

# Variational Harmonic Method for Parameterization of Computational Domain in 2D Isogeometric Analysis

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## Abstract

*In isogeometric analysis, parameterization of computational domain has great effects as mesh generation in finite element analysis. In this paper, based on the concept of harmonic map from the computational domain to parametric domain, a variational approach is proposed to construct the parameterization of computational domain for 2D isogeometric analysis. Different from the previous elliptic mesh generation method in finite element analysis, the proposed method focus on isogeometric version, and converts the elliptic PDE into a nonlinear optimization problem. A regular term is integrated into the optimization formulation to achieve more uniform grid near convex (concave) parts of the boundary. Several examples are presented to show the efficiency of the proposed method.*

## 1. Introduction

The isogeometric analysis (IGA for short) method proposed by Hughes et al. in [19] can be employed to overcome the gap between CAD and CAE. The approach uses the same type of mathematical representation (spline representation), both for the geometry and for the physical solutions, and thus avoid this costly forth and back transformations. Moreover it reduces the number of parameters needed to describe the geometry, which is of particular interest for shape optimization.

Mesh generation, which generates a discrete mesh of a computational domain from a given CAD object, is a key and the most time-consuming step in finite element analysis (FEA for short). It consumes about 80% of the overall design and analysis process [6] in automotive, aerospace and ship industry. Parameterization of computational domain in IGA, which corresponds to the mesh generation in FEA, also has some impact on analysis result and efficiency. In particular, arbitrary refinements can be performed on the computational mesh in FEA, but in IGA if we compute with tensor product B-splines, we can only perform refinement operations in  $u$  direction and  $v$  direction by knot insertion or degree

elevation. Hence, parameterization of computational domain is also being important for IGA.

In IGA, the parameterization of a computational domain is determined by control points, knot vectors and the degrees of B-spline objects. For IGA problem of two dimensions, the knot vectors and the degree of the computational domain are determined by the given boundary curves. That is, given boundary curves, the quality of parameterization of computational domain is determined by the positions of inner control points. Hence, finding a good placement of the inner control points inside the computational domain, is a key issue. A basic requirement of the resulting parameterization for IGA is that it doesn't have self-intersections, so that it is an injective map from the parametrization domain to the computational domain.

In the field of mesh generation in FEA, a general method is based on partial differential equations. The grid points are the solution of an elliptic partial differential equation system with Dirichlet boundary conditions on all boundaries. There are several advantages in elliptic mesh generation. The theory of partial differential equations guarantees that the mapping between physical and transformed regions will be one-to-one. Another important property is the inherent smoothness in the solution of elliptic systems. A disadvantage of elliptic method is that there will be some non-uniform grid elements near convex (concave) parts of the boundary.

In this paper, a variational harmonic spline method is proposed to obtain injective parameterization with high quality. Different from the previous elliptic mesh generation method in FEA, the proposed method focus on isogeometric version (the coordinates of interior control points are unknown variables), and converts the elliptic PDE into a nonlinear constraint optimization problem. A regular term is integrated into the optimization formulation to achieve more uniform grid near convex (concave) parts of the boundary.

The remainder of the paper is organized as follows. Section 2 reviews the related work in IGA. Section 3 describes the variational harmonic method for parameterization of computational domain. Some examples and comparisons

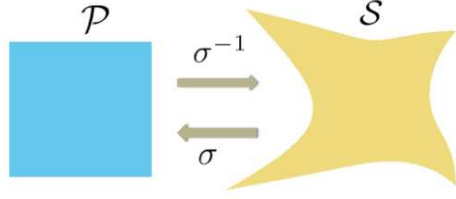


Figure 1. The mapping  $\sigma$  from physical domain  $\mathcal{S}$  to parametric domain  $\mathcal{P}$ .

based on the isogeometric heat conduction problem are presented in Section 4. Finally, we conclude this paper and outline future works in Section 5.

## 2. Related work

In this section, we will review some related works in IGA and parameterization of computational domains.

IGA was firstly proposed by Hughes et al. [19] in 2005 to achieve the seamless integration of CAD and FEA. Since then, many researchers in the fields of computational mechanical and geometric computation were involved in this topic. The current work on isogeometric analysis can be classified into three categories: (1) application of IGA to various simulation and analysis problems [2][4][11][18]; (2) application of various modeling tools in geometric computation to IGA [6][12][8][23]; (3) accuracy and efficiency improvement of IGA framework by reparameterization and refinement operations [1][3][9][10][15][21][24][25].

