

Some works of Furtwängler and Vandiver revisited and Fermat's last theorem

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ABSTRACT. From some works of P. Furtwängler and H.S. Vandiver, we put the basis of a new cyclotomic approach to Fermat's last theorem for $p > 3$ and to a stronger version called SFLT, by introducing governing fields of the form $\mathbb{Q}(\mu_{q-1})$ for prime numbers q . We prove for instance that if there exist infinitely many primes q , $q \not\equiv 1 \pmod{p}$, $q^{p-1} \not\equiv 1 \pmod{p^2}$, such that for $q \mid q$ in $\mathbb{Q}(\mu_{q-1})$, we have $q^{1-c} = \mathfrak{a}^p(\alpha)$ with $\alpha \equiv 1 \pmod{p^2}$ (where c is the complex conjugation), then Fermat's last theorem holds for p .

More generally, the main purpose of the paper is to show that the existence of nontrivial solutions for SFLT implies some strong constraints on the arithmetic of the fields $\mathbb{Q}(\mu_{q-1})$. From there, we give sufficient conditions of nonexistence that would require further investigations to lead to a proof of SFLT, and we formulate various conjectures. This text must be considered as a basic tool for future researches (probably of analytic or geometric nature).

RÉSUMÉ. Reprenant des travaux de P. Furtwängler et H.S. Vandiver, nous posons les bases d'une nouvelle approche cyclotomique du dernier théorème de Fermat pour $p > 3$ et d'une version plus forte appelée SFLT, en introduisant des corps gouvernants de la forme $\mathbb{Q}(\mu_{q-1})$ pour q premier. Nous prouvons par exemple que s'il existe une infinité de nombres premiers q , $q \not\equiv 1 \pmod{p}$, $q^{p-1} \not\equiv 1 \pmod{p^2}$, tels que pour $q \mid q$ dans $\mathbb{Q}(\mu_{q-1})$, on ait $q^{1-c} = \mathfrak{a}^p(\alpha)$ avec $\alpha \equiv 1 \pmod{p^2}$ (où c est la conjugaison complexe), alors le théorème de Fermat est vrai pour p .

Plus généralement, le but principal de l'article est de montrer que l'existence de solutions non triviales pour SFLT implique de fortes contraintes sur l'arithmétique des corps $\mathbb{Q}(\mu_{q-1})$. A partir de là, nous donnons des conditions suffisantes de non existence qui nécessiteraient des investigations supplémentaires pour conduire à une preuve de SFLT, et nous formulons diverses conjectures. Ce texte doit être considéré comme un outil de base pour de futures recherches (probablement analytiques ou géométriques).

This second version includes some corrections in the English language, an in depth study of the case $p = 3$ (especially Theorem 8), further details on some conjectures, and some minor mathematical improvements.

1. Introduction

This paper is devoted to the study of the following phenomenon. Consider the maximal abelian extension $\overline{\mathbb{Q}}^{\text{nr}}$ of \mathbb{Q} , unramified (= nonramified) at a given prime $p > 2$; from class field theory, we get $\overline{\mathbb{Q}}^{\text{nr}} = \bigcup_{n, p \nmid n} \mathbb{Q}(\mu_n)$. Then denote by $H_{\overline{\mathbb{Q}}^{\text{nr}}}$ the maximal p -ramified (i.e., unramified outside p)

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abelian p -extension of $\overline{\mathbb{Q}}^{\text{nr}}$; this extension is given by $\bigcup_{n, p \nmid n} H_{\mathbb{Q}(\mu_n)}$ where $H_{\mathbb{Q}(\mu_n)}$ is the maximal p -ramified abelian p -extension of $\mathbb{Q}(\mu_n)$.

Then consider $H_{\overline{\mathbb{Q}}^{\text{nr}}[p]} := \bigcup_{n, p \nmid n} H_{\mathbb{Q}(\mu_n)[p]}$, the maximal p -elementary p -ramified extension of $\overline{\mathbb{Q}}^{\text{nr}}$, union of the corresponding maximal p -elementary p -ramified extensions of $\mathbb{Q}(\mu_n)$.

We have found that any nontrivial solution (u, v) of a classical diophantine equation, associated to Fermat's equation, and called the SFLT equation¹, implies some constraints on the law of decomposition of every prime $q \neq p$ in $H_{\overline{\mathbb{Q}}^{\text{nr}}[p]}/\overline{\mathbb{Q}}^{\text{nr}}$.

These constraints may be characterized at some finite steps via the law of decomposition of q in a canonical family \mathcal{F}_n of p -cyclic subextensions of $H_{\mathbb{Q}(\mu_n)[p]}/\mathbb{Q}(\mu_n)$, where $n \mid q - 1$ depends on q, u, v (see Theorem 4).

Some aspects needed to prove this relation can be found in some former technics of Furtwängler and Vandiver, in a different viewpoint from ours, to try to give a classical cyclotomic proof of Fermat's last theorem (FLT).

Of course the problem is now empty for Fermat's equation, except if we wish to prove FLT by this way; but we will see that for the SFLT equation the result is unknown for $p > 3$ (but conjecturally similar) and, moreover, leads to infinitely many solutions for $p = 3$. But as we will show, the case $p = 3$ is exceptional and we will explain in Section 9 for what reasons.

Unfortunately, we have no deep results to propose, but only some material which may be helpful for those interested in going further.

2. Generalities on the method – The ω -SFLT equation

2.1. Prerequisites on Fermat's last theorem. Let p be a prime number, $p > 2$. Let a, b, c be pairwise relatively prime nonzero integers, such that $a^p + b^p + c^p = 0$.

We can find for instance in [Gr1, Ri, Wa] the following obvious properties concerning such a speculative counterexample to FLT, where ζ is a primitive p th root of unity, $K := \mathbb{Q}(\zeta)$, $\mathfrak{p} := (\zeta - 1)\mathbb{Z}[\zeta]$, and $N_{K/\mathbb{Q}}$ is the norm map in K/\mathbb{Q} (for a detailed proof, a more complete bibliography, and an analysis of the classical cyclotomic approach to FLT, we refer to [Gr1]):

(i) We have (where $\nu \geq 0$ is the p -adic valuation of c)²:

$$a + b = c_0^p \text{ or } p^{\nu p - 1} c_0^p \text{ with } \nu \geq 2, \text{ and } N_{K/\mathbb{Q}}(a + b\zeta) = c_1^p \text{ or } p c_1^p,$$

¹Equation $(u + v\zeta)\mathbb{Z}[\zeta] = \mathfrak{w}_1^p$ or $\mathfrak{p}\mathfrak{w}_1^p$, in integers u, v with $\text{g.c.d.}(u, v) = 1$, equivalent to $N_{K/\mathbb{Q}}(u + v\zeta) = w_1^p$ or pw_1^p , where $\zeta := e^{2i\pi/p}$, $K := \mathbb{Q}(\zeta)$, $\mathfrak{p} := (\zeta - 1)\mathbb{Z}[\zeta]$ (see Conjecture 1). Remark that the important condition $\text{g.c.d.}(u, v) = 1$ implies \mathfrak{w}_1 prime to \mathfrak{p} .

Note that if $uv = 0$, the condition $\text{g.c.d.}(u, v) = 1$ implies $(u, v) = (\pm 1, 0)$ or $(0, \pm 1)$.

²If $\nu \geq 1$, then $\alpha := \frac{a+c\zeta}{a+c\zeta^{-1}}$ is a pseudo-unit (i.e., the p th power of an ideal), congruent to 1 modulo \mathfrak{p} ; so, from [Gr1, Theorem 2.2, Remark 2.3, (ii)], α is locally a p th power in K giving easily $\alpha \equiv 1 \pmod{\mathfrak{p}^{p+1}}$, then $\frac{c(\zeta - \zeta^{-1})}{a+c\zeta^{-1}} \equiv 0 \pmod{\mathfrak{p}^{p+1}}$, hence $c \equiv 0 \pmod{p^2}$.

with $-c = c_0 c_1$ or $p^\nu c_0 c_1$, and $p \nmid c_0 c_1$. By permutation, since $p \nmid ab$, we have the following analogous relations:

$$\begin{aligned} b + c &= a_0^p, & N_{K/\mathbb{Q}}(b + c\zeta) &= a_1^p, & \text{with } -a &= a_0 a_1, \\ c + a &= b_0^p, & N_{K/\mathbb{Q}}(c + a\zeta) &= b_1^p, & \text{with } -b &= b_0 b_1. \end{aligned}$$

(ii) We have:

$$(a + b\zeta) \mathbb{Z}[\zeta] = \mathfrak{c}_1^p \text{ or } \mathfrak{p} \mathfrak{c}_1^p, \text{ with } N_{K/\mathbb{Q}}(\mathfrak{c}_1) = c_1 \mathbb{Z},$$

where \mathfrak{c}_1 is an integer ideal of K prime to \mathfrak{p} , and the analogous relations:

$$\begin{aligned} (b + c\zeta) \mathbb{Z}[\zeta] &= \mathfrak{a}_1^p, & \text{with } N_{K/\mathbb{Q}}(\mathfrak{a}_1) &= a_1 \mathbb{Z}, \\ (c + a\zeta) \mathbb{Z}[\zeta] &= \mathfrak{b}_1^p, & \text{with } N_{K/\mathbb{Q}}(\mathfrak{b}_1) &= b_1 \mathbb{Z}. \end{aligned}$$

(iii) The positive numbers a_1, b_1, c_1 have prime divisors all congruent to 1 modulo p .

Lemma 1. We can choose $x, y, z \in \{a, b, c\}$ in the following manner:

(i) First case of FLT, $p > 3$:

$$\begin{aligned} y - x &\not\equiv 0, & y + x &\not\equiv 0 & (\text{mod } p), \\ y - z &\not\equiv 0, & y + z &\not\equiv 0 & (\text{mod } p), \\ & & x + z &\not\equiv 0 & (\text{mod } p). \end{aligned}$$

(ii) First case of FLT, $p = 3$:

$$\begin{aligned} y - x &\equiv 0, & y + x &\not\equiv 0 & (\text{mod } 3), \\ y - z &\equiv 0, & y + z &\not\equiv 0 & (\text{mod } 3), \\ x - z &\equiv 0, & x + z &\not\equiv 0 & (\text{mod } 3). \end{aligned}$$

(iii) Second case of FLT, $p \geq 3$:

$$\begin{aligned} & & y &\equiv 0 & (\text{mod } p), \\ y - x &\not\equiv 0, & y + x &\not\equiv 0 & (\text{mod } p), \\ y - z &\not\equiv 0, & y + z &\not\equiv 0 & (\text{mod } p), \\ x - z &\not\equiv 0, & x + z &\equiv 0 & (\text{mod } p). \end{aligned}$$

Proof. Consider the differences $a - b, b - c, c - a$ in the first case of FLT. If two of them are divisible by p , we obtain $a \equiv b \equiv c \not\equiv 0 \pmod{p}$, then since $a + b + c \equiv 0 \pmod{p}$, we get $3a \equiv 0 \pmod{p}$ which implies $p = 3$. So, if $p > 3$, there exist two differences having the first required property, and called $y - x, y - z$.

The second condition is satisfied for any sum and any $p \geq 3$.

The case $p = 3$ in the first case of FLT is clear since $a \equiv b \equiv c \equiv \pm 1 \pmod{3}$.

In the second case of FLT, we take $y = c \equiv 0 \pmod{p}$ so that all the conditions in (iii) are satisfied (we put $y = c$ instead of $z = c$, to get, for $x + y\zeta$, a p -primary pseudo-unit instead of a number $x + y\zeta \in \mathfrak{p}$). \square

Note that for $p > 3$ in the first case, $x - z$ may be divisible by p under some circumstances (e.g. under the necessary condition $2^{p-1} \equiv 1 \pmod{p^2}$) since, from $x^p + y^p + z^p = 0$, we get $2z^p + y^p \equiv 0 \pmod{p^2}$.

2.2. Statement of a stronger conjecture than FLT. We have given in [Gr1] a conjecture which implies FLT and which is not covered by Wiles proof; we recall here the statement, which will be called the strong Fermat last theorem (SFLT).

Conjecture 1. *Let p be a prime number, $p > 2$. Then for $u, v \in \mathbb{Z}$, with $\text{g.c.d.}(u, v) = 1$, the equation:*

$$(u + v\zeta)\mathbb{Z}[\zeta] = \mathfrak{w}_1^p \text{ or } \mathfrak{p}\mathfrak{w}_1^p$$

(depending on whether $u + v \not\equiv 0 \pmod{p}$ or not), equivalent to:

$$N_{K/\mathbb{Q}}(u + v\zeta) = w_1^p \text{ or } pw_1^p, \quad w_1 = N_{K/\mathbb{Q}}(\mathfrak{w}_1) \in 1 + p\mathbb{Z},$$

where \mathfrak{w}_1 is an ideal of K (necessarily prime to \mathfrak{p}), has no solution for $p > 3$ except the trivial ones: $u + v\zeta = \pm 1, \pm\zeta, \pm(1 + \zeta)$, and $\pm(1 - \zeta)$. \square

The difference between FLT and SFLT is the following. A solution of Fermat's equation $u^p + v^p + w^p = 0$ comes from a solution of $(u + v\zeta)\mathbb{Z}[\zeta] = \mathfrak{w}_1^p$ or $\mathfrak{p}\mathfrak{w}_1^p$ (with the same u, v as above), if and only if there exists $w_0 \in \mathbb{Z}$ such that $u + v = w_0^p$ or $p^{p-1}w_0^p$ since $N_{K/\mathbb{Q}}(u + v\zeta) = w_1^p$ or pw_1^p , giving $w := -w_0w_1$ or $-p^{p-1}w_0w_1$ for a solution of Fermat's equation.

As for FLT we can speak of the *first case* of the conjecture (or of the equation) when:

$$uv(u + v) \not\equiv 0 \pmod{p}$$

and of the *second case* when:

$$uv \equiv 0 \pmod{p}$$

(which implies u or $v \equiv 0 \pmod{p^2}$ as for Fermat's equation); then the case:

$$u + v \equiv 0 \pmod{p}$$

will be called the *special case* for SFLT.

For the first case of SFLT, we have not necessarily $u - v \not\equiv 0 \pmod{p}$,³ except for $p = 3$ since $uv(u + v) \not\equiv 0 \pmod{3}$ implies $u \equiv v \equiv \pm 1 \pmod{3}$, hence $u - v \equiv 0 \pmod{3}$. See the forthcoming Remark 1 for $p = 3$.

In the sequel, we will assume that (x, y, z) is a solution of Fermat's equation such that the conditions of Lemma 1 are satisfied (i.e., $y - x$ and $y - z$ are prime to p when $p > 3$, and if $p \mid xyz$, we suppose that $p \mid y$).

³If $u - v \equiv 0 \pmod{p}$, then $\alpha := \frac{u\zeta + v}{u + v\zeta}$ is a pseudo-unit congruent to 1 modulo p ; so, from [Gr1, Theorem 2.2, Remark 2.3, (ii)], α is locally a p th power giving $\alpha \equiv 1 \pmod{p^{p+1}}$, then $\frac{(u-v)(\zeta-1)}{u+v\zeta} \equiv 0 \pmod{p^{p+1}}$, hence $u - v \equiv 0 \pmod{p^2}$. This is valid in the Fermat case if $x - z \equiv 0 \pmod{p}$, and gives $x - z \equiv 0 \pmod{p^2}$.

In that case we will have two similar counterexamples to the above conjecture: $(x + y\zeta)\mathbb{Z}[\zeta] = \mathfrak{z}_1^p$, $(z + y\zeta)\mathbb{Z}[\zeta] = \mathfrak{x}_1^p$ (this concerns the first or second case of SFLT). Then it will exist the third counterexample $(x + z\zeta)\mathbb{Z}[\zeta] = \mathfrak{y}_1^p$ (if $p \nmid y$) or $\mathfrak{p}\mathfrak{y}_1^p$ (if $p \mid y$).

More precisely, the first case of SFLT implies the first case of FLT, the second or the special case of SFLT implies the second case of FLT, and FLT holds as soon as first and second cases, or first and special cases of SFLT, hold.

Remark 1. Conjecture 1 is false for $p = 3$ since for $\zeta = j$ of order 3 we have the six kind of parametric formulas giving all solutions:

$u + vj = j^h (s + tj)^3$, or $j^h (1 - j)(s + tj)^3$, $s, t \in \mathbb{Z}$, $s + t \not\equiv 0 \pmod{3}$, g.c.d. $(s, t) = 1$, and $0 \leq h < 3$. These solutions concern all the cases:

– first case (for which $u - v \equiv 0 \pmod{9}$):

- $(u, v) = (-s^3 - t^3 + 3s^2t, -s^3 - t^3 + 3st^2)$, from $u + vj = j^2 (s + tj)^3$;

– second case (for which u or $v \equiv 0 \pmod{9}$):

- $(u, v) = (3st^2 - 3s^2t, s^3 + t^3 - 3s^2t)$, from $u + vj = j (s + tj)^3$;
- $(u, v) = (s^3 + t^3 - 3st^2, 3s^2t - 3st^2)$, from $u + vj = (s + tj)^3$;

– special cases (for which $u + v \equiv 0 \pmod{3}$):

- $(u, v) = (s^3 + t^3 + 3s^2t - 6st^2, -s^3 - t^3 + 6s^2t - 3st^2)$, from $u + vj = (1 - j)(s + tj)^3$;
- $(u, v) = (s^3 + t^3 - 6s^2t + 3st^2, 2s^3 + 2t^3 - 3s^2t - 3st^2)$, from $u + vj = j(1 - j)(s + tj)^3$;
- $(u, v) = (-2s^3 - 2t^3 + 3s^2t + 3st^2, -s^3 - t^3 - 3s^2t + 6st^2)$, from $u + vj = j^2(1 - j)(s + tj)^3$.

The special cases are not similar since for the first solution $u + v \equiv 0 \pmod{9}$ and for the others, $u + v \equiv \pm 3(s^3 + t^3) \equiv \pm 3(s + t) \equiv \pm 3 \pmod{9}$.

Contrary to the case of Fermat's equation, we will not take into account the symmetries of the writing of the solutions (u, v) , especially for the second case (this will be important in Section 9) but we will not distinguish (u, v) from $(-u, -v)$. \square

Thus a proof of SFLT must eliminate, in a natural way, the case $p = 3$ which is an obstruction for the relevance of the method developed here. We will explain later (Section 9) for what reasons this case is exceptional and finally does not matter, a priori, for the general theory; we are obliged to differ this justification because we need many general material. Meanwhile, for a more comprehensive information, we do not always suppose $p > 3$ in the development of the first parts of the study.

2.3. The cyclotomic field $\mathbb{Q}(\zeta)$ and the character ω . We first recall the algebraic context concerning the cyclotomic field $K = \mathbb{Q}(\zeta)$.

Definition 1. (i) Let $g := \text{Gal}(K/\mathbb{Q})$ and let ω be the character of Teichmüller of g (i.e., the character with values in $\mu_{p-1}(\mathbb{Q}_p)$ such that for the $s_k \in g$ defined by $s_k(\zeta) = \zeta^k$, $k \not\equiv 0 \pmod{p}$, $\omega(s_k)$ (also denoted $\omega(k)$) is the unique $(p-1)$ th root of unity in \mathbb{Q}_p , congruent to k modulo p).

(ii) The idempotent corresponding to ω is:

$$e_\omega := \frac{1}{p-1} \sum_{s \in g} \omega^{-1}(s) s = \frac{1}{p-1} \sum_{k=1}^{p-1} \omega^{-1}(k) s_k \in \mathbb{Z}_p[g].$$

(iii) We represent e_ω in $\mathbb{Z}[g]$ modulo p and still denote it e_ω (this means that $e_\omega s_k \equiv \omega(k) e_\omega \equiv k e_\omega \pmod{p\mathbb{Z}[g]}$ and that $e_\omega(1 - e_\omega) \in p\mathbb{Z}[g]$).

Put $e_\omega := \sum_{k=1}^{p-1} u_k s_k$, $u_k \in \mathbb{Z}$, $u_k \equiv \frac{1}{p-1} \omega^{-1}(k) \equiv \frac{k^{-1}}{p-1} \pmod{p}$.

We have $\omega^{-1}(s_{p-k}) = -\omega^{-1}(s_k)$ since $\omega(s_{-1}) = -1$; thus we can suppose that $u_{p-k} = -u_k$ for $1 \leq k \leq \frac{p-1}{2}$. Then we have $e_\omega = (1 - s_{-1}) e'_\omega$ with $e'_\omega = \sum_{k=1}^{\frac{p-1}{2}} u_k s_k$.

In that case, if an element A of a multiplicative $\mathbb{Z}[g]$ -module \mathcal{M} is fixed by the complex conjugation s_{-1} of K , we then have $A^{e_\omega} = 1_{\mathcal{M}}$ (the unit element of \mathcal{M}).

(iv) We have $\zeta^{e_\omega} = \zeta$ for any representative e_ω . □

Example 1. For $p = 3$ we have $e_\omega = \frac{1}{2}(1 - s)$, with $s := s_{-1}$. Thus a representative with integer coefficients may be $e_\omega = s - 1$.

For $p = 5$, we have for instance $e_\omega = -1 + 2s_2 - 2s_3 + s_4 = -1 + 2s + s^2 - 2s^3 = (1 - s^2)(2s - 1)$, with $s := s_2$. □

Recall that the unit group E of K is equal to $\langle \zeta \rangle \oplus E^+$, where E^+ is the group of units of the maximal real subfield K^+ of K (see [Wa, Prop. 1.5]).

Thus if $\varepsilon = \zeta^h \varepsilon^+$, $\varepsilon^+ \in E^+$, we get $\varepsilon^{e_\omega} = \zeta^h$.

2.4. The principles of the method – The fundamental relation.

The purpose of this text is to examine some properties of the arithmetic of the fields $\mathbb{Q}(\mu_{q-1})$, in relation with a nontrivial solution of the SFLT equation:

$$(u + v\zeta) \mathbb{Z}[\zeta] = \mathfrak{w}_1^p \text{ or } \mathfrak{p} \mathfrak{w}_1^p,$$

with $\text{g.c.d.}(u, v) = 1$, for prime numbers q such that $q \nmid uv$ and the order n of $\frac{v}{u}$ modulo q is prime to p .

The cases where $n \leq 2$ (i.e., $q \mid u^2 - v^2$) are particular, especially in the case where (u, v) is a part of a solution (x, y, z) of Fermat's equation, and give Furtwängler's theorems [Fur] (see Corollaries 2 and 3 to Lemma 3 for a generalization of Furtwängler's theorems for the SFLT equation, and Remark 3 for the classical case of the FLT equation; see also [Mih] in the

context of a Nagell–Ljunggren equation, which is the particular case of the SFLT equation with $v = 1$).

The cases where n is divisible by any nontrivial power of p give technical complications and are of a different nature. Some complements in this direction are developed in [Que] where similar studies are proposed.

Lemma 2. *Let u, v be relatively prime integers, let $n \geq 1$, and let q be a prime number. Then the two following properties are equivalent:*

- (i) $q \nmid n$ and $q \mid \Phi_n(u, v) := \prod_{\xi' \text{ of order } n} (u \xi' - v)$;
- (ii) $q \nmid uv$ and $\frac{v}{u}$ is of order n modulo q .

Proof. Suppose that $q \mid \Phi_n(u, v)$ and $q \nmid n$. Then $q \nmid uv$ since $\Phi_n(u, v)$ is an homogeneous form $u^{\phi(n)} \pm \dots \pm v^{\phi(n)}$ in coprime integers u, v , where $\phi(n)$ is the Euler indicator.

For ξ of order n fixed, the ideal $(q, u\xi - v)$ of the field $\mathbb{Q}(\mu_n)$ is a prime ideal dividing q because of the relation $q \mid \Phi_n(u, v) = \prod_{\xi' \text{ of order } n} (u \xi' - v)$; moreover $(q, u\xi - v)$ is of degree 1, unramified in $\mathbb{Q}(\mu_n)/\mathbb{Q}$ (since $q \nmid n$), thus we get $q \equiv 1 \pmod{n}$ and the fact that $\frac{v}{u}$ is of order n modulo q .

If $q \nmid uv$ and $\frac{v}{u}$ is of order n modulo q , then $u^n - v^n \equiv 0 \pmod{q}$. From the relation $u^n - v^n = \prod_{d \mid n} \Phi_d(u, v)$ we deduce that there exists $m \mid n$ such that $q \mid \Phi_m(u, v)$, which implies $q \mid u^m - v^m$, hence $m = n$ by definition of the order; since we have $(\frac{v}{u})^q \equiv \frac{v}{u} \pmod{q}$, it is clear that the order n cannot be divisible by q , proving the lemma. \square

Corollary 1. *Consider the set of numbers of the form $\Phi_n(u, v)$ when n varies in $\mathbb{N} \setminus \{0\}$.*

Then a prime number q divides one of the numbers $\Phi_n(u, v)$, $n \not\equiv 0 \pmod{q}$, if and only if $q \nmid uv$. When these conditions $(q \nmid n, q \mid \Phi_n(u, v))$ are satisfied, then n is unique. \square

If q is an arbitrary given prime number, to have $q \mid \Phi_n(u, v)$ with $n > 2$ and $q \nmid n$, we must first verify that $q \nmid uv(u^2 - v^2)$ and then compute the order n of $\frac{v}{u}$ modulo q which is then a divisor of $q - 1$.⁴

Definition 2. Let $q \neq p$ be a prime number.

