New degeneration of Fay's identity and its application to integrable systems

C. Kalla*

March 31, 2011

Abstract

In this paper we prove a new degenerated version of Fay's trisecant identity. The new identity is applied to construct new algebro-geometric solutions of the multi-component nonlinear Schrödinger equation. This approach also provides an independent derivation of known algebro-geometric solutions to the Davey-Stewartson equations.

Contents

*e-mail: Caroline.Kalla@u-bourgogne.fr; address: Institut de Mathématiques de Bourgogne, Université de Bourgogne, 9 avenue Alain Savary, 21078 Dijon, France

1 Introduction

The well known trisecant identity discovered by Fay is a far-reaching generalization of the addition theorem for elliptic theta functions (see [9]). This identity states that, for any points a, b, c, d on a compact Riemann surface of genus $g > 0$, and for any $\mathbf{z} \in \mathbb{C}^g$, there exist constants c_1, c_2 and c_3 such that

$$c_1\,\Theta\left(\mathbf{z} + \int_c^a\right)\Theta\left(\mathbf{z} + \int_b^d\right) + c_2\,\Theta\left(\mathbf{z} + \int_b^a\right)\Theta\left(\mathbf{z} + \int_c^d\right) = c_3\,\Theta(\mathbf{z})\,\Theta\left(\mathbf{z} + \int_c^a + \int_b^d\right), \quad (1.1)$$

where Θ is the multi-dimensional theta function (2.2); here and below we use the notation $\int_a^b$ for the Abel map (2.5) between a and b. This identity plays an important role in various domains of mathematics, as for example in the theory of Jacobian varieties [2], in conformal field theory [19], and in operator theory [15]. Moreover, as it was realized by Mumford, theta-functional solutions of certain integrable equations as Korteweg-de Vries (KdV), Kadomtsev-Petviashvili (KP), or Sine-Gordon (SG), may be derived from Fay's trisecant identity and its degenerations (see [16]).

In the present paper we apply Mumford's approach to the Davey-Stewartson equations and the multi-component nonlinear Schrödinger equation.

The first main result of this paper is a new degeneration of Fay's identity (1.1). This new identity holds for two distinct points a, b on a compact Riemann surface of genus $g > 0$, and any $\mathbf{z} \in \mathbb{C}^g$:

$$D'_a \ln \frac{\Theta(\mathbf{z} + \int_a^b)}{\Theta(\mathbf{z})} + D_a^2 \ln \frac{\Theta(\mathbf{z} + \int_a^b)}{\Theta(\mathbf{z})} + \left(D_a \ln \frac{\Theta(\mathbf{z} + \int_a^b)}{\Theta(\mathbf{z})} - K_1\right)^2 + 2\,D_a^2 \ln \Theta(\mathbf{z}) + K_2 = 0, \quad (1.2)$$

where K_1 and K_2 are scalars independent of $\mathbf{z}$ but dependent on the points a and b; here D_a and D'_a denote operators of directional derivatives along the vectors $\mathbf{V}_a$ and $\mathbf{W}_a$ (2.8). In particular, this identity implies that the following function of the variables x and t

$$\psi(x,t) = A\,\frac{\Theta(\mathbf{Z} - \mathbf{d} + \int_a^b)}{\Theta(\mathbf{Z} - \mathbf{d})}\,\exp\{\mathrm{i}\,(-K_1\,x + K_2\,t)\}, \quad (1.3)$$

where $\mathbf{Z} = \mathrm{i}\mathbf{V}_a\,x + \mathrm{i}\mathbf{W}_a\,t$ and $A \in \mathbb{C}, \mathbf{d} \in \mathbb{C}^g$ are arbitrary constants, is a solution of the linear Schrödinger equation

$$\mathrm{i}\,\frac{\partial \psi}{\partial t} + \frac{\partial^2 \psi}{\partial x^2} + 2\,u\,\psi = 0, \quad (1.4)$$

with the potential $u(x,t) = D_a^2 \ln \Theta(\mathbf{Z})$. When this potential is related to the function ψ by $u(x,t) = \rho\,|\psi|^2$, with $\rho = \pm 1$, the function ψ (1.3) becomes a solution of the nonlinear Schrödinger equation (NLS)

$$\mathrm{i}\,\frac{\partial \psi}{\partial t} + \frac{\partial^2 \psi}{\partial x^2} + 2\rho\,|\psi|^2\,\psi = 0. \tag{1.5}$$

This is the starting point of our construction of algebro-geometric solutions of the Davey-Stewartson equations and the multi-component nonlinear Schrödinger equation. The nonlinear Schrödinger equation (1.5) is a famous nonlinear dispersive partial differential equation with many applications, e.g. in hydrodynamics (deep water waves), plasma physics and nonlinear fiber optics. Integrability of this equation was established by Zakharov and Shabat in [21]. Algebro-geometric solutions of (1.5) were found by Its in [10]; the geometric theory of these solutions was developed by Previato [17].

There exist various ways to generalize the NLS equation. The first is to increase the number of spatial dimensions to two. This leads to the Davey-Stewartson equations (DS),

$$\begin{aligned} \mathrm{i}\,\psi_t + \psi_{xx} - \alpha^2\,\psi_{yy} + 2\,(\Phi + \rho\,|\psi|^2)\,\psi = 0, \\ \Phi_{xx} + \alpha^2\,\Phi_{yy} + 2\rho\,|\psi|^2_{xx} = 0, \end{aligned} \tag{1.6}$$

where $\alpha = \mathrm{i}, 1$ and $\rho = \pm 1$; $\psi(x,y,t)$ and $\Phi(x,y,t)$ are functions of the real variables x, y and t, the latter being real valued and the former being complex valued. In what follows, DS1$^\rho$ denotes the Davey-Stewartson equation when $\alpha = \mathrm{i}$, and DS2$^\rho$ the Davey-Stewartson equation when $\alpha = 1$. The Davey-Stewartson equation (1.6) was introduced in [5] to describe the evolution of a three-dimensional wave package on water of finite depth. Complete integrability of the equation was shown in [1]. If solutions ψ and Φ of (1.6) do not depend on the variable y the first equation in (1.6) reduces to the NLS equation (1.5) under appropriate boundary conditions for the function $\Phi + \rho\,|\psi|^2$ in the limit when x tends to infinity.

Algebro-geometric solutions of the Davey-Stewartson equations (1.6) were previously obtained in [13] using the formalism of Baker-Akhiezer functions. In both [13] and the present paper, solutions of (1.6) are constructed from solutions of the complexified system which, after the change of coordinates $\xi = \frac{1}{2}(x - \mathrm{i}\alpha y)$ and $\eta = \frac{1}{2}(x + \mathrm{i}\alpha y)$, with $\alpha = \mathrm{i}, 1$, reads

$$\begin{aligned} \mathrm{i}\,\psi_t + \frac{1}{2}(\psi_{\xi\xi} + \psi_{\eta\eta}) + 2\,\varphi\,\psi = 0, \\ -\mathrm{i}\,\psi^*_t + \frac{1}{2}(\psi^*_{\xi\xi} + \psi^*_{\eta\eta}) + 2\,\varphi\,\psi^* = 0, \\ \varphi_{\xi\eta} + \frac{1}{2}((\psi\psi^*)_{\xi\xi} + (\psi\psi^*)_{\eta\eta}) = 0, \end{aligned} \tag{1.7}$$

where $\varphi := \Phi + \psi\psi^*$. This system reduces to (1.6) under the reality condition:

$$\psi^* = \rho\,\overline{\psi}. \tag{1.8}$$

The second main result of our paper is an independent derivation of the solutions [13] using the degenerated Fay identity (1.2). Algebro-geometric data associated to these solutions are $\{\mathcal{R}_g, a, b, k_a, k_b\}$, where $\mathcal{R}_g$ is a compact Riemann surface of genus $g > 0$, a and b are two distinct points on $\mathcal{R}_g$, and

k_a, k_b are arbitrary local parameters near a and b. These solutions read

$$\psi(\xi,\eta,t) = A\,\frac{\Theta(\mathbf{Z}-\mathbf{d}+\int_a^b)}{\Theta(\mathbf{Z}-\mathbf{d})}\,\exp\left\{\mathrm{i}\left(-G_1\,\xi - G_2\,\eta + G_3\,\tfrac{t}{2}\right)\right\},$$

$$\psi^*(\xi,\eta,t) = -\frac{\kappa_1\kappa_2\,q_2(a,b)}{A}\,\frac{\Theta(\mathbf{Z}-\mathbf{d}-\int_a^b)}{\Theta(\mathbf{Z}-\mathbf{d})}\,\exp\left\{\mathrm{i}\left(G_1\,\xi + G_2\,\eta - G_3\,\tfrac{t}{2}\right)\right\},$$

$$\varphi(\xi,\eta,t) = \frac{1}{2}(\ln\,\Theta(\mathbf{Z}-\mathbf{d}))_{\xi\xi} + \frac{1}{2}(\ln\,\Theta(\mathbf{Z}-\mathbf{d}))_{\eta\eta} + \frac{1}{4}h,$$

where the scalars G_i, $q_2(a,b)$ depend on the points $a, b \in \mathcal{R}_g$, and $\kappa_1, \kappa_2, A, h \in \mathbb{C}$, $\mathbf{d} \in \mathbb{C}^g$ are arbitrary constants; the g-dimensional vector $\mathbf{Z}$ is a linear function of the variables ξ, η and t. The reality condition (1.8) imposes constraints on the associated algebro-geometric data. In particular, the Riemann surface $\mathcal{R}_g$ has to be real. The approach used in [13] to study reality conditions (1.8) is based on properties of Baker-Akhiezer functions. Our present approach based on identity (1.2) allows to construct solutions of DS1$^\rho$ and DS2$^\rho$ corresponding to Riemann surfaces of more general topological type than in [13].

Another way to generalize the NLS equation is to increase the number of dependent variables in (1.5). This leads to the multi-component nonlinear Schrödinger equation

$$\mathrm{i}\,\frac{\partial\psi_j}{\partial t} + \frac{\partial^2\psi_j}{\partial x^2} + 2\left(\sum_{k=1}^{n} s_k|\psi_k|^2\right)\psi_j = 0, \qquad j = 1,\ldots,n, \tag{1.9}$$

denoted by n-NLS^s, where $s = (s_1,\ldots,s_n)$, $s_k = \pm 1$. Here $\psi_j(x,t)$ are complex valued functions of the real variables x and t. The case $n = 1$ corresponds to the NLS equation. The integrability of the two-component nonlinear Schrödinger equation (1.9) in the case $s = (1,1)$ was first established by Manakov [14]; integrability for the multi-component case with any $n \geq 2$ and $s_k = \pm 1$ was established in [18]. Algebro-geometric solutions of the two-component NLS equation with signature $(1,1)$ were investigated in [8] using the Lax formalism and Baker-Akhiezer functions; these solutions are expressed in terms of theta functions of special trigonal spectral curves.

The third main result of this paper is the construction of smooth algebro-geometric solutions of the multi-component nonlinear Schrödinger equation (1.9) for arbitrary $n \geq 2$, obtained by using (1.2). We first find solutions to the complexified system

$$\begin{aligned} &\mathrm{i}\,\frac{\partial\psi_j}{\partial t} + \frac{\partial^2\psi_j}{\partial x^2} + 2\left(\sum_{k=1}^{n}\psi_k\,\psi_k^*\right)\psi_j = 0, \\ -&\mathrm{i}\,\frac{\partial\psi_j^*}{\partial t} + \frac{\partial^2\psi_j^*}{\partial x^2} + 2\left(\sum_{k=1}^{n}\psi_k\,\psi_k^*\right)\psi_j^* = 0, \qquad j = 1,\ldots,n, \end{aligned} \tag{1.10}$$

where $\psi_j(x,t)$ and $\psi_j^*(x,t)$ are complex valued functions of the real variables x and t. This system reduces to the n-NLS^s equation (1.9) under the reality conditions

$$\psi_j^* = s_j\,\overline{\psi_j}, \qquad j = 1,\ldots,n. \tag{1.11}$$

Algebro-geometric data associated to the solutions of (1.10) are given by $\{\mathcal{R}_g, f, z_a\}$, where $\mathcal{R}_g$ is a compact Riemann surface of genus $g > 0$, f is a meromorphic function of degree $n+1$ on $\mathcal{R}_g$ and

$z_a \in \mathbb{CP}^1$ is a non critical value of the meromorphic function f such that $f^{-1}(z_a) = \{a_1, \ldots, a_{n+1}\}$. Then the solutions $\{\psi_j\}_{j=1}^n$ and $\left\{\psi_j^*\right\}_{j=1}^n$ of system (1.10) read

$$\psi_j(x,t) = A_j \, \frac{\Theta(\mathbf{Z} - \mathbf{d} + \int_{a_{n+1}}^{a_j})}{\Theta(\mathbf{Z} - \mathbf{d})} \, \exp\left\{\mathrm{i}\,(-E_j\, x + F_j\, t)\right\},$$

$$\psi_j^*(x,t) = \frac{q_2(a_{n+1}, a_j)}{A_j} \, \frac{\Theta(\mathbf{Z} - \mathbf{d} - \int_{a_{n+1}}^{a_j})}{\Theta(\mathbf{Z} - \mathbf{d})} \, \exp\left\{\mathrm{i}\,(E_j\, x - F_j\, t)\right\},$$

where the scalars E_j, F_j, $q_2(a_{n+1}, a_j)$ depend on the points a_{n+1}, $a_j \in \mathcal{R}_g$, and $A_j \in \mathbb{C}$, $\mathbf{d} \in \mathbb{C}^g$ are arbitrary constants; here the g-dimensional vector $\mathbf{Z}$ is a linear function of the variables x and t. Imposing the reality conditions (1.11), we describe explicitly solutions for the focusing case $s = (1, \ldots, 1)$ and the defocusing case $s = (-1, \ldots, -1)$ associated to a real branched covering of the Riemann sphere. In particular, our solutions of the focusing case are associated to a covering without real branch points. Our general construction, being applied to the two-component case, gives solutions with more parameters than in [8] for fixed genus of the spectral curve. Moreover, we provide smoothness conditions for our solutions.

The paper is organized as follows: in section 2 we recall some facts about the theory of Riemann surfaces, and derive a new degeneration of Fay's identity. With this degeneration, we give in Section 3 an independent derivation of smooth theta-functional solutions of the Davey Stewartson equations; this approach also provides an explicit description of the constants appearing in the solutions in terms of theta functions. In Section 4, we construct new smooth theta-functional solutions of the multi-component NLS equation, and describe explicitely solutions of the focusing and defocusing cases. We also discuss the reduction from n-NLS to (n-1)-NLS, stationary solutions of n-NLS, and the link between solutions of n-NLS and solutions of the KP1 equation. Appendix A contains various facts from the theory of real Riemann surfaces. Appendix B contains an auxiliary computation required in the construction of algebro-geometric solutions of DS and n-NLS equations.

2 New degeneration of Fay's identity

In this section we recall some facts from the classical theory of Riemann surfaces [9] and derive a new corollary of Fay's trisecant identity.

2.1 Theta functions

Let $\mathcal{R}_g$ be a compact Riemann surface of genus $g > 0$. Denote by $(\mathcal{A}_1, \ldots, \mathcal{A}_g, \mathcal{B}_1, \ldots, \mathcal{B}_g)$ a canonical homology basis, and by $(\omega_1, \ldots, \omega_g)$ the dual basis of holomorphic differentials normalized via

$$\int_{\mathcal{A}_k} \omega_j = 2\mathrm{i}\pi \delta_{jk}, \quad j, k = 1, \ldots, g. \tag{2.1}$$

The matrix $\mathbb{B} = \left(\int_{\mathcal{B}_k} \omega_j\right)$ of $\mathcal{B}$-periods of the normalized holomorphic differentials $\{\omega_j\}_{j=1}^g$ is symmetric and has a negative definite real part. The theta function with (half integer) characteristics $\delta = [\delta' \delta'']$ is defined by

$$\Theta[\delta](\mathbf{z}|\mathbb{B}) = \sum_{\mathbf{m} \in \mathbb{Z}^g} \exp\left\{\tfrac{1}{2}\langle \mathbb{B}(\mathbf{m} + \delta'), \mathbf{m} + \delta' \rangle + \langle \mathbf{m} + \delta', \mathbf{z} + 2\mathrm{i}\pi\delta'' \rangle\right\}; \tag{2.2}$$

here $\mathbf{z} \in \mathbb{C}^g$ is the argument and $\delta', \delta'' \in \left\{0, \frac{1}{2}\right\}^g$ are the vectors of characteristics; $\langle , \rangle$ denotes the scalar product $\langle \mathbf{u}, \mathbf{v} \rangle = \sum_i u_i\, v_i$ for any $\mathbf{u},\, \mathbf{v} \in \mathbb{C}^g$. The theta function $\Theta[\delta](\mathbf{z})$ is even if the characteristic δ is even i.e, $4\,\langle \delta', \delta'' \rangle$ is even, and odd if the characteristic δ is odd i.e., $4\,\langle \delta', \delta'' \rangle$ is odd. An even characteristic is called nonsingular if $\Theta[\delta](\mathbf{0}) \neq 0$, and an odd characteristic is called nonsingular if the gradient $\nabla\Theta[\delta](\mathbf{0})$ is non-zero. The theta function with characteristics is related to the theta function with zero characteristics (denoted by Θ) as follows

$$\Theta[\delta](\mathbf{z}) = \Theta(\mathbf{z} + 2\mathrm{i}\pi\delta'' + \mathbb{B}\delta')\, \exp\left\{\tfrac{1}{2}\langle \mathbb{B}\delta', \delta' \rangle + \langle \mathbf{z} + 2\mathrm{i}\pi\delta'', \delta' \rangle\right\}. \tag{2.3}$$

Let Λ be the lattice $\Lambda = \{2\mathrm{i}\pi\mathbf{N} + \mathbb{B}\mathbf{M},\ \mathbf{N}, \mathbf{M} \in \mathbb{Z}^g\}$ generated by the $\mathcal{A}$ and $\mathcal{B}$-periods of the normalized holomorphic differentials $\{\omega_j\}_{j=1}^g$. The complex torus $J = J(\mathcal{R}_g) = \mathbb{C}^g/\Lambda$ is called the Jacobian of the Riemann surface $\mathcal{R}_g$. The theta function with characteristics (2.2) has the following quasi-periodicity property

$$\begin{aligned} &\Theta[\delta](\mathbf{z} + 2\mathrm{i}\pi\mathbf{N} + \mathbb{B}\mathbf{M}) \\ &= \Theta[\delta](\mathbf{z}) \exp\left\{-\tfrac{1}{2}\langle \mathbb{B}\mathbf{M}, \mathbf{M} \rangle - \langle \mathbf{z}, \mathbf{M} \rangle + 2\mathrm{i}\pi(\langle \delta', \mathbf{N} \rangle - \langle \delta'', \mathbf{M} \rangle)\right\}. \end{aligned} \tag{2.4}$$

Denote by μ the Abel map $\mu : \mathcal{R}_g \longmapsto J$ defined by

$$\mu(p) = \int_{p_0}^{p} \omega, \tag{2.5}$$

for any $p \in \mathcal{R}_g$, where $p_0 \in \mathcal{R}_g$ is the base point of the application, and $\omega = (\omega_1, \ldots, \omega_g)$ is the vector of the normalized holomorphic differentials. In the whole paper we use the notation $\int_a^b = \mu(b) - \mu(a)$.

2.2 Fay's identity and previously known degenerations

Let us introduce the prime-form which is given by

$$E(a, b) = \frac{\Theta[\delta](\int_b^a)}{h_\delta(a) h_\delta(b)}, \tag{2.6}$$

$a,\, b \in \mathcal{R}_g$; $h_\delta(a)$ is a spinor defined by $h_\delta^2(a) = \sum_{j=1}^{g} \frac{\partial\Theta[\delta]}{\partial z_j}(0)\omega_j(a)$, where $\delta = [\delta'\delta'']$ is a non-singular odd characteristic (the prime form is independent of the choice of the characteristic δ). Fay's trisecant identity has the form

$$\begin{aligned} E(a,b)E(c,d)\Theta\left(\mathbf{z} + \int_c^a\right)\Theta\left(\mathbf{z} + \int_b^d\right) \\ + E(a,c)E(d,b)\,\Theta\left(\mathbf{z} + \int_b^a\right)\Theta\left(\mathbf{z} + \int_c^d\right) \\ = E(a,d)E(c,b)\,\Theta(\mathbf{z})\,\Theta\left(\mathbf{z} + \int_c^a + \int_b^d\right), \end{aligned} \tag{2.7}$$

where $a, b, c, d \in \mathcal{R}_g$ and all integration contours do not intersect cycles of the canonical homology basis. Let us now discuss degenerations of identity (2.7).