The topic of this paper belongs to the third field. As far as we know, there are few works on the parametrizations of computational domains for IGA. T. Martin et al. [21] proposed a method to fit a genus-0 triangular mesh by B-spline volume parameterization, based on discrete volumetric harmonic functions; this can be used to build computational domains for 3D IGA problems. A variational approach for constructing NURBS parameterization of swept volumes is proposed by M. Aigner et al [1]. Many free-form shapes in CAD systems, such as blades of turbines and propellers, are covered by this kind of volumes. E. Cohen et.al. [9] proposed the concept of *analysis-aware modeling*, in which the parameters of CAD models should be selected to facilitate isogeometric analysis. They also demonstrated the influence of parameterization of computational domains by several examples. Approximate implicitization technique is used for parametrization of computational domain in [22]. In [24][25], *r*-refinement method for generating optimal analysis-aware parameterization of computational domain is proposed based on shape optimization method. However, it only works for specified analysis problems. In this paper, we propose a general method to generate analysis-suitable parameterization of computational domain in IGA.

## 3. Variational harmonic method for parameterization of computational domain

### 3.1. Problem statement

Consider a simply connected bounded domain  $\mathcal{S}$  in two dimensional space with Cartesian coordinates  $(x; y)^T$ . Suppose that  $\mathcal{S}$  is bounded by four B-spline curves  $\mathbf{S}(\xi, 0)$ ,  $\mathbf{S}(\xi, 1)$ ,  $\mathbf{S}(1, \eta)$ ,  $\mathbf{S}(0, \eta)$ . The parametric domain  $\mathcal{P}$  of  $\mathcal{S}$  should be a rectangular in two dimensional space with coordinates  $\xi, \eta$ , which is determined by the knot vector of boundary B-spline curves in  $\xi$  and  $\eta$  direction. The mapping from parametric space  $\mathcal{P}$  to physical space  $\mathcal{S}$  can be described as a B-spline surface  $\mathbf{S}(\xi, \eta) = (x(\xi, \eta), y(\xi, \eta)) = \sum_{i=0}^n \sum_{j=0}^m N_i^p(\xi) N_j^q(\eta) \mathbf{p}_{i,j}$  with given four B-spline curves as boundaries.  $N_i^p(\xi)$  and  $N_j^q(\eta)$  are B-spline basis functions,  $\mathbf{p}_{i,j}$  are control points. Assume that this mapping is prescribed which maps the boundary of  $\mathcal{P}$  one-to-one on the boundary of  $\mathcal{S}$ . The parameterization problem of computational domain can be stated as: given four boundary B-spline curves, find the placement of inner control points such that the resulted planar B-spline surface is a good computational domain for isogeometric analysis.

### 3.2. Harmonic mapping and variational harmonic function

Harmonic mapping, which is a one-to-one transformation both for 2D and 3D regions, will be used in our parameterization method. Let  $\sigma : \mathcal{S} \mapsto \mathcal{P}$  be a harmonic mapping from  $\mathcal{S}$  to  $\mathcal{P}$ . From the theory of harmonic mapping, the inverse mapping  $\sigma^{-1} : \mathcal{P} \mapsto \mathcal{S}$  should be one-to-one (see Figure.1 as an example). In this section, we will convert the harmonic conditions into some constraints the inner control points of planar B-spline parameterization should satisfy.

If the mapping  $\sigma : \mathcal{S} \mapsto \mathcal{P}$  is a harmonic mapping, we have

$$\Delta \xi(x, y) = \xi_{xx} + \xi_{yy} = 0$$

$$\Delta \eta(x, y) = \eta_{xx} + \eta_{yy} = 0$$

By chain rules, we have

$$d\xi = \xi_x dx + \xi_y dy = \xi_x(x_\xi d\xi + x_\eta d\eta) + \xi_y(y_\xi d\xi + y_\eta d\eta)$$

$$d\eta = \eta_x dx + \eta_y dy = \eta_x(x_\xi d\xi + x_\eta d\eta) + \eta_y(y_\xi d\xi + y_\eta d\eta)$$

From above two equation, we have

$$\xi_x x_\xi + \xi_y y_\xi = \eta_x x_\eta + \eta_y y_\eta = 1 \quad (1)$$

$$\xi_x x_\eta + \xi_y y_\eta = \eta_x x_\xi + \eta_y y_\xi = 0 \quad (2)$$

By solving above two linear systems, we obtain

$$\xi_x = \frac{y_\eta}{J}, \xi_y = \frac{-x_\eta}{J}$$

and

$$\eta_x = \frac{-y_\xi}{J}, \eta_y = \frac{x_\xi}{J}$$

where

$$J = x_\xi y_\eta - x_\eta y_\xi$$

is jacobian of transformation from  $\mathcal{P}$  to  $\mathcal{S}$ .