- (i) Fermat quotients. Let f be the residue degree of q in K/\mathbb{Q} and let $\kappa := \frac{q^f - 1}{p}$. Since $f \mid p - 1$, we have $\kappa \equiv 0 \pmod{p}$ if and only if $q^{p-1} \equiv 1 \pmod{p^2}$.

⁴It is clear that the trivial solutions $u + v\zeta = \pm 1, \pm\zeta, \pm(1 \pm \zeta)$ of the SFLT equation are precisely such that $uv(u^2 - v^2) = 0$, in which case such primes q do not exist, which has perhaps a significant meaning.

The integer $\bar{\kappa} := \frac{q^{p-1}-1}{p}$ is called the Fermat quotient of q . We have the relation $\bar{\kappa} \equiv \frac{p-1}{f} \kappa \equiv -\frac{1}{p} \log(q) \pmod{p}$, where \log is the p -adic logarithm.

(ii) Power residue symbols. Let us recall the definition and properties of the p th power residue symbols $\left(\frac{\cdot}{\cdot}\right)$ in K and $M := \mathbb{Q}(\mu_n)K$, $n \mid q-1$, with values in μ_p . Let \mathfrak{q} be a prime ideal dividing q in $\mathbb{Q}(\mu_n)$.

If $\alpha \in M$ is prime to $\mathfrak{Q} \mid \mathfrak{q}$ in M , then let $\bar{\alpha}$ be the image of α in the residue field $Z_M/\mathfrak{Q} \simeq Z_K/\mathfrak{q}_K \simeq \mathbb{F}_{q^f}$ for $\mathfrak{q}_K = Z_K \cap \mathfrak{Q}$ (indeed, q totally splits in M/K); since $\zeta \in Z_M$, the image $\bar{\zeta}$ of ζ is of order p (since $\zeta \not\equiv 1 \pmod{\mathfrak{Q}}$) and we can put $\bar{\alpha}^\kappa = \bar{\zeta}^\mu$, $\mu \in \mathbb{Z}/p\mathbb{Z}$, which defines the p th power residue symbol $\left(\frac{\alpha}{\mathfrak{Q}}\right)_M := \zeta^\mu$; this symbol is equal to 1 if and only if α is a local p th power at \mathfrak{Q} (see e.g. [Gr2, I.3.2.1, Ex. 1]).

With this definition, for any automorphism $\tau \in \text{Gal}(M/\mathbb{Q})$ one obtains, from $\alpha^\kappa \equiv \zeta^\mu \pmod{\mathfrak{Q}}$, $\tau\alpha^\kappa \equiv \tau\zeta^\mu \pmod{\tau\mathfrak{Q}}$, thus:

$$\left(\frac{\tau\alpha}{\tau\mathfrak{Q}}\right)_M = \tau\left(\frac{\alpha}{\mathfrak{Q}}\right)_M = \left(\frac{\alpha}{\mathfrak{Q}}\right)_M^{\omega(\tau)} = \zeta^{\mu\omega(\tau)}.$$

If $\alpha \in K$, for any $\mathfrak{q}_K \mid \mathfrak{q}$ in K we get $\left(\frac{\alpha}{\mathfrak{q}_K}\right)_K = \left(\frac{\alpha}{\mathfrak{Q}}\right)_M$ for any $\mathfrak{Q} \mid \mathfrak{q}_K$ in M .

In particular this implies $\left(\frac{\zeta}{\mathfrak{q}_K}\right)_K = \zeta^\kappa$ (the symbol of ζ does not depend on the choice of $\mathfrak{q}_K \mid \mathfrak{q}$). \square

We return to the context of the SFLT equation $(u+v\zeta)\mathbb{Z}[\zeta] = \mathfrak{w}_1^p$ or $\mathfrak{p}\mathfrak{w}_1^p$, g.c.d. $(u, v) = 1$ (the second case corresponds to $p \mid uv$ and the special case to $p \mid u+v$).

Put $\gamma_\omega := (u+v\zeta)^{e_\omega}$ for a solution (u, v) of the above SFLT equation.

In the context of a solution (x, y, z) of Fermat's equation we will have analogous computations with $\gamma_\omega := (x+y\zeta)^{e_\omega}$ and the relation $(x+y\zeta)\mathbb{Z}[\zeta] = \mathfrak{z}_1^p$, and also with $\gamma'_\omega := (z+y\zeta)^{e_\omega}$ and the relation $(z+y\zeta)\mathbb{Z}[\zeta] = \mathfrak{x}_1^p$. Then in the first case, $\gamma''_\omega := (x+z\zeta)^{e_\omega}$ with the relation $(x+z\zeta)\mathbb{Z}[\zeta] = \mathfrak{y}_1^p$ can be used knowing that $z-x$ may be divisible by p . In the second case, γ''_ω is of \mathfrak{p} -valuation 1 since $(x+z\zeta)\mathbb{Z}[\zeta] = \mathfrak{p}\mathfrak{y}_1^p$ and this gives a special case of the equation associated to SFLT.

We know, from Stickelberger, that the ω -component of the p -class group of K is trivial (also an application of the reflection theorem, see [Gr2, II.5.4.6.3]); so the ideal class $\mathcal{d}(\mathfrak{w}_1)^{e_\omega}$ is trivial.⁵

Write:

$$\mathfrak{w}_1^{e_\omega} = \delta_\omega \mathbb{Z}[\zeta], \quad \delta_\omega \in K^\times.$$

Then we have:

$$\gamma_\omega := (u+v\zeta)^{e_\omega} = \varepsilon_\omega \delta_\omega^p \quad \text{or} \quad (\zeta-1)^{e_\omega} \varepsilon_\omega \delta_\omega^p,$$

⁵ Since here the class of \mathfrak{w}_1 is of order 1 or p , the choice of any representative e_ω does not affect this property.

where $\varepsilon_\omega \in E$. To simplify, we put $\pi := \zeta - 1$.

Lemma 3. (The fundamental relation). *Let (u, v) be a solution of the equation $(u + v\zeta)\mathbb{Z}[\zeta] = \mathfrak{w}_1^p$ or $\mathfrak{p}\mathfrak{w}_1^p$, $\text{g.c.d.}(u, v) = 1$ (since the cases where $uv(u+v) = 0$ are obvious directly, we exclude them).*

(i) *In the nonspecial cases $(u + v \not\equiv 0 \pmod{p})$ for $p \geq 3$, we have $\gamma_\omega = (\frac{u}{v} + \zeta)^{e_\omega} = (1 + \frac{v}{u}\zeta)^{e_\omega} = (1 + \frac{v}{u+v}\pi)^{e_\omega} \in \zeta^{\frac{v}{u+v}} \cdot K^{\times p}$.*

(ii) *In the special case $(u + v \equiv 0 \pmod{p})$ for $p > 3$, we have $\gamma_\omega = (1 + \frac{v}{u}\zeta)^{e_\omega} \in \zeta^{\frac{1}{2}} \cdot K^{\times p}$.⁶*

(iii) *In the special case $(u + v \not\equiv 0 \pmod{3})$ for $p = 3$, then $\gamma_\omega = (1 + \frac{v}{u}\zeta)^{e_\omega} \in \zeta^{\frac{1}{2} - \frac{u+v}{3v}} \cdot K^{\times 3}$.*

Proof. (i) We have $\gamma_\omega = \varepsilon_\omega \delta_\omega^p$ with $\varepsilon_\omega = \zeta^h \varepsilon^+$, $\varepsilon^+ \in E^+$, for some h ; then applying again e_ω we get $\gamma_\omega^{e_\omega} = \varepsilon_\omega^{e_\omega} \delta_\omega^{e_\omega p} \in \zeta^h \cdot K^{\times p}$. Since $e_\omega^2 \equiv e_\omega \pmod{p\mathbb{Z}[g]}$, any factor $A^{e_\omega^2}$ may be written A^{e_ω} up to a p th power; thus $\gamma_\omega \in \zeta^h \cdot K^{\times p}$. Since $u + v\zeta = (u + v)(1 + \frac{v}{u+v}\pi)$, $(u + v\zeta)^{e_\omega} \in \zeta^h \cdot K^{\times p}$ is equivalent to $(1 + \frac{v}{u+v}\pi)^{e_\omega} \in \zeta^h \cdot K^{\times p}$; then using [Gr1, Remark 3.4]:

$$(1 + \frac{v}{u+v}\pi)^{e_\omega} \equiv 1 + \frac{v}{u+v}\pi \pmod{\pi^2},$$

we get immediately $h \equiv \frac{v}{u+v} \pmod{p}$.

Similarly we have $u + v\zeta = v(\frac{u}{v} + \zeta) = u(1 + \frac{v}{u}\zeta)$ for which $(u + v\zeta)^{e_\omega} = (\frac{u}{v} + \zeta)^{e_\omega} = (1 + \frac{v}{u}\zeta)^{e_\omega}$, proving the point (i).

(ii) Suppose that $u + v \equiv 0 \pmod{p}$; put $\frac{u}{v} = -1 + \lambda p$, then $\frac{u}{v} + \zeta = \pi + \lambda p = \pi\alpha$, where $\alpha := 1 + \frac{\lambda p}{\pi} \equiv 1 \pmod{\pi^{p-2}}$.

Then we get $\gamma_\omega := (u + v\zeta)^{e_\omega} = (1 + \frac{v}{u}\zeta)^{e_\omega} = (\frac{u}{v} + \zeta)^{e_\omega} = \pi^{e_\omega} \alpha^{e_\omega}$. But from the relation $(u + v\zeta)\mathbb{Z}[\zeta] = (\pi)\mathfrak{w}_1^p$, we obtain $(u + v\zeta)^{e_\omega} \in \pi^{e_\omega} \zeta^h K^{\times p}$, for some h , giving $\alpha^{e_\omega} \in \zeta^h K^{\times p}$ hence $h \equiv 0 \pmod{p}$ in that case since $p > 3$. Then $(1 + \frac{v}{u}\zeta)^{e_\omega} \in \pi^{e_\omega} K^{\times p}$.

Put $\alpha \sim \beta$ in K^\times if $\alpha\beta^{-1} \in K^{\times p}$. From $(\zeta - 1)(\zeta + 1) = \zeta^2 - 1$, we get:

$$(\zeta - 1)^{e_\omega} (\zeta + 1)^{e_\omega} = (\zeta^2 - 1)^{e_\omega} = (\zeta - 1)^{2e_\omega} \sim (\zeta - 1)^{2e_\omega},$$

giving $(\zeta + 1)^{e_\omega} \sim (\zeta - 1)^{e_\omega}$. But $\zeta + 1 = \zeta^{\frac{1}{2}}(\zeta^{\frac{1}{2}} + \zeta^{-\frac{1}{2}})$ yields $(\zeta + 1)^{e_\omega} \sim \zeta^{\frac{1}{2}}$ since $\zeta^{\frac{1}{2}} + \zeta^{-\frac{1}{2}} \in K^+$. Then we have the relation $(\zeta - 1)^{e_\omega} \sim (\zeta + 1)^{e_\omega} \sim \zeta^{\frac{1}{2}}$, hence the point (ii) of the lemma.

(iii) If $p = 3$ in the special case, we get from the computations in the proof of (ii), $\gamma_\omega = \pi^{e_\omega} \alpha^{e_\omega} \in \pi^{e_\omega} \zeta^h K^{\times 3}$, for some h , with $\alpha = 1 + \frac{3\lambda}{\pi}$ and $\lambda = \frac{u+v}{3v}$. Thus $\alpha = 1 + (\zeta^2 - 1)\frac{u+v}{3v} \equiv 1 - \pi\frac{u+v}{3v} \pmod{\pi^2}$, giving the congruence $h \equiv -\frac{u+v}{3v} \pmod{3}$ and $\gamma_\omega \in \zeta^{\frac{1}{2} - \frac{u+v}{3v}} \cdot K^{\times 3}$. \square

⁶Where $\zeta^{\frac{1}{2}}$ is the unique p th root of unity such that $(\zeta^{\frac{1}{2}})^2 = \zeta$; this convention will be used in a systematic way in the paper.

In the second case of SFLT we have $\gamma_\omega \in K^{\times p}$ (resp. $\zeta \cdot K^{\times p}$) if $p \mid v$ (resp. $p \mid u$) since in this case $\frac{v}{u+v} \equiv 0$ (resp. $\frac{v}{u+v} \equiv 1$) (mod p). Note that the condition $u + v \equiv 0$ (mod 9), when $p = 3$ in the special case, is not necessarily satisfied for the SFLT equation (use Remark 1) but is true when (u, v) is a part of a solution (u, y, v) or (v, y, u) of Fermat's equation when $3 \mid y$ and more generally when $p > 3$, $p \mid y$ (see Subsection 2.1, (i)).

Corollary 2. (Generalization of the first theorem of Furtwängler). *Let $q \neq p$ be a prime number such that $q \mid uv$ for a nontrivial solution of the equation $(u + v\zeta)\mathbb{Z}[\zeta] = \mathfrak{w}_1^p$ or $\mathfrak{p}\mathfrak{w}_1^p$, g.c.d. $(u, v) = 1$.*

Then, in the nonspecial cases, $u\kappa \equiv 0$ (resp. $v\kappa \equiv 0$) (mod p) if $q \mid u$ (resp. $q \mid v$), for $p \geq 3$; in the first case we get $\kappa \equiv 0$ (mod p).

For $p > 3$ in the special case, then $\kappa \equiv 0$ (mod p). For $p = 3$ in the special case, we get $\frac{u-2v}{3v}\kappa \equiv 0$ (resp. $\frac{v-2u}{3v}\kappa \equiv 0$) (mod 3) if $q \mid u$ (resp. $q \mid v$); thus if $u + v \equiv 0$ (mod 9), then $\kappa \equiv 0$ (mod 3). If $u + v = 3e$, $e \not\equiv 0$ (mod 3), then $\kappa \equiv 0$ (mod 3) if $q \mid u$ (resp. $q \mid v$) when $u \equiv e$ (resp. $v \equiv e$) (mod 3).

Proof. We have $(u+v\zeta)^{e_\omega} \in \zeta^h \cdot K^{\times p}$ with $h \equiv \frac{v}{u+v}$ (mod p) in the nonspecial cases, $p \geq 3$, $h \equiv \frac{1}{2}$ (mod p) in the special case, $p > 3$, and $h \equiv \frac{1}{2} - \frac{u+v}{3v}$ (mod 3) in the special case, $p = 3$.

Let \mathfrak{q}_K be any prime ideal of K dividing q . We use the p th power residue symbol in K (see Definition 2, (ii)).

Since $u+v\zeta \equiv v\zeta$ (resp. $u+v\zeta \equiv u$) (mod q) if $q \mid u$ (resp. $q \mid v$), we get $\left(\frac{(u+v\zeta)^{e_\omega}}{\mathfrak{q}_K}\right)_K = \zeta^\kappa$ (resp. 1) if $q \mid u$ (resp. $q \mid v$); but we have $\left(\frac{\zeta^h}{\mathfrak{q}_K}\right)_K = \zeta^{\frac{v}{u+v}\kappa}$ (resp. $\zeta^{\frac{1}{2}\kappa}$, $\zeta^{(\frac{1}{2} - \frac{u+v}{3v})\kappa}$) in the nonspecial cases (resp. in the special case, $p > 3$, $p = 3$).

This gives in the nonspecial cases for $q \mid u$, $\frac{v}{u+v}\kappa \equiv \kappa$ (mod p) equivalent to $\frac{u}{u+v}\kappa \equiv 0$ (mod p), hence $u\kappa \equiv 0$ (mod p). If $q \mid v$, we get $\frac{v}{u+v}\kappa \equiv 0$ (mod p) giving $v\kappa \equiv 0$ (mod p).

The special case for $p > 3$ yields to $\frac{1}{2}\kappa \equiv \kappa$ (resp. $\frac{1}{2}\kappa \equiv 0$) (mod p) if $q \mid u$ (resp. $q \mid v$), giving $\kappa \equiv 0$ (mod p) in any case.

For $p = 3$ in the special case we get $(\frac{1}{2} - \frac{u+v}{3v})\kappa \equiv \kappa$ (resp. $(\frac{1}{2} - \frac{u+v}{3v})\kappa \equiv 0$) (mod 3) if $q \mid u$ (resp. $q \mid v$), giving $\frac{u-2v}{3}\kappa \equiv 0$ (resp. $\frac{v-2u}{3}\kappa \equiv 0$) (mod 3). The case $u + v \equiv 0$ (mod 9) is clear as well as the case $u + v \equiv \pm 3$ (mod 9). \square

Corollary 3. (Generalization of the second theorem of Furtwängler). *Let $q \neq p$ be a prime number such that $q \mid u^2 - v^2$ for a nontrivial solution of the equation $(u + v\zeta)\mathbb{Z}[\zeta] = \mathfrak{w}_1^p$ or $\mathfrak{p}\mathfrak{w}_1^p$, g.c.d. $(u, v) = 1$.*

Then, in the nonspecial cases, $(v - u)\kappa \equiv 0$ (mod p) for $p \geq 3$; in particular, in the second case, $\kappa \equiv 0$ (mod p). In the first case for $p = 3$, the information is empty since $u \equiv v \equiv \pm 1$ (mod 3).

For $p > 3$ in the special case, the information is empty. For $p = 3$ in the special case we get $\frac{u+v}{3v} \kappa \equiv 0 \pmod{3}$. Thus $\kappa \equiv 0 \pmod{3}$ as soon as $v + u \not\equiv 0 \pmod{9}$.

Proof. We have $(u+v\zeta)^{e_\omega} \in \zeta^h \cdot K^{\times p}$ with $h \equiv \frac{v}{u+v} \pmod{p}$ in the nonspecial cases, $p \geq 3$, $h \equiv \frac{1}{2} \pmod{p}$ in the special case, $p > 3$, and $h \equiv \frac{1}{2} - \frac{u+v}{3v}$ in the special case, $p = 3$.

Then we have $(1 + \frac{v}{u}\zeta)^{e_\omega} \zeta^{-\frac{1}{2}} \in \zeta^{\bar{h}} \cdot K^{\times p}$ with $\bar{h} \equiv \frac{1}{2} \frac{v-u}{u+v} \pmod{p}$ in the nonspecial cases, $\bar{h} \equiv 0 \pmod{p}$ in the special case, $p > 3$, and $\bar{h} \equiv -\frac{u+v}{3v} \pmod{3}$ in the special case, $p = 3$.

Let \mathfrak{q}_K be any prime ideal of K dividing q . If $q \mid u^2 - v^2$, then $\frac{v}{u} \equiv \pm 1 \pmod{q}$ and we get $(1 + \frac{v}{u}\zeta)^{e_\omega} \zeta^{-\frac{1}{2}} \equiv (1 \pm \zeta)^{e_\omega} \zeta^{-\frac{1}{2}} \equiv 1 \pmod{\mathfrak{q}_K}$ since $(1 \pm \zeta)^{e_\omega} \sim \zeta^{\frac{1}{2}}$ (see proof of Lemma 3). Thus we obtain $\bar{h} \kappa \equiv 0 \pmod{p}$ in every case.

The nonspecial cases yield to $\frac{v-u}{u+v} \kappa \equiv 0 \pmod{p}$, hence $\kappa \equiv 0 \pmod{p}$ if $u - v \not\equiv 0 \pmod{p}$. Thus the case $p = 3$ is empty since $u \equiv v \equiv \pm 1 \pmod{3}$.

The special case for $p > 3$ is empty. The special case for $p = 3$ gives $\frac{v+u}{3v} \kappa \equiv 0 \pmod{3}$. \square

2.5. Consequences of Lemma 3. We make the following comments on the fundamental Lemma 3 and its corollaries to introduce the ω -SFLT equation and suitable cyclotomic units.

For *arbitrary* relatively prime integers u, v , we still have (excluding the obvious cases where $uv(u+v) = 0$):

$$\gamma_\omega := (u+v\zeta)^{e_\omega} = \left(\frac{u}{v} + \zeta\right)^{e_\omega} = \left(1 + \frac{v}{u}\zeta\right)^{e_\omega} = \left(1 + \frac{v}{v+u}\pi\right)^{e_\omega}$$

and the various congruences of Lemma 3, $\gamma_\omega \equiv \zeta^h \pmod{\pi^2}$, with $h = \frac{v}{u+v}$ (nonspecial cases, $p \geq 3$), $h = \frac{1}{2}$ (special case, $p > 3$), and $h = \frac{1}{2} - \frac{u+v}{3v}$ (special case, $p = 3$). Then we obtain $\gamma_\omega \zeta^{-h} \equiv 1 \pmod{\pi^2}$, which implies easily that $\gamma_\omega \zeta^{-h}$ is a p -primary number (use [Gr1, Lemma 3.15]); but since this number is not in general the p th power of an ideal it may not be a global p th power.⁷

So, from class field theory, there exist infinitely many prime ideals \mathfrak{q}_K of K , prime to uv , such that $(1 + \frac{v}{u}\zeta)^{e_\omega} \zeta^{-h}$ is not a local p th power at \mathfrak{q}_K , except if we have a counterexample (u, v) to SFLT in which case such primes do not exist.

The p th power residue symbol of this number is invariant by conjugation of \mathfrak{q}_K since:

$$\left(\left(1 + \frac{v}{u}\zeta\right)\zeta^{-h}\right)^{e_\omega \kappa} \equiv \zeta' \pmod{\mathfrak{q}_K}$$

⁷In the case where $(u+v\zeta)\mathbb{Z}[\zeta] = \mathfrak{w}_1^p$, Lemma 3 shows that $(1 + \frac{v}{u}\zeta)^{e_\omega} \zeta^{-h} \in K^{\times p}$; so in this particular case, where $(1 + \frac{v}{u}\zeta)^{e_\omega} \zeta^{-h}$ is a pseudo-unit, local p th power at p , we get a necessary and sufficient condition to get a global p th power.

implies, by conjugation by $s_k \in g$:

$$\left(\left(1 + \frac{v}{u} \zeta^k \right) \zeta^{-kh} \right)^{e_\omega \kappa} \sim \left(\left(1 + \frac{v}{u} \zeta \right) \zeta^{-h} \right)^{k e_\omega \kappa} \equiv \zeta'^k \pmod{s_k(\mathfrak{q}_K)}$$

equivalent (up to p th powers) to:

$$\left(\left(1 + \frac{v}{u} \zeta \right) \zeta^{-h} \right)^{e_\omega \kappa} \equiv \zeta' \pmod{s_k(\mathfrak{q}_K)};$$

so the symbol only depends on q , the prime number under \mathfrak{q}_K which does not divide uv .

We suppose $\frac{v}{u}$ of order n modulo q (which is equivalent to $q \mid \Phi_n(u, v)$ and $q \nmid n$), and we suppose n prime to p .

Let \mathfrak{q} be a prime ideal above q in $\mathbb{Q}(\mu_n)$. We have equality of the p th power residue symbols of $\left(1 + \frac{v}{u} \zeta \right)^{e_\omega} \zeta^{-h}$ at any \mathfrak{q}_K in K , and of the cyclotomic unit $(1 + \xi \zeta)^{e_\omega} \zeta^{-h}$ at \mathfrak{Q} in $\mathbb{Q}(\mu_n)K$, where ξ is a suitable n th root of unity and \mathfrak{Q} is any prime ideal above \mathfrak{q} in $\mathbb{Q}(\mu_n)K$ (ξ is characterized by the congruence $\xi \equiv \frac{v}{u} \pmod{\mathfrak{q}}$ in $\mathbb{Q}(\mu_n)$, and \mathfrak{q}_K must be $\mathfrak{Q} \cap \mathbb{Z}[\zeta]$).

Of course h is a priori unknown (but constant with respect to q) and the local study of $(1 + \xi \zeta)^{e_\omega} \zeta^{-h}$ is ineffective in general, but we may use some partial informations, as the following ones in the context of FLT.

Let (x, y, z) be a solution of Fermat's equation (first or second cases).

a) Take for instance $u := x$, $v := y$ (so we are in the nonspecial cases of SFLT) which gives $h = \frac{y}{y+x}$.

- If ζ is not a local p th power at \mathfrak{q}_K (which is equivalent to $\kappa \not\equiv 0 \pmod{p}$), we will consider the p th power residue symbol at \mathfrak{Q} of the real cyclotomic unit $\eta_1 := (1 + \xi \zeta)^{e_\omega} \zeta^{-\frac{1}{2}}$ (see Definition 4 in Subsection 3.1) which must be that of $\zeta^{h-\frac{1}{2}} = \zeta^{\frac{1}{2} \frac{y-x}{y+x}}$. For FLT we have some informations on the differences like $y - x$, $y - z$, which are prime to p for $p > 3$ or $p = 3$ in the second case; in these cases a contradiction to the existence of such a solution of Fermat's equation is that the unit $(1 + \xi \zeta)^{e_\omega} \zeta^{-\frac{1}{2}}$ be a local p th power at \mathfrak{Q} or does not give the "good" symbol.

For $p = 3$ in the first case, we know that $x \equiv y \equiv z \equiv \pm 1 \pmod{3}$; so a contradiction is that this unit be not a local 3th power at \mathfrak{Q} .

- If ζ is a local p th power at \mathfrak{q}_K (which is equivalent to $\kappa \equiv 0 \pmod{p}$), a contradiction is that the unit η_1 be not a local p th power at \mathfrak{Q} .

b) In the second case of FLT ($p \mid y$) with $u = x$, $v = z$ (special case of SFLT) we have different but similar reasonings using the value of $h - \frac{1}{2}$ given by Lemma 3 for $p \geq 3$ since $x + z \equiv 0 \pmod{9}$ when $p = 3$.