Let $k_a(p)$ denote a local parameter near $a \in \mathcal{R}_g$, where p lies in a neighbourhood of a. Consider the following expansion of the normalized holomorphic differentials ω_j near a,

$$\omega_j(p) = \left(V_{a,j} + W_{a,j}\, k_a(p) + U_{a,j}\, \frac{k_a(p)^2}{2!} + \ldots\right)\, \mathrm{d}k_a(p), \tag{2.8}$$

where $V_{a,j}$, $W_{a,j}$, $U_{a,j} \in \mathbb{C}$. Let us denote by D_a the operator of directional derivative along the vector $\mathbf{V}_a = (V_{a,1}, \ldots, V_{a,g})$:

$$D_a F(\mathbf{z}) = \sum_{j=1}^{g} \partial_{z_j} F(\mathbf{z}) V_{a,j} = \langle \nabla F(\mathbf{z}), \mathbf{V}_a \rangle\,, \tag{2.9}$$

where $F : \mathbb{C}^g \longrightarrow \mathbb{C}$ is an arbitrary function, and denote by D'_a the operator of directional derivative along the vector $\mathbf{W}_a = (W_{a,1}, \ldots, W_{a,g})$:

$$D'_a F(\mathbf{z}) = \sum_{j=1}^{g} \partial_{z_j} F(\mathbf{z}) W_{a,j} = \langle \nabla F(\mathbf{z}), \mathbf{W}_a \rangle\,.$$

Then for any $\mathbf{z} \in \mathbb{C}^g$ and any distinct points $a, b \in \mathcal{R}_g$, the following well-known degenerated version of Fay's identity holds (see [16])

$$D_a D_b\, \ln \Theta(\mathbf{z}) \;=\; q_1(a,b) \;+\; q_2(a,b)\, \frac{\Theta(\mathbf{z} + \int_a^b)\, \Theta(\mathbf{z} + \int_b^a)}{\Theta(\mathbf{z})^2}, \tag{2.10}$$

where the scalars $q_1(a,b)$ and $q_2(a,b)$ are given by

$$q_1(a,b) = D_a D_b \ln \Theta[\delta](\int_a^b), \tag{2.11}$$

$$q_2(a,b) = \frac{D_a\, \Theta[\delta](0)\, D_b\, \Theta[\delta](0)}{\Theta[\delta](\int_a^b)^2}, \tag{2.12}$$

where δ is a non-singular odd characteristic. Notice that $q_1(a,b)$ and $q_2(a,b)$ depend on the choice of local parameters k_a and k_b near a and b respectively.

2.3 New degeneration of Fay's identity

Algebro-geometric solutions of the Davey-Stewartson equations and the multi-component NLS equation constructed in this paper are obtained by using the following new degenerated version of Fay's identity.

Theorem 2.1. *Let a, b be distinct points on a compact Riemann surface $\mathcal{R}_g$ of genus g. Fix local parameters k_a and k_b in a neighbourhood of a and b respectively. Denote by δ a non-singular odd characteristic. Then for any $\mathbf{z} \in \mathbb{C}^g$,*

$$D'_a \ln \frac{\Theta(\mathbf{z} + \int_a^b)}{\Theta(\mathbf{z})} + D_a^2 \ln \frac{\Theta(\mathbf{z} + \int_a^b)}{\Theta(\mathbf{z})} + \left(D_a \ln \frac{\Theta(\mathbf{z} + \int_a^b)}{\Theta(\mathbf{z})} - K_1(a,b)\right)^2 + 2\, D_a^2 \ln \Theta(\mathbf{z}) + K_2(a,b) = 0, \tag{2.13}$$

where the scalars $K_1(a,b)$ and $K_2(a,b)$ are given by

$$K_1(a,b) = \frac{1}{2}\frac{D'_a\, \Theta[\delta](0)}{D_a\, \Theta[\delta](0)} + D_a \ln \Theta[\delta](\int_a^b)\,, \tag{2.14}$$

and

$$K_2(a,b) = -D_a' \ln \Theta(\int_a^b) - D_a^2 \ln \left(\Theta(\int_a^b)\Theta(0)\right) - \left(D_a \ln \Theta(\int_a^b) - K_1(a,b)\right)^2. \tag{2.15}$$

Proof. We start from the following lemma

Lemma 2.1. *Let $b, c \in \mathcal{R}_g$ be distinct points. Fix local parameters k_b and k_c in a neighbourhood of b and c respectively. Then for any $\mathbf{z} \in \mathbb{C}^g$,*

$$D_c \left[-D_b' \ln \frac{\Theta(\mathbf{z} + \int_c^b)}{\Theta(\mathbf{z})} + D_b^2 \ln \frac{\Theta(\mathbf{z} + \int_c^b)}{\Theta(\mathbf{z})} + \left(D_b \ln \frac{\Theta(\mathbf{z} + \int_c^b)}{\Theta(\mathbf{z})} + K_1(b,c) \right)^2 + 2\, D_b^2 \ln \Theta(\mathbf{z}) \right] = 0, \tag{2.16}$$

where the scalar $K_1(b,c)$ is defined in (2.14).

Proof of Lemma 2.1. Let us introduce the notations $\Theta_{ab} = \Theta(\mathbf{z} + \int_a^b \omega)$ and $\Theta = \Theta(\mathbf{z})$. Differentiating (2.7) twice with respect to the local parameter $k_d(p)$, where p lies in a neighbourhood of d, and taking the limit $d \to b$, we obtain

$$D_b' \ln \Theta + D_b^2 \ln \Theta + (D_b \ln \Theta)^2 + \frac{p_3}{p_2}\, D_b \ln \Theta_{ca} - \frac{p_3}{p_2} D_b \ln \Theta \tag{2.17}$$

$$= \frac{p_1 p_3}{p_2} - 2\, D_b \ln \Theta_{ca}\, D_b \ln \Theta_{cb} + 2\, D_b \ln \Theta\, D_b \ln \Theta_{cb} + 2\, p_1\, D_b \ln \Theta_{cb}$$

$$-p_4 - 2\, p_1\, D_b \ln \Theta_{ca} + D_b' \ln \Theta_{ca} + D_b^2 \ln \Theta_{ca} + (D_b \ln \Theta_{ca})^2,$$

where we took into account the relation

$$\partial_{k_d}^2\, \Theta(\mathbf{z} + \int_b^d)\big|_{d=b} \;=\; D_b' \Theta(\mathbf{z}) + D_b^2 \Theta(\mathbf{z}).$$

The quantities $p_j = p_j(a,b,c)$, for $j = 1,2,3,4$, are given by

$$p_1(a,b,c) = -\frac{E(c,b)}{E(a,b)} \partial_{k_x} \frac{E(a,x)}{E(c,x)}\Big|_{x=b}, \quad p_2(a,b,c) = \frac{E(a,c)}{E(a,b)} \partial_{k_x} \frac{E(x,b)}{E(c,x)}\Big|_{x=b}, \tag{2.18}$$

$$p_3(a,b,c) = \frac{E(a,c)}{E(a,b)} \partial_{k_x}^2 \frac{E(x,b)}{E(c,x)}\Big|_{x=b}, \quad p_4(a,b,c) = -\frac{E(c,b)}{E(a,b)} \partial_{k_x}^2 \frac{E(a,x)}{E(c,x)}\Big|_{x=b}. \tag{2.19}$$

Differentiating (2.17) with respect to the local parameter $k_a(p)$, where p lies in a neighbourhood of a, and taking the limit $a \to c$, we get

$$D_c D_b' \ln \Theta + D_c D_b^2 \ln \Theta - 2\, D_c D_b \ln \Theta\, D_b \ln \frac{\Theta_{cb}}{\Theta} + 2\, q_1\, D_b \ln \frac{\Theta_{cb}}{\Theta} - \frac{p_3}{p_2}\, D_c D_b \ln \Theta + K \;=\; 0, \tag{2.20}$$

where the scalar K depends on the points b, c, but not on the vector $\mathbf{z} \in \mathbb{C}^g$. Here the scalars q_1, p_2 and p_3 are defined in (2.11), (2.18), and (2.19) respectively. The change of variable $\mathbf{z} \leftrightarrow -\mathbf{z} + \int_b^c$ in (2.20) leads to

$$D_c D_b' \ln \Theta_{cb} - D_c D_b^2 \ln \Theta_{cb} - 2\, D_c D_b \ln \Theta_{cb}\, D_b \ln \frac{\Theta_{cb}}{\Theta} + 2\, q_1\, D_b \ln \frac{\Theta_{cb}}{\Theta} - \frac{p_3}{p_2}\, D_c D_b \ln \Theta_{cb} + K = 0. \tag{2.21}$$

Now (2.16) is obtained by substracting (2.20) and (2.21). □

To proof Theorem 2.1, make the change of variable $\mathbf{z} \mapsto -\mathbf{z} + \int_b^c$ in (2.17) and add $2\, D_b^2 \ln \Theta$ to each side of the equality to get

$$-D_b' \ln \frac{\Theta_{cb}}{\Theta} + D_b^2 \ln \frac{\Theta_{cb}}{\Theta} + \left(D_b \ln \frac{\Theta_{cb}}{\Theta} + \frac{1}{2} \frac{p_3}{p_2} \right)^2 - \frac{1}{4} \left(\frac{p_3}{p_2} \right)^2 - \frac{p_1 p_3}{p_2} + 2\, D_b^2 \ln \Theta$$

$$= -D_b' \ln \frac{\Theta_{ab}}{\Theta} + D_b^2 \ln \frac{\Theta_{ab}}{\Theta} + \left(D_b \ln \frac{\Theta_{ab}}{\Theta} + \frac{1}{2} \left(\frac{p_3}{p_2} + 2p_1 \right) \right)^2 - \frac{1}{4} \left(\frac{p_3}{p_2} + 2\, p_1 \right)^2 - p_4 + 2\, D_b^2 \ln \Theta.$$

By Lemma 2.1, the directional derivative of the left hand side of the previous equality along the vector $\mathbf{V}_c$ equals zero. Hence for any distinct points $a, b, c \in \mathcal{R}_g$, we get

$$D_c \Big[-D_b' \ln \frac{\Theta_{ab}}{\Theta} + D_b^2 \ln \frac{\Theta_{ab}}{\Theta} + \left(D_b \ln \frac{\Theta_{ab}}{\Theta} + \frac{1}{2} \left(\frac{p_3}{p_2} + 2p_1 \right) \right)^2 + 2\, D_b^2 \ln \Theta \Big] = 0. \tag{2.22}$$

Moreover, from (2.18), (2.19) and (2.6), it can be seen that the expression $\frac{1}{2}(\frac{p_3}{p_2} + 2p_1)$ does not depend on the point c and equals $K_1(b, a)$ given by (2.14). Now let us introduce the following function of the variable $\mathbf{z} \in \mathbb{C}^g$

$$f_{(b,a)}(\mathbf{z}) = -D_b' \ln \frac{\Theta_{ab}}{\Theta} + D_b^2 \ln \frac{\Theta_{ab}}{\Theta} + \left(D_b \ln \frac{\Theta_{ab}}{\Theta} + K_1(b, a) \right)^2 + 2\, D_b^2 \ln \Theta.$$

Then (2.22) can be rewritten as $D_c\, f_{(b,a)}(\mathbf{z}) = 0$ for any $\mathbf{z} \in \mathbb{C}^g$ and for all $c \in \mathcal{R}_g$, $c \neq b$ (because also $D_a\, f_{(b,a)}(\mathbf{z}) = 0$ by Lemma 2.1). Due to the fact that on each Riemann surface $\mathcal{R}_g$, there exists a positive divisor $d_1 + \ldots + d_g$ of degree g such that vectors $\frac{\omega(d_1)}{\mathrm{d}k_{d_1}}, \ldots, \frac{\omega(d_g)}{\mathrm{d}k_{d_g}}$ are linearly independent (see [11], Lemma 5), the function $f_{(b,a)}(\mathbf{z})$ is constant with respect to $\mathbf{z}$; we denote this constant by $-K_2(b, a)$:

$$f_{(b,a)}(\mathbf{z}) = -K_2(b, a) \tag{2.23}$$

for any $\mathbf{z} \in \mathbb{C}^g$. Interchanging a and b, and changing the variable $\mathbf{z} \leftrightarrow -\mathbf{z}$ in (2.23) we get (2.13). The expression (2.15) for the scalar $K_2(a, b)$ follows from (2.23) putting $\mathbf{z} = 0$. □

3 Algebro-geometric solutions of the Davey-Stewartson equations

Here we derive algebro-geometric solutions of the Davey-Stewartson equations (1.6) using the degeneration (2.13) of Fay's identity. Let us introduce the function $\phi := \Phi + \rho|\psi|^2$, where $\rho = \pm 1$, and the differential operators

$$D_1 = \partial_{xx} - \alpha^2 \partial_{yy}, \quad D_2 = \partial_{xx} + \alpha^2 \partial_{yy}.$$

Introduce also the characteristic coordinates

$$\xi = \frac{1}{2}(x - \mathrm{i}\alpha y), \quad \eta = \frac{1}{2}(x + \mathrm{i}\alpha y), \quad \alpha = \mathrm{i}, 1.$$

In these coordinates the Davey Stewartson equations (1.6) become

$$\begin{aligned} \mathrm{i}\,\psi_t + D_1\psi + 2\,\phi\,\psi = 0, \\ D_2\phi + \rho\, D_1|\psi|^2 = 0, \end{aligned} \tag{3.1}$$

where the differential operators D_1 and D_2 are given by

$$D_1 = \frac{1}{2}(\partial_\xi^2 + \partial_\eta^2), \quad D_2 = \partial_\xi\partial_\eta.$$

In what follows, DS1$^\rho$ denotes the Davey-Stewartson equation when $\alpha = \mathrm{i}$ (in this case ξ and η are both real), and DS2$^\rho$ the Davey-Stewartson equation when $\alpha = 1$ (in this case ξ and η are pairwise conjugate).

3.1 Solutions of the complexified Davey-Stewartson equations

To construct algebro-geometric solutions of (3.1), let us first introduce the complexified Davey-Stewartson equations

$$\begin{aligned} \mathrm{i}\,\psi_t + \frac{1}{2}(\psi_{\xi\xi} + \psi_{\eta\eta}) + 2\,\varphi\,\psi = 0, \\ -\mathrm{i}\,\psi_t^* + \frac{1}{2}(\psi_{\xi\xi}^* + \psi_{\eta\eta}^*) + 2\,\varphi\,\psi^* = 0, \\ \varphi_{\xi\eta} + \frac{1}{2}((\psi\psi^*)_{\xi\xi} + (\psi\psi^*)_{\eta\eta}) = 0, \end{aligned} \tag{3.2}$$

where $\varphi := \Phi + \psi\psi^*$. This system reduces to (3.1) under the *reality condition*:

$$\psi^* = \rho\,\overline{\psi}, \tag{3.3}$$

which leads to $\varphi = \phi$. Theta functional solutions of system (3.2) are given by

Theorem 3.1. *Let $\mathcal{R}_g$ be a compact Riemann surface of genus $g > 0$, and let $a, b \in \mathcal{R}_g$ be distinct points. Take arbitrary constants $\mathbf{d} \in \mathbb{C}^g$ and A, κ_1, $\kappa_2 \in \mathbb{C} \setminus \{0\}$, $h \in \mathbb{C}$. Denote by ℓ a contour connecting a and b which does not intersect cycles of the canonical homology basis. Then for any ξ, η, $t \in \mathbb{C}$, the following functions ψ, ψ^* and φ are solutions of system (3.2)*

$$\begin{aligned} \psi(\xi,\eta,t) &= A\,\frac{\Theta(\mathbf{Z} - \mathbf{d} + \mathbf{r})}{\Theta(\mathbf{Z} - \mathbf{d})}\,\exp\left\{\mathrm{i}\left(-G_1\,\xi - G_2\,\eta + G_3\,\tfrac{t}{2}\right)\right\}, \\ \psi^*(\xi,\eta,t) &= -\frac{\kappa_1\kappa_2\,q_2(a,b)}{A}\,\frac{\Theta(\mathbf{Z} - \mathbf{d} - \mathbf{r})}{\Theta(\mathbf{Z} - \mathbf{d})}\,\exp\left\{\mathrm{i}\left(G_1\,\xi + G_2\,\eta - G_3\,\tfrac{t}{2}\right)\right\}, \\ \varphi(\xi,\eta,t) &= \frac{1}{2}(\ln\Theta(\mathbf{Z} - \mathbf{d}))_{\xi\xi} + \frac{1}{2}(\ln\Theta(\mathbf{Z} - \mathbf{d}))_{\eta\eta} + \frac{1}{4}h. \end{aligned} \tag{3.4}$$

Here $\mathbf{r} = \int_\ell \omega$, where ω is the vector of normalized holomorphic differentials, and

$$\mathbf{Z} = \mathrm{i}\left(\kappa_1\,\mathbf{V}_a\,\xi - \kappa_2\,\mathbf{V}_b\,\eta + (\kappa_1^2\,\mathbf{W}_a - \kappa_2^2\,\mathbf{W}_b)\,\tfrac{t}{2}\right), \tag{3.5}$$

where the vectors $\mathbf{V}_a$, $\mathbf{V}_b$ *and* $\mathbf{W}_a$, $\mathbf{W}_b$ *were introduced in (2.8). The scalars* G_1, G_2, G_3 *are given by*

$$G_1 = \kappa_1\, K_1(a,b), \qquad G_2 = \kappa_2\, K_1(b,a), \tag{3.6}$$

$$G_3 = \kappa_1^2\, K_2(a,b) + \kappa_2^2\, K_2(b,a) + h, \tag{3.7}$$

and scalars $q_2(a,b)$, $K_1(a,b)$, $K_2(a,b)$ *are defined in (2.12), (2.14), (2.15) respectively.*

Proof. Substitute functions (3.4) in the first equation of system (3.2) to get

$$\kappa_1^2\, D_a' \ln \frac{\Theta(\mathbf{Z}-\mathbf{d}+\mathbf{r})}{\Theta(\mathbf{Z}-\mathbf{d})} + \kappa_1^2\, D_a^2 \ln \frac{\Theta(\mathbf{Z}-\mathbf{d}+\mathbf{r})}{\Theta(\mathbf{Z}-\mathbf{d})} + 2\,\kappa_1^2\, D_a^2 \ln \Theta(\mathbf{Z}-\mathbf{d}) + \, G_3 - h$$
$$+ \left(\kappa_1\, D_a \ln \frac{\Theta(\mathbf{Z}-\mathbf{d}+\mathbf{r})}{\Theta(\mathbf{Z}-\mathbf{d})} - G_1\right)^2 + \left(\kappa_2\, D_b \ln \frac{\Theta(\mathbf{Z}-\mathbf{d}+\mathbf{r})}{\Theta(\mathbf{Z}-\mathbf{d})} + G_2\right)^2$$
$$-\kappa_2^2\, D_b' \ln \frac{\Theta(\mathbf{Z}-\mathbf{d}+\mathbf{r})}{\Theta(\mathbf{Z}-\mathbf{d})} + \kappa_2^2\, D_b^2 \ln \frac{\Theta(\mathbf{Z}-\mathbf{d}+\mathbf{r})}{\Theta(\mathbf{Z}-\mathbf{d})} + 2\,\kappa_2^2\, D_b^2 \ln \Theta(\mathbf{Z}-\mathbf{d}) = 0.$$

By (2.13), the last equality holds for any $\mathbf{z} \in \mathbb{C}^g$, and in particular for $\mathbf{z} = \mathbf{Z} - \mathbf{d}$. In the same way, it can be checked that functions (3.4) satisfy the second equation of system (3.2). Moreover, from (2.10) we get

$$(\psi\psi^*)_{\xi\xi} = \kappa_1^3\, \kappa_2\, D_a^3 D_b \ln\, \Theta(\mathbf{Z}-\mathbf{d}), \quad (\psi\psi^*)_{\eta\eta} = \kappa_1\, \kappa_2^3\, D_a D_b^3 \ln\, \Theta(\mathbf{Z}-\mathbf{d}).$$

Therefore, taking into account that

$$\varphi_{\xi\eta} = -\frac{1}{2}\left(\kappa_1^3\, \kappa_2\, D_a^3 D_b \ln\, \Theta(\mathbf{Z}-\mathbf{d}) + \kappa_1\, \kappa_2^3\, D_a D_b^3 \ln\, \Theta(\mathbf{Z}-\mathbf{d})\right),$$

the functions (3.4) satisfy the last equation of system (3.2). □

The solutions (3.4) depend on the Riemann surface $\mathcal{R}_g$, the points $a, b \in \mathcal{R}_g$, the vector $\mathbf{d} \in \mathbb{C}^g$, the constants $\kappa_1, \kappa_2 \in \mathbb{C}\setminus\{0\}$, $h \in \mathbb{C}$, and the local parameters k_a and k_b near a and b. The transformation of the local parameters given by

$$k_a \longrightarrow \beta\, k_a + \mu_1\, k_a^2 + O\left(k_a^3\right),$$
$$k_b \longrightarrow \beta\, k_b + \mu_2\, k_b^2 + O\left(k_b^3\right), \tag{3.8}$$

where β, μ_1, μ_2 are arbitrary complex numbers ($\beta \neq 0$), leads to a different family of solutions of the complexified system (3.2). These new solutions are obtained via the following transformations:

$$\psi(\xi,\eta,t) \longrightarrow \psi\left(\beta\,\xi + \beta\lambda_1\, t, \beta\,\eta + \beta\lambda_2\, t, \beta^2\, t\right)\, \exp\left\{-\mathrm{i}\left(\lambda_1\,\xi + \lambda_2\,\eta + \left(\lambda_1^2 + \lambda_2^2 - \alpha\right)\, \tfrac{t}{2}\right)\right\}, \tag{3.9}$$
$$\psi^*(\xi,\eta,t) \longrightarrow \beta^2\, \psi^*\left(\beta\,\xi + \beta\lambda_1\, t, \beta\,\eta + \beta\lambda_2\, t, \beta^2\, t\right)\, \exp\left\{\mathrm{i}\left(\lambda_1\,\xi + \lambda_2\,\eta + \left(\lambda_1^2 + \lambda_2^2 - \alpha\right)\, \tfrac{t}{2}\right)\right\},$$
$$\phi(\xi,\eta,t) \longrightarrow \beta^2\, \phi\left(\beta\,\xi + \beta\lambda_1\, t, \beta\,\eta + \beta\lambda_2\, t, \beta^2\, t\right) + \frac{\alpha}{4}, \tag{3.10}$$

where $\lambda_i = \kappa_i\, \mu_i\, \beta^{-1}$ and $\alpha = h(1-\beta^2)$.

