_5, 4)$ stable graph with minimum size.*

Proof. By Lemma 5.7, let G be a $(K_5, 4)$ stable graph with 36 edges.

If G has a vertex of degree at least 8 then $G - \{v\}$ is a $(K_5, 4)$ stable graph with at most 28 edges. Hence $G - \{v\}$ must have exactly 28 edges and $d(v) = 8$. Since, by Proposition 2.7 $G - \{v\}$ is a K_8, G itself is a K_9.

We can thus assume that the maximum degree of G is at most 7. If some vertex u has degree 4, let v be one of its neighbours. We know, by Lemma 2.2 that $d(v) \geq 6$. By deleting v, we get a graph $G - \{v\}$ which is a $(K_5, 3)$ stable graph. In that graph, the edges incident with u are not contained in a K_5 since the degree of u is now 3. We can thus delete these edges and we obtain a $(K_5, 3)$ stable graph with 27 edges, a contradiction with Proposition 2.7.

Hence the degree of each vertex is 5, 6 or 7.

In the following Claims Q_1 and Q_2 denote any two induced K_5 of G.

Claim 5.8.1. $|V(Q_1) \cap V(Q_2)| \neq 1$.

Proof Assume that $|V(Q_1) \cap V(Q_2)| = 1$ then the vertex in the intersection must have degree at least 8, which is impossible. □

Claim 5.8.2. *Assume that Q_1 and Q_2 are vertex disjoint and let $xy \in E(G)$ (if any) such that $x \in V(Q_1)$ and $y \in V(Q_2)$. Then we can find a vertex $x' \in V(Q_1)$ and a vertex $y' \in V(Q_2)$ such that $\{x, x', y, y'\}$ is contained in an induced K_5 of G. Moreover the 5^{th} vertex of this K_5 must be contained in $V(Q_1) \cup V(Q_2)$.*

Proof

Since G is a minimum $(K_5, 4)$ stable graph, the edge xy must be contained in a K_5 (say Q). By Claim 5.8.1 Q contains at least one vertex more in Q_1 (say x') and one vertex more in Q_2 (say y'). Let a be the 5^{th} vertex of Q and assume that $a \notin V(Q_1) \cup V(Q_2)$. $G - \{a\}$ is a $(K_5, 3)$ stable graph but it is not minimum since the edges between $\{x, x'\}$ and $\{y, y'\}$ cannot be contained in a K_5. By deleting these 4 edges in $G - \{a\}$ we get a

$(K_5, 3)$ stable with 28 edges. By Proposition 2.7, $G - \{a\}$ is isomorphic to K_8, which is impossible. □

Claim 5.8.3. $|V(Q_1) \cap V(Q_2)| \neq 2$.

Proof Assume that $V(Q_1) \cap V(Q_2) = \{x, y\}$. Let us remark that these two vertices have degree 7. Let $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$ be the sets of remaining vertices of Q_1 and Q_2 respectively.

Assume that some edge is missing between $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$ (say $u_1v_1 \notin E(G)$). Then $G_1 = G - \{u_2, v_2, v_3\}$ is a $(K_5, 1)$ stable graph in which the vertices x and y are not contained in any K_5. Hence $G_2 = G_1 - \{x, y\}$ is a $(K_5, 1)$ stable graph. Since $d_G(v_1) \leq 7$, the degree of v_1 in G_2 is at most 3. Hence v_1 is not contained in any K_5 and $G_3 = G_2 - \{v_1\}$ is $(K_5, 1)$ stable graph.

case 1 : *The edge u_1u_3 is not contained in a K_5.*

Then $G_4 = G_3 \setminus \{u_1, u_3\}$ is a $(K_5, 1)$ stable graph. By Proposition 2.3, G_4 contains at least 15 edges. Since $V(Q_1) \cup V(Q_2)$ contains 19 edges, we need to find two more edges. By Claim 5.8.2 no edge can connect $V(Q_1) \cup V(Q_2)$ to G_4. Whatever is the place of these edges, $G - \{x, y\}$ is a $(K_5, 2)$ stable graph, where no vertex in $\{u_1, u_2, u_3\}$ nor in $\{v_1, v_2, v_3\}$ can be contained in a K_5. Hence $G - (V(Q_1) \cup V(Q_2))$ is a $(K_5, 2)$ stable graph and must contain at least 21 edges by Proposition 2.5. That is G must contain at least 40 edges, a contradiction.

case 2 : *The edge u_1u_3 is contained in a K_5.*

That means that u_1 and u_3 have 3 neighbours outside $V(Q_1) \cup V(Q_2)$. In the same way, we can consider that u_2 has also three such neighbours (take $G_1 = G - \{u_3, v_2, v_3\}$) as well as v_1, v_2 and v_3 by symmetry. Hence G_3 contains the 19 edges of $V(Q_1) \cup V(Q_2)$ and 18 edges connecting $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$ to the vertices outside, a contradiction.