The hope in this attempt is that, the arithmetical properties of the fields $\mathbb{Q}(\mu_n) \subseteq \mathbb{Q}(\mu_{q-1})$ being a priori independent of the SFLT problem, they may give valuable indications on the local properties of $\eta_1 = (1 + \xi \zeta)^{e_\omega} \zeta^{-\frac{1}{2}}$, especially in an analytic point of view. In some sense the fields $\mathbb{Q}(\mu_{q-1})$ will play the role of governing fields for this problem. Indeed, under a

solution of the SFLT equation, the residue symbol above \mathfrak{q} of this unit is, *independently of the choice of q* , equal to the symbol of a *constant* power of ζ , which may be absurd.

These cyclotomic fields have been introduced by Vandiver in some papers as [Van1, Van2, Van3] to generalize classical results of Kummer and some congruences giving Furtwängler's theorems and Wieferich's criteria; these papers essentially depend on the Stickelberger element $S := \frac{1}{p} \sum_{k=1}^{p-1} k s_k^{-1}$, related to the generalized Bernoulli numbers and the annihilation of the p -class group of K .

Meanwhile some of the relations of Lemma 3 are considered by Vandiver for other purposes than our's. In Vandiver papers, analytic results (like Čebotarev's theorem) or class field theory are not used, and it seems that no method of contradiction can be deduced from these computations which are essentially *local at p* , and it has been explained in [Gr1] the probable inefficiency of such local studies. Our present work is mainly global and does not concern the arithmetic of K as in the historical researches.

Lemma 4. *The equation $(u + v\zeta)\mathbb{Z}[\zeta] = \mathfrak{w}_1^p$ or $\mathfrak{p}\mathfrak{w}_1^p$, in integers u, v , with $\text{g.c.d.}(u, v) = 1$, is equivalent to the equation $(u + v\zeta)^{e_\omega} \in \zeta^h K^{\times p}$ with $h \equiv \frac{v}{u+v} \pmod{p}$ in the nonspecial cases, $p \geq 3$, $h \equiv \frac{1}{2} \pmod{p}$ in the special case, $p > 3$, and $h \equiv \frac{1}{2} - \frac{u+v}{3v} \pmod{3}$ in the special case, $p = 3$.*

Proof. A direction being proved (Lemma 3), let $\mathfrak{l} \neq \mathfrak{p}$ be any prime ideal dividing the ideal $(u + v\zeta)\mathbb{Z}[\zeta]$; the use of the congruences $u + v\zeta \equiv 0 \pmod{\mathfrak{l}}$ implies that $(u + v\zeta)\mathbb{Z}[\zeta] = \mathfrak{p}^\delta \prod_{\ell} \mathfrak{l}^{\alpha_\ell}$, $\delta = 0$ or 1 , $\alpha_\ell \geq 1$, for distinct prime numbers ℓ with a single $\mathfrak{l} | \ell$ (otherwise, using appropriate conjugations, we get $u \equiv v \equiv 0 \pmod{\mathfrak{l}}$). Moreover, it shows that \mathfrak{l} is of degree 1 and that g operates transitively on the set of conjugates of \mathfrak{l} ; hence, since $\mathfrak{p}^{e_\omega} = \mathbb{Z}[\zeta]$ and since $(u + v\zeta)^{e_\omega} \mathbb{Z}[\zeta] = \prod_{\ell} \mathfrak{l}^{\alpha_\ell e_\omega}$ is a p th power by assumption, we get $\alpha_\ell \equiv 0 \pmod{p}$ for all ℓ . \square

From any relation $(u' + v'\zeta)^{e_\omega} \in \langle \zeta \rangle \cdot K^{\times p}$, $u', v' \in \mathbb{Z}$, we deduce the solution $(u, v) := \frac{1}{\text{g.c.d.}(u', v')} (u', v')$ of the SFLT equation with a unique h .

We call the second equation the ω -SFLT equation; the corresponding form of the SFLT conjecture for $p > 3$ seems reasonable as soon as p is sufficiently large since it enunciates (for $uv(u^2 - v^2) \neq 0$) that there exists a sum $\sum_{k=1}^{p-1} \lambda_k \zeta^k$, $\lambda_k \in \mathbb{Q}$, whose p th power is of the form:

$$(u\zeta^{-\frac{v}{u+v}} + v\zeta^{\frac{u}{u+v}})^{e_\omega} \quad (\text{resp. of the form } (u\zeta^{-\frac{1}{2}} + v\zeta^{\frac{1}{2}})^{e_\omega}),$$

depending on two coefficients u, v instead of $p - 1$ in general. It will be interesting to have the response at least for $p = 5$.

So for $p > 3$ the truth of SFLT would imply FLT; in this paper we concentrate our attention mainly on SFLT, using the simpler ω -SFLT context which does not concern mainly the arithmetic of K , the nerve center of the unsuccessful classical theory.

For a recent critical history on FLT see [Co]. For some complements on these cyclotomic technics, see [He1, He2, Ter, Ri].

3. Introduction of the governing fields $\mathbb{Q}(\mu_{q-1})$

3.1. Furtwängler and Vandiver revisited. Consider the SFLT equation:

$$(u + v\zeta)\mathbb{Z}[\zeta] = \mathfrak{w}_1^p \text{ or } \mathfrak{p}\mathfrak{w}_1^p,$$

in integers u, v , with $\text{g.c.d.}(u, v) = 1$, independently of FLT.

Let q be a prime number such that $q \nmid uv$ and such that $\frac{v}{u}$ is of order n modulo q (which is equivalent from Lemma 2 to $q \nmid n$ and $q \mid \Phi_n(u, v)$), with n prime to p .

In another point of view, for a given n prime to p , the primes $q \mid \Phi_n(u, v)$ are solutions (i.e., $\frac{v}{u}$ is of order n modulo q), if and only if $q \nmid n$. In practice the condition $q \nmid n$ is always satisfied because we are only concerned with large primes q , so that $q \equiv 1 \pmod{n}$.

Consider the following diagram, where $L := \mathbb{Q}(\mu_n)$, $M := LK$, and $G = \text{Gal}(M/L) \simeq g$ (we have $L \cap K = \mathbb{Q}$):

$$\begin{array}{ccc} L = \mathbb{Q}(\mu_n) & \xrightarrow{G} & M \\ \downarrow & & \downarrow \\ \mathbb{Q} & \xrightarrow{g} & K = \mathbb{Q}(\zeta) \end{array}$$

Definition 3. The prime $q \equiv 1 \pmod{n}$ being totally split in L/\mathbb{Q} , if \mathfrak{q} is a prime ideal of L over q , there exists a *unique* primitive n th root of unity ξ such that $\xi \equiv \frac{v}{u} \pmod{\mathfrak{q}}$. Reciprocally, if ξ is a primitive n th root of unity, there exists a *unique* prime ideal \mathfrak{q} of L over q such that $\xi \equiv \frac{v}{u} \pmod{\mathfrak{q}}$. This ideal is $(q, u\xi - v)$ and will also be denoted \mathfrak{q}_ξ (it depends on u, v).

We associate with q (for u, v fixed) a pair (ξ, \mathfrak{q}) where the prime ideal $\mathfrak{q} := \mathfrak{q}_\xi$ above q and the primitive n th root of unity ξ are characterized by the congruence $\xi \equiv \frac{v}{u} \pmod{\mathfrak{q}}$ in L .

This pair is defined up to \mathbb{Q} -conjugation since $\xi \equiv \frac{v}{u} \pmod{\mathfrak{q}_\xi}$ is equivalent to $\xi^t \equiv \frac{v}{u} \pmod{\mathfrak{q}_\xi^t = \mathfrak{q}_{\xi^t}}$, for all $t \in \text{Gal}(L/\mathbb{Q})$. We obtain an equivalence relation. The class only depends on q for u, v given. \square

Taking a representative pair, we will fix ξ (for instance $\xi = \exp(2i\pi/n)$) which defines \mathfrak{q}_ξ .

Since $\frac{v}{u}$ modulo q is unknown but well-defined, we must note that, in what follows, the class is ineffective among $\phi(n)$ possible classes, and for each n prime to p dividing $q-1$. This explains that, in some circumstances, we will have to take $q \not\equiv 1 \pmod{p}$ since, if not, it is not possible to assert that $\frac{v}{u}$ is of order modulo q prime to p .

Definition 4. For the given n th root of unity ξ , $n \not\equiv 0 \pmod{p}$, we consider the cyclotomic number of M associated to ξ :⁸

$$\eta := \eta(\xi) := (1 + \xi \zeta) \zeta^{-\frac{1}{2}} \in M,$$

where $\zeta^{\frac{1}{2}}$ is the unique p th root of unity such that $(\zeta^{\frac{1}{2}})^2 = \zeta$ (so, the exponent $\frac{1}{2}$ is seen as a p -adic integer or as an element of $(\mathbb{Z}/p\mathbb{Z})^\times$). This is coherent with the context of p -Kummer theory above M .

Then we put:

$$\eta_1 := \eta^{e_\omega} = (1 + \xi \zeta)^{e_\omega} \zeta^{-\frac{1}{2}} \in M;$$

we have $\eta_1 \in M^+$, where M^+ is the maximal real subfield of M : indeed, if c is the complex conjugation, then:

$$\eta_1^c = (1 + \xi^{-1} \zeta^{-1})^{e_\omega} \zeta^{\frac{1}{2}} = ((1 + \xi \zeta) \xi^{-1} \zeta^{-1} \zeta^{\frac{1}{2}})^{e_\omega} = (1 + \xi \zeta)^{e_\omega} \zeta^{-\frac{1}{2}} = \eta_1,$$

since $\xi^{e_\omega} = 1$ and $\zeta'^{e_\omega} = \zeta'$ for any $\zeta' \in \mu_p$. \square

We note that η_1 is a cyclotomic unit and that $\eta_1 \equiv 1 \pmod{\pi Z_M}$.

Starting from $\xi \equiv \frac{v}{u} \pmod{\mathfrak{q}}$ and extending \mathfrak{q} to M we obtain:⁹

$$\eta_1 \equiv (1 + \frac{v}{u} \zeta)^{e_\omega} \zeta^{-\frac{1}{2}} \pmod{\prod_{\Omega|\mathfrak{q}} \Omega}.$$

We note that these prime ideals Ω of M may be written Ω_ξ since they are above \mathfrak{q}_ξ ; for ξ fixed, they are conjugated by the elements of G .

From Lemma 3, we get $(1 + \frac{v}{u} \zeta)^{e_\omega} = \zeta^{\frac{v}{v+u}} \cdot \delta_\omega^p$ (nonspecial cases, $p \geq 3$) or $\zeta^{\frac{1}{2}} \cdot \delta_\omega^p$ (special case, $p > 3$) or $\zeta^{\frac{1}{2} - \frac{v+u}{3v}} \cdot \delta_\omega^3$ (special case, $p = 3$), with $\delta_\omega \in K^\times$, giving $\eta_1 \equiv \zeta^{\frac{1}{2} \frac{v-u}{v+u}} \cdot \delta_\omega^p$ or δ_ω^p or $\zeta^{\frac{1}{2} \frac{v+u}{3v}} \cdot \delta_\omega^3 \pmod{\prod_{\Omega|\mathfrak{q}} \Omega}$.

From the congruences on η_1 , Definition 2, Corollaries 2 and 3, we then have obtained in the only context of SFLT the following result which includes the case $p = 3$:

Theorem 1. Let p be a prime number, $p \geq 3$. Suppose given the relation $(u + v \zeta) \mathbb{Z}[\zeta] = \mathfrak{w}_1^p$ or $\mathfrak{p} \mathfrak{w}_1^p$ in coprime integers u, v , where \mathfrak{w}_1 is an ideal of $K := \mathbb{Q}(\zeta)$, $\zeta^p = 1$, $\zeta \neq 1$, and $\mathfrak{p} := (\zeta - 1) \mathbb{Z}[\zeta]$.

Let $q \neq p$, $q \nmid uv$, be a prime number such that $\frac{v}{u}$ is of order n modulo q , with n prime to p ; put $\eta := (1 + \xi \zeta) \zeta^{-\frac{1}{2}}$, $\eta_1 := \eta^{e_\omega}$, where ξ is a primitive n th root of unity (see Definition 4). Put $\mathfrak{q} := (q, u \xi - v)$ in $L := \mathbb{Q}(\mu_n)$.

We get in $M := LK$:

⁸ We know that $1 + \xi \zeta$ is a (cyclotomic) unit except if $-\xi \zeta$ is of prime power order, which is the case if and only if $\xi = -1$ (i.e., $n = 2$) in which case $1 + \xi \zeta$ is a p -unit.

⁹ Warning: if for instance $\frac{v}{u}$ is of order dp modulo q , with $p \nmid d$, q is totally split in M/\mathbb{Q} and we have the congruence $\frac{v}{u} \equiv \xi =: \psi \zeta_1 \pmod{\Omega}$, for some $\Omega | q$ in M , where ψ is of order d and ζ_1 of order p ; but in the relation $1 + \frac{v}{u} \zeta \equiv 1 + \xi \zeta \pmod{\Omega}$, the root $\xi = \psi \zeta_1$ is not invariant by G so that the congruence $(1 + \frac{v}{u} \zeta)^{e_\omega} \equiv (1 + \xi \zeta)^{e_\omega} \pmod{\Omega}$ does not exist. From $\gamma := 1 + \frac{v}{u} \zeta$ we get $s_k(\gamma) := 1 + \frac{v}{u} \zeta^k$ and if $e_\omega = \sum_k u_k s_k$ we obtain instead $\gamma^{e_\omega} = (1 + \frac{v}{u} \zeta)^{e_\omega} \equiv \prod_k (1 + \psi \zeta_1 \zeta^k)^{u_k} \pmod{\Omega}$, in which the term $1 + \psi$ is not always a cyclotomic unit (see [Que]).

$$\begin{aligned} \left(\frac{\eta_1}{\Omega}\right)_M &= \zeta^{\frac{1}{2} \frac{v-u}{v+u} \kappa}, \forall \Omega \mid \mathfrak{q}, \text{ in the nonspecial cases } (p \nmid u+v), p \geq 3,^{10} \\ \left(\frac{\eta_1}{\Omega}\right)_M &= 1, \forall \Omega \mid \mathfrak{q}, \text{ in the special case } (p \mid u+v), p > 3, \\ \left(\frac{\eta_1}{\Omega}\right)_M &= \zeta^{\frac{1}{2} \frac{v+u}{3v} \kappa}, \forall \Omega \mid \mathfrak{q}, \text{ in the special case, } p = 3. \quad \square \end{aligned}$$

These relations show that $\left(\frac{\eta_1}{\Omega}\right)_M$ only depends on the Fermat quotient of q once u, v are given. Note that the class of the pairs (η_1^t, Ω^t) , $t \in \text{Gal}(M/K)$, for any choice of $\Omega \mid \mathfrak{q}$ in M , corresponds canonically to the class of the (ξ^t, \mathfrak{q}^t) , since we have the relation $\left(\frac{\eta_1}{\Omega}\right)_M^t = \left(\frac{\eta_1}{\Omega}\right)_M = \left(\frac{\eta_1^t}{\Omega^t}\right)_M$, where $\Omega^t \mid \mathfrak{q}^t$, and $\eta_1^t = (1 + \xi^t \zeta)^{e_\omega} \zeta^{-\frac{1}{2}}$ (see Definitions 2, 3, and 4).

The symbol $\left(\frac{\eta_1}{\Omega}\right)_M$ may be different from $\left(\frac{\eta_1}{\Omega^t}\right)_M$, $t \neq 1$, since there is no local information on $\frac{1 + \xi^t \zeta}{1 + \xi \zeta}$. But as we have seen, $\left(\frac{\eta_1}{\Omega}\right)_M = \left(\frac{\eta_1}{s\Omega}\right)_M$ for any $s \in G$.

Remark 2. Since for g.c.d. $(u, v) = 1$ the equation $(u + v\zeta)\mathbb{Z}[\zeta] = \mathfrak{w}_1^p$ or $\mathfrak{p}\mathfrak{w}_1^p$ is equivalent to the equation $N_{K/\mathbb{Q}}(u + v\zeta) = w_1^p$ or pw_1^p , we deduce from $u + v\zeta \equiv u(1 + \xi\zeta) \pmod{\Omega}$ for all $\Omega \mid \mathfrak{q}$, that (the case $n = 2$ giving $\left(\frac{u}{\mathfrak{q}_K}\right)_K = \left(\frac{v}{\mathfrak{q}_K}\right)_K = \left(\frac{p}{\mathfrak{q}_K}\right)_K$ in the nonspecial cases, = 1 otherwise):

$$N_{M/L}(u + v\zeta) \equiv N_{M/L}(u(1 + \xi\zeta)) \equiv u^{p-1} \frac{1 + \xi^p}{1 + \xi} = u^{p-1} (1 + \xi)^{t_p - 1} \pmod{\Omega},$$

for all $\Omega \mid \mathfrak{q}$, where t_p is the Frobenius automorphism of p in L . This gives:

$$\left(\frac{(1 + \xi)^{t_p - 1}}{\Omega}\right)_M = \left(\frac{u}{\Omega}\right)_M = \left(\frac{v}{\Omega}\right)_M \quad (\text{resp. } = \left(\frac{pu}{\Omega}\right)_M = \left(\frac{pv}{\Omega}\right)_M),$$

in the nonspecial cases (resp. the special case), for all $\Omega \mid \mathfrak{q}$, with $\mathfrak{q} = \mathfrak{q}_\xi$. \square

From a solution (x, y, z) of Fermat's equation, we get the three relations (see Subsection 2.1):

$$(x + y\zeta)\mathbb{Z}[\zeta] = \mathfrak{z}_1^p, \quad (z + y\zeta)\mathbb{Z}[\zeta] = \mathfrak{x}_1^p, \quad (x + z\zeta)\mathbb{Z}[\zeta] = \eta_1^p \text{ or } \mathfrak{p}\eta_1^p.$$

For $p > 3$, the conditions:

$$p \nmid x^2 - y^2, \quad p \nmid z^2 - y^2, \quad p \nmid x + z \quad (\text{resp. } p \nmid x - z),$$

in the first (resp. second) case, and the conditions:

$$p \nmid x^2 - y^2, \quad p \nmid z^2 - y^2,$$

in the second case, are satisfied by choice of the notations (see Lemma 1).

¹⁰In the first case of SFLT for $p > 3$ we may have $u - v \equiv 0 \pmod{p}$ (then $u - v \equiv 0 \pmod{p^2}$) contrary to FLT with $v := y$ and $u := x$ or z . For $p = 3$, $u - v \equiv 0 \pmod{9}$ in the first case.

If $n \leq 2$ about the relation $(x + y\zeta)\mathbb{Z}[\zeta] = \mathfrak{z}_1^p$ or $(z + y\zeta)\mathbb{Z}[\zeta] = \mathfrak{x}_1^p$ (i.e., $q|x^2 - y^2$ or $q|z^2 - y^2$), then $M = K$, $\Omega = \mathfrak{q}_K|q$ in K . Since $\eta_1 = (1 \pm \zeta)^{e_\omega} \zeta^{-\frac{1}{2}} \in K^{\times p}$, we get:

$$\zeta^{\frac{1}{2} \frac{y-x}{y+x} \kappa} = 1 \quad \text{or} \quad \zeta^{\frac{1}{2} \frac{y-z}{y+z} \kappa} = 1.$$

Then these two values of n give again the second theorem of Furtwängler [Fur] in the context of FLT for $p > 3$, that is the fact that when $q|x^2 - y^2$ or $q|z^2 - y^2$, then $\zeta^\kappa = 1$, which means $\kappa \equiv 0 \pmod{p}$ (see Corollaries 2 and 3 generalizing the FLT situation to SFLT).

We have the same conclusion in the first case of FLT, with the supplementary condition $p \nmid x - z$, if $q|x^2 - z^2$ (in the second case of FLT, this does not work for (x, z) since $p|x + z$).

Remark 3. (Furtwängler's theorems and FLT; see e.g. [Gr1, Appendix] or [Ri, IX, 3]). Let (x, y, z) be a solution of Fermat's equation for $p > 3$, under the conditions of Lemma 1.

(i) Recall that the first theorem of Furtwängler giving Wieferich criteria is that for any prime number $q \neq p$, if $q|x$ or z (or y in the first case), then $\kappa \equiv 0 \pmod{p}$.

Of course, if $q|x + y$ or $z + y$ (or $x + z$ in the first case), then from Subsection 2.1, (i) with obvious notations, $q|z_0$ or x_0 (or y_0 in the first case), giving $\kappa \equiv 0 \pmod{p}$ (from the first theorem of Furtwängler) what we can call the first part of the second theorem of Furtwängler, the second part being that if $q|x - y$ or $z - y$ (or $x - z$ in the second case), then $\kappa \equiv 0 \pmod{p}$.

(ii) If $q|x$ or z (or y in the first case) when $q \not\equiv 1 \pmod{p}$, then from Subsection 2.1, (iii), $q|x_0$ or z_0 (or y_0 in the first case). Then we deduce that $q^p|y + z = x_0^p$ or $x + y = z_0^p$ (or $x + z = y_0^p$ in the first case). This means, since $q \nmid yz$ or xy (or xz in the first case), that $\frac{y}{z}$ or $\frac{y}{x}$ (or $\frac{x}{z}$ in the first case) is of order 2 modulo q , giving again the first part of the second theorem of Furtwängler and $\kappa \equiv 0 \pmod{p}$.

The two results are not independent in the case $q \not\equiv 1 \pmod{p}$. For some other remarks on Furtwängler's theorems, see [Que].

(iii) So, if we choose $q \not\equiv 1 \pmod{p}$ such that $\kappa \not\equiv 0 \pmod{p}$, this implies that $q \nmid xyz$ in the first case of FLT, and $q \nmid xz$ in the second case of FLT. Thus, under these assumptions on q , the hypothesis $q \nmid xyz$ (in the first case) or $q \nmid xz$ (in the second case) are useless for the development of our method and give effective tests in practice.

It remains the case $q|y$ in the second case ($p|y$). When $q \not\equiv 1 \pmod{p}$, $q|y_0$, then $q|x + z$; we obtain that $q \nmid xz$ and $q|x + z$ but we cannot conclude, except that the root ξ'' associated to $\frac{x}{z}$ is -1 . To eliminate the case $q|y$ in the second case we must suppose q large enough, which is not effective.

(iv) In any case of FLT we have the following result (see [Ri, IV.3]). If $q \neq p$ divides y and does not divide $x + z$ then $q \equiv 1 \pmod{p^2}$. Indeed, since $q \nmid x + z = y_0^p$ or $p^{\nu} p^{-1} y_0^p$, we have $q \mid y_1$ and $q = 1 + dp$.

Suppose that $p \nmid d$; since $y + x = z_0^p$ and $y + z = x_0^p$, we see that x and z are p th powers modulo q and that $x^d \equiv z^d \equiv 1 \pmod{q}$ giving $x^d - z^d \equiv 0$ with $x^p + z^p \equiv 0 \pmod{q}$. Since d is even this may be written $x^d \equiv (-z)^d$ and $x^p \equiv (-z)^p \pmod{q}$ with $\text{g.c.d.}(d, p) = 1$ which yields to $x \equiv -z \pmod{q}$ (absurd). So $q \equiv 1 \pmod{p^2}$.

This result is valid by cyclic permutation of x, y, z , only in the first case of FLT since p (in $p^{\nu} p^{-1} y_0^p$) may not be a p th power modulo q . \square

If we suppose that (x, y, z) (with the choices of Lemma 1) is a solution of Fermat's equation, we obtain, from Theorem 1 and the fact that in the second case for $p = 3$, $x + z \equiv 0 \pmod{9}$ (special case $(u, v) = (x, z)$ with $u + v \equiv 0 \pmod{9}$):

Corollary 4. *Suppose that the prime $q \neq p$ is given such that $q \nmid xyz$ and such that $\frac{y}{x}, \frac{y}{z}, \frac{x}{z}$ are of orders n, n', n'' (modulo q) prime to p .*

Let ξ, ξ', ξ'' in $\mathbb{Q}(\mu_{q-1})$, of orders n, n', n'' , and let $\mathfrak{q}, \mathfrak{q}', \mathfrak{q}''$ dividing q in $L = \mathbb{Q}(\mu_n), L' = \mathbb{Q}(\mu_{n'}), L'' = \mathbb{Q}(\mu_{n''})$, built from $\frac{y}{x}, \frac{y}{z}, \frac{x}{z}$; then consider the corresponding cyclotomic units $\eta_1, \eta'_1, \eta''_1$. We then have:

(i) *First case of FLT for $p > 3$:*

$$\left(\frac{\eta_1}{\Omega}\right)_M = \zeta^{\frac{1}{2} \frac{y-x}{y+x} \kappa}, \quad \left(\frac{\eta'_1}{\Omega'}\right)_{M'} = \zeta^{\frac{1}{2} \frac{y-z}{y+z} \kappa}, \quad \left(\frac{\eta''_1}{\Omega''}\right)_{M''} = \zeta^{\frac{1}{2} \frac{x-z}{x+z} \kappa},$$

with $y - x \not\equiv 0$ and $y - z \not\equiv 0 \pmod{p}$.¹¹

(ii) *First case of FLT for $p = 3$:*

$$\left(\frac{\eta_1}{\Omega}\right)_M = \left(\frac{\eta'_1}{\Omega'}\right)_{M'} = \left(\frac{\eta''_1}{\Omega''}\right)_{M''} = 1.$$

(iii) *Second case of FLT for $p \geq 3$ ($y \equiv x + z \equiv 0 \pmod{p}$):*

$$\left(\frac{\eta_1}{\Omega}\right)_M = \zeta^{-\frac{1}{2} \kappa}, \quad \left(\frac{\eta'_1}{\Omega'}\right)_{M'} = \zeta^{-\frac{1}{2} \kappa}, \quad \left(\frac{\eta''_1}{\Omega''}\right)_{M''} = 1. \quad \square$$

Remark 4. (a) Suppose that we are in the first case of FLT for $p > 3$; let $q \neq p$ be a prime number such that $\kappa \not\equiv 0 \pmod{p}$, and let n and n' be the orders of $\frac{y}{x}$ and $\frac{y}{z}$ modulo q ; we suppose that $p \nmid n n'$ (we have $n, n' > 2$ from the second theorem of Furtwängler, and from Remark 3, (i), on the first theorem of Furtwängler, we know that $q \nmid xyz$).