3.2 Reality condition and solutions of the DS1$^\rho$ equation

Let us consider the DS1$^\rho$ equation

$$\begin{aligned} \mathrm{i}\,\psi_t + \frac{1}{2}(\partial_\xi^2 + \partial_\eta^2)\psi + 2\,\phi\,\psi = 0, \\ \partial_\xi\partial_\eta\,\phi + \rho\,\frac{1}{2}(\partial_\xi^2 + \partial_\eta^2)|\psi|^2 = 0, \end{aligned} \tag{3.11}$$

where $\rho = \pm 1$. Here ξ, η, t are real variables. Algebro-geometric solutions of (3.11) are constructed from solutions ψ, ψ^* (3.4) of the complexified system, under the reality condition $\psi^* = \rho\,\overline{\psi}$.

Let $\mathcal{R}_g$ be a real compact Riemann surface with an anti-holomorphic involution τ. Denote by $\mathcal{R}_g(\mathbb{R})$ the set of fixed points of the involution τ (see Appendix A.1). Let us choose the homology basis satisfying (A.2). Then the solutions of (3.11) are given by

Theorem 3.2. *Let $a, b \in \mathcal{R}_g(\mathbb{R})$ be distinct points with local parameters satisfying $\overline{k_a(\tau p)} = k_a(p)$ for any p lying in a neighbourhood of a, and $\overline{k_b(\tau p)} = k_b(p)$ for any p lying in a neighbourhood of b. Denote by $\{\mathcal{A}, \mathcal{B}, \ell\}$ the standard generators of the relative homology group $H_1(\mathcal{R}_g, \{a, b\})$ (see Appendix A.2). Let $\mathbf{d}_R \in \mathbb{R}^g$, $\mathbf{T} \in \mathbb{Z}^g$, and define $\mathbf{d} = \mathbf{d}_R + \frac{\mathrm{i}\pi}{2}(diag(\mathbb{H}) - 2\,\mathbf{T})$. Morover, take θ, $h, \in \mathbb{R}$, $\tilde{\kappa}_1$, $\kappa_2 \in \mathbb{R}\setminus\{0\}$ and put*

$$\kappa_1 = -\rho\,\tilde{\kappa}_1^2\,\kappa_2\,q_2(a,b)\,\exp\left\{\tfrac{1}{2}\,\langle \mathbb{B}\mathbf{M}, \mathbf{M}\rangle + \langle \mathbf{r} + \mathbf{d}, \mathbf{M}\rangle\right\}, \tag{3.12}$$

where $\mathbf{M} \in \mathbb{Z}^g$ is defined in (A.13). Then the following functions ψ and ϕ are solutions of the DS1$^\rho$ equation

$$\psi(\xi,\eta,t) = |A|\,e^{\mathrm{i}\theta}\,\frac{\Theta(\mathbf{Z} - \mathbf{d} + \mathbf{r})}{\Theta(\mathbf{Z} - \mathbf{d})}\,\exp\left\{\mathrm{i}\left(-G_1\,\xi - G_2\,\eta + G_3\,\tfrac{t}{2}\right)\right\}, \tag{3.13}$$

$$\phi(\xi,\eta,t) = \frac{1}{2}\,(\ln\,\Theta(\mathbf{Z} - \mathbf{d}))_{\xi\xi} + \frac{1}{2}\,(\ln\,\Theta(\mathbf{Z} - \mathbf{d}))_{\eta\eta} + \frac{1}{4}h, \tag{3.14}$$

where $|A| = |\tilde{\kappa}_1\,\kappa_2\,q_2(a,b)|\,\exp\{\langle \mathbf{d}_R, \mathbf{M}\rangle\}$. Here $\mathbf{r} = \int_\ell \omega$, and the vector $\mathbf{Z}$ is defined in (3.5). Scalars $q_2(a,b)$, G_1, G_2 and G_3 are defined in (2.12), (3.6) and (3.7) respectively.

The case where $\mathbf{V}_a + \mathbf{V}_b = 0$ and $\kappa_1 = \kappa_2$ is treated at the end of this section. It corresponds to solutions of the nonlinear Schrödinger equation.

Proof. Let us check that under the conditions of the theorem, the functions ψ and ψ^* (3.4) satisfy the reality conditions (3.3). First of all, invariance with respect to the anti-involution τ of the points a and b implies the reality of vector (3.5):

$$\overline{\mathbf{Z}} = \mathbf{Z}. \tag{3.15}$$

In fact, using the expansion (2.8) of the normalized holomorphic differentials ω_j near a we get

$$\overline{\tau^*\omega_j}(a)(p) = (\overline{V_{a,j}} + \overline{W_{a,j}}\,k_a(p) + \ldots)\,\mathrm{d}k_a(p),$$

for any point p lying in a neighbourhood of a. Then by (A.3), the vectors $\mathbf{V}_a$ and $\mathbf{W}_a$ appearing in expression (3.5) satisfy

$$\overline{\mathbf{V}_a} = -\mathbf{V}_a, \quad \overline{\mathbf{W}_a} = -\mathbf{W}_a. \tag{3.16}$$

The same holds for the vectors $\mathbf{V}_b$ and $\mathbf{W}_b$, which leads to (3.15). Moreover, from (A.3) and (A.13) we get

$$\bar{\mathbf{r}} = -\mathbf{r} - 2\mathrm{i}\pi\mathbf{N} - \mathbb{B}\mathbf{M}, \tag{3.17}$$

where $\mathbf{N}$, $\mathbf{M} \in \mathbb{Z}^g$ are defined in (A.13) and satisfy

$$2\,\mathbf{N} + \mathbb{H}\mathbf{M} = 0. \tag{3.18}$$

From (2.13), it is straightforward to see that the scalars $K_1(a,b)$ and $K_2(a,b)$ defined by (2.14) and (2.15) satisfy

$$\overline{K_1(a,b)} = K_1(a,b) - \langle \mathbf{V}_a, \mathbf{M}\rangle\,, \qquad \overline{K_2(a,b)} = K_2(a,b) + \langle \mathbf{W}_a, \mathbf{M}\rangle\,, \tag{3.19}$$

which implies

$$\overline{G_1} = G_1 - \kappa_1 \langle \mathbf{V}_a, \mathbf{M}\rangle\,, \qquad \overline{G_2} = G_2 - \kappa_2 \langle \mathbf{V}_b, \mathbf{M}\rangle\,, \qquad \overline{G_3} = G_3 + \kappa_1^2 \langle \mathbf{W}_a, \mathbf{M}\rangle + \kappa_2^2 \langle \mathbf{W}_b, \mathbf{M}\rangle\,.$$

Therefore, the reality condition (3.3) together with (3.4) leads to

$$|A|^2 = -\rho\,\kappa_1\kappa_2\, q_2(a,b)\, \frac{\Theta(\mathbf{Z} - \mathbf{d} - \mathbf{r})\,\Theta(\mathbf{Z} - \overline{\mathbf{d}} + \mathrm{i}\pi\,\mathrm{diag}(\mathbb{H}))}{\Theta(\mathbf{Z} - \overline{\mathbf{d}} - \mathbf{r} + \mathrm{i}\pi\,\mathrm{diag}(\mathbb{H}))\,\Theta(\mathbf{Z} - \mathbf{d})} \times \exp\left\{\tfrac{1}{2}\langle \mathbb{B}\mathbf{M}, \mathbf{M}\rangle + \left\langle \mathbf{r} + \overline{\mathbf{d}} - \mathrm{i}\pi\mathrm{diag}(\mathbb{H}), \mathbf{M}\right\rangle\right\}, \tag{3.20}$$

taking into account the action (A.5) of the complex conjugation on the theta function, and the quasi-periodicity (2.4) of the theta function. Let us choose a vector $\mathbf{d} \in \mathbb{C}^g$ such that

$$\overline{\mathbf{d}} \equiv \mathbf{d} - \mathrm{i}\pi\,\mathrm{diag}(\mathbb{H}) \mod (2\mathrm{i}\pi\mathbb{Z}^g + \mathbb{B}\,\mathbb{Z}^g),$$

which is, since $\overline{\mathbf{d}} - \mathbf{d}$ is purely imaginary, equivalent to $\overline{\mathbf{d}} = \mathbf{d} - \mathrm{i}\pi\,\mathrm{diag}(\mathbb{H}) + 2i\pi\mathbf{T}$, for some $\mathbf{T} \in \mathbb{Z}^g$. Here we used the action (A.4) of the complex conjugation on the matrix of $\mathcal{B}$-periods $\mathbb{B}$, and the fact that $\mathbb{B}$ has a negative definite real part. Hence, the vector $\mathbf{d}$ can be written as

$$\mathbf{d} = \mathbf{d}_R + \frac{\mathrm{i}\pi}{2}(\mathrm{diag}(\mathbb{H}) - 2\,\mathbf{T}), \tag{3.21}$$

for some $\mathbf{d}_R \in \mathbb{R}^g$ and $\mathbf{T} \in \mathbb{Z}^g$. Therefore, all theta functions in (3.20) cancel out and (3.20) becomes

$$|A|^2 = -\rho\,\kappa_1\kappa_2\, q_2(a,b)\, \exp\left\{\tfrac{1}{2}\langle \mathbb{B}\mathbf{M}, \mathbf{M}\rangle + \langle \mathbf{r} + \mathbf{d}, \mathbf{M}\rangle\right\}. \tag{3.22}$$

The reality of the right hand side of equality (3.22) can be deduced from formula (B.11) for the argument of $q_2(a,b)$. Moreover, it is straightforward to see from (3.21) and (3.18) that $\exp\{\langle \mathbf{d}, \mathbf{M}\rangle\}$ is also real. Since κ_1, κ_2 are arbitrary real constants, we can choose κ_1 as in (3.12), which leads to

$$|A|^2 = \left(\tilde{\kappa}_1\,\kappa_2\, q_2(a,b)\, \exp\left\{\tfrac{1}{2}\langle \mathbb{B}\mathbf{M}, \mathbf{M}\rangle + \langle \mathbf{r} + \mathbf{d}, \mathbf{M}\rangle\right\}\right)^2 = |\tilde{\kappa}_1\,\kappa_2\, q_2(a,b)|^2\, \exp\left\{2\,\langle \mathbf{d}_R, \mathbf{M}\rangle\right\}.$$

□

Functions ψ and ϕ given in (3.13) and (3.14) describe a family of algebro-geometric solutions of (3.11) depending on: a real Riemann surface $(\mathcal{R}_g, \tau)$, two distinct points $a, b \in \mathcal{R}_g(\mathbb{R})$, local parameters k_a, k_b which satisfy $\overline{k_a(\tau p)} = k_a(p)$ and $\overline{k_b(\tau p)} = k_b(p)$, and arbitrary constants $\mathbf{d}_R \in \mathbb{R}^g$, $\mathbf{T} \in \mathbb{Z}^g$,

$\theta, h, \in \mathbb{R}$, $\tilde{\kappa}_1$, $\kappa_2 \in \mathbb{R}\setminus\{0\}$. Note that by periodicity properties of the theta function, without loss of generality, the vector $\mathbf{T}$ can be chosen in the set $\{0,1\}^g$. The case where the Riemann surface is dividing and $\mathbf{T}=0$ is of special importance, because the related solutions are smooth, as explained in the next proposition.

Since the theta function is entire, singularities of the functions ψ and ϕ can appear only at the zeros of their denominator. Following Vinnikov's result [20] we obtain

Proposition 3.1. *Solutions (3.13) and (3.14) are smooth if the curve $\mathcal{R}_g$ is dividing and $\mathbf{d}\in\mathbb{R}^g$. Assume that solutions (3.13) and (3.14) are smooth for any vector $\mathbf{d}$ lying in a component T_v (A.25) of the Jacobian, then the curve is dividing and $\mathbf{d}\in\mathbb{R}^g$.*

Proof. By (3.15) and (3.21), the vector $\mathbf{Z}-\mathbf{d}$ belongs to the set S_1 introduced in (A.23). Hence by Proposition A.3, the solutions are smooth if the curve is dividing (in this case diag($\mathbb{H}$)=0), and if the argument $\mathbf{Z}-\mathbf{d}$ of the theta function in the denominator is real, which by (3.15) leads to the choice $\mathbf{d}\in\mathbb{R}^g$ (and then $\mathbf{T}=0$ in Theorem 3.2).

The following assertions were proved in [20]: let $\mathcal{R}_g(\mathbb{R})\neq\emptyset$; if $\mathcal{R}_g$ is non dividing, then $T_v\cap(\Theta)\neq\emptyset$ for all v, where (Θ) denotes the set of zeros of the theta function; if $\mathcal{R}_g$ is dividing, then $T_v\cap\Theta\neq\emptyset$ if and only if $v\neq 0$. It follows that if solutions are smooth for any vector $\mathbf{d}$ lying in a component T_v (A.25) of the Jacobian, then the curve is dividing and $v=0$. Hence $\mathbf{d}\in T_0$ where $T_0=\mathbb{R}^g$. □

3.3 Reality condition and solutions of the DS2$^\rho$ equation

Let us consider the DS2$^\rho$ equation

$$\begin{aligned} &\mathrm{i}\,\psi_t+\frac{1}{2}(\partial_\xi^2+\partial_\eta^2)\psi+2\,\phi\,\psi=0,\\ &\partial_\xi\partial_\eta\,\phi+\rho\,\frac{1}{2}(\partial_\xi^2+\partial_\eta^2)|\psi|^2=0, \end{aligned} \tag{3.23}$$

where $\rho=\pm1$. Here t is a real variable and variables ξ, η satisfy $\overline{\xi}=\eta$. Analogously to the case where ξ and η are real variables (see Section 3.2), algebro-geometric solutions of (3.23) are constructed from solutions ψ, ψ^* (3.4) of the complexified system by imposing the reality condition $\psi^*=\rho\,\overline{\psi}$.

Let $\mathcal{R}_g$ be a real compact Riemann surface with an anti-holomorphic involution τ. Let us choose the homology basis satisfying (A.2). Then the solutions of (3.23) are given by

Theorem 3.3. *Let $a,b\in\mathcal{R}_g$ be distinct points such that $\tau a=b$, with local parameters satisfying $\overline{k_b(\tau p)}=k_a(p)$ for any point p lying in a neighbourhood of a. Denote by $\{\mathcal{A},\mathcal{B},\ell\}$ the standard generators of the relative homology group $H_1(\mathcal{R}_g,\{a,b\})$ (see Appendix A.2). Let $\mathbf{T}$, $\mathbf{L}\in\mathbb{Z}^g$ satisfy*

$$2\,\mathbf{T}+\mathbb{H}\,\mathbf{L}=diag(\mathbb{H}), \tag{3.24}$$

and define $\mathbf{d}=\frac{1}{2}\mathrm{Re}(\mathbb{B})\,\mathbf{L}+\mathrm{i}\mathbf{d}_I$, for some $\mathbf{d}_I\in\mathbb{R}^g$. Moreover, take θ, $h\in\mathbb{R}$ and $\kappa_1,\kappa_2\in\mathbb{C}\setminus\{0\}$ such that $\overline{\kappa_1}=\kappa_2$. Let us consider the following functions ψ and ϕ:

$$\psi(\xi,\eta,t)=|A|\,e^{\mathrm{i}\theta}\,\frac{\Theta(\mathbf{Z}-\mathbf{d}+\mathbf{r})}{\Theta(\mathbf{Z}-\mathbf{d})}\,\exp\left\{\mathrm{i}\left(-G_1\,\xi-G_2\,\eta+G_3\,\tfrac{t}{2}\right)\right\}, \tag{3.25}$$

$$\phi(\xi,\eta,t)=\frac{1}{2}\,(\ln\,\Theta(\mathbf{Z}-\mathbf{d}))_{\xi\xi}+\frac{1}{2}\,(\ln\,\Theta(\mathbf{Z}-\mathbf{d}))_{\eta\eta}+\frac{1}{4}h, \tag{3.26}$$

where $|A| = |\kappa_1|\,|q_2(a,b)|^{1/2}\,\exp\left\{-\frac{1}{2}\langle \mathrm{Re}(\mathbf{r}), \mathbf{L}\rangle\right\}$. *Then,*

1. *if* ℓ *intersects the set of real ovals of* $\mathcal{R}_g$ *only once, and if this intersection is transversal, functions* ψ *and* ϕ *are solutions of DS2*$^\rho$ *whith* $\rho = e^{\mathrm{i}\pi\langle \mathbf{N},\mathbf{L}\rangle}$,
2. *if* ℓ *does not cross any real oval, functions* ψ *and* ϕ *are solutions of DS2*$^\rho$ *whith* $\rho = -e^{\mathrm{i}\pi\langle \mathbf{N},\mathbf{L}\rangle}$.

Here $\mathbf{r} = \int_\ell \omega$, *the vector* $\mathbf{Z}$ *is defined in (3.5) and vector* $\mathbf{N} \in \mathbb{Z}^g$ *is defined in (A.6). Scalars* $q_2(a,b)$, G_1, G_2 *and* G_3 *are defined in (2.12), (3.6) and (3.7) respectively.*

Proof. Analogously to the proof of Theorem 3.2, let us check that under the conditions of the theorem, the functions ψ^* and ψ (3.2) satisfy the reality condition (3.3). First of all, due to the fact that points a and b are interchanged by τ, the vector $\mathbf{Z}$ (3.5) satisfies

$$\overline{\mathbf{Z}} = -\mathbf{Z}. \tag{3.27}$$

In fact, using the expansion (2.8) of the normalized holomorphic differentials ω_j near a we get

$$\overline{\tau^*\omega_j}(a)(p) = (\overline{V_{b,j}} + \overline{W_{b,j}}\,k_a(p) + \ldots)\,\mathrm{d}k_a(p),$$

for any point p lying in a neighbourhood of a. Then by (A.3) the vectors $\mathbf{V}_a$, $\mathbf{V}_b$ and $\mathbf{W}_a$, $\mathbf{W}_b$ appearing in the vector $\mathbf{Z}$ satisfy

$$\overline{\mathbf{V}_a} = -\mathbf{V}_b, \quad \overline{\mathbf{W}_a} = -\mathbf{W}_b, \tag{3.28}$$

which leads to (3.27). From (A.3) and (A.6) we get

$$\overline{\mathbf{r}} = \mathbf{r} - 2\mathrm{i}\pi\,\mathbf{N}, \tag{3.29}$$

where $\mathbf{N} \in \mathbb{Z}^g$ is defined in (A.6). By Proposition B.3, the scalar $q_2(a,b)$ is real. From (2.13), it is straightforward to see that the scalars $K_1(a,b)$ and $K_2(a,b)$, defined in (2.14) and (2.15), satisfy

$$\overline{K_1(a,b)} = K_1(b,a), \qquad \overline{K_2(a,b)} = K_2(b,a),$$

which leads to $\overline{G_1} = G_2$ and $G_3 \in \mathbb{R}$. Therefore, the reality condition (3.3) together with (3.4) leads to

$$|A|^2 = -\rho\,|\kappa_1|^2\,q_2(a,b)\,\frac{\Theta(\mathbf{Z}-\mathbf{d}-\mathbf{r})\,\Theta(\mathbf{Z}+\overline{\mathbf{d}}+\mathrm{i}\pi\,\mathrm{diag}(\mathbb{H}))}{\Theta(\mathbf{Z}+\overline{\mathbf{d}}-\mathbf{r}+\mathrm{i}\pi\,\mathrm{diag}(\mathbb{H}))\,\Theta(\mathbf{Z}-\mathbf{d})}, \tag{3.30}$$

taking into account (A.5). Let us choose a vector $\mathbf{d} \in \mathbb{C}^g$ such that

$$\overline{\mathbf{d}} = -\mathbf{d} - \mathrm{i}\pi\,\mathrm{diag}(\mathbb{H}) + 2\mathrm{i}\pi\mathbf{T} + \mathbb{B}\,\mathbf{L},$$

for some vector $\mathbf{T}$, $\mathbf{L} \in \mathbb{Z}^g$. The reality of the vector $\overline{\mathbf{d}} + \mathbf{d}$ together with (A.4) imply

$$\mathbf{d} = \frac{1}{2}\mathrm{Re}(\mathbb{B})\,\mathbf{L} + \mathrm{i}\mathbf{d}_I \tag{3.31}$$

for some $\mathbf{d}_I \in \mathbb{R}^g$, where $2\,\mathbf{T} + \mathbb{H}\,\mathbf{L} = \mathrm{diag}(\mathbb{H})$. With this choice of vector $\mathbf{d}$, (3.30) becomes

$$|A|^2 = -\rho\,|\kappa_1|^2\,q_2(a,b)\,e^{-\langle \mathbf{r},\mathbf{L}\rangle}. \tag{3.32}$$

Moreover, from (3.29) we deduce that equality (3.32) holds only if

$$\rho = -\mathrm{sign}(q_2(a,b))\,e^{-\mathrm{i}\pi\langle \mathbf{N},\mathbf{L}\rangle}.$$

The sign of $q_2(a,b)$ in the case where $\tau a = b$ is given in Proposition B2, which completes the proof. □

Corollary 3.1. *From Theorem 3.3 we deduce that*

1. *if $\mathcal{R}_g$ is dividing and each component of* $\mathbf{L}$ *is even, functions (3.25) and (3.26) are solutions of DS2$^+$,*

2. *if $\mathcal{R}_g$ does not have real ovals and each component of* $\mathbf{L}$ *is even, functions (3.25) and (3.26) are solutions of DS2$^-$.*

Remark 3.1. To construct solutions associated to non-dividing Riemann surfaces, we first observe from (3.24) that all components of the vector $\mathbf{L}$ cannot be even, since for non dividing Riemann surfaces the vector diag($\mathbb{H}$) contains odd coefficients (see Appendix A.1). In this case, the vector $\mathbf{N}$ has to be computed to determine the sign $\rho = -e^{\mathrm{i}\pi\langle\mathbf{N},\mathbf{L}\rangle}$ in the reality condition. This vector $\mathbf{N}$ is defined by the action of τ on the relative homology group $H_1(\mathcal{R}_g, \{a, b\})$ (see (A.6)). It follows that we do not have a general expression for this vector.