We can thus suppose that every vertex in $\{u_1, u_2, u_3\}$ is joined to every vertex in $\{v_1, v_2, v_3\}$. That means that $V(Q_1) \cup V(Q_2)$ is a connected component of G and induces a K_8. No connected component distinct from this K_8 can contain a K_5, which is impossible.

□

Claim 5.8.4. $|V(Q_1) \cap V(Q_2)| \neq 3$ *or* G *is isomorphic to* $K_7 + K_6$.

Proof Assume that $|V(Q_1) \cap V(Q_2)| = \{x, y, z\}$. Let $\{u_1, u_2\}$ and $\{v_1, v_2\}$ be the sets of remaining vertices of Q_1 and Q_2 respectively.

Then $G_1 = G - \{x, y, z\}$ is a $(K_5, 1)$ stable graph in which the vertices u_1, u_2, v_1, v_2 are not contained in any K_5 by Claims 5.8.1 and 5.8.3. That means that $G_2 = G - (V(Q_1) \cup V(Q_2))$ is a $(K_5, 1)$ stable graph. If $w \in V(Q_1) \cup V(Q_2)$ is adjacent to some vertex $w^{'}$ in G_2 then a K_5 using that edge forces 4 more edges more between these two subgraphs, a contradiction since G would have at least 37 edges (by Proposition 2.3 G_2 has at least 15 edges).

If some edge is missing between $\{u_1, u_2\}$ and $\{v_1, v_2\}$ (say $u_1v_1 \notin E(G)$), then $G_3 = G - \{u_2, v_2\}$ is a $(K_5, 2)$ stable graph where x, y, z, u_1, v_1 are not contained in any K_5. The graph G_3 is still $(K_5, 2)$ stable. Hence, by Proposition 2.5 G must have at least 38 edges, a contradiction.

We can thus suppose that $V(Q_1) \cup V(Q_2)$ induces a K_7. The remaining part of G is the $(K_5, 1)$ stable graph G_2 described above. This graph must have exactly 15 edges. Hence, G_2 is isomorphic to K_6 by Proposition 2.3. That means that G is isomorphic to $K_7 + K_6$. □

Claim 5.8.5. $|V(Q_1) \cap V(Q_2)| \neq 4$ *or* G *is isomorphic to* $K_7 + K_6$.

Proof Assume on the contrary that $|V(Q_1) \cap V(Q_2)| = \{x, y, z, t\}$ and G is not isomorphic to $K_7 + K_6$. Let u and v be the remaining vertices of Q_1 and Q_2 respectively.

Let r be a neighbour of u, if any, outside $V(Q_1) \cup V(Q_2)$. Let Q_3 be a K_5 containing the edge ur. Then $V(Q_1) \cap V(Q_3)$ contains 4 vertices (Claims 5.8.1 and 5.8.3) but $V(Q_2) \cap V(Q_3)$ contains 3 vertices, a contradiction.

Since $d(u) \geq 5$, we must have $uv \in E(G)$ (and, moreover, $d(u) = d(v) = 5$).

case 1 : *There are neighbours of* $\{x, y, z, t\}$ *outside* $V(Q_1) \cup V(Q_2)$.

Let s be such a neighbour of x,. The edge xs being contained in a K_5, this K_5 must have 4 common vertices with Q_1 and 4 common vertices with Q_2 (Claims 5.8.1, 5.8.3 and 5.8.4). Hence, s must be adjacent to the 4 vertices of $V(Q_1) \cap V(Q_2)$ and $\{x, y, z, t, s\}$ induces a K_5 with 4 common vertices with Q_1 and 4 common vertices with Q_2. By the above remark, we have $us \in E(G)$ as well as $vs \in E(G)$ and $V(Q_1) \cup V(Q_2)$ induces a K_7. By deleting 3 vertices of this component, the resulting graph is $(K_5, 1)$ stable with 15 edges, and hence is isomorphic to K_6.

case 2 : *There are no neighbours of* $\{x, y, z, t\}$ *outside* $V(Q_1) \cup V(Q_2)$.