If we find, with independent reasons, that at least one of the symbols $\left(\frac{\eta_1}{\Omega}\right)_M$ or $\left(\frac{\eta'_1}{\Omega'}\right)_{M'}$ is trivial, this is absurd since by definition $x - y \not\equiv 0$ and $z - y \not\equiv 0 \pmod{p}$ under a solution of Fermat's equation (cf. (i)).

¹¹ Recall that for $p > 3$ we have no information on $x - z$ modulo p in the first case, so that we cannot consider the third symbol in some reasonings using the above property.

The reasoning on the third symbol does not work since $x - z$ can be divisible by p .

(b) For $p = 3$ in the first case, all the right members are trivial under a solution of the first case of FLT and the above reasoning is different but a contradiction arises as soon as an independent fact implies the nontriviality of one of these symbols (cf. (ii)).

(c) In the second case for $p \geq 3$, when $\kappa \not\equiv 0 \pmod{p}$, we know that $q \nmid xz$. Since $p \nmid nn'$, we deduce that $p \nmid n''$. The symbol $\left(\frac{\eta_1''}{\Omega''}\right)_{M''}$ is trivial under a solution of Fermat's equation (cf. (iii)) and a contradiction arises if not.

To have a similar reasoning as for the first case with the two other nontrivial symbols associated to ξ and ξ' , we need the condition $q \nmid y$, so that we must suppose q large enough (in practice, to get a contradiction, we need the existence of infinitely many q such that at least one of the symbols $\left(\frac{\eta_1}{\Omega}\right)_M, \left(\frac{\eta_1'}{\Omega'}\right)_{M'}$ is trivial).

(d) If $\kappa \equiv 0 \pmod{p}$, in any case all the symbols are trivial under a solution of Fermat's equation. So a contradiction supposes that for infinitely many such q we get, independently, nontrivial symbols.

(e) We can use the above remarks to give the following reciprocal statements, for $p > 3$ to simplify; we suppose that any solution (x, y, z) of Fermat's equation satisfies the conventions of Lemma 1.

Let ξ be a primitive n th root of unity with $n \not\equiv 0 \pmod{p}$, let $\eta_1 := (1 + \xi \zeta)^{e\omega} \zeta^{-\frac{1}{2}}$ (Definition 4), and let $q \equiv 1 \pmod{n}$. Consider an arbitrary ideal $\mathfrak{q} \mid q$ in $L := \mathbb{Q}(\mu_n)$ and any prime ideal $\Omega \mid \mathfrak{q}$ in $M := LK$.

We suppose given integers u, v with $\text{g.c.d.}(u, v) = 1$, such that $q \nmid uv$ and $\frac{v}{u} \equiv \xi \pmod{\mathfrak{q}}$.

- If $\kappa \not\equiv 0 \pmod{p}$ and $\left(\frac{\eta_1}{\Omega}\right)_M = 1$, we then have:

If $u + v \not\equiv 0 \pmod{p}$, (u, v) cannot be a part of a solution $(x, y, z) = (u, v, z), (v, u, z), (x, v, u)$, or (x, u, v) of Fermat's equation.

- If $\kappa \not\equiv 0 \pmod{p}$ and $\left(\frac{\eta_1}{\Omega}\right)_M \neq 1$, we then have:

If $u + v \equiv 0 \pmod{p}$, (u, v) cannot be a part of a solution $(x, y, z) = (u, y, v)$ or (v, y, u) of the second case of Fermat's equation.

- If $\kappa \equiv 0 \pmod{p}$ and $\left(\frac{\eta_1}{\Omega}\right)_M \neq 1$, we then have:

The pair (u, v) cannot be a part of a solution (x, y, z) of any case of Fermat's equation. \square

Proposition 1. Let (x, y, z) be a solution of Fermat's equation; then let $q \nmid xyz$ be such that $\frac{y}{x}, \frac{y}{z}, \frac{x}{z}$ are of orders n, n', n'' (modulo q) prime to p , and let $\tilde{L} := \mathbb{Q}(\mu_d)$ where $q = 1 + dp^r, r \geq 0, p \nmid d$.

Let ξ, ξ', ξ'' , of orders n, n', n'' , be such that $\xi \equiv \frac{y}{x} \pmod{\mathfrak{q}_\xi}$, $\xi' \equiv \frac{y}{z} \pmod{\mathfrak{q}_{\xi'}}$, $\xi'' \equiv \frac{x}{z} \pmod{\mathfrak{q}_{\xi''}}$ in L, L', L'' , respectively.

Then there exist a prime ideal $\tilde{\mathfrak{q}} \mid q$ of \tilde{L} and $t', t'' \in \text{Gal}(\tilde{L}/\mathbb{Q})$ such that the following congruences hold:

$$(i) \quad \xi^{t'} \equiv \frac{-\xi}{\xi+1} \pmod{\tilde{\mathfrak{q}}},$$

$$(ii) \quad \xi^{t''} \equiv \frac{-1}{\xi+1} \pmod{\tilde{\mathfrak{q}}}.$$

Proof. Since L, L', L'' are subfields of \tilde{L} , there exist prime ideals $\tilde{\mathfrak{q}}_0, \tilde{\mathfrak{q}}'_0, \tilde{\mathfrak{q}}''_0$ of \tilde{L} dividing $\mathfrak{q}_\xi, \mathfrak{q}_{\xi'}, \mathfrak{q}_{\xi''}$, respectively, such that:

$$\xi \equiv \frac{y}{x} \pmod{\tilde{\mathfrak{q}}_0}, \quad \xi' \equiv \frac{y}{z} \pmod{\tilde{\mathfrak{q}}'_0}, \quad \xi'' \equiv \frac{x}{z} \pmod{\tilde{\mathfrak{q}}''_0}.$$

The ideals $\tilde{\mathfrak{q}}'_0$ and $\tilde{\mathfrak{q}}''_0$ are some conjugates of $\tilde{\mathfrak{q}}_0$ and there exist $t', t'' \in \text{Gal}(\tilde{L}/\mathbb{Q})$ such that $\xi \equiv \frac{y}{x}, \xi^{t'} \equiv \frac{y}{z}, \xi^{t''} \equiv \frac{x}{z} \pmod{\tilde{\mathfrak{q}}_0}$.

From $x^p + y^p + z^p = 0$ we get $\left(\frac{x}{y}\right)^p + \left(\frac{z}{y}\right)^p = -1$ giving:

$$\xi^{-p} + (\xi^{t'})^{-p} \equiv -1 \pmod{\tilde{\mathfrak{q}}_0}.$$

Since $p \nmid d$, we can use the inverse of the Frobenius automorphism t_p of p in \tilde{L}/\mathbb{Q} , which gives easily the relation (i) (for $\tilde{\mathfrak{q}} := t_p^{-1}(\tilde{\mathfrak{q}}_0)$).

From the obvious relation $\xi^{t''} \xi^{t'-t''} \xi \equiv 1 \pmod{\tilde{\mathfrak{q}}_0}$, which implies the equality $\xi^{t''} \xi^{t'-t''} \xi = 1$, we obtain the point (ii) since $\xi \neq -1$.¹² \square

Corollary 5. Let $m := \text{l.c.m.}(n', n'')$; then we have $\phi(m) > \frac{\log(q)}{\log(3)}$.

Proof. We have $\xi^{t''} + \xi^{t'} + 1 \equiv 0 \pmod{\tilde{\mathfrak{q}}}$; then $\xi^{t''} + \xi^{t'} + 1 \in \mathbb{Q}(\mu_m)$ by definition of m , and $N_{\mathbb{Q}(\mu_m)/\mathbb{Q}}(\xi^{t''} + \xi^{t'} + 1) = qN$, $N \geq 1$. Since $N_{\mathbb{Q}(\mu_m)/\mathbb{Q}}(\xi^{t''} + \xi^{t'} + 1) < 3^{\phi(m)}$, we get $N < \frac{1}{q} 3^{\phi(m)}$, giving the result. \square

Same results for $m' := \text{l.c.m.}(n, n')$ and $m'' := \text{l.c.m.}(n, n'')$.

Corollary 6. We can choose the representative pairs $(\xi, \mathfrak{q}), (\xi', \mathfrak{q}'), (\xi'', \mathfrak{q}'')$ such that $\xi' \equiv \frac{-\xi}{\xi+1}$ and $\xi'' \equiv \frac{-1}{\xi+1} \pmod{\tilde{\mathfrak{q}}}$ for a suitable $\tilde{\mathfrak{q}} \mid q$ in \tilde{L} .

In such a way, we have $\xi'' = \xi^{-1} \xi'$. \square

3.2. Case of an odd character $\chi \neq \omega$. We suppose that χ is an odd character of g distinct from ω ; then $\chi = \omega^k$, k odd, $k \not\equiv 1 \pmod{p-1}$, which excludes the case $p = 3$.

As for the case $k = 1$, we can represent modulo p the corresponding idempotent by an element in $\mathbb{Z}[g]$ of the form $e_\chi = (1 - s_{-1}) e'_\chi$, $e'_\chi \in \mathbb{Z}[g]$ (see Subsection 2.3).

We suppose that the χ -component of the p -class group of K is trivial; for this, a necessary and sufficient condition is that the Bernoulli number B_{p-k} be prime to p (see e.g. [Gr1, Section 2] for more details).

¹²The case $\xi = -1$ means $y + x = z^p \equiv 0 \pmod{q}$, i.e., $q \mid z$, which is excluded; in the same way, $\xi' \neq -1, \xi'' \neq -1$.

So, for any relation of the form $(u + v\zeta)\mathbb{Z}[\zeta] = \mathfrak{w}_1^p$ or $\mathfrak{p}\mathfrak{w}_1^p$ where g.c.d. $(u, v) = 1$, we get immediately:

$$(u + v\zeta)^{e_\chi} = \delta_\chi^p, \delta_\chi \in K^\times,$$

since any χ -unit of K is a p th power for χ odd distinct from ω (moreover in the special case $(1 - \zeta)^{e_\chi}$ is a χ -unit).

It is clear that Lemma 4 is valid for the character χ and that the two equations are equivalent.

The relation $(u + v\zeta)^{e_\chi} = \delta_\chi^p$ may be considered as the χ -SFLT equation associated to SFLT under the triviality of the χ -class group.

As in the previous subsection, let $q \neq p$ be a prime number such that $q \nmid uv$ and $\frac{v}{u}$ is of order n modulo q , with n prime to p (see Lemma 2).

Then let ξ of order n and $\mathfrak{q} := \mathfrak{q}_\xi | q$ in $L = \mathbb{Q}(\mu_n)$, characterized by the relation $\xi \equiv \frac{v}{u} \pmod{\mathfrak{q}}$. From $\eta = (1 + \xi\zeta)\zeta^{-\frac{1}{2}}$, put $\eta_k := \eta^{e_\chi} \in M$, where $M := LK$; then $\eta_k = (1 + \xi\zeta)^{e_\chi}$, since $\zeta^{e_\chi} = 1$. Thus $\eta_k \in M^+$ and $\eta_k \in K^{\times p}$ if $n \leq 2$.

We deduce the congruence in M :

$$\eta_k \equiv (1 + \frac{v}{u}\zeta)^{e_\chi} = \delta_\chi^p \pmod{\prod_{\Omega|q} \Omega}.$$

We then have the relation $\left(\frac{\eta_k}{\Omega}\right)_M = 1$, for all $\Omega | \mathfrak{q}$, so that, in this situation, a contradiction to the existence of a nontrivial solution of the SFLT equation is that this symbol be nontrivial for some q .

Here the value of κ does not enter.

From Kummer duality, the extension $M(\sqrt[p]{\eta_k})/M$ is splitted, by means of a p -cyclic extension, over the extension LK_{χ^*} , where $\chi^* = \omega^{1-k}$ and K_{χ^*} is the subfield of K fixed by the kernel of χ^* ; this field K_{χ^*} is real. Of course, $LK_{\chi^*} = L$ if and only if $K_{\chi^*} = \mathbb{Q}$, i.e., $\chi = \omega$.

This criterion may be used for any odd character $\chi \neq \omega$ such that the χ -component of the p -class group of K is trivial, which may have some interest. In some sense, it is similar to the case $\kappa \equiv 0 \pmod{p}$ of the preceding case $\chi = \omega$, the symbols being trivial independently of q .

But unfortunately, the corresponding extensions $M(\sqrt[p]{\eta_k})/L$ are metabelian (nonabelian) extensions and do not define intrinsic arithmetic properties of the field L as with the use of the single character ω to which we will return to study its properties.

3.3. Computation of the \mathbb{F}_p -dimension of a group of units. Since η_1 is considered in $(E_M/E_M^p)^{e_\omega}$, it is necessary to precise the \mathbb{F}_p -dimension of this group. The computation is the same for any odd character χ (this may be useful for Subsection 3.2).

Proposition 2. *Let $M = LK$, where $L = \mathbb{Q}(\mu_n)$, $n > 2$, $p \nmid n$. Let E_M be the group of units of M and let $\chi = \omega^k$ be an odd character of g .*

Then the \mathbb{F}_p -dimension of $(E_M/E_M^p \cdot \mu_p)^{e_\chi}$ is equal to $\frac{1}{2}[L : \mathbb{Q}] = \frac{1}{2}\phi(n)$.

Proof. Put $\Gamma := \text{Gal}(M/\mathbb{Q}) = G \oplus H$ where $G := \text{Gal}(M/L)$ and where $H := \text{Gal}(M/K)$. Let $\widehat{\Gamma} = \widehat{G} \oplus \widehat{H}$ be the group of irreducible characters of Γ ; for any $\psi \in \widehat{\Gamma}$, let ε_ψ be the idempotent:

$$\varepsilon_\psi := \frac{1}{|\Gamma|} \sum_{\sigma \in \Gamma} \psi^{-1}(\sigma) \sigma \in \mathbb{C}_p[\Gamma].$$

If $\psi = \omega^i \cdot \theta$, $\omega^i \in \widehat{G}$, $1 \leq i \leq p-1$, $\theta \in \widehat{H}$, then $\varepsilon_\psi = \varepsilon_{\omega^i} \cdot \varepsilon_\theta$.

From the Dirichlet–Herbrand theorem on units (see e.g. [Gr2, I.3.7]) we know that the representation $\mathbb{C}_p \oplus (\mathbb{C}_p \otimes_{\mathbb{Z}} E_M)$ is given by the representation of permutation:

$$\mathbb{C}_p[\Gamma] \frac{1}{2}(1+c) = \bigoplus_{\psi \text{ even}} \mathbb{C}_p[\Gamma] \varepsilon_\psi.$$

Then, since the character χ is odd, $(\mathbb{C}_p \oplus (\mathbb{C}_p \otimes_{\mathbb{Z}} E_M))^{\varepsilon_\chi} = (\mathbb{C}_p \otimes_{\mathbb{Z}} E_M)^{\varepsilon_\chi}$ is the representation $\bigoplus_{\psi \text{ even}} \mathbb{C}_p[\Gamma] \varepsilon_\psi \cdot \varepsilon_\chi$.

Put $\psi = \omega^i \cdot \theta$; then $\varepsilon_\psi = \varepsilon_{\omega^i} \cdot \varepsilon_\theta$ and $\varepsilon_\psi \cdot \varepsilon_\chi = 0$ except if $i = k$. The sum is over $\psi = \chi \theta$ with θ odd since ψ must be even. Then:

$$(\mathbb{C}_p \otimes_{\mathbb{Z}} E_M)^{\varepsilon_\chi} \simeq \bigoplus_{\theta \in \widehat{H}, \text{ odd}} \mathbb{C}_p[\Gamma] \varepsilon_{\chi \cdot \theta}.$$

We deduce that the \mathbb{C}_p -dimension of $(\mathbb{C}_p \otimes_{\mathbb{Z}} E_M)^{\varepsilon_\chi}$ is $\frac{1}{2}[L : \mathbb{Q}]$. Hence the proposition follows since $\varepsilon_\chi \equiv e_\chi \pmod{p\mathbb{Z}_p[g]}$. \square

In particular, the \mathbb{F}_p -dimension of $(E_M/E_M^p \cdot \mu_p)^{e_\omega}$ is equal to $\frac{1}{2}[L : \mathbb{Q}]$. Thus the subgroup of $(E_M/E_M^p \cdot \mu_p)^{e_\omega}$ generated by the images of the units $t\eta_1$, $t \in \text{Gal}(M/K)/\langle t_{-1} \rangle$, is of \mathbb{F}_p -dimension less or equal to $\frac{1}{2}[L : \mathbb{Q}] = \frac{1}{2}\phi(n)$.

4. Study of the cyclotomic units η_1 and the extensions F_ξ

In this Section we use some classical elements of Kummer theory and of decomposition of a Kummer extension over a subfield.

4.1. The cyclotomic unit η_1 . We consider, independently of any relation of the form $(u + v\zeta)\mathbb{Z}[\zeta] = \mathfrak{w}_1^p$ or $\mathfrak{p}\mathfrak{w}_1^p$, the cyclotomic number:

$$\eta := (1 + \xi\zeta)\zeta^{-\frac{1}{2}},$$

where ξ is a primitive n th root of unity with $p \nmid n$, and the real cyclotomic unit $\eta_1 := \eta^{e_\omega}$ defined in Definition 4. We exclude the cases $n = 1$ and $n = 2$ seen above for which $\eta_1 \in K^{\times p}$.

For $n > 2$, $L := \mathbb{Q}(\xi)$ is an imaginary cyclotomic field, hence we can consider the biquadratic extension M/L^+K^+ , where $M := LK$; then M^+ is the subfield of M of relative degree 2, distinct from LK^+ and from KL^+ .

Let f be the residue degree of q in K/\mathbb{Q} . We note that the residue degree of q in M^+/\mathbb{Q} is equal to f .

Since η_1 is a unit, the extension $M(\sqrt[p]{\eta_1})/M$ is p -ramified (i.e., unramified outside p). Put $\pi = \zeta - 1$; π is still an uniformizing parameter at p in M (indeed, \mathfrak{p} is not ramified in M/K). We have:

$$\eta \equiv 1 + \xi + \frac{1}{2}(\xi - 1)\pi \pmod{\pi^2},$$

giving by the usual computation modulo π^2 :

$$\eta_1 := \eta^{e_\omega} \equiv 1 + \frac{1}{2} \frac{\xi - 1}{\xi + 1} \pi \pmod{\pi^2};$$

since $n > 2$, $\frac{\xi - 1}{\xi + 1}$ is a local unit at p , showing that η_1 is not p -primary; thus in particular, the extension $M(\sqrt[p]{\eta_1})/M$ is cyclic of degree p .

Kummer theory shows that the conductor of $M(\sqrt[p]{\eta_1})/M$ is \mathfrak{p}^p extended to M (see [Gr2, II.1.6.3]). In some sense, $M(\sqrt[p]{\eta_1})/M$ is maximally wildly p -ramified and has the same conductor as $M(\sqrt[p]{\zeta})/M$.

Moreover, this extension does not depend on the choice of ζ since we have:

$$((1 + \xi \zeta^k)(\zeta^k)^{-\frac{1}{2}})^{e_\omega} = ((1 + \xi \zeta) \zeta^{-\frac{1}{2}})^{s_k e_\omega},$$

with $s_k e_\omega \equiv k e_\omega \pmod{p\mathbb{Z}[g]}$ for any k prime to p , giving the same radical.

4.2. The abelian extension F_ξ/L . By definition of the character ω , whose reflect is $\omega^* = \chi_0$ (the unit character), the extension $M(\sqrt[p]{\eta_1})/M$ is splitted over L by means of a cyclic p -ramified extension F_ξ , of degree p over $L = \mathbb{Q}(\mu_n)$ (i.e., $F_\xi M = M(\sqrt[p]{\eta_1})$).

This extension, as extension of L , only depends on ξ of order n . The family $(F_{\xi^t})_{\xi^t \text{ of order } n}$ is canonical.

Since η_1 is real, $\eta_1 = (1 + \xi^{-1} \zeta^{-1})^{e_\omega} \zeta^{\frac{1}{2}}$ which defines the same extension as $(1 + \xi^{-1} \zeta)^{e_\omega} \zeta^{-\frac{1}{2}}$ as we have seen at the end of Subsection 4.1. Then we get $F_\xi = F_{\xi^{-1}}$. In the cases $n \leq 2$, we have $L = \mathbb{Q}$, $\eta_1 \in K^{\times p}$, and $F_{\pm 1} = \mathbb{Q}$.

It is easy to see that for any $t \in \text{Gal}(L/\mathbb{Q})$ we have the relation $F_{\xi^t} = tF_\xi$, where by abuse of notation tF_ξ means $t'F_\xi$ for any \mathbb{Q} -automorphism t' of F_ξ such that $t'|_L = t$. Indeed, we have in the same way, $t'(\sqrt[p]{\eta_1}) = \sqrt[p]{t(\eta_1)}$ (up to a p th root of unity) where $t(\eta_1) = (1 + \xi^t \zeta)^{e_\omega} \zeta^{-\frac{1}{2}}$.¹³

Suppose now that we have chosen a prime number q such that $q \equiv 1 \pmod{n}$, $p \nmid n$, and let \mathfrak{q} be a fixed prime ideal above q in L ; later, we will have $\mathfrak{q} = \mathfrak{q}_\xi$ when ξ is associated to the usual integers u, v , but in this subsection \mathfrak{q} is arbitrary.

¹³We use the same notations for the elements of the Galois groups $\text{Gal}(M/K)$ and $\text{Gal}(L/\mathbb{Q})$, then $G = \text{Gal}(M/L)$ and $g = \text{Gal}(K/\mathbb{Q})$ and similarly for $\text{Gal}(M(\sqrt[p]{\eta_1})/M)$ and $\text{Gal}(F_\xi/L)$.

Consider the symbol $\left(\frac{\eta_1}{\Omega}\right)_M$ (which is independent of the choice of $\mathfrak{Q} | \mathfrak{q}$ in M); this symbol is equal to 1 if and only if the image of η_1 in the residue field Z_M/Ω is a p th power, thus if and only if Ω splits in $M(\sqrt[p]{\eta_1})/M$ (Hensel's Lemma) which is equivalent to the splitting of \mathfrak{q} in F_ξ/L .

Let H_L be the maximal abelian p -ramified p -extension of L ; it contains all the extensions $F_{\xi'}$, ξ' of order n , the cyclotomic \mathbb{Z}_p -extension $L_\infty = L\mathbb{Q}_\infty$ of L which is abelian over \mathbb{Q} , and $\frac{1}{2}[L : \mathbb{Q}]$ other independent \mathbb{Z}_p -extensions of L . This extension H_L will be studied in more details in Section 5.

Since q totally splits in L/\mathbb{Q} , the decomposition field of q in L_∞/\mathbb{Q} is $L_e = L\mathbb{Q}_e$, where $\mathbb{Q}_e \subset \mathbb{Q}_\infty$ is the stage of degree p^e over \mathbb{Q} where $q^f = 1 + p^{e+1}d$, $e \geq 0$, $p \nmid d$; note that $e = 0$ is equivalent to $\kappa \not\equiv 0 \pmod{p}$.

5. Study of the extensions H_L/L and F_n/L

In this section we recall some class field theory results concerning the abelian p -ramification over L .

5.1. Class field theory and p -ramification. Let H_L be the maximal abelian p -ramified p -extension of $L := \mathbb{Q}(\mu_n)$ in the case $n > 2$, $p \nmid n$ (so that L is an imaginary cyclotomic field of even degree) and let $H_{L[p]} \subseteq H_L$ be the maximal p -elementary p -ramified extension of L .

We consider its Galois group as a vector space over \mathbb{F}_p .

Its dimension is given by the following Šafarevič formula (see e.g. [Gr2, II.5.4.1, (ii)]):

$$\dim_{\mathbb{F}_p}(\text{Gal}(H_{L[p]}/L)) = \dim_{\mathbb{F}_p}(V_L/L^{\times p}) + \frac{1}{2}[L : \mathbb{Q}] + 1,$$

where V_L is the group of pseudo-units of L which are local p th powers at each place dividing p in L .

Lemma 5. *The conductor of $H_{L[p]}/L$ divides (p^2) as ideal of L .*

Proof. From Hensel's Lemma, since $p > 2$ is not ramified in L/\mathbb{Q} ($p \nmid n$), the modulus (p^2) is sufficient for any $\alpha \in L^\times$, $\alpha \equiv 1 \pmod{p^2}$, to be locally a p th power at each place dividing p in L . \square

Thus $H_{L[p]}$ is contained in the ray class field $L(p^2)$ and this yields:

$$\text{Gal}(H_{L[p]}/L) \simeq I/I^p R,$$

where I is the group of fractional ideals of L prime to p and R is the ray group modulo p^2 , i.e., $\{(\alpha) \in I, \alpha \equiv 1 \pmod{p^2}\}$.