To ensure the smoothness of solutions (3.25) and (3.26) for all complex conjugate ξ, η, and $t \in \mathbb{R}$, the function $\Theta(\mathbf{Z} - \mathbf{d})$ of the variables ξ, η, t must not vanish. Following the work by Dubrovin and Natanzon [6] on smoothness of algebro-geometric solutions of the Kadomtsev Petviashvili (KP1) equation in the case where $\mathcal{R}_g$ admits real ovals we get

Proposition 3.2. *Functions (3.25) and (3.26) are smooth solutions of DS2$^+$ if the curve is an M-curve and $\mathbf{d} \in \mathrm{i}\,\mathbb{R}^g$. Assume that the curve admits real ovals and functions (3.25), (3.26) are smooth solutions of DS2$^\rho$ for any vector $\mathbf{d}$ lying in a component $\tilde{T}_v$ (A.27) of the Jacobian, then the curve is an M-curve, $\mathbf{d} \in \mathrm{i}\,\mathbb{R}^g$ and $\rho = +1$.*

Proof. By (3.27) and (3.31) the vector $\mathbf{Z} - \mathbf{d}$ belongs to the set S_2 introduced in (A.24). Hence by Proposition A.4, the solutions are smooth if the curve is an M-curve and $\mathbf{Z} - \mathbf{d} \in \mathrm{i}\,\mathbb{R}^g$ which implies $\mathbf{d} \in \mathrm{i}\,\mathbb{R}^g$ by (3.27) (and therefore $\mathbf{L} = \mathbf{T} = 0$). □

Remark 3.2. Smoothness of solutions of the DS2$^-$ equation was investigated in [13]. It is proved that solutions are smooth if and only if the associated Riemann surface does not have real ovals, and if there are no pseudo-real functions of degree $g-1$ on it (i.e. functions which satisfy $\overline{f(\tau p)} = -f(p)^{-1}$).

3.4 Reduction of the DS1$^\rho$ equation to the NLS equation

Solutions of the nonlinear Schrödinger equation (1.5) can be derived from solutions of the Davey Stewartson equations, when the associated Riemann surface is hyperelliptic.

Proposition 3.3. *Let $\mathcal{R}_g$ be a hyperelliptic curve of genus g which admits an anti-holomorphic involution τ. Denote by σ the hyperelliptic involution defined on $\mathcal{R}_g$. Let $a, b \in \mathcal{R}_g(\mathbb{R})$ with local parameters satisfying $\overline{k_a(\tau p)} = k_a(p)$ for p near a, and $\overline{k_b(\tau p)} = k_b(p)$ for p near b. Moreover, assume that $\sigma a = b$ and $k_a(p) = k_b(\sigma p)$. Then, taking $\kappa_1 = \kappa_2 = 1$, the function ψ in (3.13) is a solution of the equation*

$$\mathrm{i}\,\psi_t + \psi_{\xi\xi} + 2\,\left(\rho\,|\psi|^2 + q_1(a,b) + \tfrac{1}{4}h\right)\,\psi = 0,$$

which can be transformed to the NLS$^\rho$ equation

$$\mathrm{i}\,\tilde{\psi}_t + \tilde{\psi}_{\xi\xi} + 2\,\rho\,|\tilde{\psi}|^2\,\tilde{\psi} = 0,$$

by the substitution

$$\tilde{\psi}(\xi,t)=\psi(\xi,t)\,\exp\left\{-2\mathrm{i}\left(q_1(a,b)+\tfrac{1}{4}h\right)\,t\right\}.$$

If all branch points of $\mathcal{R}_g$ are real, $\tilde{\psi}$ is a smooth solution of NLS$^-$. If they are all pairwise conjugate, $\tilde{\psi}$ is a smooth solution of NLS$^+$.

Proof. If $a,b\in\mathcal{R}_g$ are such that $\sigma a=b$ and the local parameters satisfy $k_a(p)=k_b(\sigma p)$, one has

$$\mathbf{V}_a+\mathbf{V}_b=0,\qquad \mathbf{W}_a+\mathbf{W}_b=0. \tag{3.33}$$

To verify (3.33), we use the action $\sigma\mathcal{A}_k=-\mathcal{A}_k$ of the involution σ on the $\mathcal{A}$-cycles of the homology basis. Hence by (2.1) we have

$$2\mathrm{i}\pi\delta_{jk}=\int_{\sigma\mathcal{A}_k}\sigma^*\omega_j=-\int_{\mathcal{A}_k}\sigma^*\omega_j.$$

It follows that the holomorphic differential $-\sigma^*\omega_j$ satisfies the normalization condition (2.1), which implies, by virtue of uniqueness of the normalized holomorphic differentials,

$$\sigma^*\omega_j=-\omega_j.$$

Using (2.8) we obtain

$$\begin{aligned}\sigma^*\omega_j(a)(p)&=\left(V_{b,j}+W_{b,j}\,k_b(\sigma p)+\ldots\right)\,\mathrm{d}k_b(\sigma p)\\&=\left(V_{b,j}+W_{b,j}\,k_a(p)+\ldots\right)\,\mathrm{d}k_a(p),\end{aligned}$$

which implies $\mathbf{V}_a+\mathbf{V}_b=0$ and $\mathbf{W}_a+\mathbf{W}_b=0$.

Therefore, when the Riemann surface associated to solutions of DS1$^\rho$ is hyperelliptic, assuming that a and b satisfy $\sigma a=b$, and $\kappa_1=\kappa_2=1$, by (3.33) and (2.10), under the reality condition $\psi^*=\rho\,\overline{\psi}$, the function ϕ in (3.14) satisfies

$$\phi(\xi,\eta,t)=\rho\,|\psi|^2+q_1(a,b)+\frac{1}{4}h.$$

Hence the function ψ (3.13) becomes a solution of the equation

$$\mathrm{i}\,\psi_t+\psi_{\xi\xi}+2\,\left(\rho\,|\psi|^2+q_1(a,b)+\tfrac{1}{4}h\right)\,\psi=0,$$

with $\rho=\pm1$, depending on the reality of the branch points as explained in Section 4. □

Solutions of the NLS equation obtained in this way coincide with those in [3].

4 Algebro-geometric solutions of the multi-component NLS equation

In this section, we present another application of the degenerated Fay identity (2.13), which leads to new theta-functional solutions of the multi-component nonlinear Schrödinger equation (n-NLS^s)

$$\mathrm{i}\,\frac{\partial\psi_j}{\partial t}+\frac{\partial^2\psi_j}{\partial x^2}+2\left(\sum_{k=1}^{n}s_k|\psi_k|^2\right)\,\psi_j=0,\qquad j=1,\ldots,n, \tag{4.1}$$

where $s=(s_1,\ldots,s_n)$, $s_i=\pm1$. Here $\psi_j(x,t)$ are complex valued functions of the real variables x and t.

4.1 Solutions of the complexified n-NLS equation

Consider first the complexified version of the n-NLS^s equation, which is a system of $2n$ equations of $2n$ dependent variables $\left\{\psi_j,\,\psi_j^*\right\}_{j=1}^n$

$$\mathrm{i}\,\frac{\partial\psi_j}{\partial t}+\frac{\partial^2\psi_j}{\partial x^2}+2\left(\sum_{k=1}^n\psi_k\,\psi_k^*\right)\psi_j=0,$$
$$-\mathrm{i}\,\frac{\partial\psi_j^*}{\partial t}+\frac{\partial^2\psi_j^*}{\partial x^2}+2\left(\sum_{k=1}^n\psi_k\,\psi_k^*\right)\psi_j^*=0,\qquad j=1,\ldots,n, \tag{4.2}$$

where $\psi_j(x,t)$ and $\psi_j^*(x,t)$ are complex valued functions of the real variables x and t. This system reduces to the n-NLS^s equation (4.1) under the *reality conditions*

$$\psi_j^*=s_j\,\overline{\psi_j},\qquad j=1,\ldots,n. \tag{4.3}$$

Theta functional solutions of the system (4.2) are given by

Theorem 4.1. *Let $\mathcal{R}_g$ be a compact Riemann surface of genus $g>0$ and let f be a meromorphic function of degree $n+1$ on $\mathcal{R}_g$. Let $z_a\in\mathbb{C}$ be a non critical value of f, and consider the fiber $f^{-1}(z_a)=\{a_1,\ldots,a_{n+1}\}$ over z_a. Choose the local parameters k_{a_j} near a_j as $k_{a_j}(p)=f(p)-z_a$, for any point $p\in\mathcal{R}_g$ lying in a neighbourhood of a_j. Let $\mathbf{d}\in\mathbb{C}^g$ and $A_j\neq 0$ be arbitrary constants. Then the following functions $\{\psi_j\}_{j=1}^n$ and $\left\{\psi_j^*\right\}_{j=1}^n$ are solutions of the system (4.2)*

$$\psi_j(x,t)=A_j\,\frac{\Theta(\mathbf{Z}-\mathbf{d}+\mathbf{r}_j)}{\Theta(\mathbf{Z}-\mathbf{d})}\,\exp\left\{\mathrm{i}(-E_j\,x+\,F_j\,t)\right\},$$
$$\psi_j^*(x,t)=\frac{q_2(a_{n+1},a_j)}{A_j}\,\frac{\Theta(\mathbf{Z}-\mathbf{d}-\mathbf{r}_j)}{\Theta(\mathbf{Z}-\mathbf{d})}\,\exp\left\{\mathrm{i}(E_j\,x-\,F_j\,t)\right\}. \tag{4.4}$$

Here $\mathbf{r}_j=\int_{a_{n+1}}^{a_j}\omega$, where ω is the vector of normalized holomorphic differentials, and

$$\mathbf{Z}=\mathrm{i}\,\mathbf{V}_{a_{n+1}}\,x+\mathrm{i}\,\mathbf{W}_{a_{n+1}}\,t. \tag{4.5}$$

The vectors $\mathbf{V}_{a_{n+1}}$ and $\mathbf{W}_{a_{n+1}}$ are defined in (2.8), and the scalars E_j, F_j are given by

$$E_j=K_1(a_{n+1},a_j),\qquad F_j=K_2(a_{n+1},a_j)-2\sum_{k=1}^n q_1(a_{n+1},a_k). \tag{4.6}$$

The scalars $q_2(a_{n+1},a_j)$, $K_1(a_{n+1},a_j)$, $K_2(a_{n+1},a_j)$ and $q_1(a_{n+1},a_k)$ are defined in (2.12), (2.14), (2.15) and (2.11) respectively.

Proof. We start with the following technical lemma.

Lemma 4.1. *Let $\mathcal{R}_g$ be a compact Riemann surface of genus $g>0$ and let $a_1,\ldots,a_{n+1}$ be distinct points on $\mathcal{R}_g$. Then the vectors $\mathbf{V}_{a_j}$ for $j=1,\ldots,n+1$ are linearly dependent if and only if there exists a meromorphic function f of degree $n+1$ on $\mathcal{R}_g$, and $z_a\in\mathbb{CP}^1$ such that $f^{-1}(z_a)=\{a_1,\ldots,a_{n+1}\}$.*

Proof of Lemma 4.1. Assume that there exist $\alpha_1, \dots, \alpha_{n+1} \in \mathbb{C}^*$ such that $\sum_{k=1}^{n+1} \alpha_k \mathbf{V}_{a_k} = 0$. The left hand side of this equality equals the vector of $\mathcal{B}$-periods (see e.g. [4]) of the normalized differential of the second kind $\Omega = \sum_{k=1}^{n+1} \alpha_k \Omega_{a_k}^{(2)}$. Hence all periods of the differential Ω vanish, which implies that the Abelian integral $p \longmapsto \int_{p_0}^{p} \Omega$ is a meromorphic function of degree $n+1$ on $\mathcal{R}_g$ having simple poles at $a_1, \dots, a_{n+1}$.

Conversely, assume that there exists a meromorphic function f of degree $n+1$ on $\mathcal{R}_g$, and $z_a \in \mathbb{C}$ such that $f^{-1}(z_a) = \{a_1, \dots, a_{n+1}\}$ (the case $z_a = \infty$ can be treated in the same way). The function $h(p) = (f(p) - z_a)^{-1}$ is a meromorphic function of degree $n+1$ on $\mathcal{R}_g$ having simple poles at $a_1, \dots, a_{n+1}$ only. Therefore all periods of the differential $\mathrm{d}h$ vanish. Let $p_0 \in \mathcal{R}_g$ satisfy $h(p_0) = 0$. Using Riemann's bilinear identity [4] we get

$$\int_{\partial F_g} \omega_j \int_{p_0}^{p} \mathrm{d}h = \int_{\partial F_g} \omega_j h(p) = 0,$$

where F_g denotes the simply connected domain with the boundary $\partial F_g = \sum_{j=1}^{g} (\mathcal{A}_j + \mathcal{A}_j^{-1} + \mathcal{B}_j + \mathcal{B}_j^{-1})$. By Cauchy's theorem, taking local parameters k_{a_j} near a_j such that $k_{a_j}(p) = f(p) - z_a$ for any point $p \in \mathcal{R}_g$ lying in a neighbourhood of a_j, we deduce that $\sum_{k=1}^{n+1} \mathbf{V}_{a_k} = 0$. □

To prove Theorem 4.1, substitute the functions (4.4) into the first equation of (4.2) to get

$$D'_{a_{n+1}} \ln \frac{\Theta(\mathbf{z} + \mathbf{r}_1)}{\Theta(\mathbf{z})} + D^2_{a_{n+1}} \ln \frac{\Theta(\mathbf{z} + \mathbf{r}_1)}{\Theta(\mathbf{z})} + \left(D_{a_{n+1}} \ln \frac{\Theta(\mathbf{z} + \mathbf{r}_1)}{\Theta(\mathbf{z})} - E_1 \right)^2$$

$$+ F_1 - 2 \sum_{k=1}^{n} q_2(a_{n+1}, a_k) \frac{\Theta(\mathbf{z} + \mathbf{r}_k)\Theta(\mathbf{z} - \mathbf{r}_k)}{\Theta(\mathbf{z})^2} = 0. \tag{4.7}$$

It can be shown that equation (4.7) holds as follows: in (2.13), let us choose $a = a_{n+1}$ and $b = a_1$ to obtain

$$D'_{a_{n+1}} \ln \frac{\Theta(\mathbf{z} + \mathbf{r}_1)}{\Theta(\mathbf{z})} + D^2_{a_{n+1}} \ln \frac{\Theta(\mathbf{z} + \mathbf{r}_1)}{\Theta(\mathbf{z})} + \left(D_{a_{n+1}} \ln \frac{\Theta(\mathbf{z} + \mathbf{r}_1)}{\Theta(\mathbf{z})} - K_1 \right)^2 + K_2 + 2\, D^2_{a_{n+1}} \ln \Theta(\mathbf{z}) = 0, \tag{4.8}$$

for any $\mathbf{z} \in \mathbb{C}^g$, and in particular for $\mathbf{z} = \mathbf{Z} - \mathbf{d}$; here we used the notation $K_i = K_i(a_{n+1}, a_1)$ for $i = 1, 2$. By Lemma 4.1 the sum $\sum_{k=1}^{n+1} \mathbf{V}_{a_k}$ equals zero, which implies

$$\sum_{k=1}^{n+1} D_{a_k} = 0.$$

Substituting $D_{a_{n+1}}$ instead of $-\sum_{k=1}^{n} D_{a_k}$ in (4.8) and using (2.10) we obtain (4.7), where

$$E_1 = K_1, \qquad F_1 = K_2 - 2 \sum_{k=1}^{n} q_1(a_{n+1}, a_k).$$

In the same way, it can be proved that the functions in (4.13) satisfy the $2n-1$ other equations of the system (4.2). □

The solutions (4.4) of the complexified sytem (4.2) depend on the Riemann surface $\mathcal{R}_g$, the meromorphic function f of degree $n+1$, a non critical value $z_a \in \mathbb{C}$ of f, and arbitrary constants $\mathbf{d} \in \mathbb{C}^g$,

$A_j \neq 0$. The transformation of the local parameters given by

$$k_{a_j} \longrightarrow \beta\, k_{a_j} + \mu\, k_{a_j}^2 + O\left(k_{a_j}^3\right), \tag{4.9}$$

where β, μ are arbitrary complex numbers ($\beta \neq 0$), leads to a different family of solutions of the complexified system (4.2). These new solutions are obtained via the following transformations:

$$\begin{aligned} \psi_j(x,t) &\longrightarrow \psi_j\left(\beta\, x + 2\beta\lambda\, t, \beta^2\, t\right)\, \exp\left\{-\mathrm{i}\left(\lambda\, x + \lambda^2\, t\right)\right\}, \\ \psi_j^*(x,t) &\longrightarrow \beta^2\, \psi_j^*\left(\beta\, x + 2\beta\lambda\, t, \beta^2\, t\right)\, \exp\left\{\mathrm{i}\left(\lambda\, x + \lambda^2\, t\right)\right\}, \end{aligned} \tag{4.10}$$

where $\lambda = \mu\, \beta^{-1}$.

4.2 Reality conditions

Algebro-geometric solutions of the n-NLS^s equation (4.1) are constructed from solutions (4.4) of the complexified system by imposing the reality conditions $\psi_j^* = s_j\, \overline{\psi_j}$ (4.3).

Let $\mathcal{R}_g$ be a real compact Riemann surface with an anti-holomorphic involution τ. Let us choose the homology basis satisfying (A.2). A meromorphic function f on $\mathcal{R}_g$ is called real if $f(\tau p) = \overline{f(p)}$ for any $p \in \mathcal{R}_g$.

In the next proposition we derive theta-functional solutions of (4.1). The signs s_j appearing in the reality conditions (4.3) are expressed in terms of certain intersection indices on $\mathcal{R}_g$. These intersection indices are defined as follows: let f be a real meromorphic function of degree $n+1$ on $\mathcal{R}_g$. Let $z_a \in \mathbb{R}$ be a non critical value of f, and assume that the fiber $f^{-1}(z_a) = \{a_1, \ldots, a_{n+1}\}$ over z_a belongs to the set $\mathcal{R}_g(\mathbb{R})$. Let $\tilde{a}_{n+1}, \tilde{a}_j \in \mathcal{R}_g(\mathbb{R})$ lie in a neighbourhood of a_{n+1} and a_j respectively such that $f(\tilde{a}_{n+1}) = f(\tilde{a}_j)$. Denote by $\tilde{\ell}_j$ an oriented contour connecting $\tilde{a}_{n+1}$ and $\tilde{a}_j$, and having the following decomposition in $H_1(\mathcal{R}_g \setminus \{a_{n+1}, a_j\})$ (see Appendix A.2.2)

$$\tau\tilde{\ell}_j = \tilde{\ell}_j + \mathcal{A}\mathbf{N}_j + \mathcal{B}\mathbf{M}_j + \alpha_j\, \mathcal{S}_{a_j}, \tag{4.11}$$

for some $\alpha_j \in \mathbb{Z}$, where vectors $\mathbf{N}_j, \mathbf{M}_j \in \mathbb{Z}^g$ are the same as in (A.13). Then

$$\alpha_j = (\tau\tilde{\ell}_j - \tilde{\ell}_j) \circ \ell_j, \tag{4.12}$$

between the closed contour $\tau\tilde{\ell}_j - \tilde{\ell}_j$ and the contour ℓ_j; this intersection is computed in the relative homology group $H_1(\mathcal{R}_g, \{a_{n+1}, a_j\})$.