From above results and  $\frac{\partial}{\partial x} = \xi_x \frac{\partial}{\partial \xi}$ ,  $\frac{\partial}{\partial y} = \xi_y \frac{\partial}{\partial \xi}$ , we have

$$\begin{aligned} J\Delta\xi(x, y) &= J\frac{\partial}{\partial x}\xi_x + J\frac{\partial}{\partial y}\xi_y \\ &= (J\xi_x\frac{\partial}{\partial \xi} + J\eta_x\frac{\partial}{\partial \eta})\xi_x + (J\xi_y\frac{\partial}{\partial \xi} + J\eta_y\frac{\partial}{\partial \eta})\xi_y \\ &= (y_\eta\frac{\partial}{\partial \xi} - y_\xi\frac{\partial}{\partial \eta})\left(\frac{y_\xi}{J}\right) + (-x_\eta\frac{\partial}{\partial \xi} + x_\xi\frac{\partial}{\partial \eta})\left(\frac{-x_\eta}{y}\right) = 0 \end{aligned}$$

After differential computation, we have

$$-y_\eta Lx + x_\eta Ly = 0$$

where

$$L = (x_\eta^2 + y_\eta^2)\frac{\partial^2}{\partial \xi^2} - 2(x_\xi x_\eta + y_\xi y_\eta)\frac{\partial^2}{\partial \xi \partial \eta} + (x_\xi^2 + y_\xi^2)\frac{\partial^2}{\partial \eta^2}$$

Similarly, from  $J\Delta\eta(x, y) = 0$ , we have

$$y_\xi Lx - x_\xi Ly = 0$$

Then we have

$$Lx(\xi, \eta) = Ly(\xi, \eta) = 0$$

that is,

$$\|LS(\xi, \eta)\|^2 = (Lx)^2 + (Ly)^2 = 0$$

This is a nonlinear system in terms of inner control points of the planar B-spline parameterization. The final grid obtained by this method usually has non-uniform elements near convex(concave) boundary region. In order to solve this problem, we consider the uniform issues and minimize the following energy function,

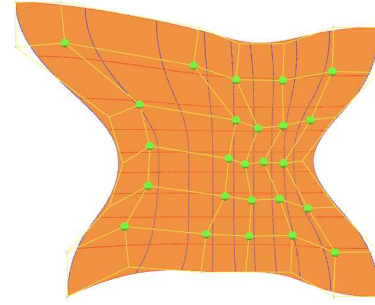
$$\begin{aligned} &\iint \lambda_1(\|S_{\xi\xi}\|^2 + \|S_{\eta\eta}\|^2 + 2\|S_{\xi\eta}\|^2) \\ &+ \lambda_2(\|S_\xi\|^2 + \|S_\eta\|^2)dudv \end{aligned}$$

Then by adding the harmonic mapping term into above energy function, we obtain a new energy function for parameterization of computational domain

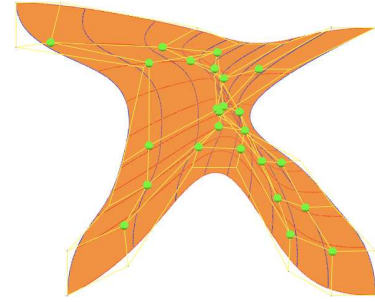
$$\begin{aligned} &\iint \|LS(\xi, \eta)\|^2 + \lambda_1(\|S_{\xi\xi}\|^2 + \|S_{\eta\eta}\|^2 \\ &+ 2\|S_{\xi\eta}\|^2) + \lambda_2(\|S_\xi\|^2 + \|S_\eta\|^2)dudv \quad (3) \end{aligned}$$

where  $\lambda_1$  and  $\lambda_2$  are positive weights to control the final parameterization results.

The Discrete Coons method is used to construct initial inner control points, and the Steepest Descent method is employed to minimize the energy function. The details will be described in the following two subsections.



(a)



(b)

Figure 2. Examples of B-spline surface constructed by discrete Coons method: (a) example without self-intersection ; (b) example with self-intersections. Green control points are inner control points to be constructed.