5.2. The subextension F_n . Let t_{-1} be the element of order 2 of the group $\text{Gal}(M/L^+K)$ and $s_{-1} \in G$ be the element of order 2 of $\text{Gal}(M/K^+L)$ (the complex conjugation is $c = s_{-1}t_{-1}$ as generator of $\text{Gal}(M/M^+)$).

Since we have the relations $\eta_1^c = \eta_1$, $\eta_1^{s_{-1}} = \eta^{e_\omega \cdot s_{-1}} = \eta_1^{-1}$, giving the relation $\eta_1^{t_{-1}} = \eta_1^{-1}$, we deduce that:

$$\text{Gal}(M(\sqrt[p]{\eta_1})/L^+K) \simeq \text{Gal}(F_\xi/L^+) \simeq D_{2p},$$

the dihedral group of order $2p$.¹⁴

In other words, $\text{Gal}(L/L^+)$ acts on $\text{Gal}(F_\xi/L)$ by $\sigma^{t_{-1}} := t'_{-1} \cdot \sigma \cdot t_{-1} = \sigma^{-1}$ for all $\sigma \in \text{Gal}(F_\xi/L)$ and any extension t'_{-1} of t_{-1} in $\text{Gal}(F_\xi/L^+)$.

It will be necessary to consider the compositum of all the extensions $M(\sqrt[p]{\eta_1})$ when ξ (of order n) varies. Indeed, in the situation of a nontrivial solution (u, v) of the SFLT equation, for any $n > 2$, $p \nmid n$, the root ξ such that $\xi \equiv \frac{v}{u} \pmod{\mathfrak{q}}$, for $\mathfrak{q} = \mathfrak{q}_\xi$, is ineffective and the properties of the symbols $\left(\frac{\eta}{\Omega}\right)_M$, $\Omega \mid \mathfrak{q}$ in M , for the pairs (η_1, Ω) , can be studied in this extension.

Let F_n be the compositum of the corresponding extensions $F_{\xi'}$, ξ' of order n , so that F_n is also the compositum of the $F_{\xi t}$, $t \in \text{Gal}(M/K)$, ξ fixed; since $t_{-1}\eta_1 = \eta_1^{-1}$, we can consider the $t\eta_1$ with t modulo $\langle t_{-1} \rangle$ (this is coherent with the relation $F_\xi = F_{\xi^{-1}}$).

We have the equality $F_n M = M(\sqrt[p]{\langle t\eta_1 \rangle_{t \bmod \langle t_{-1} \rangle}})$.

Then as above $\text{Gal}(L/L^+)$ acts on $\text{Gal}(F_n/L)$ by $\sigma^{t_{-1}} = \sigma^{-1}$ for all $\sigma \in \text{Gal}(F_n/L)$.

Lemma 6. *The Galois closure of F_ξ over \mathbb{Q} is F_n which is linearly disjoint from L_∞/L .*

Proof. Over the field K , the Galois closure of $M(\sqrt[p]{\eta_1})$ is given by the radical $\langle t\eta_1 \rangle_{t \bmod \langle t_{-1} \rangle}$ with $t\eta_1 = (1 + \xi^t \zeta)^{e_\omega} \zeta^{-\frac{1}{2}}$, $t \in \text{Gal}(M/K)$, giving the first part of the lemma. The relation $L_1 \subseteq F_n$ is equivalent to:

$$M(\sqrt[p]{\zeta}) \subseteq M(\sqrt[p]{\langle t\eta_1 \rangle_{t \bmod \langle t_{-1} \rangle}}),$$

then to the existence of a relation of the form $\prod_{t \bmod \langle t_{-1} \rangle} (t\eta_1)^{\lambda_t} = \zeta \delta^p$, $\lambda_t \in \mathbb{Z}$, $\delta \in M^\times$; but since the left member is real, the use of complex conjugation implies $\zeta^2 \in M^{\times p}$, which is absurd. \square

¹⁴Let $A := \text{Gal}(M/L^+) = G \oplus \langle t_{-1} \rangle$. Let χ_1 be the character of A defined by $\chi_1(s) = 1$ for all $s \in G$ and $\chi_1(t_{-1}) = -1$. Put $\chi = \omega \chi_1$; it is easy to see that χ is the character of the radical $\langle \eta_1 \rangle M^{\times p}/M^{\times p}$ as A -module, since $\eta_1 = \eta^{e_\omega}$ and $\eta_1^{t_{-1}} = \eta_1^{-1}$. From Kummer duality, the character of $\text{Gal}(M(\sqrt[p]{\eta_1})/M)$ is $\chi^* := \omega \chi^{-1} = \chi_1$ proving that $\text{Gal}(M(\sqrt[p]{\eta_1})/L^+) \simeq G \times \text{Gal}(F_\xi/L^+)$, with $\text{Gal}(F_\xi/L^+) \simeq D_{2p}$. We also have $\text{Gal}(M(\sqrt[p]{\eta_1})/M^+) \simeq D_{2p}$.

Remark 5. The \mathbb{F}_p -dimension of the above radical depends on the study of the relation $\prod_{t \bmod \langle t_{-1} \rangle} (t \eta_1)^{\lambda_t} \in M^{\times p}$; this yields to (see Subsection 4.1):

$$\prod_{t \bmod \langle t_{-1} \rangle} \left(1 + \frac{1}{2} \frac{\xi^t - 1}{\xi^t + 1} \pi \right)^{\lambda_t e_\omega} \equiv 1 + \left(\sum_{t \bmod \langle t_{-1} \rangle} \lambda_t \frac{1}{2} \frac{\xi^t - 1}{\xi^t + 1} \right) \pi \pmod{\pi^2}.$$

Thus if the numbers $\frac{\xi^t - 1}{\xi^t + 1}$, $t \bmod \langle t_{-1} \rangle$, are linearly independent modulo p , we get the dimension $\frac{1}{2}[L : \mathbb{Q}]$ and $\dim_{\mathbb{F}_p}(\text{Gal}(F_n/L)) = \frac{1}{2}[L : \mathbb{Q}] = \frac{1}{2}\phi(n)$.

Since η_1 is a cyclotomic unit of M , the classical study of the whole group of cyclotomic units of M (of finite index in E_M) may give the exact \mathbb{F}_p -dimension of the radical (see Washington book, Chap.8); but this study depends, in a complicate manner, on the Galois group of M/\mathbb{Q} and the law of decomposition of the prime divisors of n in this extension. \square

5.3. Canonical decomposition of $\text{Gal}(H_L[p]/L)$. Consider the Galois group $C_L := \text{Gal}(H_L[p]/L)$ as a module over $\mathbb{F}_p[\text{Gal}(L/L^+)]$. Write:

$$C_L = C_L^+ \oplus C_L^-, \text{ with } C_L^+ := C_L^{\frac{1}{2}(1+t_{-1})}, \quad C_L^- := C_L^{\frac{1}{2}(1-t_{-1})}.$$

We denote by $H_L^-[p]$ the subfield of $H_L[p]$ fixed by C_L^+ and by $H_L^+[p]$ the subfield of $H_L[p]$ fixed by C_L^- . We then have $F_n \subseteq H_L^-[p]$ and the diagram:

$$\begin{array}{ccc} & C_L^- & \\ & \text{-----} & \\ H_L^+[p] & & H_L[p] \\ & & \downarrow \\ & & C_L^+ \\ & & \downarrow \\ L & \text{-----} & H_L^-[p] \end{array}$$

Lemma 7. Put $\overline{V}_L := V_L/L^{\times p}$ (see Subsection 5.1) and $\overline{V}_L = \overline{V}_L^+ \oplus \overline{V}_L^-$ as above. Then $\overline{V}_L^+ \simeq V_{L^+}/(L^+)^{\times p}$ giving:

$$\dim_{\mathbb{F}_p}(C_L^+) = \dim_{\mathbb{F}_p}(\overline{V}_L^+) + 1; \quad \dim_{\mathbb{F}_p}(C_L^-) = \dim_{\mathbb{F}_p}(\overline{V}_L^-) + \frac{1}{2}[L : \mathbb{Q}].$$

Proof. Since $p \neq 2$, we have $C_L^+ \simeq \text{Gal}(H_{L^+}[p]/L^+)$ for which the Šafarevič formula is $\dim_{\mathbb{F}_p}(C_L^+) = \dim_{\mathbb{F}_p}(\overline{V}_L^+) + 1$, proving the lemma. \square

By this way, the case where $\dim_{\mathbb{F}_p}(\text{Gal}(F_n/L)) = \frac{1}{2}[L : \mathbb{Q}]$ is compatible with the \mathbb{F}_p -dimension of C_L^- since when the invariant C_L^- is minimal (which is equivalent to $\dim_{\mathbb{F}_p}(\overline{V}_L^-) = 0$) then $F_n = H_L^-[p]$ as soon as the $t\eta_1$, $t \bmod \langle t_{-1} \rangle$, are independent in $M^\times/M^{\times p}$.

Note that the group of pseudo-units $Y_L := \{\alpha \in L^\times, (\alpha) = \mathfrak{a}^p\}$ is elucidated by the following obvious exact sequence:

$$1 \longrightarrow E_L/E_L^p \longrightarrow \overline{Y}_L \longrightarrow {}_p\mathcal{C}_L \longrightarrow 1,$$

where \mathcal{C}_L is the p -class group of L , ${}_p\mathcal{C}_L$ the subgroup of \mathcal{C}_L of classes killed by p , E_L the group of units of L , and $\overline{Y}_L := Y_L/L^{\times p}$.

For L^+ we get the analogous exact sequence:

$$1 \longrightarrow E_{L^+}/E_{L^+}^p \longrightarrow \overline{Y}_{L^+} \longrightarrow {}_p\mathcal{C}_{L^+} \longrightarrow 1.$$

We have, with usual notations \pm , the relations $(E_L/E_L^p)^+ \simeq E_{L^+}/E_{L^+}^p$ and $(E_L/E_L^p)^- = 1$, so that $\overline{Y}_L^- \simeq {}_p\mathcal{C}_L^-$ and $\overline{V}_L^- \subseteq \overline{Y}_L^-$ only depends on the minus part of the p -class group of L and is often trivial.

The group $\overline{V}_L^+ \simeq \overline{V}_{L^+} \subseteq \overline{Y}_{L^+}$ depends on the p -class group of L^+ (in general trivial) and more essentially on the units locally p th power at p in the group of units E_{L^+} of L^+ which is of \mathbb{Z} -rank $\frac{1}{2}[L : \mathbb{Q}] - 1$; but $\varepsilon \in E_{L^+}$ is a local p th power at each place dividing p if and only if $\varepsilon^{p^\delta - 1} \equiv 1 \pmod{p^2}$, where $\delta \mid \frac{1}{2}\phi(n)$ is the residue degree of p in L^+ , which is also very rare, giving often a trivial \overline{V}_L^+ .

Remark 6. Suppose that the group \overline{V}_L is trivial.¹⁵ Then $\dim_{\mathbb{F}_p}(C_L^+) = 1$ and $\dim_{\mathbb{F}_p}(C_L^-) = \frac{1}{2}[L : \mathbb{Q}]$. In this case H_L is the compositum of the \mathbb{Z}_p -extensions of L which is of the form $H_L^+ H_L^-$ where $H_L^+ = L_\infty$ is the cyclotomic \mathbb{Z}_p -extension of L and H_L^- the compositum of $\frac{1}{2}[L : \mathbb{Q}]$ independent relative \mathbb{Z}_p -extensions of L (i.e., which are pro-diedral over L^+).

Then $H_L[p]$ is the compositum of the first stages of these \mathbb{Z}_p -extensions, the extension $H_L^+[p]$ is L_1 , and $H_L^-[p]M$ may be the Kummer extension defined by the radical generated by the $t\eta_1$ as soon as its \mathbb{F}_p -dimension is $\frac{1}{2}[L : \mathbb{Q}]$. See Subsection 3.3 about these questions of dimensions. \square

5.4. Conclusion. We have established, from Corollary 4 and Remark 4 (Subsection 3.1), that, under a solution of Fermat's equation ($p > 3$), for infinitely many particular prime numbers q in the case $\kappa \not\equiv 0 \pmod{p}$, there exist privilegiate pairs $(F_\xi, \mathfrak{q}_\xi)$, $(F_{\xi'}, \mathfrak{q}_{\xi'})$ for the first case (resp. $(F_\xi, \mathfrak{q}_\xi)$, $(F_{\xi'}, \mathfrak{q}_{\xi'})$, $(F_{\xi''}, \mathfrak{q}_{\xi''})$ for the second case), defined up to conjugation, with p -cyclic p -ramified extensions F_ξ/L , $F_{\xi'}/L'$, $F_{\xi''}/L''$ and prime ideals \mathfrak{q}_ξ , $\mathfrak{q}_{\xi'}$, $\mathfrak{q}_{\xi''}$, for which $\mathfrak{q}_\xi, \mathfrak{q}_{\xi'}$ are inert for the first case (resp. $\mathfrak{q}_\xi, \mathfrak{q}_{\xi'}$ are inert and $\mathfrak{q}_{\xi''}$ splits, for the second case) in the corresponding extensions F_ξ/L , $F_{\xi'}/L'$, $F_{\xi''}/L''$.

In the case $\kappa \equiv 0 \pmod{p}$, for all the above pairs, the ideals split in the corresponding extensions.

This intricacy may be in contradiction, for mostly primes q , since the arithmetical properties of the governing fields $\mathbb{Q}(\mu_{q-1})$ are independent of the Fermat problem; more precisely, a general philosophy is that the decomposition groups of prime ideals in Galois extensions do not fulfill any other laws than standard ones, and may be analyzed in a statistical point of view (see Section 7 for a direct study of these aspects).

¹⁵This situation is by definition equivalent to the p -rationality of the field L (see [Gr2, IV.3.5] for some equivalent conditions).

About this, we will explain in Section 9 that the case $p = 3$ is precisely an exceptional counterexample to the above claim, since some constraints do exist; but we will show that these constraints are not in contradiction with statistical considerations because of the structure of the *infinite* set of solutions.

One may object that F_ξ comes from the radical:

$$\langle (1 + \xi \zeta)^{e_\omega} \zeta^{-\frac{1}{2}} \rangle M^{\times p}$$

over M , which is directly associated to a problem of SFLT type, and in a standard algebraic point of view the above circumstances on the laws of decomposition may be *equivalent* to a contradiction to SFLT. Thus it will be necessary to obtain some analytic or geometrical informations on the splitting of q in $H_{L[p]}/L$ (especially in the canonical family $(F_{\xi'}/L)_{\xi'}$ of order n) so as to prove that the above particularities do not exist.

6. A sufficient condition proving Fermat's last theorem

In this section we study a sufficient condition for FLT, which only involves congruential properties of prime ideals over q in $\mathbb{Q}(\mu_{q-1})$.

6.1. Main result. We suppose that $p > 3$ and that the primes q considered are such that $f > 1$ and $\kappa := \frac{q^f - 1}{p} \not\equiv 0 \pmod{p}$; we will then use Remark 3 using Furtwängler's theorems. Thus any divisor n of $q - 1$ is prime to p .

From a nontrivial solution (u, v) of the SFLT equation, for which $\frac{v}{u}$ is of order $n > 2$ modulo $q \nmid uv$, we consider the pair (ξ, \mathfrak{q}_ξ) , defined up to \mathbb{Q} -conjugation in $L := \mathbb{Q}(\mu_n)$ (see Definition 3).

The integer n and the pair are ineffective since if we fix an ideal $\mathfrak{q} | q$ in L , the root ξ such that $\mathfrak{q} = \mathfrak{q}_\xi$ is unknown, or if we fix a primitive n th root ξ , then the ideal $\mathfrak{q} | q$ such that $\mathfrak{q} = \mathfrak{q}_\xi$ is unknown.

Let \mathfrak{Q}_ξ be any prime ideal of $M := LK$ above \mathfrak{q}_ξ . Then the pair $(\eta_1, \mathfrak{Q}_\xi)$, where $\eta_1 = (1 + \xi \zeta)^{e_\omega} \zeta^{-\frac{1}{2}} \in M^+$, is also unknown in the same manner.

So if we ensure that, for instance for \mathfrak{q}_0 fixed arbitrarily in L , for $\mathfrak{Q}_0 | \mathfrak{q}_0$ in M , $\left(\frac{\eta_1}{t\mathfrak{Q}_0}\right)_M = 1$ for all $t \in \text{Gal}(M/K)/\langle t_{-1} \rangle$, then in particular for the "good" value of the pair (η_1, \mathfrak{Q}) (i.e., such that $\mathfrak{Q} | \mathfrak{q}_\xi$), we get:

$$\left(\frac{\eta_1}{\mathfrak{Q}}\right)_M = \zeta^{\frac{1}{2} \frac{v-u}{v+u} \kappa} = 1,$$

giving $v - u \equiv 0 \pmod{p}$ which is absurd in the case of a solution (x, y, z) of Fermat's equation by choice of the difference $v - u = \pm(y - x)$ or $\pm(y - z)$ (see Corollary 4, (i)).

The problem is to know if there exist such prime numbers q with F_n in the splitting field of \mathfrak{q} in $H_{L[p]}/L$, for all $n | q - 1$, $n > 2$. If so, this will prove FLT (in the first case we know that $q \nmid xyz$ and a single q is sufficient; for

the second case where we know that $q \nmid xz$, it is necessary to have infinitely many such primes q to be certain that $q \nmid y$.

For the proof of SFLT, we must suppose $u - v \not\equiv 0 \pmod{p}$ in the first case.

If F_n is in the splitting field of \mathfrak{q} , then this does not depend on the choice of $\mathfrak{q} \mid q$ in L , which is a convenient simplification. In other words, q totally splits in F_n/\mathbb{Q} .

Since $F_n \subseteq H_L^-[p]$, a sufficient condition to have the total splitting of \mathfrak{q} in F_n is that the Frobenius φ of \mathfrak{q} in $H_L[p]/L$ be an element of C_L^+ , which is equivalent to $\varphi^{t-1} = \varphi$, hence to $\varphi^{t-1-1} = 1$. Note that φ is of order p since its restriction to L_1 is of order p by assumption.

The image of $\varphi \in C_L$ by the isomorphism $\text{Gal}(H_L[p]/L) \simeq I/I^p R$ of class field theory, is given by the class of \mathfrak{q} in $I/I^p R$; thus the condition $\varphi^{t-1-1} = 1$ is equivalent to $\mathfrak{q}^{t-1-1} \in I^p R$, i.e.,

$$\mathfrak{q}^{t-1-1} = \mathfrak{a}^p(\alpha), \quad \alpha \equiv 1 \pmod{p^2},$$

for an ideal \mathfrak{a} of L . We must realize this for any divisor $n > 2$ of $q - 1$.

For $\tilde{n} := q - 1$, $\tilde{L} := \mathbb{Q}(\mu_{q-1})$, we suppose that the above condition $\tilde{\mathfrak{q}}^{\tilde{t}-1-1} = \tilde{\mathfrak{a}}^p(\tilde{\alpha})$, $\tilde{\alpha} \equiv 1 \pmod{p^2}$, is satisfied (for $\tilde{\mathfrak{q}} \mid q$ in \tilde{L}/\mathbb{Q}).

Then let $n \mid q - 1$, $n > 2$; since $L = \mathbb{Q}(\mu_n)$ is imaginary, L^+ is fixed by the restriction t_{-1} of \tilde{t}_{-1} to L , and taking the norm $N_{\tilde{L}/L}$ we get:

$$N_{\tilde{L}/L}(\tilde{\mathfrak{q}}^{\tilde{t}-1-1}) = N_{\tilde{L}/L}(\tilde{\mathfrak{a}})^p N_{\tilde{L}/L}(\tilde{\alpha}).$$

Since q is totally split in \tilde{L} , we have by definition $N_{\tilde{L}/L}(\tilde{\mathfrak{q}}) = \mathfrak{q}$ for some $\mathfrak{q} \mid q$ in L , and the above relation is of the form $\mathfrak{q}^{t-1-1} = \mathfrak{a}^p(\alpha)$, with $\alpha \equiv 1 \pmod{p^2}$, as expected (this coherent choice of the ideals \mathfrak{q} is possible since the required condition of splitting at each stage is independent of the choice of the ideal). So the whole condition for our purpose is given by the single condition for $n = q - 1$, $L = \mathbb{Q}(\mu_{q-1})$.

We have obtained the following criterion, where c is the complex conjugation:

Theorem 2. *Let p be a prime number, $p > 3$. If there exists at least a prime number q , $q \not\equiv 1 \pmod{p}$, $q^{p-1} \not\equiv 1 \pmod{p^2}$, such that for a prime ideal $\mathfrak{q} \mid q$ in $\mathbb{Q}(\mu_{q-1})$, we have $\mathfrak{q}^{1-c} = \mathfrak{a}^p(\alpha)$ for an ideal \mathfrak{a} and an element α of $\mathbb{Q}(\mu_{q-1})$ with $\alpha \equiv 1 \pmod{p^2}$,¹⁶ then the first case of FLT (or the*

¹⁶ Since the multiplicative groups of the residue fields of L at p are of order prime to p , in any writing $\mathfrak{a}^p(\alpha)$ we can suppose $\alpha = 1 + p\beta$, β p -integer of L . The condition $\mathfrak{q}^{1-c} = \mathfrak{a}^p(\alpha)$, $\alpha \equiv 1 \pmod{p^2}$, is equivalent to $\mathfrak{q}^{1-c} = \mathfrak{a}^p(1 + p\beta)$, where $\beta \equiv \beta^+ \pmod{p}$, for a p -integer β^+ of L^+ ; indeed, this last condition implies $\mathfrak{q}^{2(1-c)} = \mathfrak{a}^{(1-c)p}(1 + p\beta)^{1-c}$ where $(1 + p\beta)^{1-c} \equiv 1 + p(1-c)\beta \equiv 1 \pmod{p^2}$, which gives the result thanks to a Bézout relation between 2 and p .

The condition $\mathfrak{q}^{1-c} = \mathfrak{a}^p(\alpha)$, $\alpha \equiv 1 \pmod{p^2}$, is also equivalent to $\mathfrak{q} = \mathfrak{b}^{1+c}\mathfrak{a}'^p(\alpha')$, $\alpha' \equiv 1 \pmod{p^2}$; indeed, a direction being trivial, from $\mathfrak{q}^{1-c} = \mathfrak{a}^p(\alpha)$ we get $\mathfrak{q}^2 = \mathfrak{q}^{1+c}\mathfrak{a}^p(\alpha)$.

first case of SFLT under the supplementary condition $u - v \not\equiv 0 \pmod{p}$) holds for p .

The second case of FLT (or of SFLT) holds as soon as there exist infinitely many such primes q . \square

From the Čebotarev's theorem, there exist infinitely many prime ideals \mathfrak{l} of $\mathbb{Q}(\mu_{q-1})$ such that their Frobenii $\varphi_{\mathfrak{l}}$ lie in $C_{\mathbb{Q}(\mu_{q-1})}^+$ (which is at least of dimension 1); the problem is to be sure that there is no obstruction to the fact that it is sometimes possible for $\mathfrak{l} = \mathfrak{q} | q$.

It is clear that such a set of prime numbers q would be of Dirichlet density 0, as for the set of prime numbers q , such that the ring $\mathbb{Z}[\mu_{q-1}]$ contains a principal ideal of norm q , a result proved by Lenstra in [Len, Cor. 7.6].

Theorem 2 may be of empty use due to an excessive condition on the primes q . So we intend, in the forthcoming subsection, to try to give a weaker form of this result (see Conjecture 2).

Proposition 2 shows that the extension $F_{q-1} \subseteq H_L^-[p]$, for $L = \mathbb{Q}(\mu_{q-1})$, is of degree less or equal to $\frac{1}{2}[L : \mathbb{Q}] = \frac{1}{2}\phi(q-1)$. So, if the torsion group \overline{V}_L is trivial, the equality $F_{q-1} = H_L^-[p]$ is possible and the sufficient condition of Theorem 2 is also necessary; thus if there is any hope of success of the method, this condition cannot be improved in practice.

6.2. Some related viewpoints. We will examine if some effective (or numerical) aspects allow us to justify the method of proof of FLT based on Theorem 2 for $p > 3$.

a) In this first approach, we fix q and $\mathfrak{q} | q$ in $L = \mathbb{Q}(\mu_{q-1})$, and we try to find some suitable values of p for which $\varphi_{\mathfrak{q}} \in C_{\mathbb{Q}(\mu_{q-1})}^+$.

Suppose that $\mathfrak{q}^k = (\alpha)$ in $L = \mathbb{Q}(\mu_{q-1})$ for some $k > 0$ and suppose that we find $d > 0$ such that: $\alpha^d \equiv \alpha^+ \pmod{p^2}$, for some prime p such that $p \nmid kd$, and some $\alpha^+ \in L^+$; then $\alpha^{d(1-c)} \equiv 1 \pmod{p^2}$ giving a solution of the problem for the prime p (then a posteriori k may be chosen as the order of the ideal class of \mathfrak{q} and d as a suitable divisor of the order of the multiplicative group of the residue field of L at p).

Of course this relation looks like the general problem of the Fermat quotients of algebraic numbers as studied by Hatada in [Hat]. Considering the work of Hatada and others, a serious conjecture would be that there exist infinitely many solutions p for q fixed.

Since the numerical values of p are out of range of any computer, this conjectural property is not of a practical use, but connect FLT to deep properties of algebraic numbers.