Theta functional solutions of (4.1) are given by

Proposition 4.1. *Let f be a real meromorphic function of degree $n+1$ on $\mathcal{R}_g$. Let $z_a \in \mathbb{R}$ be a non critical value of f, and assume that the fiber $f^{-1}(z_a) = \{a_1, \ldots, a_{n+1}\}$ over z_a belongs to the set $\mathcal{R}_g(\mathbb{R})$. Choose the local parameters k_{a_j} near a_j as $k_{a_j}(p) = f(p) - z_a$, for any point $p \in \mathcal{R}_g$ lying in a neighbourhood of a_j. Denote by $\{\mathcal{A}, \mathcal{B}, \ell_j\}$ the standard generators of the relative homology group $H_1(\mathcal{R}_g, \{a_{n+1}, a_j\})$ (see Appendix A.2.2). Let $\mathbf{d}_R \in \mathbb{R}^g$, $\mathbf{T} \in \mathbb{Z}^g$, and define $\mathbf{d} = \mathbf{d}_R + \frac{\mathrm{i}\pi}{2}(diag(\mathbb{H}) - 2\,\mathbf{T})$. Morover, take $\theta \in \mathbb{R}$. Then the following functions $\{\psi_j\}_{j=1}^n$ are solutions of n-NLS^s (4.1)*

$$\psi_j(x,t) = |A_j|\, e^{\mathrm{i}\theta}\, \frac{\Theta(\mathbf{Z} - \mathbf{d} + \mathbf{r}_j)}{\Theta(\mathbf{Z} - \mathbf{d})}\, \exp\left\{\mathrm{i}(-E_j\, x + F_j\, t)\right\}, \tag{4.13}$$

where $\mathbf{Z} = \mathrm{i}\,\mathbf{V}_{a_{n+1}}\,x + \mathrm{i}\,\mathbf{W}_{a_{n+1}}\,t$, *and*

$$|A_j| = |q_2(a_{n+1}, a_j)|^{1/2}\, \exp\left\{\tfrac{1}{2}\left\langle \mathbf{d}_R, \mathbf{M}_j\right\rangle\right\}. \tag{4.14}$$

Here $\mathbf{r}_j = \int_{\ell_j}\omega$, *the vectors* $\mathbf{V}_{a_{n+1}}, \mathbf{W}_{a_{n+1}}$ *are defined in (2.8), and the vector* $\mathbf{M}_j \in \mathbb{Z}^g$ *is defined by the action of* τ *on the relative homology group* $H_1\left(\mathcal{R}_g^{(n+1)}, \{a_{n+1}, a_j\}\right)$ *(see (A.13)). The scalars* $q_2(a_{n+1}, a_j)$ *and* E_j, F_j *are introduced in (2.12) and (4.6) respectively. The signs* $s_1, \ldots, s_n$ *are given by*

$$s_j = \exp\left\{\mathrm{i}\pi(1+\alpha_j) + \mathrm{i}\pi\left\langle \mathbf{T}, \mathbf{M}_j\right\rangle\right\}, \tag{4.15}$$

where the intersection indices $\alpha_j \in \mathbb{Z}$ *are defined in (4.12).*

Proof. The proof follows the lines of Section 3.2, where similar statements were proven for the DS1$^\rho$ equation. First of all, invariance with respect to the anti-involution τ of the point a_{n+1} implies the reality of the vector $\mathbf{Z} = \mathrm{i}\,\mathbf{V}_{a_{n+1}}\,x + \mathrm{i}\,\mathbf{W}_{a_{n+1}}\,t$. Moreover, from (A.3) and (A.13) we get

$$\overline{\mathbf{r}_j} = -\mathbf{r}_j - 2\mathrm{i}\pi\mathbf{N}_j - \mathbb{B}\mathbf{M}_j. \tag{4.16}$$

where $\mathbf{N}_j$, $\mathbf{M}_j \in \mathbb{Z}^g$ are defined in (A.13) and satisfy

$$2\,\mathbf{N}_j + \mathbb{H}\mathbf{M}_j = 0. \tag{4.17}$$

For $j = 1, \ldots, n$, the action of the complex conjugation on the scalars $K_1(a_{n+1}, a_j)$ and $K_2(a_{n+1}, a_j)$ is given by (3.19), and one can directy see from (2.10) that $q_1(a_{n+1}, a_j)$ is real. Hence we get

$$\overline{E_j} = E_j - \left\langle \mathbf{V}_{a_{n+1}}, \mathbf{M}_j\right\rangle, \qquad \overline{F_j} = F_j + \left\langle \mathbf{W}_{a_{n+1}}, \mathbf{M}_j\right\rangle. \tag{4.18}$$

Under the assumptions of the theorem and by (B.11), the argument of $q_2(a_{n+1}, a_j)$ is given by

$$\arg(q_2(a_{n+1}, a_j)) = \pi\left(1 + \alpha_j + \tfrac{1}{2}\left\langle \mathbb{H}\mathbf{M}_j, \mathbf{M}_j\right\rangle\right) - \frac{1}{2\mathrm{i}}\left(\left\langle \mathbb{B}\,\mathbf{M}_j, \mathbf{M}_j\right\rangle + 2\left\langle \mathbf{r}_j, \mathbf{M}_j\right\rangle\right). \tag{4.19}$$

Therefore, the reality conditions (4.3) together with (4.4) lead to

$$\begin{aligned}|A_j|^2 = s_j\,|q_2(a_{n+1}, a_j)|\,&\frac{\Theta(\mathbf{Z} - \mathbf{d} - \mathbf{r}_j)\,\Theta(\mathbf{Z} - \overline{\mathbf{d}} + \mathrm{i}\pi\,\mathrm{diag}(\mathbb{H}))}{\Theta(\mathbf{Z} - \overline{\mathbf{d}} - \mathbf{r}_j + \mathrm{i}\pi\,\mathrm{diag}(\mathbb{H}))\,\Theta(\mathbf{Z} - \mathbf{d})}\\ &\times \exp\left\{\mathrm{i}\pi\left(1 + \alpha_j + \tfrac{1}{2}\left\langle \mathbb{H}\mathbf{M}_j, \mathbf{M}_j\right\rangle\right) + \left\langle \overline{\mathbf{d}} - \mathrm{i}\pi\mathrm{diag}(\mathbb{H}), \mathbf{M}_j\right\rangle\right\},\end{aligned} \tag{4.20}$$

if one takes into account (A.5) and (2.4). Let us choose a vector $\mathbf{d} \in \mathbb{C}^g$ such that

$$\overline{\mathbf{d}} \equiv \mathbf{d} - \mathrm{i}\pi\,\mathrm{diag}(\mathbb{H}) \pmod{2\mathrm{i}\pi\mathbb{Z}^g + \mathbb{B}\,\mathbb{Z}^g}.$$

Since $\overline{\mathbf{d}} - \mathbf{d}$ is purely imaginary we have

$$\overline{\mathbf{d}} = \mathbf{d} - \mathrm{i}\pi\,\mathrm{diag}(\mathbb{H}) + 2\mathrm{i}\pi\mathbf{T}, \tag{4.21}$$

for some $\mathbf{T} \in \mathbb{Z}^g$, where we have used (A.4) and the fact that $\mathbb{B}$ has a non-degenerate real part. It follows that the vector $\mathbf{d}$ can be written as

$$\mathbf{d} = \mathbf{d}_R + \frac{\mathrm{i}\pi}{2}(\mathrm{diag}(\mathbb{H}) - 2\,\mathbf{T}), \tag{4.22}$$

for some $\mathbf{d}_R \in \mathbb{R}^g$ and $\mathbf{T} \in \mathbb{Z}^g$. Therefore, (4.20) becomes

$$|A_j|^2 = s_j \, |q_2(a_{n+1}, a_j)| \exp\left\{ \mathrm{i}\pi(1+\alpha_j) + \tfrac{\mathrm{i}\pi}{2} \langle \mathbb{H}\mathbf{M}_j, \mathbf{M}_j \rangle + \langle \mathbf{d}, \mathbf{M}_j \rangle \right\}, \tag{4.23}$$

which by (4.22) leads to (4.14). Moreover we deduce from (4.22) and (4.23) that

$$s_j = \exp\left\{ \mathrm{i}\pi(1+\alpha_j) + \tfrac{\mathrm{i}\pi}{2} \langle \mathbb{H}\mathbf{M}_j + \mathrm{diag}(\mathbb{H}), \mathbf{M}_j \rangle - \mathrm{i}\pi \langle \mathbf{T}, \mathbf{M}_j \rangle \right\}.$$

From (4.17) and the definition of the matrix $\mathbb{H}$ (see Appendix A.1), it can be deduced that the quantity $\frac{1}{2} \langle \mathbb{H}\mathbf{M}_j + \mathrm{diag}(\mathbb{H}), \mathbf{M}_j \rangle$ is even in each case, which yields (4.15). □

Functions ψ_j given in (4.13) describe a family of algebro-geometric solutions of (4.1) depending on: a real Riemann surface $(\mathcal{R}_g, \tau)$, a real meromorphic function f on $\mathcal{R}_g$ of degree $n+1$, a non critical value $z_a \in \mathbb{R}$ of f such that the fiber over z_a belongs to the set $\mathcal{R}_g(\mathbb{R})$, and arbitrary constants $\mathbf{d}_R \in \mathbb{R}^g$, $\mathbf{T} \in \mathbb{Z}^g$, $\theta \in \mathbb{R}$. Note that the periodicity properties of the theta function imply without loss of generality that the vector $\mathbf{T}$ can be chosen in the set $\{0,1\}^g$. The case where the Riemann surface $\mathcal{R}_g$ is dividing and $\mathbf{T} = 0$ is of special importance, because the related solutions are smooth, as explained in Proposition 3.1. In this case, the sign s_j (4.15) is given by $s_j = \exp\{\mathrm{i}\pi(1+\alpha_j)\}$.

4.3 Solutions of n-NLS$^+$ and n-NLS$^-$

Here, we consider the two most physically significant situations: the completely focusing multi-component system n-NLS$^+$ (which corresponds to $s = (1, \ldots, 1)$), and the completely defocusing system n-NLS$^-$ (which corresponds to $s = (-1, \ldots, -1)$).

Starting from a pair $(\mathcal{R}_g, f)$, where $\mathcal{R}_g$ is a Riemann surface of genus g, and where f is a meromorphic function of degree $n+1$ on $\mathcal{R}_g$, which has $n+1$ simple poles, we construct an $n+1$-sheeted branched covering of $\mathbb{CP}^1$, which we denote by $\mathcal{R}_{g,n+1}$. The ramification points of the covering correspond to critical points of f; we assume that all of them are simple.

For any point $a \in \mathcal{R}_{g,n+1}$ which is not a critical point or a pole of the meromorphic function f, we use the local parameter $k_a(p) = f(p) - f(a)$, for any point p in a neighbourhood of a.

According to [7], by an appropriate choice of the set of generators $\{\gamma_j\}_{j=1}^{2g+2n}$ of the fundamental group $\pi_1(\mathbb{CP}^1 \setminus \{z_1, \ldots, z_{2g+2n}\}, z_0)$ of the base, which satisfy $\gamma_1 \ldots \gamma_{2g+2n} = id$, the covering $\mathcal{R}_{g,n+1}$ can be represented as follows: consider the hyperelliptic covering of genus g and attach to it $n-1$ spheres as shown in Figure 1. More precisely, the generators γ_j can be chosen in such way that the loop γ_j encircles only the point z_j; the corresponding elements $\sigma_j \in \mathbf{S}_{n+1}$ (where $\mathbf{S}_{n+1}$ denotes the symmetric group of order $n+1$) of the monodromy group of the covering are given by

$$\begin{aligned} \sigma_j &= (n+1, n), & j &= 1, \ldots, 2g+2, \\ \sigma_{2g+2+2k+1} = \sigma_{2g+2+2k+2} &= (n-k, n-k-1), & k &= 0, \ldots, n-2. \end{aligned}$$

We denote by $x_1, \ldots, x_{2g+2n} \in \mathcal{R}_{g,n+1}$ the critical points of the meromorphic function f, and by $z_j = f(x_j) \in \mathbb{C}$ the critical values. Assume that the branch points $\{z_j\}_{j=1}^{2g+2n}$ are real or pairwise conjugate, and order them as follows:

$$\mathrm{Re}(z_1) \leq \ldots \leq \mathrm{Re}(z_{2g+2n}).$$

Let us introduce an anti-holomorphic involution τ on $\mathcal{R}_{g,n+1}$, which acts as the complex conjugation on each sheet.

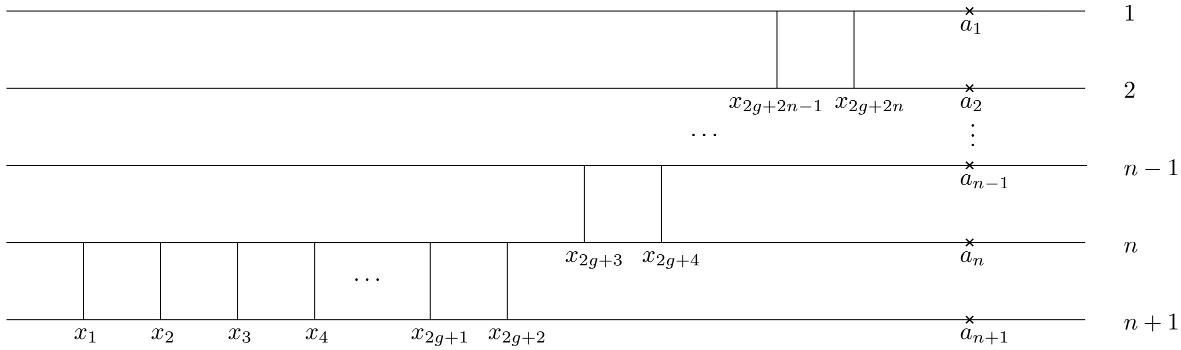

Figure 1: *Hurwitz diagram of the covering* $\mathcal{R}_{g,n+1}$.

4.3.1 Solutions of n-NLS$^+$.

Here we construct solutions of the n-NLS$^+$ system

$$\mathrm{i}\,\frac{\partial \psi_j}{\partial t} + \frac{\partial^2 \psi_j}{\partial x^2} + 2\left(\sum_{k=1}^{n} |\psi_k|^2\right)\psi_j = 0, \qquad j = 1,\dots,n. \tag{4.24}$$

Let us first describe the covering and the homology basis used in the construction of the solutions.

Assume that all branch points of the covering $\mathcal{R}_{g,n+1}$ are pairwise conjugate. Denote this covering by $\mathcal{R}^+_{g,n+1}$, refering to the focusing system (4.24). The covering $\mathcal{R}^+_{g,n+1}$ admits two real ovals if the genus g is odd, and only one if g is even. Each of them consists of a closed contour on the covering having a real projection into the base. It is straightforward to see that the covering $\mathcal{R}^+_{g,n+1}$ is dividing (see Appendix A.1): two points which have respectively a positive and a negative imaginary projection onto $\mathbb{C}$, cannot be connected by a contour which does not cross a real oval. Hence the set of fixed points of the anti-holomorphic involution τ separates the covering into two connected components.

Now let us choose the canonical homology basis such that all basic cycles belong to sheets $n+1$ and n, and such that the anti-holomorphic involution τ acts on them as in (A.2). By the previous topological description of $\mathcal{R}^+_{g,n+1}$, the matrix $\mathbb{H}$ involved in (A.2) looks as:

$$\mathbb{H} = \begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & 1 & \\ & & & 1 & 0 & \\ & & & & & 0 \end{pmatrix} \quad \text{if } g \text{ is odd,}$$

$$\mathbb{H} = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & 1 & 0 \end{pmatrix} \quad \text{if } g \text{ is even.}$$

The canonical homology basis is described explicitely in Figure 2 for odd genus, and in Figure 3 for even genus.

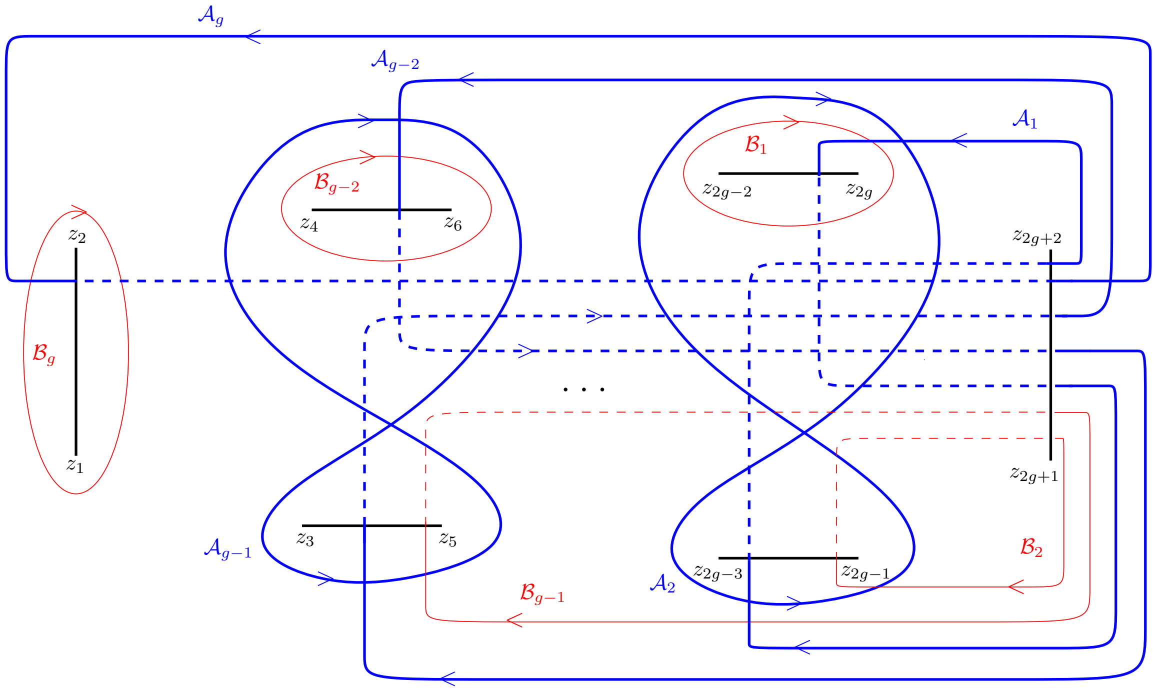

Figure 2: *Homology basis on the covering $\mathcal{R}^+_{g,n+1}$ when the genus g is odd. The solid line indicates the sheet $n+1$, and the dashed line sheet n.*

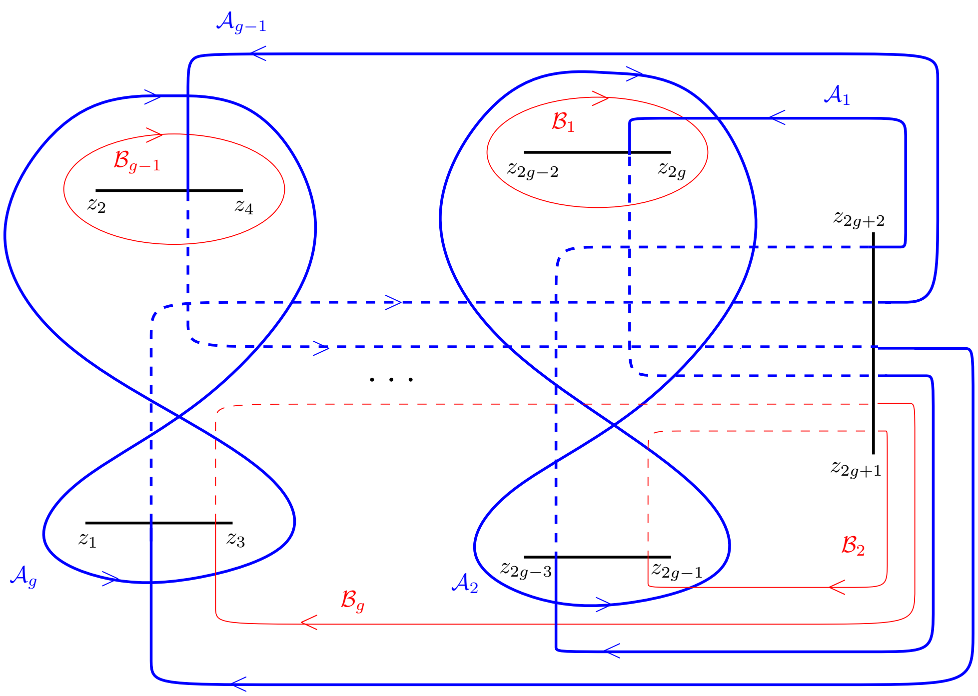

Figure 3: *Homology basis on the covering $\mathcal{R}^+_{g,n+1}$ when the genus g is even. The solid line indicates the sheet $n+1$, and the dashed line sheet n.*

As proved in the following theorem, among all coverings having a monodromy group described in Figure 1, only the covering $\mathcal{R}^+_{g,n+1}$ leads to algebro-geometric solutions of the focusing system (4.24).