### 3.3. Initial construction of inner control points

In order to solve this constraint optimization problem, an initial construction of inner control points is required. We rely on the discrete Coons method presented in [16] to generate inner control points as initial value from boundary control points. See Figure 2 (a) for an example.

Given the boundary control points  $\mathbf{p}_{0,j}, \mathbf{p}_{n,j}, \mathbf{p}_{i,0}, \mathbf{p}_{i,m}$ ,  $i = 0, \dots, n, j = 0, \dots, m$ , the inner control points  $\mathbf{p}_{i,j}$  ( $i = 1, \dots, n-1, j = 1, \dots, m-1$ ) can be constructed by the discrete Coons method as follows:

$$\begin{aligned} \mathbf{p}_{i,j} &= (1 - \frac{i}{n})\mathbf{p}_{0,j} + \frac{i}{n}\mathbf{p}_{n,j} + (1 - \frac{j}{m})\mathbf{p}_{i,0} + \frac{j}{m}\mathbf{p}_{i,m} \\ &- [1 - \frac{i}{n} \quad \frac{i}{n}] \begin{bmatrix} \mathbf{p}_{0,0} & \mathbf{p}_{0,m} \\ \mathbf{p}_{n,0} & \mathbf{p}_{n,m} \end{bmatrix} \begin{bmatrix} 1 - \frac{j}{m} \\ \frac{j}{m} \end{bmatrix} \end{aligned}$$

Since the sum of the coefficients equals 1, the resulting inner control points lie in the convex hull of the boundary control points. For some given boundary curves, this construction may cause some self-intersections, and lead to an improper parameterization for IGA. See Figure 2 (b) for such an example.

### 3.4. Optimization method

In the proposed approach, we minimize the objective function (3), by moving inner control points of the computational domain. Therefore, we consider as optimization variables the coordinates of the inner control points and as cost function the error of the IGA solution. The optimization algorithm used for this study is a classical steepest-descent method in conjunction with a back-tracking line-search. For this exercise, the gradient of the cost function is approximated using a centered finite-differencing scheme.

Each iteration  $k$  of the optimization algorithm can be summarized as follows, starting from a point  $x_k$  in the variable space:

- 1) Evaluation of perturbed points  $x_k + \epsilon e_k$
- 2) Estimation of the gradient  $\nabla f(x_k)$  by finite-difference
- 3) Define search direction  $d_k = -\nabla f(x_k)$
- 4) Line search : find  $\rho$  such as  $f(x_k + \rho d_k) < f(x_k)$

These steps are carried out until a stopping criterion is satisfied.

### 3.5. Overview of variational harmonic method

In general, the proposed variational harmonic method can be summarized as follows:

**Input:** four coplanar boundary B-spline curves

**Output:** inner control points and the corresponding planar B-spline surfaces

- Construct the initial inner control points as in subsection 3.3;
- Solve the following optimization problem by using steepest-descent method

$$\iint \|LS(\xi, \eta)\|^2 + \lambda_1(\|S_{\xi\xi}\|^2 + \|S_{\eta\eta}\|^2 + 2\|S_{\xi\eta}\|^2) + \lambda_2(\|S_\xi\|^2 + \|S_\eta\|^2) dudv$$

- Generate the corresponding planar B-spline surface  $S(\xi, \eta)$  as computational domain.

## 4. Examples and comparison

In this section, we aim at presenting several examples to show the efficiency of the proposed method. We will also give a comparison study between the original parameterization and the final parameterization constructed by the proposed method by a heat conduction problem.

### 4.1. Test model — heat conduction problem

For ease of presentation, we consider the two dimensional second order elliptic PDE with homogeneous Dirichlet boundary condition as an illustrative model problem :

$$\begin{aligned} -\Delta U(\mathbf{x}) &= f(\mathbf{x}) & \text{in } \Omega \\ U(\mathbf{x}) &= 0 & \text{on } \partial\Omega \end{aligned} \quad (4)$$