The condition $\mathfrak{q}^{1-c} = \mathfrak{a}^p(\alpha)$, $\alpha = 1 + p\beta$, is satisfied as soon as the class of \mathfrak{q}^{1-c} is of order prime to p , which is a weak condition; it remains to get the stronger condition $\beta \equiv \beta^+ \pmod{p}$.

Meanwhile, we have found the following example which gives a very partial illustration but shows that there is, a priori, no systematic obstruction for this question.

Example 2. Let $q = 5$ and $p = 463$. We then have $L = \mathbb{Q}(\mu_4) = \mathbb{Q}(i)$, where $i := \sqrt{-1}$, and $\mathfrak{q} = (2 + i)$. We see that q is totally inert in K (i.e., $f = 462$) and that p is also inert in L .

We obtain the following numerical informations:

- $(5^{463-1} - 1)/463 \not\equiv 0 \pmod{463}$ (i.e., $\kappa \not\equiv 0 \pmod{p}$),
- $(2 + i)^{463+1} \equiv 43990 \pmod{463^2}$.

This implies immediately:

$$\mathfrak{q}^{1-c} = \left(\frac{2+i}{2-i}\right) \text{ and } \mathfrak{q}^{(p+1)(1-c)} = \left(\frac{2+i}{2-i}\right)^{p+1} \equiv 1 \pmod{p^2},$$

giving the relation $\mathfrak{q}^{1-c} = \mathfrak{a}^p(\alpha)$ with $\mathfrak{a} = \mathfrak{q}^{c-1}$ and $\alpha \equiv 1 \pmod{p^2}$. \square

b) In a slightly different point of view, we must consider that in general, for a solution (u, v) of the SFLT equation, the order n of $\frac{v}{u}$ modulo q may be a strict divisor of $q - 1$, even if it is clear directly that n tends to infinity with q (Corollary 5).

Thus we have the following comments.

Let m be a fixed integer, $m > 2$, $p \nmid m$. Put $K' := \mathbb{Q}(\mu_{p^2}) \supset K$, $L := \mathbb{Q}(\mu_m)$, $H := H_{\tilde{L}}^-[p]$ (see Section 5), and $H' := HK'$. Then H/\mathbb{Q} and K'/\mathbb{Q} are linearly disjoint.

Let $\varphi \in \text{Gal}(H'/H)$ of order pf , $f \mid p-1$. From the Čebotarev's theorem, there exist infinitely many prime numbers q such that, for a suitable $\mathfrak{Q}' \mid q$ in H' , the Frobenius automorphism satisfies the equality $\left(\frac{H'/\mathbb{Q}}{\mathfrak{Q}'}\right) = \varphi$.

This implies the following properties:

- $q \equiv 1 \pmod{m}$ (since q splits in L/\mathbb{Q}),
- $q^f \not\equiv 1 \pmod{p^2}$ (since q is inert in K'/K),
- q is totally split in H/L (since φ fixes H).

The condition $\mathfrak{q}^{1-c} = \mathfrak{a}^p(\alpha)$, $\alpha \equiv 1 \pmod{p^2}$, is satisfied for any prime ideal $\mathfrak{q} \mid q$ in $L = \mathbb{Q}(\mu_m)$ but not necessarily for $\tilde{\mathfrak{q}}$ in $\tilde{L} = \mathbb{Q}(\mu_{q-1})$ (i.e., the Frobenius of \mathfrak{q} in $H_L^-[p]$ fixes $H_L^-[p]$, but this is not necessarily true for the Frobenius of $\tilde{\mathfrak{q}}$ in $H_{\tilde{L}}^-[p]$ giving possible inertia in $H_{\tilde{L}}^-[p]/\tilde{L}H_L^-[p]$).

The order of $\frac{v}{u}$ modulo q is $n \mid q - 1$ and not necessarily m , and the obvious analogue of Theorem 2 applies only if $n \mid m$. In other words, we try to replace the order $q - 1$ (probably too big under the condition that the Frobenius of \mathfrak{q} lies in $C_{\mathbb{Q}(\mu_{q-1})}^+$) by a strict divisor m (depending on q), for infinitely many q for which we hope that the Frobenius of \mathfrak{q} lies in $C_{\mathbb{Q}(\mu_m)}^+$.

Then, under a nontrivial solution (u, v) of the SFLT equation ($u - v \not\equiv 0 \pmod{p}$), there is an obstruction to the fact that there exists at least a pair (m, q) (m and q defined as above with a Frobenius in $C_{\mathbb{Q}(\mu_m)}^+$) such that a divisor n of m is the order of $\frac{v}{u}$ modulo q .

This remark may constitute a way of access to a proof of FLT by means of analytic investigations and we can propose the following independent conjecture.

Conjecture 2. *Let p be a prime number, $p > 3$, and let $\rho = \frac{v}{u}$, with $\text{g.c.d.}(u, v) = 1$, be a rational distinct from 0 and ± 1 .*

There exists a divisor function $m : \mathbb{N} \setminus \{0\} \rightarrow \mathbb{N} \setminus \{0\}$ (i.e., such that $m(e) \mid e$ for all $e \in \mathbb{N} \setminus \{0\}$) such that there exist infinitely many prime numbers q , with $\kappa \not\equiv 0 \pmod{p}$, totally split in $F_{m(q-1)}$ (see Subsection 5.2), for which the order n of ρ modulo q divides $m(q-1)$. \square

Since n tends to infinity with q , this means that $m(q-1)$ is unbounded with q . The existence of infinitely many primes q satisfying the conditions of Theorem 2 is equivalent to the conjecture with $m(e) = e$ for all e .

The existence of such a function depends on two phenomena:

- (i) The order of magnitude of the primes q discussed above from the Čebotarev's theorem.
- (ii) The minimal value of the order modulo q of a given rational ρ .

Example 3. For $p = 5$, $m = 4$, we have $L = \mathbb{Q}(i)$, and an obvious family of ideals \mathfrak{q} of L such that $\mathfrak{q}^{1-c} = (\alpha)$, $\alpha \equiv 1 \pmod{25}$, is given by the following expression:

$$\mathfrak{q} = (e + 5a + 25bi) \mathbb{Z}[i], \quad e \in \{1, 2, 3, 4\}, \quad a, b \in \mathbb{Z},$$

e, a, b being such that $(e + 5a)^2 + (25b)^2$ is a prime number q .

The prime numbers $q < 10000$, $q \not\equiv 1 \pmod{5}$ and $q^4 \not\equiv 1 \pmod{25}$, of the above form, are the following: 769, 1109, 1409, 2069, 2389, 2789, 3229, 3329, 3989, 5309, 5689, 6469, 6709, 7069, 7829, 8329, 8369, 8429. \square

It is clear that such a construction does exist for any p and any $m > 2$, and the question is the following: p, u , and v being given, is it possible to find in such infinite lists of prime numbers (corresponding to arbitrary values of m), a prime q for which the order of $\frac{v}{u}$ modulo q is a divisor of m (which is equivalent to $q \mid u^m - v^m$)? Note that for each m , only a finite number of q in the list can be solution.

The existence of one solution (m, q) gives the proof of the first case of FLT for p and the existence of infinitely many solutions (m, q) gives a complete proof of FLT for p .

6.3. Explicit formula for the p th power residue symbol $\left(\frac{\eta_1}{\mathfrak{Q}}\right)_M$. We suppose that $q \neq p$ is a given prime, and that $n \mid q - 1$ is such that $p \nmid n$. Let ξ of order n and let \mathfrak{q} be a prime ideal of $L = \mathbb{Q}(\mu_n)$ dividing q .

We consider the real cyclotomic unit $\eta_1 := (1 + \xi \zeta)^{e_\omega} \zeta^{-\frac{1}{2}}$ (see Definition 4). Recall that for $n \leq 2$, $\eta_1 \in K^{\times p}$, so we suppose $n > 2$.

Let c be the complex conjugation. We suppose in this subsection that the ideal class of \mathfrak{q}^{1-c} is in the p th power of the class group of L , which is equivalent to $\mathfrak{q}^{1-c} = \mathfrak{a}^p (\alpha)$ for an ideal \mathfrak{a} of L and an $\alpha \in L^\times$ prime to p . This condition is also equivalent to $\mathfrak{q} = \mathfrak{b}^{1+c} \mathfrak{a}'^p (\alpha')$ for ideals \mathfrak{a}' , \mathfrak{b} of L and an $\alpha' \in L^\times$.

We can suppose $\alpha \equiv 1 \pmod{p}$ (see the footnote in Theorem 2), so that we get $\mathfrak{q}^{1-c} = \mathfrak{a}^p (1 + p\beta)$, β p -integer in L . Taking the absolute norm gives $N_{L/\mathbb{Q}}(1 + p\beta) = N_{L/\mathbb{Q}}(\mathfrak{a})^{-p}$ which is a rational congruent to 1 modulo p^2 . Thus since $N_{L/\mathbb{Q}}(1 + p\beta) \equiv 1 + p \operatorname{Tr}_{L/\mathbb{Q}}(\beta) \pmod{p^2}$, where $\operatorname{Tr}_{L/\mathbb{Q}}$ is the absolute trace, we obtain $\operatorname{Tr}_{L/\mathbb{Q}}(\beta) \equiv 0 \pmod{p}$. This remark will be used later.

We note that, as for the context of Theorem 2, if $q-1 =: dp^r$, $p \nmid d$, and if the condition $\tilde{\mathfrak{q}}^{1-c} = \tilde{\mathfrak{a}}^p (1+p\tilde{\beta})$ is satisfied for $\tilde{n} = d$ and $\tilde{\mathfrak{q}} \mid q$ in $\tilde{L} = \mathbb{Q}(\mu_{\tilde{n}})$, then it is satisfied for any divisor $n > 2$ of \tilde{n} and the corresponding ideal $\mathfrak{q} = N_{\tilde{L}/L}(\tilde{\mathfrak{q}})$; we then have $\beta \equiv \operatorname{Tr}_{\tilde{L}/L}(\tilde{\beta}) \pmod{p}$.

In $M = LK$ we have:

$$\left(\frac{\eta_1}{(\mathfrak{q})^{1-c}}\right)_M = \left(\frac{\eta_1}{\prod_{\mathfrak{Q} \mid \mathfrak{q}} \mathfrak{Q}^{1-c}}\right)_M = \prod_{\mathfrak{Q} \mid \mathfrak{q}} \left(\frac{\eta_1}{\mathfrak{Q}^{1-c}}\right)_M = \left(\frac{\eta_1}{\mathfrak{Q}}\right)_M^{2 \frac{p-1}{f}},$$

where f is the residue degree of q in K/\mathbb{Q} ; indeed, we have:

$$\left(\frac{\eta_1}{\mathfrak{Q}^{1-c}}\right)_M = \left(\frac{\eta_1}{\mathfrak{Q}}\right)_M \cdot \left(\frac{\eta_1}{\mathfrak{Q}^c}\right)_M^{-1} = \left(\frac{\eta_1}{\mathfrak{Q}}\right)_M \cdot c \left(\frac{\eta_1}{\mathfrak{Q}}\right)_M^{-1} = \left(\frac{\eta_1}{\mathfrak{Q}}\right)_M^2,$$

since η_1 is real, hence the result since the symbol of η_1 does not depend on the choice of \mathfrak{Q} above \mathfrak{q} . But $\left(\frac{\eta_1}{(\mathfrak{q})^{1-c}}\right)_M = \left(\frac{\eta_1}{(\mathfrak{a}^p)(\alpha)}\right)_M = \left(\frac{\eta_1}{(\alpha)}\right)_M$. Then using the general reciprocity law (see e.g. [Gr2, II.7.4.4]) we obtain, since η_1 is a unit:

$$\left(\frac{\eta_1}{\alpha}\right)_M = \left(\frac{\eta_1}{\alpha}\right)_M \left(\frac{\alpha}{\eta_1}\right)_M^{-1} = \prod_{\mathfrak{P} \mid p} (\eta_1, \alpha)_{\mathfrak{P}}^{-1},$$

product over the prime ideals \mathfrak{P} of M above p ; since M/L is totally ramified at p , we will write by abuse $(\eta_1, \alpha)_{\mathfrak{p}}$ for these Hilbert symbols, where $\mathfrak{p} \mid p$ in L , knowing that they are defined on $M^\times \times M^\times$ with values in μ_p .¹⁷

¹⁷Warning: in the literature, two definitions are possible, which give the Hilbert symbol or its inverse; this is the case with the reference [Ko] used below, by comparison with our's (see e.g. [Gr2, II.7.3.1]).

Thus we have obtained:

$$\left(\frac{\eta_1}{\Omega}\right)_M = \prod_{\mathfrak{p}|p} (\eta_1, \alpha)_{\mathfrak{p}}^{\frac{f}{2}}.$$

We refer now to the Brückner–Vostokov explicit formula proved in [Ko, 6.2, Th. 2.99] by giving some details for the convenience of the reader, using similar notations.

Consider the uniformizing parameter $\pi := \zeta - 1$ of the completions $M_{\mathfrak{P}}$ of M at $\mathfrak{P} | \mathfrak{p} | p$. The inertia field is $L_{\mathfrak{p}}$. We need the formal series $t(x) := 1 - (1+x)^p$ since $1 - \zeta = -\pi$ here, for which $t(x)^{-1}$ is the Laurent series:

$$-\frac{1}{x^p} \left(1 - p \left(\frac{c_1}{x} + \dots + \frac{c_{p-1}}{x^{p-1}}\right) + p^2 \left(\frac{c_1}{x} + \dots + \frac{c_{p-1}}{x^{p-1}}\right)^2 - \dots\right),$$

where the c_i are integers.

We associate with $\eta_1 \equiv 1 + \theta \pi \pmod{\pi^2}$, where $\theta := \frac{1}{2} \frac{\xi-1}{\xi+1}$ (see Subsection 4.1), and with $\alpha = 1 + p\beta$, the series:

$$\begin{aligned} F(x) &\equiv 1 + \theta x \pmod{(x^2)}, \\ G(x) &:= 1 + p\beta \text{ (a constant series)}, \end{aligned}$$

such that $F(\pi) \equiv \eta_1 \pmod{\pi^2}$ and $G(\pi) = \alpha$. Recall that \log is the p -adic logarithm and dlog the logarithmic derivative; so $\text{dlog}(G) = 0$ giving:

$$(F, G) = -\frac{1}{p^2} \cdot \log\left(\frac{G^p}{\sigma_p(G)}\right) \cdot \text{dlog}(\sigma_p(F)),$$

where σ_p is the Frobenius automorphism on $L_{\mathfrak{p}}$ extended to series by putting $\sigma_p(x) := x^p$.

Thus $\sigma_p(G) = 1 + p\sigma_p(\beta)$, $\sigma_p(F) \equiv 1 + \sigma_p(\theta)x^p \pmod{(x^{2p})}$, giving:

$$\begin{aligned} \log\left(\frac{G^p}{\sigma_p(G)}\right) &\equiv -p\sigma_p(\beta) \pmod{p^2} \\ \text{dlog}(\sigma_p(F)) &\equiv p\sigma_p(\theta)x^{p-1} \pmod{(x^{2p}, px^{2p-1})}, \end{aligned}$$

and finally:

$$(F, G) \equiv \sigma_p(\theta\beta)x^{p-1} \pmod{(px^{p-1}, x^{2p-1}, \frac{x^{2p}}{p})}.$$

Then the residue of $t(x)^{-1}(F, G)$ is that of:

$$-\frac{1}{x^p} \sigma_p(\theta\beta)x^{p-1} = -\frac{1}{x} \sigma_p(\theta\beta) \pmod{\left(\frac{p}{x}, x^{p-1}, \frac{x^p}{p}\right)},$$

hence it is $-\sigma_p(\theta\beta) \pmod{p}$ since the generator $\frac{x^p}{p}$ of the above ideal gives rise to a residue only with a term of the form $\frac{c}{x^{p+1}}$ of $t(x)^{-1}$ (to give $\frac{c}{px}$) in which case c is a multiple of p^2 (see the expression of $t(x)^{-1}$).

To conclude we have to take the absolute local trace (which eliminates the action of the Frobenius):

$$\text{Tr}_{M_{\mathfrak{P}}/\mathbb{Q}_p}(-\theta\beta) = (p-1) \text{Tr}_{L_{\mathfrak{p}}/\mathbb{Q}_p}(-\theta\beta) \equiv \text{Tr}_{L_{\mathfrak{p}}/\mathbb{Q}_p}(\theta\beta) \pmod{p}.$$

Then $(\eta_1, \alpha)_{\mathfrak{p}} = \zeta^{-\text{Tr}_{L_{\mathfrak{p}}/\mathbb{Q}_p}(\frac{1}{2}\frac{\xi-1}{\xi+1}\beta)}$ because of our definition of the Hilbert symbol, and $\prod_{\mathfrak{p}} (\eta_1, \alpha)_{\mathfrak{p}} = \zeta^{-\sum_{\mathfrak{p}} \text{Tr}_{L_{\mathfrak{p}}/\mathbb{Q}_p}(\frac{1}{2}\frac{\xi-1}{\xi+1}\beta)} = \zeta^{-\text{Tr}_{L/\mathbb{Q}}(\frac{1}{2}\frac{\xi-1}{\xi+1}\beta)}$, the global trace being the sum of the local ones.

We have $\frac{1}{2}\frac{\xi-1}{\xi+1}\beta = (\frac{1}{2} - \frac{1}{\xi+1})\beta$, so the final expression of the trace is $-\text{Tr}_{L/\mathbb{Q}}(\frac{\beta}{\xi+1})$ since that of β is zero modulo p .

This yields to $(\frac{\eta_1}{\Omega})_M = \prod_{\mathfrak{p}} (\eta_1, \alpha)_{\mathfrak{p}}^{\frac{f}{2}} = \zeta^{\frac{1}{2}f \text{Tr}_{L/\mathbb{Q}}(\frac{\beta}{\xi+1})}$.

We have obtained the following explicit formula.

Theorem 3. *Let $q \neq p$ be a prime number, let $n | q - 1$ be such that $p \nmid n$ and $n > 2$. Let ξ of order n and let \mathfrak{q} be any prime ideal of $L = \mathbb{Q}(\mu_n)$ dividing q . We suppose that the ideal class of \mathfrak{q}^{1-c} is the p th power of a class of L , which is equivalent to $\mathfrak{q}^{1-c} = \mathfrak{a}^p (1 + p\beta)$ for an ideal \mathfrak{a} of L and β p -integer in L .¹⁸ Put $\eta_1 := (1 + \xi \zeta)^{e_{\omega}} \zeta^{-\frac{1}{2}}$ (see Definition 4).*

Then for any $\Omega | \mathfrak{q}$ in $M := LK$, $(\frac{\eta_1}{\Omega})_M = \zeta^{\frac{1}{2}f \text{Tr}_{L/\mathbb{Q}}(\frac{\beta}{\xi+1})}$, where f is the residue degree of q in K/\mathbb{Q} and $\text{Tr}_{L/\mathbb{Q}}$ the absolute trace in L/\mathbb{Q} . \square

This gives again the situation of Theorem 2 when $\beta \equiv \beta^+ \pmod{p}$, $\beta^+ \in L^+$, since we then have:

$$\text{Tr}_{L/\mathbb{Q}}(\frac{\beta}{\xi+1}) \equiv \text{Tr}_{L^+/\mathbb{Q}}(\frac{\beta^+}{\xi+1} + \frac{\beta^+}{\xi^c+1}) \equiv \text{Tr}_{L^+/\mathbb{Q}}(\beta^+) \equiv 0 \pmod{p},$$

since $\text{Tr}_{L/\mathbb{Q}}(\beta) \equiv 0 \pmod{p}$.

This theorem confirms the independence, with the SFLT problem, of the class field theory properties of the fields $\mathbb{Q}(\mu_n)$. Meanwhile, under a nontrivial solution of the SFLT equation, for suitable values of q , $n | q - 1$ and ξ of order n , the quantity $\text{Tr}_{L/\mathbb{Q}}(\frac{\beta_{\xi}}{\xi+1})$, where β_{ξ} corresponds to \mathfrak{q}_{ξ} , is imposed, which yields to infinitely many conditions. But as usual we need to explain how the case $p = 3$ interferes appropriately with the arithmetic of the fields $\mathbb{Q}(\mu_n)$ (see Section 9).

Remark 7. Suppose, as in Theorem 3, that $\mathfrak{q}^{1-c} = \mathfrak{a}^p (1 + p\beta)$ for an ideal \mathfrak{a} of L and β p -integer in $L = \mathbb{Q}(\mu_n)$, with $n | q - 1$ such that $p \nmid n$ and $n > 2$. To obtain that \mathfrak{q} is totally split in F_n/L , we study the equivalent condition $(\frac{\eta_t^i}{\Omega})_M = 1$ for all $t \in \text{Gal}(M/K)/\langle t_{-1} \rangle$; from the theorem this is equivalent to $\text{Tr}_{L/\mathbb{Q}}(\frac{\beta}{\xi^t+1}) \equiv 0 \pmod{p}$ for all $t \in \text{Gal}(L/\mathbb{Q})/\langle t_{-1} \rangle$.

This can be written in the following two forms:

$$\sum_{\tau \in \text{Gal}(L/\mathbb{Q})} \frac{\beta^{\tau}}{\xi^{t\tau}+1} \equiv 0 \pmod{p}, \text{ for all } t \in \text{Gal}(L/\mathbb{Q})/\langle t_{-1} \rangle.$$

¹⁸ As we know, this condition is also equivalent to $\mathfrak{q} = \mathfrak{b}^{1+c} \mathfrak{a}'^p (1 + p\beta')$ for ideals \mathfrak{a}' , \mathfrak{b} of L and β' p -integer in L . It is satisfied as soon as the class of \mathfrak{q}^{1-c} is of order prime to p .

$$\sum_{\tau \in \text{Gal}(L/\mathbb{Q})} \frac{\beta^{t\tau}}{\xi^{\tau+1}} \equiv 0 \pmod{p}, \text{ for all } t \in \text{Gal}(L/\mathbb{Q})/\langle t_{-1} \rangle.$$

So we obtain linear systems (with “variables” β^τ and $\frac{1}{\xi^{\tau+1}}$, respectively), whose matrices have $\phi(n)$ columns and $\frac{1}{2}\phi(n)$ lines, and the rank over \mathbb{F}_p of the first matrix (less than or equal to $\frac{1}{2}\phi(n)$) gives a more precise approach of the required conditions on β ; the condition $\beta \equiv \beta^+ \pmod{p}$ is sufficient (use the second system) but not necessary as soon as the rank of the matrix is less than $\frac{1}{2}\phi(n)$.

Let Z'_L be the ring of p -integers of L . Then the knowledge of the image of β in Z'_L/pZ'_L summarizes all the needed local properties of η_1 at the prime q . Since Z'_L/pZ'_L is the product of the residue fields of L at the primes $\mathfrak{p} | p$ in L , any analytic approach is available. \square

Example 4. Take $p = 5$, $n = 4$, and $q \neq 5$ prime congruent to 1 modulo 4. Put $q = a^2 + b^2$ as usual; then $\mathfrak{q} = (a + ib)$ and $\mathfrak{q}^4 = (A + iB)$, with $A = a^4 + b^4 - 6a^2b^2$, $B = 4ab(a^2 - b^2)$. We then have:

$$\mathfrak{q}^{1-c} = \mathfrak{q}^{5(1-c)} \left(\frac{A - iB}{A + iB} \right) =: \mathfrak{q}^{5(1-c)} (1 + 5\beta).$$

Since $A + iB \equiv 1 \pmod{5}$, we get $A \equiv 1$ and $B \equiv 0 \pmod{5}$, and a straightforward computation gives:

$$\beta \equiv -\frac{8iab(a^2 - b^2)}{5} \text{ and } \frac{\beta}{i+1} \equiv -\frac{4(i+1)ab(a^2 - b^2)}{5} \pmod{5},$$

which yields to $\frac{1}{2}\text{Tr}_{L/\mathbb{Q}}\left(\frac{\beta}{i+1}\right) \equiv -\frac{1}{2}\frac{8ab(a^2 - b^2)}{5} \pmod{5}$, hence:

$$\left(\frac{\eta_1}{\Omega}\right)_M = \zeta^f \frac{ab(a^2 - b^2)}{5}.$$

So the symbol is trivial if and only if $ab(a^2 - b^2) \equiv 0 \pmod{25}$. We find the values $q = 313$ ($a = 13$, $b = 12$), $q = 317$ ($a = 14$, $b = 11$), ...

For $q = 457$ ($a = 21$, $b = 4$), we have $\kappa \equiv 0 \pmod{5}$. A case with $25 | ab$ is given by $q = 641$ ($a = 25$, $b = 4$).

The symbol is nontrivial for the values $q = 13$ ($a = 3$, $b = 2$) where $\left(\frac{\eta_1}{\Omega}\right)_M = \zeta^4$, $q = 17$ ($a = 4$, $b = 1$) where $\left(\frac{\eta_1}{\Omega}\right)_M = \zeta^3, \dots$ \square

7. Decomposition law of q in $H_{\mathbb{Q}(\mu_{q-1})}/\mathbb{Q}(\mu_{q-1})$ and conjectures

In this section we study in full generality the situation that we have encountered in the previous sections.

7.1. Law of ρ -decomposition relative to the family \mathcal{F}_n . Let $p > 2$ be a fixed prime number and let $\rho = \frac{v}{u}$, with $\text{g.c.d.}(u, v) = 1$, be a fixed rational distinct from 0 and ± 1 . We do not suppose any relation of SFLT type between u and v .