Theorem 4.2. *Consider the covering $\mathcal{R}^+_{g,n+1}$ and the canonical homology basis discussed above. Fix $z_a \in \mathbb{R}$ such that $z_a > Re(z_j)$ for $j = 1, \dots, 2g+2n$. Consider the fiber $f^{-1}(z_a) = \{a_1, \dots, a_{n+1}\}$ over z_a, where $a_j \in \mathcal{R}^+_{g,n+1}(\mathbb{R})$ belongs to sheet j (each of the a_j is invariant under the involution τ). Let $\mathbf{d} \in \mathbb{R}^g$ and $\theta \in \mathbb{R}$. Then the following functions $\{\psi_j\}_{j=1}^n$ are smooth solutions of n-NLS$^+$:*

$$\psi_j(x,t) = |A_j|\, e^{\mathrm{i}\theta}\, \frac{\Theta(\mathbf{Z} - \mathbf{d} + \mathbf{r}_j)}{\Theta(\mathbf{Z} - \mathbf{d})}\, \exp\left\{\mathrm{i}(-E_j\, x + F_j\, t)\right\}, \tag{4.25}$$

where $\mathbf{Z} = \mathrm{i}\,\mathbf{V}_{a_{n+1}}\, x + \mathrm{i}\,\mathbf{W}_{a_{n+1}}\, t$. Here $\mathbf{r}_j = \int_{a_{n+1}}^{a_j} \omega$, the vectors $\mathbf{V}_{a_{n+1}}$, $\mathbf{W}_{a_{n+1}}$ are defined in (2.8), and the vector $\mathbf{M}_j \in \mathbb{Z}^g$ is defined in (A.13), according to the action of τ on the relative homology group $H_1\left(\mathcal{R}^+_{g,n+1}, \{a_{n+1}, a_j\}\right)$. The scalars $|A_j|$ and E_j, F_j are given by (4.14) and (4.6) respectively.

Proof. Let us check that the conditions of the theorem imply that functions ψ_j in (4.13) are solutions of n-NLS^s for $s = (1, \dots, 1)$. Since the matrix $\mathbb{H}$ associated to the covering $\mathcal{R}^+_{g,n+1}$ satisfies $\mathrm{diag}(\mathbb{H}) = 0$, and $\mathbf{d} \in \mathbb{R}^g$ (i.e. $\mathbf{T} = 0$), the quantities $\{s_j\}_{j=1}^n$ (4.15) become

$$s_j = \exp\left\{\mathrm{i}\pi(1 + \alpha_j)\right\}. \tag{4.26}$$

Let us first compute the intersection index α_n. Let $\tilde{a}_{n+1}, \tilde{a}_n \in \mathcal{R}^+_{g,n+1}(\mathbb{R})$ lie in a neighbourhood of a_{n+1} and a_n respectively such that $f(\tilde{a}_{n+1}) = f(\tilde{a}_n) = z_{\tilde{a}}$. Denote by $\tilde{\ell}_n$ an oriented contour connecting $\tilde{a}_{n+1}$ and $\tilde{a}_n$. Then the intersection index α_n between the closed contour $\tau\tilde{\ell}_n - \tilde{\ell}_n$ and the contour ℓ_n satisfies (see Figure 4)

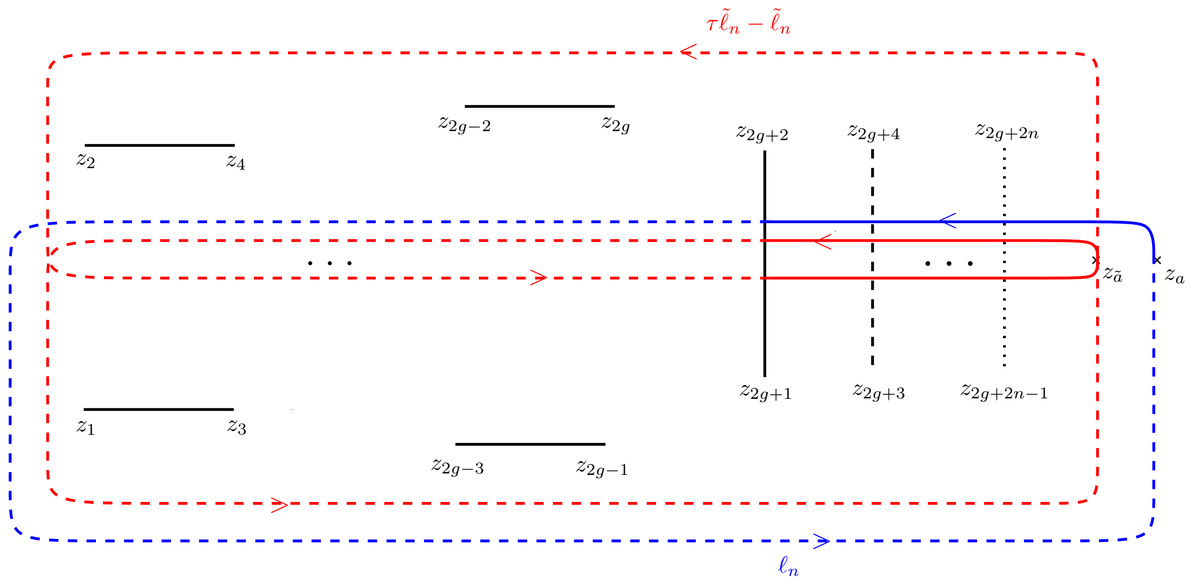

Figure 4: *The closed contour $\tau\tilde{\ell}_n - \tilde{\ell}_n \in H_1\left(\mathcal{R}^+_{g,n+1} \setminus \{a_{n+1}, a_n\}\right)$ is homologous to a closed contour which encircles the vertical cut $[z_{2g+1}, z_{2g+2}]$, then $\alpha_n = (\tau\tilde{\ell}_n - \tilde{\ell}_n) \circ \ell_n = 1$.*

$$\alpha_n = (\tau\tilde{\ell}_n - \tilde{\ell}_n) \circ \ell_n \equiv 1 \pmod 2, \tag{4.27}$$

which leads to $s_n = 1$. Intersection indices α_j for $j = 1, \ldots, n-1$ can be computed in the same way. Therefore

$$\alpha_1 \equiv \alpha_2 \equiv \ldots \equiv \alpha_n \equiv 1 \pmod 2,$$

which implies $s_j = 1$. By Proposition 3.1, smoothness of the solutions is ensured by the reality of the vector $\mathbf{Z} - \mathbf{d}$ and the fact that the curve is dividing. □

Functions ψ_j given in (4.25) describe a family of smooth algebro-geometric solutions of the focusing multi-component NLS equation depending on $g + n$ complex parameters: $z_{2k-1} \in \mathbb{C} \setminus \mathbb{R}$ for $k = 1, \ldots, g+n$; and $g + 2$ real parameters: $z_a, \theta \in \mathbb{R}$, and $\mathbf{d} \in \mathbb{R}^g$.

4.3.2 Solutions of n-NLS$^-$.

Now let us construct solutions of the system n-NLS$^-$

$$\mathrm{i}\,\frac{\partial \psi_j}{\partial t} + \frac{\partial^2 \psi_j}{\partial x^2} - 2\left(\sum_{k=1}^{n} |\psi_k|^2\right)\psi_j = 0, \qquad j = 1, \ldots, n. \tag{4.28}$$

As for the focusing case, let us first describe the covering and the homology basis used in our construction of the solutions of (4.28).

Assume that the branch points z_k of the covering $\mathcal{R}_{g,n+1}$ are real for $k = 1, \ldots, g+2$, and that the branch points z_k, z_{k+1} are pairwise conjugate for $k = 2g+3, \ldots, 2g+2n$. Denote by $\mathcal{R}^-_{g,n+1}$ this covering, refering to the defocusing system (4.28). It is straightforward to see that such a covering is an M-curve (see Appendix A.1), that is it admits a maximal number of real ovals $g+1$ with respect to the anti-holomorphic involution τ. On the other hand, it can be directly seen that $\mathcal{R}^-_{g,n+1}$ is dividing: two points which lie on the sheet $n+1$ and have respectively a positive and a negative imaginary projection onto $\mathbb{C}$ cannot be connected by a contour which does not cross a real oval.

Now let us choose the canonical homology basis such that all basic cycles belong to sheets $n+1$ and n, and which satisfies (A.2). Since the covering $\mathcal{R}^-_{g,n+1}$ is an M-curve, the matrix $\mathbb{H}$ involved in (A.2) satisfies $\mathbb{H} = 0$. Such a canonical homology basis is shown in Figure 5.

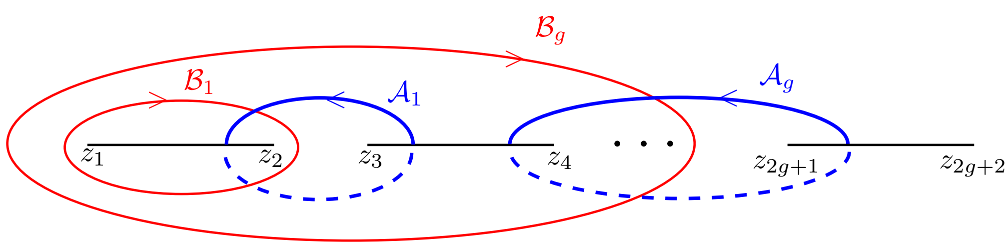

Figure 5: *Homology basis on the covering $\mathcal{R}^-_{g,n+1}$. The solid line indicates the sheet $n+1$, and the dashed line sheet n.*

In the following theorem, we construct algebro-geometric solutions of the defocusing system (4.28) associated to the covering $\mathcal{R}^-_{g,n+1}$.

Theorem 4.3. *Consider the covering $\mathcal{R}^-_{g,n+1}$ and the canonical homology basis discussed above. Fix $z_a \in \mathbb{R} \setminus \{z_1, \ldots, z_{2g+2}\}$ such that $z_a > Re(z_j)$ for $j = 1, \ldots, 2g+2n$. Consider the fiber $f^{-1}(z_a) = \{a_1, \ldots, a_{n+1}\}$ over z_a, where $a_j \in \mathcal{R}^-_{g,n+1}(\mathbb{R})$ belongs to sheet j (each of the a_j is invariant under the involution τ). Let $\mathbf{d} \in \mathbb{R}^g$ and $\theta \in \mathbb{R}$. Then the functions $\{\psi_j\}_{j=1}^n$ in (4.25) are smooth solutions of n-NLS$^-$.*

Proof. Analogously to the focusing case, one has to check that all $s_j = -1$. Since all branch points z_k are real for $k = 1, \ldots, 2g+2$, the intersection index α_n between the closed contour $\tau\tilde{\ell}_n - \tilde{\ell}_n$ and the contour ℓ_n satisfies (see Figure 6)

$$\alpha_n = (\tau\tilde{\ell}_n - \tilde{\ell}_n) \circ \ell_n \equiv 0 \pmod 2, \tag{4.29}$$

which leads to $s_n = -1$. Intersection indices α_j for $j = 1, \ldots, n-1$ can be computed in the same way, and we get

$$\alpha_1 \equiv \alpha_2 \equiv \ldots \equiv \alpha_n \equiv 0 \pmod 2,$$

which implies $s_j = -1$. Smoothness of the solutions is ensured by the reality of the vector $\mathbf{Z} - \mathbf{d}$ and the fact that the curve is dividing. □

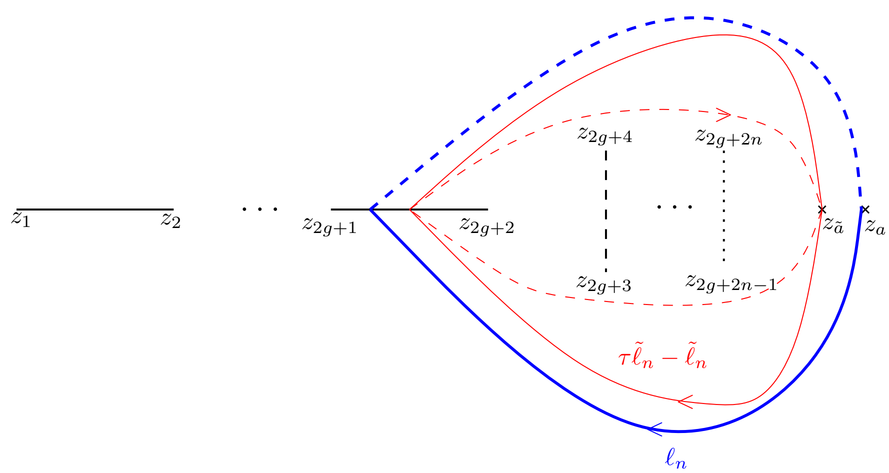

Figure 6: *The closed contour $\tau\tilde{\ell}_n - \tilde{\ell}_n \in H_1\left(\mathcal{R}^-_{g,n+1} \setminus \{a_{n+1}, a_n\}\right)$ is homologous to zero, then $\alpha_n = (\tau\tilde{\ell}_n - \tilde{\ell}_n) \circ \ell_n = 0$.*

Solutions ψ_j constructed here describe a family of smooth algebro-geometric solutions of the defocusing multi-component NLS equation depending on $n-1$ complex parameters: $z_{2g+2+2k-1} \in \mathbb{R}$ for $k = 1, \ldots, n-1$; and $3g+4$ real parameters: $z_k \in \mathbb{R}$ for $k = 1, \ldots, 2g+2$, z_a, $\theta \in \mathbb{R}$, and $\mathbf{d} \in \mathbb{R}^g$.

Remark 4.1. Smooth solutions of n-NLS^s for a vector s with mixed signs can be constructed in the same way.

4.4 Stationary solutions of n-NLS

It is well-known that the algebro-geometric solutions (4.13) on an elliptic surface describe *travelling waves*, i.e., the modulus of the corresponding solutions depends only on $x - ct$, where c is a constant.

Due to the *Galilei invariance* of the multi-component NLS equation (see (4.10)), the invariance under transformations of the form

$$\psi_j(x,t) \longrightarrow \psi_j(x+2\lambda\, t, t)\, \exp\left\{-\mathrm{i}\left(\lambda\, x + \lambda^2\, t\right)\right\},$$

where $\lambda = -\frac{1}{2}\, W_{a_{n+1}}\, (V_{a_{n+1}})^{-1}$, leads to stationary solutions (t-independent) in the transformed coordinates.

For arbitrary genus of the spectral curve, stationary solutions of the multi-component NLS equation are obtained from solutions (4.13) under the vanishing condition

$$\mathbf{W}_{a_{n+1}} = 0. \tag{4.30}$$

This condition is equivalent to the existence of a meromorphic function h of order two on $\mathcal{R}_g$, such that the point a_{n+1} is a critical point of h (this can be proved analogously to Lemma 4.1).

Therefore, stationary solutions of the multi-component NLS can be constructed from the algebro-geometric data $(\mathcal{R}_g, f, h, z_a)$, where:

- $\mathcal{R}_g$ is a real Riemann surface of genus g, and f is a real meromorphic function of order $n+1$ on $\mathcal{R}_g$,
- $z_a \in \mathbb{CP}^1$ is a non critical value of f such that $f^{-1}(z_a) = \{a_1, \ldots, a_{n+1}\}$,
- h is a real meromorphic function of order two on $\mathcal{R}_g$, and a_{n+1} is a critical point of h,
- for $j = 1, \ldots, n$, local parameters k_{a_j} near a_j are chosen to be $k_{a_j}(p) = h(p) - h(a_j)$ for any point p lying in a neighbourhood of a_j, and $k_{a_{n+1}}(p) = (h(p) - h(a_{n+1}))^{1/2}$ for any point p lying in a neighbourhood of a_{n+1}.

With this choice of local parameters, we get $f(p) - z_a = \beta_j\, k_{a_j}(p) + \mu_j\; k_{a_j}(p)^2 + O\left(k_{a_j}(p)^3\right)$, for any point $p \in \mathcal{R}_g$ which lies in a neighbourhood of a_j, where $\beta_j, \mu_j \in \mathbb{R}$. Hence solutions (4.13) can be rewritten using this choice of local parameters and then are expressed by the use of the scalars β_j and μ_j.

Moreover, choosing a_{n+1} as a critical point of h, we get (4.30). In this case, the modulus of solutions (4.13) do not depend on the variable t.

4.5 Reduction of n-NLS to (n-1)-NLS

It is natural to ask if starting from solutions of n-NLS we can obtain solutions of (n-1)-NLS for $n > 2$. Such a reduction is possible if one of the functions ψ_j solutions of n-NLS vanishes identically.

Let $\mathcal{R}^+_{g,n+1}$ be the $(n+1)$-sheeted covering introduced in Section 4.3.1; to obtain solutions of (n-1)-NLS$^+$ from solutions of n-NLS$^+$, we consider the following degeneration of the covering $\mathcal{R}^+_{g,n+1}$: let the branch points z_{2g+2n} and $z_{2g+2n-1}$ coalesce, in such way that the first sheet gets disconnected from the other sheets (see Figure 1); denote by $\mathcal{R}^+_{g,n}$ the covering obtained in this limit.

Then the normalized holomorphic differentials on $\mathcal{R}^+_{g,n+1}$ tend to normalized holomorphic differentials on $\mathcal{R}^+_{g,n}$; on the first sheet, all holomorphic differentials tend to zero. Therefore, in this limit, each component of the vector $\mathbf{V}_{a_1}$ tends to 0.

Hence by (2.12) and (4.14), the function ψ_1 tends to zero as z_{2g+2n} and $z_{2g+2n-1}$ coalesce. Functions $\{\psi_j\}_{j=2}^n$ obtained in this limit are solutions of (n-1)-NLS$^+$ associated to the covering $\mathcal{R}^+_{g,n}$.

A similar degeneration produces a solution of (n-1)-NLS$^-$ from a solution of n-NLS$^-$.

Remark 4.2. Repeating this degeneration $n-3$ times, we rediscover (see [10]) algebro-geometric solutions of the focusing (resp. defocusing) non-linear Schrödinger equation (1.5) associated to an hyperelliptic curve with pairwise conjugate branch points (resp. real branch points).

4.6 Relationship between solutions of KP1 and solutions of n-NLS

Historically, the Korteweg-de Vries equation (KdV) and its generalization to two spatial variables, the Kadomtsev-Petviashvili equations (KP), were the most important examples of applications of methods of algebraic geometry in the 1970's (see e.g. [3]). Moreover, the KP equation is the first example of a system with two space variables for which it has been possible to completely solve the problem of reality of algebro-geometric solutions.

Here we show that starting from our solutions of the multi-component NLS equation and its complexification, we can construct a subclass of complex and real solutions of the Kadomtsev-Petviashvili equation (KP1)

$$\frac{3}{4}\,u_{yy}=\left(u_t-\frac{1}{4}\,(6\,u\,u_x-u_{xxx})\right)_x. \tag{4.31}$$

Let $\mathcal{R}_g$ be an arbitrary Riemann surface with marked point a, and let k_a be an arbitrary local parameter near a. Define vectors $\mathbf{V}_a$, $\mathbf{W}_a$, $\mathbf{U}_a$ as in (2.8) and let $\mathbf{d}\in\mathbb{C}^g$. Then, according to Krichever's theorem [12], the function

$$u(x,y,t)=2\,D_a^2\log\Theta(\mathrm{i}\,\mathbf{V}_a\,x+\mathrm{i}\,\mathbf{W}_a\,y+\mathrm{i}\,\mathbf{U}_a\,t+\mathbf{d})+2\,c \tag{4.32}$$

is a solution of KP1; here the constant c is defined by the expansion near a of the normalized meromorphic differential $\Omega_a^{(2)}(p)$ having a pole of order two at a only: $\Omega_a^{(2)}(p)=(k_a(p))^{-2}+c\,k_a(p)+...$, where p lies in a neighbourhood of a.

Let us check that if the local parameter k_a is defined by the meromorphic function f as $k_a(p)=f(p)-f(a)$, then formula (4.32) naturally arises from our construction of solutions of the n-NLS^s system. Namely, identify a with a_{n+1}. Then, due to the fact that $\sum_{j=1}^{n+1}\mathbf{V}_{a_j}=0$ (see Lemma 4.1), the solution (4.32) of KP1 can be rewritten as

$$u(x,y,t)=-2\sum_{j=1}^{n}D_{a_{n+1}}D_{a_j}\log\Theta(\mathbf{z})+2\,c,$$

where $\mathbf{z}=\mathrm{i}\,\mathbf{V}_{a_{n+1}}\,x+\mathrm{i}\,\mathbf{W}_{a_{n+1}}\,y+\mathrm{i}\,\mathbf{U}_{a_{n+1}}\,t+\mathbf{d}$. Using corollary (2.10) of Fay's identity, we get

$$u(x,y,t)=-2\sum_{j=1}^{n}\left(q_1(a_{n+1},a_j)+q_2(a_{n+1},a_j)\,\frac{\Theta(\mathbf{z}+\mathbf{r}_j)\,\Theta(\mathbf{z}-\mathbf{r}_j)}{\Theta(\mathbf{z})^2}\right)+2\,c. \tag{4.33}$$

Now let us consider solutions ψ_j, ψ_j^* (4.4) of the complexified multi-component NLS equation, and make the change of variables $(x,t)\to(x,y)$ and $\mathbf{d}\to-\mathrm{i}\,\mathbf{U}_{a_{n+1}}\,t+\mathbf{d}$. Then by (4.33), the complex-valued solutions u (4.32) of KP1 and solutions ψ_j, ψ_j^* (4.4) of the complexified n-NLS system are related by

$$u(x,y,t)=\gamma-2\sum_{j=1}^{n}\psi_j(x,y,t)\,\psi_j^*(x,y,t), \tag{4.34}$$

where

$$\gamma=-2\sum_{j=1}^{n}q_1(a_{n+1},a_j)+2\,c.$$

If we impose the reality conditions (4.3), we obtain real solutions (4.32) of KP1 from our solutions (4.25) of n-NLS^s equation

$$u(x,y,t)=\gamma-2\sum_{j=1}^{n}s_j\,|\psi_j(x,y,t)|^2. \tag{4.35}$$

Due to the fact that in our construction of solutions of the multi-component NLS equation, the local parameters are defined by the meromorphic function f, complex solutions (4.34) and real solutions (4.35) of KP1 obtained in this way form only a subclass of Krichever's solutions.