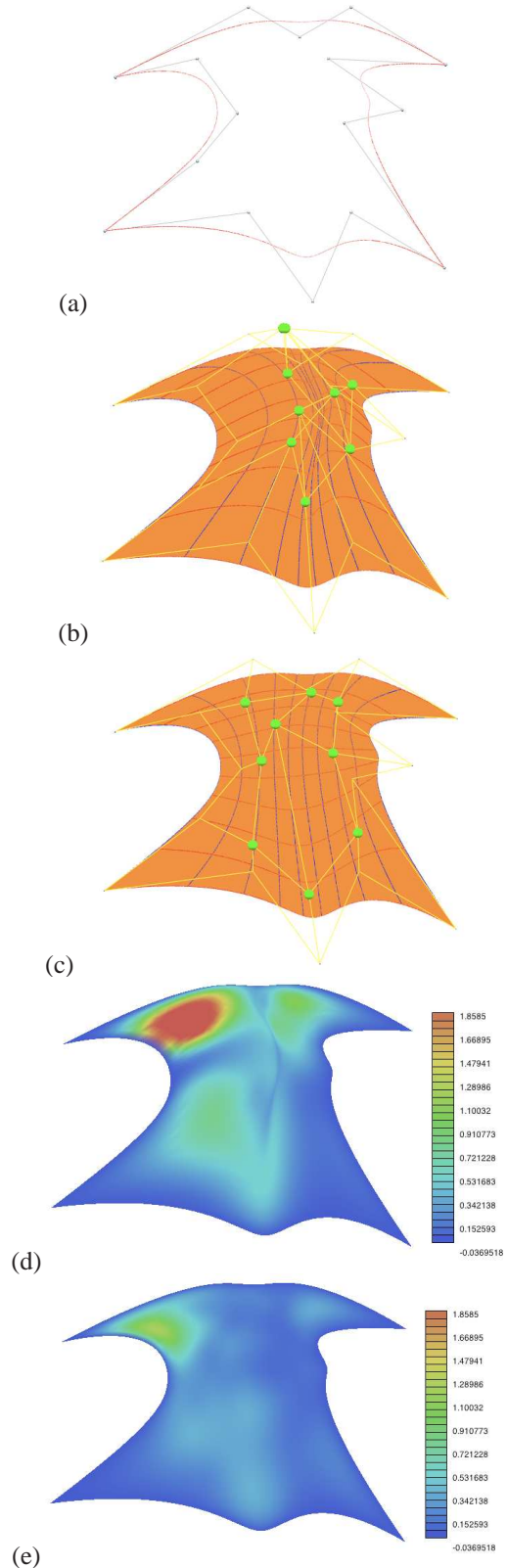


Figure 3. Example I: (a) given boundary B-spline curves ; (b) initial Coons surface; (c) final optimization result ; (d) initial simulation error; (e) final simulation error with same scale.

where  $\mathbf{x}$  are the Cartesian coordinates,  $\Omega$  is a Lipschitz domain with boundary  $\partial\Omega$ ,  $f(\mathbf{x}) \in L^2(\Omega) : \Omega \mapsto \mathbb{R}$  is a given source term, and  $U(\mathbf{x}) : \Omega \mapsto \mathbb{R}$  is the unknown solution.

Starting from a planar B-spline surface as computational domain, a general framework of an isogeometric solver for heat conduction problem (4) has been implemented as a plugin in the AXEL<sup>1</sup> platform, yielding a B-spline surface as solution field. Additional details concerning the isogeometric solver of problem (4) can be found in [13]. The proposed variational harmonic method is implemented as a part of the isogeometric toolbox of the project EXCITING<sup>2</sup>.

In this paper, we test the different parameterizations of computational domains for the heat conduction problem (4) with source term

$$f(x, y) = \frac{4\pi^2}{9} \sin\left(\frac{\pi x}{3}\right) \sin\left(\frac{\pi y}{3}\right). \quad (5)$$

For problems with unknown exact solution  $U$ , suppose that  $U_h$  is the approximation solution obtained by isogeometric method, then the discrete error  $e = U - U_h$ . We employ a posteriori error assessment proposed in [25] to compare the initial and final parameterization of the computational domain. It can be obtained by resolving the following problem,

$$\begin{aligned} \Delta e &= -f + \Delta U_h && \text{in } \Omega \\ e &= 0 && \text{on } \partial\Omega_D \end{aligned} \quad (6)$$

Obviously, The approximation error  $e$  from (6) also has a B-spline form. Some h-refinement operation should be performed to achieve more accurate results for above problem. Though it is much more expensive, it can be used as an error assessment method to show the effectiveness of the proposed construction method of computational domain.

The first example is shown in Figure 3. Figure 3 (a) presents the given boundary B-spline curves. Figure 3(b) presents the initial parametrization of computational domain constructed by discrete Coons method. The resulted parameterization has some self-intersections as shown by the isoparametric curves. Figure 3(c) shows the final parametrization of computational domain constructed by the proposed variational harmonic method. The final parameterization has no self-intersection. The corresponding simulation error obtained from (6) are illustrated in Fig. 3 (d) and 3 (e) with the same scale.