For any prime number $q \neq p$ let f be the residue degree of q in K and put $\kappa := \frac{q^f - 1}{p}$. Note that we have the relation (see Definition 2, (i)):

$$\bar{\kappa} := \frac{q^{p-1} - 1}{p} \equiv \frac{p-1}{f} \kappa \equiv -\frac{1}{p} \log(q) \pmod{p}.$$

We consider the infinite set of prime numbers:

$$Q_\rho := \{q, q \nmid uv(u^2 - v^2) \text{ and the order of } \rho \text{ modulo } q \text{ is prime to } p\}.$$

For $q \in Q_\rho$, let n be the order of ρ modulo q (by definition we have $p \nmid n$, $n > 2$); from Lemma 2, $q \in Q_\rho$ is equivalent to $q \nmid n$, $q \mid \Phi_n(u, v)$, for $n > 2$, $p \nmid n$.

We consider the fields $K := \mathbb{Q}(\mu_p)$, $L := \mathbb{Q}(\mu_n)$, and $M := LK$ which only depend on q (for ρ fixed).

We associate with q a pair (ξ, \mathfrak{q}) where the primitive n th root of unity $\xi \in L$ and the prime ideal $\mathfrak{q} \mid q$ of L are characterized by the congruence $\xi \equiv \rho \pmod{\mathfrak{q}}$; thus, $\mathfrak{q} = (q, u\xi - v)$ is also denoted \mathfrak{q}_ξ as in the previous sections (see Definition 3). As we know, this pair is defined up to \mathbb{Q} -conjugation and we obtain an equivalence relation. The class associated to q is well-defined. Of course, the classes of (ξ_1, \mathfrak{q}_1) and (ξ_2, \mathfrak{q}_2) , corresponding to different primes q_1 and q_2 , are relative to the fields $L_1 = \mathbb{Q}(\mu_{n_1})$, $n_1 \mid q_1 - 1$, and $L_2 = \mathbb{Q}(\mu_{n_2})$, $n_2 \mid q_2 - 1$, and one of the main problem would be to try to connect the two situations.

From the construction of the extensions F_ξ and $F_n \subseteq H_L^- [p]$ given in Subsections 4.2 and 5.2 via the real cyclotomic unit:

$$\eta_1 := (1 + \xi \zeta)^{e_\omega} \zeta^{-\frac{1}{2}},$$

the pair $(F_\xi, \mathfrak{q}_\xi)$ is defined up to \mathbb{Q} -conjugation since $(tF_\xi, \mathfrak{q}_\xi^t) = (F_{\xi^t}, \mathfrak{q}_{\xi^t})$ corresponds to $(\xi^t, \mathfrak{q}_{\xi^t})$; thus the class of the pair $(F_\xi, \mathfrak{q}_\xi)$ (or similarly of the pair $(\eta_1, \mathfrak{Q}_\xi \mid \mathfrak{q}_\xi)$) characterizes the class of (ξ, \mathfrak{q}_ξ) and reciprocally. Recall that $F_\xi = F_{\xi^{-1}}$ is dihedral over L^+ .

The following lemma is elementary but gives details on the action of $\text{Gal}(L/\mathbb{Q})$ on the family of Frobenii:

Lemma 8. Let $\varphi_\xi := \left(\frac{F_\xi/L}{\mathfrak{q}_\xi}\right)$ be the Frobenius automorphism of the prime ideal $\mathfrak{q}_\xi = (q, u\xi - v)$ in F_ξ/L .

Then $\varphi_{\xi^t} := \left(\frac{F_{\xi^t}/L}{\mathfrak{q}_{\xi^t}}\right) = \varphi_\xi^t := t \varphi_\xi t^{-1}$ for all $t \in \text{Gal}(L/\mathbb{Q})$.

If $t = t_{-1}$, then $\varphi_{\xi^{-1}} = \varphi_\xi^{t_{-1}} = \varphi_\xi^{-1}$ in $F_{\xi^{\pm 1}}/L$.

Proof. From the defining congruence $\varphi_\xi(\alpha) \equiv \alpha^q \pmod{\mathfrak{q}_\xi}$ for all integers α of F_ξ , we get easily $t' \varphi_\xi(\alpha) \equiv t'(\alpha)^q \pmod{\mathfrak{q}_{\xi^t}}$, for any \mathbb{Q} -isomorphism t' of F_ξ such that $t'|_L = t$. Put $t'(\alpha) =: \beta \in F_{\xi^t}$; this yields $t' \varphi_\xi t'^{-1}(\beta) \equiv \beta^q \pmod{\mathfrak{q}_{\xi^t}}$ for all integers β of F_{ξ^t} , proving the lemma by uniqueness of the Frobenius. \square

The Frobenius of \mathfrak{q}_ξ in F_ξ/L is characteristic of the class of (ξ, F_ξ) since we still have $(\xi^t, \varphi_\xi^t) = (\xi^t, \varphi_{\xi^t})$ by conjugation. This leads to give the following definition.

Definition 5. Let $\rho := \frac{v}{u}$, with g.c.d. $(u, v) = 1$, be a fixed rational, distinct from 0 and ± 1 . For $n > 2$ prime to p , let $K := \mathbb{Q}(\mu_p)$, $L = \mathbb{Q}(\mu_n)$, $M = LK$, and for ξ of order n , let F_ξ be such that $F_\xi M = M(\sqrt[p]{(1 + \xi \zeta)^{e_\omega} \zeta^{-\frac{1}{2}}})$.

(i) For any prime q , $q \nmid n$, $q \mid \Phi_n(u, v)$ (i.e., $q \nmid uv$ and ρ is of order n modulo q), and for $\mathfrak{q}_\xi = (q, u\xi - v) \mid q$, we consider the class of Frobenii:

$$\left(\frac{F_{\xi^t}/L}{\mathfrak{q}_{\xi^t}} \right) = \left(\frac{F_\xi/L}{\mathfrak{q}_\xi} \right)^t, \quad t \in \text{Gal}(L/\mathbb{Q}),$$

that we normalize in the following way:

- if $\kappa \not\equiv 0 \pmod{p}$, we put $\left[\frac{F_*/L}{\mathfrak{q}_*} \right]_{\rho, n} := \left(\left(\frac{F_{\xi^t}/L}{\mathfrak{q}_{\xi^t}} \right)_{t \in \text{Gal}(L/\mathbb{Q})}^{\frac{p}{\log(q)}} \right)$;
- if $\kappa \equiv 0 \pmod{p}$, we put $\left[\frac{F_*/L}{\mathfrak{q}_*} \right]_{\rho, n} := 1$.

(ii) Call \mathcal{F}_n the canonical family $(F_{\xi^t})_t = (F_{\xi'})_{\xi'}$ of order n defining F_n/L , where $F_n \subseteq H_L^-[p]$ is the compositum of the F_{ξ^t} , $t \in \text{Gal}(L/\mathbb{Q}) / \langle t_{-1} \rangle$.¹⁹

(iii) The symbol $\left[\frac{F_*/L}{\mathfrak{q}_*} \right]_{\rho, n}$ is called, by abuse of language, the *law of ρ -decomposition of q for the family \mathcal{F}_n* . \square

This object depending on ρ and n is, for each q , relative to a universal family \mathcal{F}_n which is independent of any hypothetic nontrivial solution of the SFLT equation.

Let σ be a generator of $\text{Gal}(F_\xi/L)$; since the Frobenius φ_ξ in F_ξ/L is well defined, it is of the form σ^r , $r \in \mathbb{Z}/p\mathbb{Z}$, so that (when $\kappa \not\equiv 0 \pmod{p}$) the symbol $\left[\frac{F_*/L}{\mathfrak{q}_*} \right]_{\rho, n}$ represents the family (or class):

$$\left(\sigma^t \right)_{t \in \text{Gal}(L/\mathbb{Q})}^{\frac{p}{\log(q)}} = \left(t \cdot \sigma \cdot t^{-1} \right)_{t \in \text{Gal}(L/\mathbb{Q})}^{\frac{p}{\log(q)}}.$$

Thus the symbol $\left[\frac{F_*/L}{\mathfrak{q}_*} \right]_{\rho, n}$ can take $p - 1$ nontrivial “values” (called the cases of ρ -inertia of q for \mathcal{F}_n , when $r \not\equiv 0 \pmod{p}$) and a trivial one (the ρ -splitting of q for \mathcal{F}_n). The case $\kappa \equiv 0 \pmod{p}$ gives the ρ -splitting of q for \mathcal{F}_n .

Note that the Frobenii $\tilde{\varphi}_{\xi^t} := \left(\frac{F_n/L}{\mathfrak{q}_{\xi^t}} \right)$, $t \in \text{Gal}(L/\mathbb{Q})$, are a priori unknown and must not be confused with $\left[\frac{F_*/L}{\mathfrak{q}_*} \right]_{\rho, n}$; they are conjugated, of order 1 or p , and the case of order 1 is very rare since it means that q totally splits in F_n/\mathbb{Q} , i.e., $\left(\frac{F_n/L}{\mathfrak{q}_{\xi^t}} \right) = 1$ for all $t \in \text{Gal}(L/\mathbb{Q})$ (situation of Theorem 2).

¹⁹ Remark that the only knowledge of n determines the field $L = \mathbb{Q}(\mu_n)$ then the family \mathcal{F}_n .

The restriction of $\tilde{\varphi}_\xi$ to F_ξ gives by definition φ_ξ . Its restrictions to the other F_{ξ^t} are the $\left(\frac{F_{\xi^t}/L}{\mathfrak{q}_\xi}\right) = \left(\frac{F_\xi/L}{\mathfrak{q}_{\xi^{t-1}}}\right)^t$.

In the previous sections, in the case $\kappa \not\equiv 0 \pmod{p}$ for $p > 3$, we have used, as a contradiction for the existence of a solution of Fermat's equation, the splitting of \mathfrak{q}_ξ in F_ξ for infinitely many values of q (taking for instance $(u, v) = (x, y), (y, x), (z, y)$, or (y, z)). Same remark for a solution (u, v) of the SFLT equation under the condition $u - v \not\equiv 0 \pmod{p}$.

This property “ $\mathfrak{q}_\xi = (q, u\xi - v)$ splits in F_ξ ”, independent of the choice of the representative pair as Lemma 8 shows, will be called by analogy the “ ρ -splitting of $q \in Q_\rho$ for \mathcal{F}_n ”, $\rho := \frac{v}{u}$. It is equivalent to $\left[\frac{F_*/L}{\mathfrak{q}_*}\right]_{\rho,n} = 1$.

Remark 8. In a probabilistic point of view, the ρ -splitting of $q \in Q_\rho$ for \mathcal{F}_n has a probability around $\frac{1}{p}$, and we can hope a strong incompatibility for analytic reasons since Q_ρ is infinite. If we ask that q be totally split in F_n , this means that each $\mathfrak{q} | q$ splits in $F_\xi = F_{-\xi}$ (for any fixed ξ) and the probability is around $\left(\frac{1}{p}\right)^{\frac{1}{2}\phi(n)}$ which tends to 0 rapidly with $q \rightarrow \infty$. \square

Put:

$$Q_\rho^{\text{spl}} := \{q \in Q_\rho, q \text{ has a } \rho\text{-splitting for } \mathcal{F}_n\}.$$

With a counterexample (u, v) to SFLT, we have, from a pair (ξ, \mathfrak{q}_ξ) , the following results proved in Theorem 1. Put $\rho := \frac{v}{u}$; we may have $u \equiv 0 \pmod{p}$ in which case ρ is not defined modulo p , but is always defined as a rational, so we preserve u and v in the congruences modulo p .

In the nonspecial cases (i.e., $v + u \not\equiv 0 \pmod{p}$):

$$\left(\frac{\eta_1}{\mathfrak{Q}}\right)_M = \zeta^{\frac{1}{2} \frac{v-u}{v+u} \kappa}, \text{ for all } \mathfrak{Q} | \mathfrak{q}_\xi, p \geq 3;$$

In the special case (i.e., $v + u \equiv 0 \pmod{p}$):

$$\begin{aligned} \left(\frac{\eta_1}{\mathfrak{Q}}\right)_M &= 1, \text{ for all } \mathfrak{Q} | \mathfrak{q}_\xi, p > 3, \\ \left(\frac{\eta_1}{\mathfrak{Q}}\right)_M &= \zeta^{\frac{1}{2} \frac{v+u}{3v} \kappa}, \text{ for all } \mathfrak{Q} | \mathfrak{q}_\xi, p = 3. \end{aligned}$$

Recall that for SFLT we cannot exclude the case $u - v \equiv 0 \pmod{p}$ contrary to FLT for $(u, v) = (x, y), (y, x), (z, y)$, or (y, z) . This explain that for SFLT (first case and $\kappa \not\equiv 0 \pmod{p}$) we cannot use, as a general contradiction, the ρ -splitting of q for \mathcal{F}_n .

This does not matter since the existence of a nontrivial solution to SFLT is equivalent to a precise law of ρ -decomposition of q for \mathcal{F}_n , i.e., a precise value of the symbol $\left[\frac{F_*/L}{\mathfrak{q}_*}\right]_{\rho,n}$ (which can be trivial even if $\kappa \not\equiv 0 \pmod{p}$ when $u - v \equiv 0 \pmod{p}$).

More precisely, we have the following lemma giving the action of the Frobenius, which determines explicitly the law of ρ -decomposition (the case $p = 3$ being immediate from Theorem 1, we assume for simplicity $p > 3$):

Lemma 9. *We suppose given, for the prime $p > 3$, a relation of the form $(u + v\zeta)\mathbb{Z}[\zeta] = \mathfrak{w}_1^p$ or $\mathfrak{p}\mathfrak{w}_1^p$, with g.c.d. $(u, v) = 1$.*

Let q be a prime number such that $q \nmid uv$, and such that the order n of $\rho := \frac{v}{u}$ modulo q is prime to p . Let $\Omega \mid \mathfrak{q}_\xi$ in M , where (ξ, \mathfrak{q}_ξ) represents the class corresponding to q .

Let $\left(\frac{M(\sqrt[q]{\eta_1})/M}{\Omega}\right)$ be the Frobenius automorphism of Ω in $M(\sqrt[q]{\eta_1})/M$, where $\eta_1 := (1 + \xi\zeta)^{e_\omega} \zeta^{-\frac{1}{2}}$. We have:

(i) *Nonspecial cases. If $v + u \not\equiv 0 \pmod{p}$, then $\left(\frac{M(\sqrt[q]{\eta_1})/M}{\Omega}\right) \cdot \sqrt[q]{\eta_1} = \zeta^{\frac{1}{2} \frac{v-u}{v+u} \kappa} \cdot \sqrt[q]{\eta_1}$.*

(ii) *Special case. If $v + u \equiv 0 \pmod{p}$, then $\left(\frac{M(\sqrt[q]{\eta_1})/M}{\Omega}\right) \cdot \sqrt[q]{\eta_1} = \sqrt[q]{\eta_1}$.*

Proof. From the defining congruence $(\sqrt[q]{\eta_1})^\sigma \equiv (\sqrt[q]{\eta_1})^{q^f} \pmod{\Omega}$, for the Frobenius automorphism $\sigma := \left(\frac{M(\sqrt[q]{\eta_1})/M}{\Omega}\right)$, we get:

$$(\sqrt[q]{\eta_1})^{\sigma-1} \equiv (\sqrt[q]{\eta_1})^{q^f-1} \equiv \eta_1^\kappa \equiv \left(\frac{\eta_1}{\Omega}\right)_M \pmod{\Omega}.$$

Hence the result since $\left(\frac{\eta_1}{\Omega}\right)_M = \zeta^{\frac{1}{2} \frac{v-u}{v+u} \kappa}$ (resp. 1) in the nonspecial cases (resp. in the special case). \square

We intend now, in the following theorem, to translate this property into a property of the symbol $\left[\frac{F_*/L}{\mathfrak{q}_*}\right]_{\rho, n}$ (see Definition 5), which will give the main phenomenon about the existence of a nontrivial solution to the SFLT equation (see also Remark 9).

Theorem 4. *Let p be a prime number, $p > 3$. We suppose given a solution of the SFLT equation $(u + v\zeta)\mathbb{Z}[\zeta] = \mathfrak{w}_1^p$ or $\mathfrak{p}\mathfrak{w}_1^p$, with g.c.d. $(u, v) = 1$.*

Let q be a prime number such that $q \nmid uv$, and such that the order n of $\rho := \frac{v}{u}$ modulo q is prime to p and > 2 .

Then the symbol $\left[\frac{F_/\mathbb{Q}(\mu_n)}{\mathfrak{q}_*}\right]_{\rho, n}$ only depends on ρ and n when q varies in $Q_\rho := \{q, q \nmid uv(u^2 - v^2) \text{ and the order of } \rho \text{ modulo } q \text{ is prime to } p\}$. In other words, the law of ρ -decomposition of $q \in Q_\rho$ for \mathcal{F}_n only depends on ρ and n .*

Proof. Let $\Omega \mid \mathfrak{q}_\xi$ in M , where (ξ, \mathfrak{q}_ξ) represents the class corresponding to q . The Frobenius automorphism of \mathfrak{q}_ξ in F_ξ/L is given, by restriction, by the relation $\left(\frac{F_\xi/L}{\mathfrak{q}_\xi}\right)^f = \left(\frac{M(\sqrt[q]{\eta_1})/M}{\Omega}\right)_{|_{F_\xi}}$. Indeed, in the projection

$\text{Gal}(M(\sqrt[p]{\eta_1})/M) \rightarrow \text{Gal}(F_\xi/L)$, the Frobenius of the prime ideal \mathfrak{Q} gives the Artin symbol of the norm in M/L of \mathfrak{Q} , which is \mathfrak{q}_ξ^f ; hence the result.

If $\kappa \not\equiv 0 \pmod{p}$, using the relation $f \kappa^{-1} \equiv -\bar{\kappa}^{-1} \pmod{p}$ (see Definition 2, (i)) we get from Lemma 9 that $\left(\frac{F_\xi/L}{\mathfrak{q}_\xi}\right)^{-\bar{\kappa}^{-1}} = \left(\frac{M(\sqrt[p]{\eta_1})/M}{\mathfrak{Q}}\right)_{|_{F_\xi}}^{\kappa^{-1}}$ only depends on ρ and n when q varies. This proves the theorem in this case since $-\bar{\kappa} \equiv \frac{1}{p} \log(q) \not\equiv 0 \pmod{p}$ (see Definition 5).

If $\kappa \equiv 0 \pmod{p}$, we get $\left(\frac{F_\xi/L}{\mathfrak{q}_\xi}\right) = 1$ in any case. □

Remark 9. We can justify the expression “only depends on ρ and n when q varies in Q_ρ ” in the following way.

Let $\bar{F}_n := L_1 F_n$, where $L_1 K = M(\sqrt[p]{\zeta})$, and let $\bar{\varphi}_\xi := \left(\frac{\bar{F}_n/L}{\mathfrak{q}_\xi}\right)$; we know that $\bar{\varphi}_\xi$ projects on φ_ξ in F_ξ/L and on $\varphi_1 := \left(\frac{L_1/L}{\mathfrak{q}_\xi}\right)$ in L_1/L . We treat the case $\kappa \not\equiv 0 \pmod{p}$, i.e., $\varphi_1 \neq 1$.

In the same manner as in the proof of the theorem, in the projection $\text{Gal}(M(\sqrt[p]{\zeta})/M) \rightarrow \text{Gal}(L_1/L)$, we obtain that:

$$\left(\frac{L_1/L}{\mathfrak{q}_\xi}\right)^{\frac{p}{\log(q)}} = \left(\frac{M(\sqrt[p]{\zeta})/M}{\mathfrak{Q}}\right)_{|_{L_1}}^{\kappa^{-1}}$$

is independent of q because of the equality $\left(\frac{M(\sqrt[p]{\zeta})/M}{\mathfrak{Q}}\right)^{\kappa^{-1}} \cdot \sqrt[p]{\zeta} = \zeta \cdot \sqrt[p]{\zeta}$.

Moreover, this is independent of the choice of ξ (of order n) since for all $t \in \text{Gal}(L/\mathbb{Q})$, $\bar{\varphi}_{\xi^t} = t \bar{\varphi}_\xi t^{-1}$ projects, in L_1/L , on $\bar{\varphi}_{\xi^t|_{L_1}} = t \bar{\varphi}_{\xi|_{L_1}} t^{-1} = t \varphi_1 t^{-1} = \varphi_1$ since $\text{Gal}(L_1/\mathbb{Q})$ is abelian.

Which justifies the normalization and the fact that, in some sense, under the existence of a nontrivial solution to the SFLT equation, the symbol $\left[\frac{F_*/L}{\mathfrak{q}_*}\right]_{\rho,n}$ does not depend on q but only on ρ and n (of course n depends on q , but not in a deep arithmetical manner). □

From Theorem 3, when the condition $\mathfrak{q}_\xi^{1-c} = \mathfrak{a}^p (1 + p \beta_\xi)$ is satisfied, for an ideal \mathfrak{a} of L and β_ξ p -integer of L , then $\left(\frac{\eta_1}{\mathfrak{Q}}\right)_M = \zeta^{\frac{1}{2}f} \text{Tr}_{L/\mathbb{Q}}\left(\frac{\beta_\xi}{\xi+1}\right)$, where $\text{Tr}_{L/\mathbb{Q}}$ is the absolute trace in L/\mathbb{Q} . So with a counterexample to SFLT we must have:

$$\text{Tr}_{L/\mathbb{Q}}\left(\frac{\beta_\xi}{\xi+1}\right) \equiv f^{-1} \frac{v-u}{v+u} \kappa \equiv \frac{v-u}{v+u} \frac{\log(q)}{p} \pmod{p}$$

(nonspecial cases, $p \geq 3$) or

$$\text{Tr}_{L/\mathbb{Q}}\left(\frac{\beta_\xi}{\xi+1}\right) \equiv 0 \pmod{p}$$

(special case, $p > 3$).

This means that, under a nontrivial counterexample to SFLT:

$$\left(\frac{F_\xi/L}{\mathfrak{q}_\xi}\right)^{\frac{p}{\log(q)}} \quad \text{and} \quad \frac{p}{\log(q)} \operatorname{Tr}_{L/\mathbb{Q}}\left(\frac{\beta_\xi}{\xi+1}\right), \quad \text{if } \kappa \not\equiv 0 \pmod{p},$$

$$\left(\frac{F_\xi/L}{\mathfrak{q}_\xi}\right) \quad \text{and} \quad \operatorname{Tr}_{L/\mathbb{Q}}\left(\frac{\beta_\xi}{\xi+1}\right), \quad \text{if } \kappa \equiv 0 \pmod{p},$$

both equivalent to the knowledge of $\left[\frac{F_*/L}{\mathfrak{q}_*}\right]_{\rho,n}$, only depend on ρ and n for prime numbers $q \in \mathbb{Q}_\rho$.

So we can hope that this fact, summarized in Theorem 4, is incompatible with the arithmetic of the cyclotomic fields $\mathbb{Q}(\mu_n)$ for $p > 3$.

Remark 10. In the context of Fermat's equation with $r = \frac{y}{x}$, $r' = \frac{y}{z}$, or $r'' = \frac{x}{z}$ (supposed of orders n, n', n'' modulo q , prime to p), we have the same conclusion as in Lemma 9 by using the units η_1, η'_1 , and η''_1 ; from the relation $x + y + z \equiv 0 \pmod{p}$, the values r', r'' can be computed \pmod{p} from r ,²⁰ and we get the following relations valid for $p \geq 3$ since in Fermat's equation, the special case corresponds to $x + z \equiv 0 \pmod{9}$.

(i) If $\kappa \not\equiv 0 \pmod{p}$, then:

$$\left(\frac{M(\sqrt[p]{\eta_1})/M}{\Omega}\right)^{\kappa^{-1}} \cdot \sqrt[p]{\eta_1} = \zeta^{\frac{1}{2}\frac{r-1}{r+1}} \cdot \sqrt[p]{\eta_1},$$

$$\left(\frac{M(\sqrt[p]{\eta'_1})/M}{\Omega^r}\right)^{\kappa^{-1}} \cdot \sqrt[p]{\eta'_1} = \zeta^{-\frac{1}{2}-r} \cdot \sqrt[p]{\eta'_1},$$

$$\left(\frac{M(\sqrt[p]{\eta''_1})/M}{\Omega^{r'}}\right)^{\kappa^{-1}} \cdot \sqrt[p]{\eta''_1} = \zeta^{-\frac{1}{2}-\frac{1}{r}} \cdot \sqrt[p]{\eta''_1}, \quad \text{if } r \not\equiv 0 \pmod{p},$$

$$\left(\frac{M(\sqrt[p]{\eta''_1})/M}{\Omega^{r'}}\right)^{\kappa^{-1}} \cdot \sqrt[p]{\eta''_1} = \sqrt[p]{\eta''_1}, \quad \text{if } r \equiv 0 \pmod{p}.$$

(ii) If $\kappa \equiv 0 \pmod{p}$, the three symbols $\left(\frac{M(\sqrt[p]{\bullet})/M}{\bullet}\right)$ are trivial. \square

7.2. Law of ρ -decomposition relative to the family $\widehat{\mathcal{F}}_n$, for $n > 2$. We still suppose $p > 3$. We have, under a counterexample (u, v) to SFLT, the following interpretation of the equality:

$$\left(\frac{\eta_1}{\Omega_\xi}\right)_M = \zeta^{\frac{1}{2}\frac{v-u}{v+u}\kappa} \left(\text{resp. } \left(\frac{\eta_1}{\Omega_\xi}\right)_M = 1\right)$$

in the nonspecial cases $v + u \not\equiv 0 \pmod{p}$ (resp. the special case $v + u \equiv 0 \pmod{p}$), which is also valid in the cases $\kappa \equiv 0$ or $u - v \equiv 0 \pmod{p}$. This will give also another formulation of Theorem 4.