I thank C. Klein, who interested me in the subject, and D. Korotkin for carefully reading the manuscript and providing valuable hints. I am grateful to B. Dubrovin and V. Shramchenko for useful discussions. This work has been supported in part by the project FroM-PDE funded by the European Research Council through the Advanced Investigator Grant Scheme, the Conseil Régional de Bourgogne via a FABER grant and the ANR via the program ANR-09-BLAN-0117-01.

A Real Riemann surfaces

In this section, we recall some facts from the theory of real compact Riemann surfaces. Following [20], we introduce a symplectic basis of cycles on $\mathcal{R}_g$ and study reality properties of various objects on the Riemann surface $\mathcal{R}_g$ associated to this basis.

A.1 Action of τ on the homology group $H_1(\mathcal{R}_g)$

A Riemann surface $\mathcal{R}_g$ is called real if it admits an anti-holomorphic involution $\tau : \mathcal{R}_g \to \mathcal{R}_g$, $\tau^2 = 1$. The connected components of the set of fixed points of the anti-involution τ are called real ovals of τ. We denote by $\mathcal{R}_g(\mathbb{R})$ the set of fixed points. Assume that $\mathcal{R}_g(\mathbb{R})$ consists of k real ovals, with $0 \leq k \leq g+1$. The curves with the maximal number of real ovals, $k = g+1$, are called M-curves.

The complement $\mathcal{R}_g \setminus \mathcal{R}_g(\mathbb{R})$ has either one or two connected components. The curve $\mathcal{R}_g$ is called a *dividing* curve (or that $\mathcal{R}_g$ divides) if $\mathcal{R}_g \setminus \mathcal{R}_g(\mathbb{R})$ has two components, and $\mathcal{R}_g$ is called *non-dividing* if $\mathcal{R}_g \setminus \mathcal{R}_g(\mathbb{R})$ is connected (notice that an M-curve is always a dividing curve).

Example A.1. *Consider the hyperelliptic Riemann surface of genus g defined by the equation*

$$\mu^2 = \prod_{k=1}^{2g+1} (\lambda - \lambda_k), \tag{A.1}$$

where the branch points $\lambda_k \in \mathbb{R}$ are ordered such that $\lambda_1 < \ldots < \lambda_{2g+1}$. On such a Riemann surface, we can define two anti-holomorphic involutions τ_1 and τ_2, given respectively by $\tau_1(\lambda, \mu) = (\overline{\lambda}, \overline{\mu})$ and $\tau_2(\lambda, \mu) = (\overline{\lambda}, -\overline{\mu})$. Projections of real ovals of τ_1 on the λ-plane coincide with the intervals $[\lambda_1, \lambda_2], \ldots, [\lambda_{2g+1}, +\infty]$, and projections of real ovals of τ_2 on the λ-plane coincide with the intervals $[-\infty, \lambda_1], \ldots, [\lambda_{2g}, \lambda_{2g+1}]$. Hence the curve (A.1) is an M-curve with respect to both anti-involutions τ_1 and τ_2.

Denote by $\{\mathcal{A}, \mathcal{B}\}$ the set of generators of the homology group $H_1(\mathcal{R}_g)$, where $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_g)^T$ and $\mathcal{B} = (\mathcal{B}_1, \ldots, \mathcal{B}_g)^T$. According to Proposition 2.2 in [20], there exists a canonical homology basis such that

$$\begin{pmatrix} \tau\mathcal{A} \\ \tau\mathcal{B} \end{pmatrix} = \begin{pmatrix} \mathbb{I}_g & 0 \\ \mathbb{H} & -\mathbb{I}_g \end{pmatrix} \begin{pmatrix} \mathcal{A} \\ \mathcal{B} \end{pmatrix}, \tag{A.2}$$

where $\mathbb{I}_g$ is the $g \times g$ unit matrix, and $\mathbb{H}$ is a $g \times g$ matrix defined as follows

1) if $\mathcal{R}_g(\mathbb{R}) \neq \emptyset$,

$$\mathbb{H} = \begin{pmatrix} 0 & 1 & & & & & & \\ 1 & 0 & & & & & & \\ & & \ddots & & & & & \\ & & & 0 & 1 & & & \\ & & & 1 & 0 & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{pmatrix} \quad \text{if } \mathcal{R}_g(\mathbb{R}) \text{ is dividing,}$$

$$\mathbb{H} = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \quad \text{if } \mathcal{R}_g(\mathbb{R}) \text{ is non-dividing,}$$

($\mathrm{rank}(\mathbb{H}) = g + 1 - k$ in both cases).

2) if $\mathcal{R}_g(\mathbb{R}) = \emptyset$, (i.e. the curve does not have real ovals), then

$$\mathbb{H} = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & 1 & 0 \end{pmatrix} \quad \text{or} \quad \mathbb{H} = \begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & 1 & \\ & & & 1 & 0 & \\ & & & & & 0 \end{pmatrix},$$

($\mathrm{rank}(\mathbb{H}) = g$ if g is even, $\mathrm{rank}(\mathbb{H}) = g - 1$ if g is odd).

Let us choose the homology basis satisfying (A.2), and study the action of τ on the normalized holomorphic differentials, and the action of the complex conjugation on the theta function with zero characteristics.

By (A.2) the $\mathcal{A}$-cycles of the homology basis are invariant under τ. Due to normalization condition (2.1) this leads to the following action of τ on the normalized holomorphic differentials

$$\overline{\tau^* \omega_j} = -\omega_j. \tag{A.3}$$

Using (A.2) and (A.3) we get the following reality property for the matrix $\mathbb{B}$ of $\mathcal{B}$-periods

$$\overline{\mathbb{B}} = \mathbb{B} - 2\mathrm{i}\pi\, \mathbb{H}. \tag{A.4}$$

By Proposition 2.3 in [20], for any $\mathbf{z} \in \mathbb{C}^g$, relation (A.4) implies

$$\overline{\Theta(\mathbf{z})} = \kappa\, \Theta(\overline{\mathbf{z}} - \mathrm{i}\pi\, \mathrm{diag}(\mathbb{H})), \tag{A.5}$$

where $\mathrm{diag}(\mathbb{H})$ denotes the vector of the diagonal elements of the matrix $\mathbb{H}$, and κ is a root of unity which depends on matrix $\mathbb{H}$ (knowledge of the exact value of κ is not needed for our purpose).

A.2 Action of τ on $H_1(\mathcal{R}_g \setminus \{a,b\})$ and $H_1(\mathcal{R}_g, \{a,b\})$

Here, we study the action of τ on the homology group $H_1(\mathcal{R}_g \setminus \{a,b\})$ of the punctured Riemann surface $\mathcal{R}_g \setminus \{a,b\}$, and the action of τ on its dual relative homology group $H_1(\mathcal{R}_g, \{a,b\})$. We consider the case where $\tau a = b$, and the case where $\tau a = a$, $\tau b = b$.

Denote by $\{\mathcal{A}, \mathcal{B}, \ell\}$ the generators of the relative homology group $H_1(\mathcal{R}_g, \{a,b\})$, where ℓ is a contour between a and b which does not intersect the canonical homology basis $\{\mathcal{A}, \mathcal{B}\}$, and denote by $\{\mathcal{A}, \mathcal{B}, \mathcal{S}_b\}$ the generators of the homology group $H_1(\mathcal{R}_g \setminus \{a,b\})$, where $\mathcal{S}_b$ is a positively oriented small contour around b such that $\mathcal{S}_b \circ \ell = 1$.

A.2.1 Case $\tau a = b$

Proposition A.1. *Let us choose the canonical homology basis in $H_1(\mathcal{R}_g)$ satisfying (A.2), and assume that $\tau a = b$. Then*

1. *the action of τ on the generators $\{\mathcal{A}, \mathcal{B}, \ell\}$ of the relative homology group $H_1(\mathcal{R}_g, \{a,b\})$ is given by*
$$\begin{pmatrix} \tau\mathcal{A} \\ \tau\mathcal{B} \\ \tau\ell \end{pmatrix} = \begin{pmatrix} \mathbb{I}_g & 0 & 0 \\ \mathbb{H} & -\mathbb{I}_g & 0 \\ \mathbf{N} & 0 & -1 \end{pmatrix} \begin{pmatrix} \mathcal{A} \\ \mathcal{B} \\ \ell \end{pmatrix}, \tag{A.6}$$
for some $\mathbf{N} \in \mathbb{Z}^g$,

2. *the action of τ on the generators $\{\mathcal{A}, \mathcal{B}, \mathcal{S}_b\}$ of the homology group $H_1(\mathcal{R}_g \setminus \{a,b\})$ is given by*
$$\begin{pmatrix} \tau\mathcal{A} \\ \tau\mathcal{B} \\ \tau\mathcal{S}_b \end{pmatrix} = \begin{pmatrix} \mathbb{I}_g & 0 & 0 \\ \mathbb{H} & -\mathbb{I}_g & \mathbf{N} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{A} \\ \mathcal{B} \\ \mathcal{S}_b \end{pmatrix}, \tag{A.7}$$

where vector $\mathbf{N} \in \mathbb{Z}^g$ is the same as in (A.6).

Proof. The action of τ on $\mathcal{A}$ and $\mathcal{B}$-cycles in (A.6) coincides with the one (A.2) in $H_1(\mathcal{R}_g)$. From (A.2), one sees that any contour in $H_1(\mathcal{R}_g)$ which is invariant under τ is a combination of $\mathcal{A}$-cycles only. In particular, the closed contour $\tau\ell + \ell \in H_1(\mathcal{R}_g)$ can be written as

$$\tau\ell + \ell = \mathbf{N}\mathcal{A}, \tag{A.8}$$

for some $\mathbf{N} \in \mathbb{Z}^g$. This proves (A.6).

Now let us prove (A.7). By (A.2), the cycles $\tau\mathcal{A}$ admit the following decomposition in $H_1(\mathcal{R}_g \setminus \{a,b\})$:

$$\tau\mathcal{A} = \mathcal{A} + \mathbf{n}\,\mathcal{S}_b, \tag{A.9}$$

for some $\mathbf{n} \in \mathbb{Z}^g$. Since τ changes the orientation of $\mathcal{R}_g$, all intersection indices change their sign under the action of τ. We get from (A.9)

$$\begin{aligned} 0 &= \mathcal{A} \circ \ell \\ &= -\tau\mathcal{A} \circ \tau\ell \\ &= -(\mathcal{A} + \mathbf{n}\,\mathcal{S}_b) \circ \tau\ell \\ &= -(\mathcal{A} + \mathbf{n}\,\mathcal{S}_b) \circ (-\ell + \mathbf{N}\mathcal{A}), \end{aligned} \tag{A.10}$$

where $\mathbf{N} \in \mathbb{Z}^g$ is defined by (A.6). The last intersection index in (A.10) equals $\mathbf{n}$, which implies $\tau\mathcal{A} = \mathcal{A}$. According to (A.2), the action of τ on $\mathcal{B}$-cycles in $H_1(\mathcal{R}_g \setminus \{a, b\})$ is given by

$$\tau\mathcal{B} = -\mathcal{B} + \mathbb{H}\mathcal{A} + \mathbf{m}\,\mathcal{S}_b, \tag{A.11}$$

for some $\mathbf{m} \in \mathbb{Z}^g$. Then

$$\begin{aligned} 0 &= \mathcal{B} \circ \ell \\ &= -\tau\mathcal{B} \circ \tau\ell \\ &= -(-\mathcal{B} + \mathbb{H}\mathcal{A} + \mathbf{m}\,\mathcal{S}_b) \circ \tau\ell \\ &= -(-\mathcal{B} + \mathbb{H}\mathcal{A} + \mathbf{m}\,\mathcal{S}_b) \circ (-\ell + \mathbf{N}\mathcal{A}), \end{aligned} \tag{A.12}$$

where $\mathbf{N}$ is defined by (A.6). The last intersection index in (A.12) equals $\mathbf{m} - \mathbf{N}$, which gives $\tau\mathcal{B} = -\mathcal{B} + \mathbb{H}\mathcal{A} + \mathbf{N}\,\mathcal{S}_b$. Finally, to prove that $\tau\mathcal{S}_b = \mathcal{S}_b$, we use the relation $\mathcal{S}_a + \mathcal{S}_b = 0$, where $\mathcal{S}_a$ is a positively oriented small contour around a, and the relation $\tau\mathcal{S}_b = -\mathcal{S}_a$. □

A.2.2 Case $\tau a = a$ and $\tau b = b$

Proposition A.2. *Let us choose the canonical homology basis in $H_1(\mathcal{R}_g)$ satisfying (A.2), and assume that $\tau a = a$ and $\tau b = b$. Then*

1. *the action of τ on the generators $\{\mathcal{A}, \mathcal{B}, \ell\}$ of the relative homology group $H_1(\mathcal{R}_g, \{a, b\})$ is given by*
$$\begin{pmatrix} \tau\mathcal{A} \\ \tau\mathcal{B} \\ \tau\ell \end{pmatrix} = \begin{pmatrix} \mathbb{I}_g & 0 & 0 \\ \mathbb{H} & -\mathbb{I}_g & 0 \\ \mathbf{N} & \mathbf{M} & 1 \end{pmatrix} \begin{pmatrix} \mathcal{A} \\ \mathcal{B} \\ \ell \end{pmatrix}, \tag{A.13}$$
where $\mathbf{N}$, $\mathbf{M} \in \mathbb{Z}^g$ are related by
$$2\,\mathbf{N} + \mathbb{H}\mathbf{M} = 0, \tag{A.14}$$

2. *the action of τ on the generators $\{\mathcal{A}, \mathcal{B}, \mathcal{S}_b\}$ of the homology group $H_1(\mathcal{R}_g \setminus \{a, b\})$ is given by*
$$\begin{pmatrix} \tau\mathcal{A} \\ \tau\mathcal{B} \\ \tau\mathcal{S}_b \end{pmatrix} = \begin{pmatrix} \mathbb{I}_g & 0 & -\mathbf{M} \\ \mathbb{H} & -\mathbb{I}_g & \mathbf{N} \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \mathcal{A} \\ \mathcal{B} \\ \mathcal{S}_b \end{pmatrix}, \tag{A.15}$$

where vectors $\mathbf{N}$, $\mathbf{M} \in \mathbb{Z}^g$ are the same as in (A.13).

Proof. The action of τ on $\mathcal{A}$ and $\mathcal{B}$-cycles in (A.13) coincides with the one (A.2) in $H_1(\mathcal{R}_g)$. From (A.2), one sees that each contour $\mathcal{C} \in H_1(\mathcal{R}_g)$ which satisfies $\tau\mathcal{C} = -\mathcal{C}$, can be represented by

$$\mathcal{C} = \tilde{\mathbf{N}}\mathcal{A} + \tilde{\mathbf{M}}\mathcal{B}, \tag{A.16}$$

where $\tilde{\mathbf{N}}$, $\tilde{\mathbf{M}} \in \mathbb{Z}^g$ are related by $2\,\tilde{\mathbf{N}} + \mathbb{H}\tilde{\mathbf{M}} = 0$. In particular, the closed contour $\tau\ell - \ell \in H_1(\mathcal{R}_g, \{a, b\})$ can be written as

$$\tau\ell - \ell = \mathbf{N}\mathcal{A} + \mathbf{M}\mathcal{B}, \tag{A.17}$$

where $\mathbf{N}$, $\mathbf{M} \in \mathbb{Z}^g$ are related by $2\,\mathbf{N} + \mathbb{H}\mathbf{M} = 0$. This proves (A.13).

Now let us prove (A.15). By (A.2), the cycles $\tau\mathcal{A}$ admit the following decomposition in $H_1(\mathcal{R}_g \setminus \{a,b\})$

$$\tau\mathcal{A} = \mathcal{A} + \mathbf{n}\,\mathcal{S}_b, \tag{A.18}$$

for some $\mathbf{n} \in \mathbb{Z}^g$. Therefore, we get from (A.18)

$$\begin{aligned} 0 &= \mathcal{A} \circ \ell \\ &= -\tau\mathcal{A} \circ \tau\ell \\ &= -(\mathcal{A} + \mathbf{n}\,\mathcal{S}_b) \circ \tau\ell \\ &= -(\mathcal{A} + \mathbf{n}\,\mathcal{S}_b) \circ (\ell + \mathbf{N}\mathcal{A} + \mathbf{M}\mathcal{B}), \end{aligned} \tag{A.19}$$

where $\mathbf{N}$, $\mathbf{M} \in \mathbb{Z}^g$ is defined by (A.13). The last intersection index in (A.19) equals $-(\mathbf{n}+\mathbf{M})$, which gives $\tau\mathcal{A} = \mathcal{A} - \mathbf{M}\,\mathcal{S}_b$. According to (A.2), the action of τ on $\mathcal{B}$-cycles in $H_1(\mathcal{R}_g \setminus \{a,b\})$ is given by

$$\tau\mathcal{B} = -\mathcal{B} + \mathbb{H}\mathcal{A} + \mathbf{m}\,\mathcal{S}_b, \tag{A.20}$$

for some $\mathbf{m} \in \mathbb{Z}^g$. Then

$$\begin{aligned} 0 &= \mathcal{B} \circ \ell \\ &= -\tau\mathcal{B} \circ \tau\ell \\ &= -(-\mathcal{B} + \mathbb{H}\mathcal{A} + \mathbf{m}\,\mathcal{S}_b) \circ \tau\ell \\ &= -(-\mathcal{B} + \mathbb{H}\mathcal{A} + \mathbf{m}\,\mathcal{S}_b) \circ (\ell + \mathbf{N}\mathcal{A} + \mathbf{M}\mathcal{B}), \end{aligned} \tag{A.21}$$

where $\mathbf{N}$, $\mathbf{M} \in \mathbb{Z}^g$ are defined by (A.13). The last intersection index in (A.21) equals $-(\mathbf{m}+\mathbf{N}+\mathbb{H}\mathbf{M})$, which by (A.14) implies $\tau\mathcal{B} = -\mathcal{B}+\mathbb{H}\mathcal{A}+\mathbf{N}\,\mathcal{S}_b$. Finally, since the anti-holomorphic involution τ inverses orientation we have $\tau\mathcal{S}_b = -\mathcal{S}_b$. This completes the proof of Proposition A.1. □

A.3 Action of τ on the Jacobian and theta divisor of real Riemann surfaces

In this part, we review known results [20], [6] about the theta divisor of real Riemann surfaces. Let us choose the canonical homology basis satisfying (A.2) and consider the Jacobian $J = J(\mathcal{R}_g)$ of the real Riemann surface $\mathcal{R}_g$. The Abel map (2.5) $\mu : \mathcal{R}_g \longmapsto J$ can be extended linearly to all divisors on $\mathcal{R}_g$, which defines a map on linear equivalence classes of divisors.

The anti-holomorphic involution τ on $\mathcal{R}_g$ gives rise to an anti-holomorphic involution on the Jacobian J: if $\mathcal{D}$ is a positive divisor of degree n on $\mathcal{R}_g$, then $\tau\,\mathcal{D}$ is the class of the point $(\int_{n\,\tau p_0}^{\tau\mathcal{D}} \omega) = (\int_{n\,p_0}^{\mathcal{D}} \tau^*\omega)$ in the Jacobian. Therefore by (A.3), τ lifts to the anti-holomorphic involution on J, denoted also by τ, given by

$$\tau\zeta = -\overline{\zeta} + n_\zeta\,\mu(\tau p_0), \quad \forall \zeta \in J, \tag{A.22}$$

where $n_\zeta \in \mathbb{Z}$, $n_\zeta \leq g$, is the degree of the divisor $\mathcal{D}$ such that $\mu(\mathcal{D}) = \zeta$.

Now consider the following two subsets of the Jacobian

$$S_1 = \{\zeta \in J;\, \zeta + \tau\,\zeta = \mathrm{i}\pi\,\mathrm{diag}(\mathbb{H})\}, \tag{A.23}$$

$$S_2 = \{\zeta \in J;\, \zeta - \tau\,\zeta = \mathrm{i}\pi\,\mathrm{diag}(\mathbb{H})\}. \tag{A.24}$$

In this section we study their intersections $S_1 \cap (\Theta)$ and $S_2 \cap (\Theta)$ with the theta divisor (Θ), the set of zeros of the theta function.

Let us introduce the following notations: $(e_i)_k = \delta_{ik}$, $\mathbb{B}_i = \mathbb{B}\,e_i$. The following proposition was proved in [20].

Proposition A.3. *The set S_1 is a disjoint union of the tori T_v defined by*

$$T_v = \{\zeta \in J;\, \zeta = 2i\pi\, (\tfrac{1}{4}\, diag(\mathbb{H}) + \tfrac{v_1}{2}\, e_{r+1} + \ldots + \tfrac{v_{g-r}}{2}\, e_g) + \beta_1\, Re(\mathbb{B}_1) + \ldots + \beta_g\, Re(\mathbb{B}_g)\,,\\ \beta_1, \ldots, \beta_r \in \mathbb{R}/2\mathbb{Z}\,,\ \beta_{r+1}, \ldots, \beta_g \in \mathbb{R}/\mathbb{Z}\}, \qquad (A.25)$$

where $v = (v_1, \ldots, v_{g-r}) \in (\mathbb{Z}/2\mathbb{Z})^{g-r}$ and r is the rank of the matrix $\mathbb{H}$. Moreover, if $\mathcal{R}_g(\mathbb{R}) \neq \emptyset$, then $T_v \cap (\Theta) = \emptyset$ if and only if the curve is dividing and $v = 0$.