Hence, $V(Q_1) \cup V(Q_2)$ is a connected component of G inducing a K_6. By deleting 2 vertices in this component, the resulting graph is $(K_5, 2)$ stable. Since the remaining vertices of $V(Q_1) \cup V(Q_2)$ in this graph are not contained in any K_5, we can delete them and the $(K_5, 2)$ stable graph obtained in this way must have 21 edges exactly. This component is a K_7 by Proposition 2.5, a contradiction. □

To end our proof, it is sufficient to say that any two induced K_5 of G must be disjoint by Claims 5.8.1, 5.8.3, 5.8.4 and 5.8.5. That means that each component of G is a K_5, which is impossible since G must have 36 edges.

□

Lemma 5.9. $Q(K_5, 5) = 42$.

Proof. Since $K_7 + K_7$ is a $(K_5, 5)$ stable graphs, we certainly have $Q(K_5, 5) \leq 42$. Let G be a $(K_5, 5)$ stable graph with minimum size and assume that $e(G) \leq 41$. Let us remark that the size of G is certainly greater than $Q(K_5, 4)$.

If G has a vertex of degree at least 6 then $G-v$ is a $(K_5, 4)$ stable graph with at most 35 edges, a contradiction with Lemma 5.7. If G has a vertex of degree 4 then, since the degree of every neighbour is at most 5, we must have, by Lemma 2.2, $Q(K_5, 5) \geq Q(K_5, 4) + 9$, a contradiction.

Hence, every vertex must have degree 5 and by Lemma 5.6, the connected component of G are isomorphic to K_6. It is easy to see that no such graph can exist. □

Lemma 5.10. *$K_7 + K_7$ is the unique $(K_5, 5)$ stable graph with minimum size.*

Proof. By Lemma 5.9, let G be a $(K_5, 5)$ stable graph with 42 edges.

If G has a vertex of degree at least 7 then $G - v$ is a $(K_5, 4)$ stable graph with at most 35 edges, a contradiction with Lemma 5.7.

If G has a vertex u of degree 4, let v be one of its neighbours. By deleting v we get a $(K_5, 4)$ stable graph where the edges incident with the vertex u are not contained in any K_5 since the degree of u in that graph is 3. By deleting these edges we get a $(K_5, 4)$ stable graph with at most 35 edges, a contradiction with Lemma 5.7.

Hence every vertex has degree 5 or 6. By Lemma 5.6, the connected components of G are complete. It is an easy task to see that the only convenient graph G is isomorphic to $K_7 + K_7$, as claimed. □

Theorem 5.11. *If G is (K_5, k) stable $(k \geq 5)$ with minimum size then* $|E(G)| = 7k + 7$.

Proof. We can check that the property holds for $k = 5$ (G is the vertex disjoint union of two K_7s by Lemma 5.10). Assume that the property holds for any k $(5 \leq k < k_0)$ and let us consider a (K_5, k_0) stable graph G with minimum size. Assume that G has at most $7k_0 + 6$ edges and let v be a vertex of maximum degree. Since $G - v$ is a $(K_5, k_0 - 1)$ stable graph, it must have $7k_0$ edges, which means that $d(v) \leq 6$. Moreover, by Proposition 1.2, we certainly have $d(v) \geq 4$.

Let z be a vertex of degree 4 in some component of G. If z has a neighbour v whose degree is 6 then $G - v$ has exactly $7k_0$ edges. Hence $G - v$ is a $(K_5, k_0 - 1)$ stable graph with minimum size. Since the degree of z is 3 in $G - v$, any edge incident with z in $G - v$ is not contained in a K_5, a contradiction.

If z has a neighbour v whose degree is 5 then $G - v$ has exactly $7k_0 + 1$ edges. $G - v$ is a $(K_5, k_0 - 1)$ stable graph. This graph does not have minimum size since the 3 remaining edges incident with z are not contained in a K_5. If we delete these 3 edges, we still have a $(K_5, k_0 - 1)$ stable graph, but the number of edges is then $7k_0 - 2$, which is impossible by the induction hypothesis.

Hence the neighbours of z have also degree 4, that means that the component containing a vertex of degree 4 is a 4 regular graph containing a K_5. That is, this component is a K_5.

Since each component containing only vertices of degree 5 or 6 are complete by Lemma 5.6, we have thus that all the connected components of G are complete. By Lemma 5.3, each component has 7 vertices or 8 vertices (recall that $k_0 \geq 5$). Assume that we have p components isomorphic to a K_7 and q isomorphic to a K_8, then $k_0 \leq 3p + 4q - 1$ and G

has $21p+28q$ edges. If $k_0 = 3p+4q-1$, we have $21p+28q = 7k_0+7$, a contradiction. If $k_0 < 3p+4q-1$ then deleting one vertex in some component leaves the graph (K_5, k_0) stable, which is impossible. $\square$

Dudek, Szymański and Zwonek proposed the following conjecture.