Another example is illustrated in Figure 4. The given boundary B-spline curves is shown in Figure 4 (a). Figure 4(b) presents the initial parametrization of computational domain constructed by discrete Coons method. There are some self-intersections on the initial parameterization as shown by the isoparametric curves. Figure 4(c) shows the final parametrization of computational domain constructed

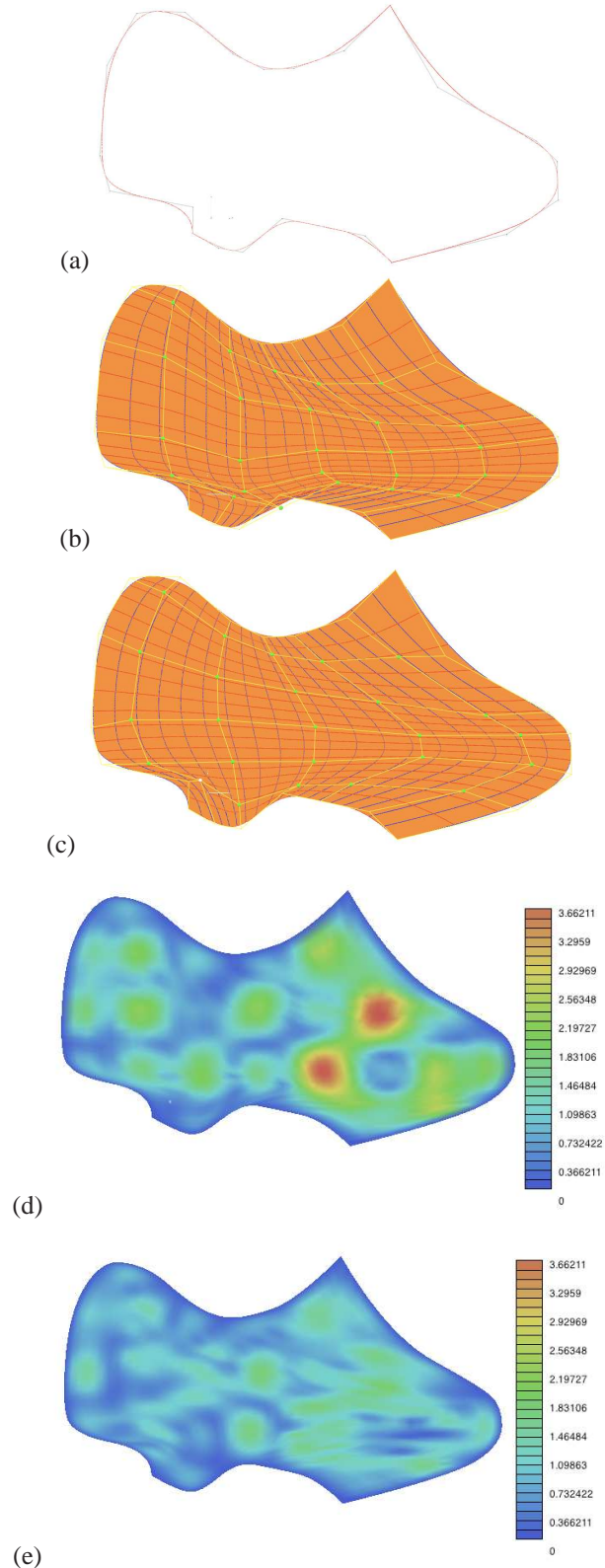


Figure 4. Example I: (a) given boundary B-spline curves ; (b) initial Coons surface; (c) final optimization result ; (d) initial simulation error; (e) final simulation error with same scale.

1. <http://axel.inria.fr/>  
 2. <http://exciting-project.eu/>

by variational harmonic method. There is no self-intersection on the final parameterization. Fig. 4 (d) and 4 (e) presents the corresponding simulation error obtained from (6) with the same scale.

Overall, the final parametrization obtained by the variational harmonic method has better quality, and can achieve better simulation results than the initial Coons parametrization.

## 5. Conclusion

Parameterization of computational domain is a key point in isogeometric analysis. In this paper, we propose a variational harmonic method to parameterize the computational domain from the theory of harmonic mapping. The resulted parameterization is injective and has high quality near convex(concave) parts of the boundary. Examples and comparison are presented to show that the proposed methods can produce analysis-suitable parameterization of computational domain for isogeometric analysis.

As part of the future work, we will generalize the proposed method to 3D case and the case with multi-patches, which are more important in practice.

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