The last statement means that among all curves which admit real ovals, the only torus T_v which does not intersect the theta-divisor is the torus T_0 corresponding to dividing curves. This torus is given by

$$T_0 = \{\zeta \in J;\, \zeta = \beta_1\, \mathrm{Re}(\mathbb{B}_1) + \ldots + \beta_g\, \mathrm{Re}(\mathbb{B}_g),\ \beta_1, \ldots, \beta_r \in \mathbb{R}/2\mathbb{Z}\,,\ \beta_{r+1}, \ldots, \beta_g \in \mathbb{R}/\mathbb{Z}\}. \qquad (A.26)$$

The following proposition was proved in [6].

Proposition A.4. *The set S_2 is a disjoint union of the tori $\tilde{T}_v$ defined by*

$$\tilde{T}_v = \{\zeta \in J\,;\, \zeta = 2i\pi\, (\alpha_1\, e_1 + \ldots + \alpha_g\, e_g) + \tfrac{v_1}{2}\, \mathbb{B}_{r+1} + \ldots + \tfrac{v_{g-r}}{2}\, \mathbb{B}_g, \alpha_1, \ldots, \alpha_g \in \mathbb{R}/\mathbb{Z}\}, \qquad (A.27)$$

where $v = (v_1, \ldots, v_{g-r}) \in (\mathbb{Z}/2\mathbb{Z})^{g-r}$ and r is the rank of the matrix $\mathbb{H}$. Moreover, if $\mathcal{R}_g(\mathbb{R}) \neq \emptyset$, then $\tilde{T}_v \cap (\Theta) = \emptyset$ if and only if the curve is an M-curve and $v = 0$.

B Computation of the argument of the fundamental scalar $q_2(a, b)$

This section is devoted to the computation of $\arg\{q_2(a, b)\}$, where $q_2(a, b)$ is defined by (2.12). As before, $\mathcal{R}_g$ denotes a real compact Riemann surface of genus g with an anti-holomorphic involution τ. The argument of $q_2(a, b)$ is computed both in the case $\tau a = b$, as well as in the case $\tau a = a$, $\tau b = b$.

B.1 Integral representation for $q_2(a, b)$

Assume that $a, b \in \mathcal{R}_g$ can be connected by a contour which does not intersect basic cycles. Hence we can define the normalized meromorphic differential of the third kind Ω_{b-a} which has residue 1 at b and residue -1 at a.

Proposition B.1. *Let a, b be distinct points on a compact Riemann surface $\mathcal{R}_g$ of genus g. Denote by k_a and k_b local parameters in a neighbourhood of a and b respectively. Then the quantity $q_2(a, b)$ defined in (2.12) admits the following integral representation*

$$q_2(a, b) = -\lim_{\substack{\tilde{b} \to b \\ \tilde{a} \to a}} \left[\Big(k_a(\tilde{a})\, k_b(\tilde{b}) \Big)^{-1} \exp\left\{ \int_{\tilde{a}}^{\tilde{b}} \Omega_{b-a}(p) \right\} \right], \qquad (B.1)$$

where the integration contour between $\tilde{a}$ and $\tilde{b}$, which in the sequel is denoted by $\tilde{\ell}$, does not cross any cycle from the canonical homology basis.

Proof. Notice that the scalar $q_2(a,b)$ does not depend on the choice of the contour $\tilde{\ell}$, assuming that $\tilde{\ell}$ lies in the fundamental polygon of the Riemann surface.

Denote by k_x a local parameter in a neighbourhood of a point $x \in \mathcal{R}_g$. To prove (B.1), recall that

$$\int_{\tilde{a}}^{\tilde{b}} \Omega_{b-a}(p) = \ln \frac{\Theta[\delta](\int_b^{\tilde{b}})}{\Theta[\delta](\int_a^{\tilde{b}})} + \ln \frac{\Theta[\delta](\int_a^{\tilde{a}})}{\Theta[\delta](\int_b^{\tilde{a}})}. \tag{B.2}$$

Since δ is an odd non singular characteristic, the expression $\frac{\Theta[\delta](\int_b^p)}{\Theta[\delta](\int_a^p)}$ has a simple zero at b and a simple pole at a. Therefore, if we consider $\tilde{a}$ lying in a neighbourhood of a, and $\tilde{b}$ lying in a neighbourhood of b, we get (with $\alpha_1, \beta_1 \neq 0$)

$$\frac{\Theta[\delta](\int_b^{\tilde{b}})}{\Theta[\delta](\int_a^{\tilde{b}})} = \alpha_1\, k_b(\tilde{b}) + o(k_b(\tilde{b})), \tag{B.3}$$

$$\frac{\Theta[\delta](\int_a^{\tilde{a}})}{\Theta[\delta](\int_b^{\tilde{a}})} = \beta_1\, k_a(\tilde{a}) + o(k_a(\tilde{a})). \tag{B.4}$$

Combining (B.2) together with (B.3) and (B.4), we obtain the following relation

$$\lim_{\substack{\tilde{b} \to b \\ \tilde{a} \to a}} \left[\left(k_a(\tilde{a})\, k_b(\tilde{b}) \right)^{-1} \exp\left\{ \int_{\tilde{a}}^{\tilde{b}} \Omega_{b-a}(p) \right\} \right] = \alpha_1 \beta_1. \tag{B.5}$$

Moreover, using the definition (2.12) of $q_2(a,b)$, it follows from (B.3) and (B.4) that $\alpha_1\beta_1 = -q_2(a,b)$, which by (B.5) completes the proof. □

B.2 Argument of $q_2(a,b)$ when $\tau a = b$

Here we compute the argument of the fundamental scalar $q_2(a,b)$ defined in (2.12) in the case where $\tau a = b$. Let us choose the homology basis satisfying (A.2).

Proposition B.2. *Let $a, b \in \mathcal{R}_g$ be distinct points such that $\tau a = b$, with local parameters satisfying the relation $\overline{k_b(\tau p)} = k_a(p)$ for any point p lying in a neighbourhood of a. Consider a contour ℓ connecting points a and b; assume that ℓ is lying in the fundamental polygon of the Riemann surface $\mathcal{R}_g$. Then the scalar $q_2(a,b)$ is real, and its sign is given by:*

1. *if ℓ intersects the set of real ovals of $\mathcal{R}_g$ only once, and if this intersection is transversal, then $q_2(a,b) < 0$,*

2. *if ℓ does not cross any real oval, then $q_2(a,b) > 0$.*

Proof. Let $\tilde{a}, \tilde{b} \in \mathcal{R}_g$ lie in a neighbourhood of a and b respectively, and $\tau\tilde{a} = \tilde{b}$. Denote by $\tilde{\ell}$ an oriented contour connecting $\tilde{a}$ and $\tilde{b}$. First, let us check that

$$\arg\{q_2(a,b)\} = \pi(1+\alpha), \tag{B.6}$$

where $\alpha = (\tau\tilde{\ell} + \tilde{\ell}) \circ \ell$. The integral representation (B.1) of $q_2(a,b)$ leads to

$$\arg\{q_2(a,b)\} = \pi + \mathrm{Im}\left(\int_{\tilde{\ell}} \Omega_{b-a}(p)\right). \tag{B.7}$$

Using the action (A.6) of τ on the $\mathcal{A}$-cycles in the homology group $H_1(\mathcal{R}_g \setminus \{a,b\})$, we get the following action of τ on the normalized meromorphic differentials of third kind Ω_{b-a}:

$$\overline{\tau^*\Omega_{b-a}} = -\Omega_{b-a}, \tag{B.8}$$

(notice that $\tau a = b$). Hence, the last term in the right hand side of (B.7) is equal to $\frac{1}{2\mathrm{i}} \int_{\tau\tilde{\ell}+\tilde{\ell}} \Omega_{b-a}(p)$. The closed contour $\tau\tilde{\ell} + \tilde{\ell}$ admits the following decomposition in $H_1(\mathcal{R}_g \setminus \{a,b\})$,

$$\tau\tilde{\ell} + \tilde{\ell} = \mathbf{N}\mathcal{A} + \alpha\,\mathcal{S}_b, \tag{B.9}$$

where $\alpha = (\tau\tilde{\ell} + \tilde{\ell}) \circ \ell$ and $\mathbf{N} \in \mathbb{Z}^g$ is defined in (A.13). Since the differential Ω_{b-a} has vanishing $\mathcal{A}$-periods, by (B.9) we obtain

$$\int_{\tau\tilde{\ell}+\tilde{\ell}} \Omega_{b-a}(p) = 2\mathrm{i}\pi\alpha, \tag{B.10}$$

which leads to (B.6). Therefore, the sign of $q_2(a,b)$ depends on the parity of the intersection index $\alpha = (\tau\tilde{\ell} + \tilde{\ell}) \circ \ell$.

Let us now consider cases (1) and (2) separatly.

Case (1). Assume that each of the contours ℓ and $\tilde{\ell}$ intersects the set of real ovals of $\mathcal{R}_g$ transversally only once, and, moreover, this intersection point is the same for ℓ and $\tilde{\ell}$; we denote it by $p_0 \in \mathcal{R}_g(\mathbb{R})$. Then the closed contour $\tau\tilde{\ell} + \tilde{\ell}$ can be decomposed into a sum of two closed contours $c\tilde{\ell}_1$ and $c\tilde{\ell}_2$, having the common point p_0, and such that τ sends the set of points $\left\{c\tilde{\ell}_1\right\}$ into the set of points $\left\{c\tilde{\ell}_2\right\}$. Therefore, if the orientation of $c\tilde{\ell}_1$ and $c\tilde{\ell}_2$ is inherited from the orientation of $\tau\tilde{\ell} + \tilde{\ell}$, we have $\tau c\tilde{\ell}_1 = c\tilde{\ell}_2$ as elements of $H_1(\mathcal{R}_g \setminus \{a,b\})$. Then,

$$c\tilde{\ell}_1 \circ \ell = -\tau c\tilde{\ell}_1 \circ \tau\ell = -c\tilde{\ell}_2 \circ (-\ell + \mathcal{A}\mathbf{N}) = c\tilde{\ell}_2 \circ \ell,$$

where we used the action (A.6) of τ on the contour ℓ, and the fact that the intersection index between $c\tilde{\ell}_2$ and $\mathcal{A}$-cycles is zero by (B.9). Hence the intersection index α satisfies

$$\alpha = (\tau\tilde{\ell} + \tilde{\ell}) \circ \ell = (c\tilde{\ell}_1 + c\tilde{\ell}_2) \circ \ell = 2,$$

which by (B.6) leads to $q_2(a,b) < 0$.

Case (2). Let $\mathcal{V}$ be a ring neighbourhood of the path $\tau\tilde{\ell} + \tilde{\ell}$, bounded by two closed paths denoted by $\partial\mathcal{V}_1$ and $\partial\mathcal{V}_2$, in such way that the path ℓ lies in $\mathcal{V}$ and $\tau\{\partial\mathcal{V}_1\} = \{\partial\mathcal{V}_2\}$. We assume that $\mathcal{V}$ is chosen such that no point of $\mathcal{V}$ is invariant under τ. Then $\mathcal{V}$ can be decomposed into two connected components denoted by $\mathcal{V}_1$ and $\mathcal{V}_2$ as follows: $\mathcal{V}_1$ is bounded by $\partial\mathcal{V}_1$ and $\tau\tilde{\ell} + \tilde{\ell}$, and $\mathcal{V}_2$ is bounded by $\partial\mathcal{V}_2$ and $\tau\tilde{\ell} + \tilde{\ell}$. Then $\tau\mathcal{V}_1 = \mathcal{V}_2$ since the set of points $\left\{\tau\tilde{\ell} + \tilde{\ell}\right\}$ is invariant under τ. In particular if $a \in \mathcal{V}_1$, then $b \in \mathcal{V}_2$. Thus the intersection index $\alpha = (\tau\tilde{\ell} + \tilde{\ell}) \circ \ell$ is odd, which leads to $q_2(a,b) > 0$. $\square$

B.3 Argument of $q_2(a,b)$ when $\tau a = a$ and $\tau b = b$

Now let us consider the case where a and b are invariant with respect to τ.

Proposition B.3. *Let $a, b \in \mathcal{R}_g(\mathbb{R})$ with local parameters satisfying $\overline{k_a(\tau p)} = k_a(p)$ for any point p lying in a neighbourhood of a and $\overline{k_b(\tau p)} = k_b(p)$ for any point p lying in a neighbourhood of b. Denote by $\{\mathcal{A}, \mathcal{B}, \ell\}$ the generators of the relative homology group $H_1(\mathcal{R}_g, \{a,b\})$ (see Section A.2). Let $\tilde{a}, \tilde{b} \in \mathcal{R}_g(\mathbb{R})$ lie in a neighbourhood of a and b respectively, and denote by $\tilde{\ell}$ an oriented contour connecting $\tilde{a}$ and $\tilde{b}$. Then the argument of the scalar $q_2(a,b)$ is given by*

$$\arg\{q_2(a,b)\} = \arg\{k_a(\tilde{a})\, k_b(\tilde{b})\} + \pi \left(1 + \alpha + \frac{1}{2}\langle \mathbb{H}\mathbf{M}, \mathbf{M}\rangle\right) - \frac{1}{2\mathrm{i}}\left(\langle \mathbb{B}\,\mathbf{M}, \mathbf{M}\rangle + 2\,\langle \mathbf{r}, \mathbf{M}\rangle\right), \qquad \text{(B.11)}$$

where α equals the intersection index $(\tau\tilde{\ell} - \tau\tilde{\ell}) \circ \ell$. Here $\mathbf{r} = \int_\ell \omega$, and $\mathbf{M} \in \mathbb{Z}^g$ is defined in (A.13).

Proof. From the integral representation (B.1) of $q_2(a,b)$ we get

$$\arg\{q_2(a,b)\} = \pi + \arg\{k_a(\tilde{a})\, k_b(\tilde{b})\} + \mathrm{Im}\left(\int_{\tilde{\ell}} \Omega_{b-a}(p)\right). \qquad \text{(B.12)}$$

Considering the action (A.15) of τ on the $\mathcal{A}$-cycles, due to the uniqueness of the normalized differential of the third kind Ω_{b-a}, we obtain

$$\overline{\tau^*\Omega_{b-a}} = \Omega_{b-a} + \sum_k M_k\, \omega_k, \qquad \text{(B.13)}$$

where ω_k are the normalized holomorphic differentials. Therefore

$$\mathrm{Im}\left(\int_{\tilde{\ell}} \Omega_{b-a}(p)\right) \equiv \frac{1}{2\mathrm{i}}\left(\int_{\tilde{\ell}} \Omega_{b-a} - \int_{\tau\tilde{\ell}} \Omega_{b-a} - \sum_k M_k \int_{\tau\tilde{\ell}} \omega_k\right).$$

The closed contour $\tau\tilde{\ell} - \tilde{\ell} \in H_1(\mathcal{R}_g)$ satisfies $\tau(\tau\tilde{\ell} - \tilde{\ell}) = -(\tau\tilde{\ell} - \tilde{\ell})$; thus by (A.16) it has the following decomposition in $H_1(\mathcal{R}_g \setminus \{a,b\})$

$$\tau\tilde{\ell} - \tilde{\ell} = \mathbf{N}\mathcal{A} + \mathbf{M}\mathcal{B} + \alpha\, \mathcal{S}_b, \qquad \text{(B.14)}$$

for some $\alpha \in \mathbb{Z}$, where $\mathbf{N}$, $\mathbf{M} \in \mathbb{Z}^g$ are defined in (A.13). Hence we get

$$\mathrm{Im}\left(\int_{\tilde{\ell}} \Omega_{b-a}(p)\right) \equiv \frac{1}{2\mathrm{i}}\left(-\int_{\mathcal{B}\mathbf{M}} \Omega_{b-a} + 2\mathrm{i}\pi\alpha - \sum_k M_k \left\{\int_{\tilde{\ell}} \omega_k + \sum_j (\mathbb{B}_{jk} - \mathrm{i}\pi\, \mathbb{H}_{jk}) M_j\right\}\right), \qquad \text{(B.15)}$$

where we used the fact that the normalized differential Ω_{b-a} has vanishing $\mathcal{A}$-periods, and that the integral over the small contour $\mathcal{S}_b$ of the holomorphic differentials is zero. Since by definition the contour ℓ does not cross any cycles of the absolute homology basis,

$$\int_{\mathcal{B}\mathbf{M}} \Omega_{b-a} = \langle \mathbf{M}, \mathbf{r}\rangle. \qquad \text{(B.16)}$$

Hence we get

$$\mathrm{Im}\left(\int_{\tilde{\ell}} \Omega_{b-a}(p)\right) \equiv \pi\alpha + \frac{\pi}{2}\langle \mathbb{H}\mathbf{M}, \mathbf{M}\rangle - \frac{1}{2\mathrm{i}}\left(\langle \mathbf{M}, \tilde{\mathbf{r}} + \mathbf{r}\rangle + \langle \mathbb{B}\mathbf{M}, \mathbf{M}\rangle\right), \qquad \text{(B.17)}$$

where $\tilde{\mathbf{r}} = \int_{\tilde{\ell}} \omega$. Considering the limit when $\tilde{a}$ tends to a and $\tilde{b}$ tends to b, we obtain (B.11). □

References

[1] D. Anker and N. C. Freeman, *On the Soliton Solutions of the Davey-Stewartson Equation for Long Waves*, Proc. R. Soc. London A **360** 529 (1978).

[2] E. Arbarello, *Fay's trisecant formula and a characterization of Jacobian Varieties*, Proceedings of Symposia in Pure Mathematics Vol. **46** (1987).

[3] E. Belokolos, A. Bobenko, V. Enolskii, A. Its, V. Matveev, *Algebro-geometric approach to nonlinear integrable equations*, Springer Series in nonlinear dynamics (1994).

[4] A. Bobenko, *Introduction to Compact Riemann Surfaces*, In Bobenko, A.I., and Klein, C. (ed.), 'Computational Approach to Riemann Surfaces', Lect. Notes Math. **2013** (2011).

[5] A. Davey and K. Stewartson, *On three-dimensional packets of surface waves*, Proc. R. Soc. Lond. A **388**, 101–110 (1974).

[6] B. Dubrovin, S. Natanzon, *Real theta-function solutions of the Kadomtsev- Petviashvili equation*, Math. USSR Irvestiya, **32**:2, 269–288 (1989).

[7] D. Eisenbud, N. Elkies, J. Harris, R. Speiser, *On the Hurwitz scheme and its monodromy,* Compositio Mathematica **77** No.1, 95-117 (1991).

[8] J. Elgin, V. Enolski, A. Its, *Effective integration of the nonlinear vector Schrödinger equation*, Physica D **225** (22), 127–152 (2007).

[9] J. Fay, *Theta functions on Rieman surfaces*, Lecture Notes in Mathematics **352** (1973).

[10] A. R. Its, *Inversion of hyperelliptic integrals and integration of nonlinear differential equations*, Vestn. Leningr. Gos. Univ. **7**, No. 2, 37–46 (1976).

[11] C. Klein, D. Korotkin, and V. Shramchenko, *Ernst equation, Fay identities and variational formulas on hyperelliptic curves*, Math. Res. Lett. **9**, 1–20 (2002).

[12] I. M. Krichever, *Algebro-geometric construction of the Zaharov-Shabat equations and their periodic solutions*, Sov. Math. Dokl., **17**, 394–397, (1976).

[13] T. Malanyuk, *Finite-gap solutions of the Davey-Stewartson equations*, J. Nonlinear Sci, **4**, No. 1, 1–21 (1994).

[14] S. Manakov, *On the theory of two-dimensional stationary self-focusing of electromagnetic waves*, Sov. Phys.ŨJETP **38**, 248 (1974).

[15] S. McCullough, *The trisecant identity and operator theory*, Integral Equations Operator Theory **25**, pp. 104–128 (1996).

[16] D. Mumford, *Tata Lectures on Theta. I and II.*, Progress in Mathematics, 28 and 43, respectively. Birkhäuser Boston, Inc., Boston, MA, 1983 and 1984.

[17] E. Previato, *Hyperelliptic quasi-periodic and soliton solutions of the nonlinear Schrödinger equation*, Duke Math. J. **52**, 329–377 (1985).

[18] R. Radhakrishnan, R. Sahadevan and M. Lakshmanan, *Integrability and singularity structure of coupled nonlinear Schrödinger equations*, Chaos, Solitons and Fractals **5**, No. 12, 2315–2327 (1995).

[19] A. Raina, *Fay's Trisecant Identity and Conformal Field Theory*, Commun. Math. Phys. **122**, 625 (1989).

[20] V. Vinnikov, *Self-adjoint determinantal representions of real plane curves*, Math. Ann. **296**, 453–479 (1993).

[21] V. Zakharov, A. Shabat, *Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media*, Soy. Phys. JETP **34**, 62–69 (1972).