Conjecture 5.12. *[2] For every integer $q \geq 5$ there is an integer $k(q)$ such that $Q(K_q, k) = (2q-3)(k+1)$ for $k \geq k(q)$.*

Theorem 5.11 proves this conjecture for $q = 5$ with $k(q) = 5$.

Theorem 5.13. *If G is (K_5, k) stable $(k \geq 5)$ with minimum size then*

- $|E(G)| = 7k+7$,
- *each connected component is isomorphic to a complete graph with 7 or 8 vertices,*
- *there are p components isomorphic to K_7 and q components isomorphic to K_8 for any choice of p and q with $3p+4q = k+1$.*

Proof. By Theorem 5.11, the first claim is true. We can check that the property of the second claim holds for $k = 5$ (G is the vertex disjoint union of two K_7s). Assume that the property holds for any k $(5 \leq k < k_0)$ and let us consider a (K_5, k_0) stable graph G with minimum size.

If G has a vertex v of degree at least 8, then $G - v$ has at most $7k_0 - 1$ edges and cannot be a $(K_5, k_0 - 1)$ stable graph, a contradiction. Thus the maximum degree of G is at most 7.

case 1 : $\exists \quad v \in V(G) \quad d_H(v) = 7$.

In that case, $G - v$ is $(K_5, k_0 - 1)$ stable graph with minimum size. By the induction hypothesis, each connected component of $G - v$ is isomorphic to a complete graph with 7 or 8 vertices. Going back to G by adding the vertex v leads to join v to a whole connected component of $G - v$, otherwise, some edge incident with v cannot be contained in a K_5, a contradiction with Proposition 1.2. The vertex v cannot be connected to 7 vertices of a K_8, otherwise we would have a near complete graph, a contradiction. Hence v is joined to the 7 vertices of a K_7 and the connected component of G containing v is a K_8.

case 2 : If some connected component of G contains vertices of degree 5 or 6 only, then, by Lemma 5.6, this component is a complete graph on at least 7 or 8 vertices (Lemma 5.3), since $k_0 > 5$.

case 3 : If some connected component of G contains a vertex v of degree 4 then, no neighbour w of v may have a degree at least 5. Otherwise, $G - w$ is a $(K_5, k_0 - 1)$ stable graph with at most $7k_0 + 2$ edges. Since the degree of v is 3 in $G - w$, the 3 edges incident with v are not contained in any K_5. We can thus delete these 3 edges from $G - w$, getting a $(K_5, k_0 - 1)$ stable graph with at most $7k_0 - 1$ edges, which is impossible by Theorem 5.11. Hence this component is $4-$regular. That is, this component is reduced to a K_5, a contradiction with Lemma 5.3 since $k_0 > 5$.

It is now a routine matter to check that the third claim holds. □

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References

[1] J.A. Bondy and U.S.R. Murty, *Graph theory*, vol. 244, Springer, Series Graduate Texts in Mathematics, 2008.

[2] A. Dudek, A. Szymański, and M. Zwonek, (H, k) *stable graphs with minimum size*, Discuss. Math. Graph Theory **28** (2008), 137–149.

[3] A. Dudek and A. Żak, *On vertex stability with regard to complete bipartite subgraphs*, Discuss. Math. Graph Theory **30** (2010), 663–669.

[4] A. Dudek and M. Zwonek, (H, k) *stable bipartite graphs with minimum size*, Discuss. Math. Graph Theory (2009), 573–581.

[5] J-L. Fouquet, H. Thuillier, J-M. Vanherpe, and A.P. Wojda, *On* (K_q, k) *stable graphs with small* k, http://hal.archives-ouvertes.fr/hal-00560277/PDF/KnkSmallk.pdf, submitted, 01 2011.

[6] P. Frankl and G.Y. Katona, *Extremal* k*-edge hamiltonian hypergraphs*, Discrete Math. **308** (2008), 1415–1424.

[7] I. Horváth and G.Y. Katona, *Extremal* P_4*-stable graphs*, Discrete Applied Mathematics (2011), Article in Press, Corrected Proof, http://dx.doi.org/10.1016/j.dam.2010.11.016.

[8] G.Y. Katona and I. Horváth, *Extremal stable graphs*, CTW, 2009, pp. 149–152.