Consider the unit:

$$\widehat{\eta}_1 := \eta_1 \zeta^{-\frac{1}{2}\frac{v-u}{v+u}} \quad (\text{resp. } \widehat{\eta}_1 := \eta_1)$$

²⁰The notations r, r' , and r'' correspond to $\rho = \frac{v}{u}$ in the equation $(u + v\zeta)\mathbb{Z}[\zeta] = \mathfrak{w}_1^p$ or $\mathfrak{p}\mathfrak{w}_1^p$, for $(u, v) = (x, y), (y, x), (z, y)$, or (y, z) , and $(u, v) = (x, z)$ or (z, x) (nonspecial cases and special case, respectively); this explains the changes of notations in the Fermat context. We obtain easily $r' \equiv \frac{-r}{r+1}, r'' \equiv \frac{-1}{r+1}$ modulo p .

in the nonspecial cases (resp. in the special case).

(i) In the nonspecial cases we have:

$$\widehat{\eta}_1 = (1 + \xi \zeta)^{e\omega} \zeta^{-\frac{1}{2} - \frac{1}{2} \frac{v-u}{v+u}} = (1 + \xi \zeta)^{e\omega} \zeta^{-\frac{v}{v+u}},$$

which is by construction such that $\left(\frac{\widehat{\eta}_1}{\Omega_\xi}\right)_M = 1$, but the unit $\widehat{\eta}_1$ is not any-more real; its definition from η_1 is independent of q under a given solution of the SFLT equation.

(ii) In the special case we obtain $\widehat{\eta}_1 := \eta_1 = (1 + \xi \zeta)^{e\omega} \zeta^{-\frac{1}{2}}$, which is real and by construction such that $\left(\frac{\widehat{\eta}_1}{\Omega_\xi}\right)_M = 1$.

The extension $M(\sqrt[p]{\widehat{\eta}_1})/M$ is splitted over L by a p -cyclic p -ramified extension \widehat{F}_ξ similar to F_ξ except that it is not dihedral over L^+ in the nonspecial cases.

We note that the relation $\widehat{\eta}_1 = \eta_1 \zeta^{-\frac{1}{2} \frac{v-u}{v+u}}$ in the nonspecial cases shows that \widehat{F}_ξ is a subfield of the compositum $F_\xi L_1$ obtained in an obvious systematic way (\widehat{F}_ξ/L is still of degree p and p -ramified since $n > 2$). But \widehat{F}_ξ is effective only if ρ is known, which is not in general the case in the nonspecial cases of the SFLT problem.

It is clear that $\widehat{F}_\xi = F_\xi$ if and only if $u^2 - v^2 \equiv 0 \pmod{p}$.

We still have $t\widehat{F}_\xi = \widehat{F}_{\xi^t}$. We call \widehat{F}_n the compositum of the \widehat{F}_{ξ^t} , $t \in \text{Gal}(L/\mathbb{Q})$. Hence $F_n L_1 = \widehat{F}_n L_1$.

We denote, as in Definition 5, by $\widehat{\mathcal{F}}_n$ the family $(\widehat{F}_{\xi^t})_{\xi^t}$ of order n .

Then under a nontrivial solution of the equation attached to SFLT, we must have the splitting of \mathfrak{q}_ξ in \widehat{F}_ξ (i.e., a ρ -splitting for $\widehat{\mathcal{F}}_n$). In other words if we define, as in Definition 5, for $\kappa \not\equiv 0 \pmod{p}$, the symbol:

$$\left[\frac{\widehat{F}_*/L}{\mathfrak{q}_*}\right]_{\rho,n} := \left(\left(\frac{\widehat{F}_{\xi^t}/L}{\mathfrak{q}_{\xi^t}}\right)^{\frac{p}{\log(q)}}\right)_{t \in \text{Gal}(L/\mathbb{Q})},$$

the analog of Theorem 4 is $\left[\frac{\widehat{F}_*/L}{\mathfrak{q}_*}\right]_{\rho,n} = 1$ for all $q \in Q_\rho$, where:

$$Q_\rho := \{q, q \nmid uv(u^2 - v^2) \text{ and the order of } \rho \text{ modulo } q \text{ is prime to } p\}.$$

A contradiction would be that there exist prime numbers q such that $\left[\frac{\widehat{F}_*/L}{\mathfrak{q}_*}\right]_{\rho,n} \neq 1$ i.e., \mathfrak{q}_ξ is inert in \widehat{F}_ξ , which is independent of the representative pair $(\widehat{F}_{\xi^t}, \mathfrak{q}_{\xi^t})$ and has a probability very near from $\frac{p-1}{p}$ since $p-1$ values of the symbol are possible. About the class of pairs $(\widehat{F}_{\xi^t}, \mathfrak{q}_{\xi^t})$, when $\left[\frac{\widehat{F}_*/L}{\mathfrak{q}_*}\right]_{\rho,n} \neq 1$, we can speak of “ ρ -inertia of q for $\widehat{\mathcal{F}}_n$ ”.

In a similar way, in the context of Fermat's equation, we deduce from the units η_1 , η'_1 , and η''_1 (see Remark 10), the units, where $r := \frac{y}{x} \not\equiv \pm 1 \pmod{p}$:

$$\begin{aligned}\widehat{\eta}_1 &:= (1 + \xi \zeta)^{e_\omega} \zeta^{\frac{-r}{r+1}}, \\ \widehat{\eta}'_1 &:= (1 + \xi' \zeta)^{e_\omega} \zeta^r, \\ \widehat{\eta}''_1 &:= (1 + \xi'' \zeta)^{e_\omega} \zeta^{\frac{1}{r}}, \text{ if } r \not\equiv 0 \pmod{p}, \\ \widehat{\eta}'''_1 &:= (1 + \xi'' \zeta)^{e_\omega} \zeta^{-\frac{1}{2}}, \text{ if } r \equiv 0 \pmod{p},\end{aligned}$$

giving a trivial p th power residue symbol at Ω , Ω' , and Ω'' , respectively.

We have the same conclusion as above for the extensions \widehat{F}_ξ/L , $\widehat{F}_{\xi'}/L'$, $\widehat{F}_{\xi''}/L''$ defined from $M(\sqrt[r]{\widehat{\eta}_1})/M$, $M'(\sqrt[r]{\widehat{\eta}'_1})/M'$, $M''(\sqrt[r]{\widehat{\eta}''_1})/M''$.

Returning to SFLT with a nontrivial solution (u, v) , we put as above:

$$\widehat{Q}_\rho^{\text{in}} := \{q \in Q_\rho, q \text{ has a } \rho\text{-inertia for } \widehat{\mathcal{F}}_n\}.$$

Lemma 10. *Suppose $p > 3$ and $\kappa \not\equiv 0 \pmod{p}$. If $u^2 - v^2 \not\equiv 0 \pmod{p}$ then we have $Q_\rho^{\text{spl}} \subseteq \widehat{Q}_\rho^{\text{in}}$. If $u^2 - v^2 \equiv 0 \pmod{p}$ then $Q_\rho^{\text{spl}} \cap \widehat{Q}_\rho^{\text{in}} = \emptyset$.*

Proof. We know that \widehat{F}_ξ is contained in the compositum $L_1 F_\xi$ and is distinct from L_1 since $\xi \neq \pm 1$.

Suppose that \widehat{F}_ξ is distinct from F_ξ ; if $q \in Q_\rho^{\text{spl}}$, \mathfrak{q}_ξ splits in F_ξ/L and the Frobenius of \mathfrak{q}_ξ in $L_1 F_\xi/L$ fixes F_ξ and since this Frobenius must be nontrivial in L_1/L ($\kappa \not\equiv 0 \pmod{p}$) then projects to a nontrivial Frobenius in \widehat{F}_ξ/L . When $\widehat{F}_\xi = F_\xi$, the result is clear. The lemma comes from the characterization of the equality $\widehat{F}_\xi = F_\xi$ (i.e., $u^2 - v^2 \equiv 0 \pmod{p}$). \square

It will be interesting to examine the problem, for any ρ , independently of any equation giving exceptional values of ρ .

The natural conjecture in this direction would be the following, which implies SFLT (we still put $K = \mathbb{Q}(\mu_p)$, $L = \mathbb{Q}(\mu_n)$, $M = LK$, to simplify the notations):

Conjecture 3. *Let p be a prime number, $p > 3$, and let $\rho = \frac{v}{u}$, with $\text{g.c.d.}(u, v) = 1$, be a rational distinct from 0 and ± 1 . Put:*

$$Q_\rho := \{q, q \nmid uv(u^2 - v^2) \text{ and the order of } \rho \text{ modulo } q \text{ is prime to } p\}.$$

For $q \in Q_\rho$, let n be the order of ρ modulo q and let $\widehat{\mathcal{F}}_n$ be the family of the cyclic extensions \widehat{F}_ξ of L , for ξ of order n , defined by the relations $\widehat{F}_\xi K = M\left(\sqrt[p]{(1 + \xi \zeta)^{e_\omega} \zeta^{-\frac{v}{v+u}}}\right)$ (resp. $\widehat{F}_\xi K = M\left(\sqrt[p]{(1 + \xi \zeta)^{e_\omega} \zeta^{-\frac{1}{2}}}\right)$ if $v + u \not\equiv 0$ (resp. $v + u \equiv 0$) \pmod{p}). Say that q has a ρ -inertia for $\widehat{\mathcal{F}}_n$ if $\left[\frac{\widehat{F}_/L}{\mathfrak{q}_*}\right]_{\rho, n} \neq 1$, i.e., $\mathfrak{q}_\xi := (q, u\xi - v)$ is inert in \widehat{F}_ξ/L (condition independent of the choice of ξ of order n).*

Then the set of primes $q \in Q_\rho$ having a ρ -inertia for $\widehat{\mathcal{F}}_n$, is infinite. \square

The extension \widehat{F}_n (depending on ρ contrary to F_n) is a subfield of the maximal p -ramified p -elementary extension $H_{L[p]}$ of L , and the arithmetical properties of $H_{L[p]}$ and of its subextensions are, a priori, independent of any diophantine problem as Fermat's equation.

Recall that to prove the first case of FLT for p , the existence of a single $q \in Q_\rho$ ($\rho = \frac{y}{x}$ for a solution (x, y, z)) having a ρ -inertia for \widehat{F}_n is sufficient, contrary to the second case which needs infinitely many such primes.

In the first case, $p \nmid xy(x^2 - y^2)$ (from Lemma 1) and so, if $\kappa \not\equiv 0 \pmod{p}$ then $q \nmid xy(x^2 - y^2)$ from the two theorems of Furtwängler (see Corollaries 2, 3, and Remark 3).

Hence $q \in Q_\rho$ as soon as $\kappa \not\equiv 0 \pmod{p}$ and $q \not\equiv 1 \pmod{p}$. These two conditions on q are effective and the first case of FLT is easier than the second one because it is generally possible to check the conjecture for small values of q . The second case supposes to find q large enough; this shows that the first case is likely a weaker conjecture.

If we examine, for logical reasons, the case $p = 3$ for SFLT, we know that for any of the six families of solutions (u, v) of the SFLT equation (see Remark 1), we have (supposing $\kappa \not\equiv 0 \pmod{3}$) and defining $\widehat{\eta}_1$ in an analogous way):

- (i) $\left(\frac{\eta_1}{\Omega_\xi}\right)_M = \zeta^{\frac{1}{2} \frac{v-u}{v+u} \kappa} = 1$, in the first case (i.e., $uv(u+v) \not\equiv 0 \pmod{3}$), which implies $u - v \equiv 0 \pmod{3}$, hence $\widehat{\eta}_1 = \eta_1$ and $\widehat{F}_\xi = F_\xi$;
- (ii) $\left(\frac{\eta_1}{\Omega_\xi}\right)_M = \zeta^{\pm \frac{1}{2} \kappa}$ in the second case (i.e., $uv \equiv 0 \pmod{3}$), thus $\widehat{\eta}_1 = \eta_1 \zeta^{\mp \frac{1}{2}}$ and $\widehat{F}_\xi \neq F_\xi$;
- (iii) $\left(\frac{\eta_1}{\Omega_\xi}\right)_M = \zeta^{\frac{1}{2} \frac{v+u}{3v} \kappa}$ in the special case (i.e., $u + v \equiv 0 \pmod{3}$) for which $\widehat{\eta}_1 = \eta_1 \zeta^{-\frac{1}{2} \frac{v+u}{3v}}$ and $\widehat{F}_\xi = F_\xi$ if and only if $v + u \equiv 0 \pmod{9}$.

If $v + u \equiv 0 \pmod{3}$ and $v + u \not\equiv 0 \pmod{9}$ then, for $\rho := \frac{v}{u}$, we get $Q_\rho^{\text{spl}} \subseteq \widehat{Q}_\rho^{\text{in}}$; if $v + u \equiv 0 \pmod{9}$ or $u - v \equiv 0 \pmod{3}$ then $Q_\rho^{\text{spl}} \cap \widehat{Q}_\rho^{\text{in}} = \emptyset$.

We see that $u - v \equiv 0 \pmod{3}$ in case (i), $uv \equiv 0 \pmod{3}$ in case (ii); for (iii), we verify from Remark 1 that $\frac{v}{u} \in \{-1, 2, 5\}$ modulo 9, which gives $\frac{1}{2} \frac{v+u}{3v} \in \{0, 1, 2\}$ modulo 3. So, for q fixed we can find solutions (u_i, v_i) giving the same order n of $\frac{v_i}{u_i}$ modulo q and any of the above value of $\frac{v}{u}$ modulo 9.

See Section 9 to go thoroughly into the exceptional case $p = 3$.

7.3. Construction of universal defining polynomials. The group g operates canonically on the field $K(Y)$ of rational fractions, where Y is an indeterminate. Consider:

$$F(Y) := (1 + Y \zeta)^{e_\omega} \zeta^{-\frac{1}{2}} \in K(Y).$$

Then if $s = s_r$ is a generator of g we have:

$$s.F(Y) := ((1+Y \zeta^s) \zeta^{-\frac{1}{2}s})^{e_\omega} = ((1+Y \zeta) \zeta^{-\frac{1}{2}})^s e_\omega = ((1+Y \zeta) \zeta^{-\frac{1}{2}})^{r e_\omega + p \Lambda},$$

since $s e_\omega = r e_\omega + p \Lambda$ for some $\Lambda \in \mathbb{Z}[g]$ (see Definition 1, (iii)). Then we obtain:

$$s.F(Y) = F(Y)^r \cdot ((1+Y \zeta) \zeta^{-\frac{1}{2}})^{p \Lambda}.$$

Consider the Kummer extension $K(Y)(\sqrt[p]{F(Y)})/K(Y)$; since this extension is abelian over $\mathbb{Q}(Y)$, the $K(Y)$ -automorphism of $K(Y)(\sqrt[p]{F(Y)})$, still denoted s , defined by $s \cdot \sqrt[p]{F(Y)} := (\sqrt[p]{F(Y)})^r \cdot ((1+Y \zeta) \zeta^{-\frac{1}{2}})^{\Lambda}$ is of order $p-1$ and it is a classical result that the trace $\Psi := \sum_{k=1}^{p-1} s^k \cdot \sqrt[p]{F(Y)}$ defines a primitive element of the subextension cyclic of degree p contained in $K(Y)(\sqrt[p]{F(Y)})/\mathbb{Q}(Y)$.

An easy way to find $\text{Irr}(\Psi, \mathbb{Q}(Y))$ is to use the Newton formulas from the computations of the traces:

$$\text{Tr}((\sqrt[p]{F(Y)})^i) := \sum_{k=1}^{p-1} s^k \cdot (\sqrt[p]{F(Y)})^i, \quad i = 1, \dots, p-1.$$

For instance, for $p = 3$, $e_\omega = s - 1$, $s = s_2$, $s e_\omega = 1 - s = -e_\omega$ (thus $r = 2$, $\Lambda = -e_\omega$), $F(Y) = (1+Y j)^{e_\omega} j = ((1+Y j) j)^{s-1}$; we have $\Psi = \left(\frac{(1+Y j^2) j}{1+Y j}\right)^{\frac{1}{3}} + \left(\frac{(1+Y j) j^2}{1+Y j^2}\right)^{\frac{1}{3}}$, for which we get $\Psi^3 = \frac{(1+Y j^2) j}{1+Y j} + \frac{(1+Y j) j^2}{1+Y j^2} + 3 \Psi$, giving the irreducible polynomial:

$$\text{Irr}(\Psi, \mathbb{Q}(Y)) = X^3 - 3X + \frac{Y^2 - 4Y + 1}{Y^2 - Y + 1},$$

or the unitary polynomial $X^3 - 3(Y^2 - Y + 1)^2 X + (Y^2 - 4Y + 1)(Y^2 - Y + 1)$ taking the representative idempotent $e_\omega = s + 2$.

Definition 6. The general case of degree p can be written:

$$\text{Irr}(\Psi, \mathbb{Q}(Y)) = A_p(Y)X^p + \dots + A_1(Y)X + A_0(Y), \quad A_i(Y) \in \mathbb{Z}[Y],$$

and will be called a universal polynomial of degree p for the SFLT problem.

For any given n th root of unity ξ , $n > 2$, the polynomial:

$$A_p(\xi)X^p + \dots + A_1(\xi)X + A_0(\xi) \in L[X], \quad L := \mathbb{Q}(\mu_n),$$

is the irreducible polynomial of the primitive element:

$$\psi := \text{Tr}_{M(\sqrt[p]{\eta_1})/F_\xi}(\sqrt[p]{\eta_1})$$

defining the extension F_ξ , with the usual notations. \square

We have the following result where we recall that, for g.c.d. $(u, v) = 1$, we have put $\Phi_n(u, v) := \prod_{\xi' \text{ of order } n} (u \xi' - v)$.

Theorem 5. Let p be a prime number, $p > 3$, and let $\rho = \frac{v}{u}$, with $\text{g.c.d.}(u, v) = 1$, be a fixed rational distinct from 0 and ± 1 ; suppose $u - v \not\equiv 0 \pmod{p}$.

(i) *Nonspecial cases* ($u + v \not\equiv 0 \pmod{p}$). Let $n > 2$ be prime to p , and let $q \nmid n$ be a prime number such that $\kappa \not\equiv 0 \pmod{p}$ and $q \mid \Phi_n(u, v)$.

If the polynomial $A_p(\rho)X^p + \dots + A_1(\rho)X + A_0(\rho)$ is not irreducible modulo q , then (u, v) cannot be a solution of the equation $(u + v\zeta)\mathbb{Z}[\zeta] = \mathfrak{w}_1^p$ attached to the nonspecial cases of SFLT.

(ii) *Special case* ($v + u \equiv 0 \pmod{p}$). Let $n > 2$ be prime to p , and let $q \nmid n$ be a prime number such that $\kappa \not\equiv 0 \pmod{p}$ and $q \mid \Phi_n(u, v)$.

If the polynomial $A_p(\rho)X^p + \dots + A_1(\rho)X + A_0(\rho)$ is irreducible modulo q , then (u, v) cannot be a solution of the equation $(u + v\zeta)\mathbb{Z}[\zeta] = \mathfrak{p}\mathfrak{w}_1^p$ attached to the special case of SFLT.

(iii) Let $n > 2$ be prime to p , and let $q \nmid n$ be a prime number such that $\kappa \equiv 0 \pmod{p}$ and $q \mid \Phi_n(u, v)$.

If the polynomial $A_p(\rho)X^p + \dots + A_1(\rho)X + A_0(\rho)$ is irreducible modulo q , then (u, v) cannot be a solution of the SFLT equation.

Proof. From Lemma 2, $q \nmid n$ and $q \mid \Phi_n(u, v)$ is equivalent to $q \nmid uv$ and ρ is of order n modulo q ; then $\rho \equiv \xi \pmod{\mathfrak{q}_\xi}$, for any choice of the n th root of unity ξ , and in case (i) there exists a root $\lambda \in \mathbb{Z}$ modulo q , of the polynomial, such that:

$$\begin{aligned} A_p(\rho)\lambda^p + \dots + A_1(\rho)\lambda + A_0(\rho) &\equiv A_p(\xi)\lambda^p + \dots + A_1(\xi)\lambda + A_0(\xi) \\ &\equiv 0 \pmod{\mathfrak{q}_\xi}, \end{aligned}$$

since q divides the left member. This means that $\text{Irr}(\psi, L)$ has the root λ modulo \mathfrak{q}_ξ and that \mathfrak{q}_ξ splits in F_ξ/L (i.e., $\left[\frac{F_\xi/L}{\mathfrak{q}_\xi}\right]_{\rho, n} = 1$).

If we suppose that (u, v) is a counterexample to SFLT, Theorem 1 in the nonspecial cases gives $\left(\frac{\eta}{\Omega}\right)_M = \zeta^{\frac{1}{2}\frac{v-u}{v+u}\kappa} \neq 1$ by assumption, equivalent to the inertia of \mathfrak{q}_ξ in F_ξ/L (contradiction).

The proofs of cases (ii) and (iii) are similar but inverted (the hypothesis implies $\left[\frac{F_\xi/L}{\mathfrak{q}_\xi}\right]_{\rho, n} \neq 1$ while $\left(\frac{\eta}{\Omega}\right)_M = 1$ for a solution in these cases). \square

In other words, the corresponding conjecture giving a proof of SFLT under the assumption $u - v \not\equiv 0 \pmod{p}$, which implies the two cases of FLT, is the following.

Conjecture 4. Let p be a prime number, $p > 3$, and let $\rho = \frac{v}{u}$, with $\text{g.c.d.}(u, v) = 1$, be a rational distinct from 0 and ± 1 ; suppose $u - v \not\equiv 0 \pmod{p}$.

(i) *Case* $u + v \not\equiv 0 \pmod{p}$. There exist infinitely many prime numbers q with $\kappa \not\equiv 0 \pmod{p}$ such that $A_p(\rho)X^p + \dots + A_1(\rho)X + A_0(\rho)$ is not

irreducible modulo q and $q \mid \Phi_n(u, v)$ for suitable values of $n > 2$ prime to p .

(ii) Case $v+u \equiv 0 \pmod{p}$. There exist infinitely many prime numbers q with $\kappa \not\equiv 0 \pmod{p}$ such that $A_p(\rho)X^p + \cdots + A_1(\rho)X + A_0(\rho)$ is irreducible modulo q and $q \mid \Phi_n(u, v)$ for suitable values of $n > 2$ prime to p .

(iii) There exist infinitely many prime numbers q with $\kappa \equiv 0 \pmod{p}$ such that $A_p(\rho)X^p + \cdots + A_1(\rho)X + A_0(\rho)$ is irreducible modulo q and $q \mid \Phi_n(u, v)$ for suitable values of $n > 2$ prime to p . \square

Of course, without an independent approach (analytic or geometric), the problem has no longer solution since the polynomial:

$$A_p(\rho)X^p + \cdots + A_1(\rho)X + A_0(\rho)$$

can be in case (i) that of a primitive element of L_1 , in which case all the primes which split in L_1/\mathbb{Q} are such that $\kappa \equiv 0 \pmod{p}$, and in cases (ii) and (iii), the polynomial may be splitted over \mathbb{Q} . Meanwhile the universal polynomial $A_p(Y)X^p + \cdots + A_1(Y)X + A_0(Y)$ has the nontrivial property that for any primitive n th root of unity ξ , $n > 2$, $A_p(\xi)X^p + \cdots + A_1(\xi)X + A_0(\xi)$ is irreducible in $\mathbb{Q}(\mu_n)[X]$ and defines a p -ramified cyclic extension of $\mathbb{Q}(\mu_n)$.

8. Normic relations for cyclotomic units

In this section we give a relation between the units η_1 and η'_1 associated to the classes of two prime numbers q and q' for which the pairs (ξ, \mathfrak{q}_ξ) , $(\xi', \mathfrak{q}_{\xi'})$ are such that the order n' of ξ' divides the order n of ξ , $p \nmid n$.

Put $n = n'd$. We introduce the following notations:

$$\begin{aligned} L &= \mathbb{Q}(\mu_n), \quad L' = \mathbb{Q}(\mu_{n'}), \\ M &= LK, \quad M' = L'K, \\ \eta_1 &= (1 + \xi \zeta)^{e_\omega} \zeta^{-\frac{1}{2}}, \quad \eta'_1 = (1 + \xi' \zeta)^{e_\omega} \zeta^{-\frac{1}{2}}; \end{aligned}$$

to fix the notations, we suppose that $\xi' = \xi^d$.

Since η_1 is a cyclotomic unit, the action of relative norms on this unit is well-known and we now recall the result in our particular context.

Proposition 3. Denote by N the relative norm $N_{M/M'}$ and by S the set of distinct prime numbers dividing d and not dividing n' .

Then we have $N(\eta_1) = (\eta'_1)^{\Lambda'}$, where $\Lambda' \equiv d \cdot \prod_{\ell \in S} (1 - \ell^{-1} t'_\ell^{-1}) \pmod{p}$, $t'_\ell \in \text{Gal}(M'/K)$ being the Artin automorphism defined by $t'_\ell(\xi') := (\xi')^\ell$.

Proof. By induction we can suppose that d is a prime number ℓ .

Let $\psi := \xi^{n'}$ which is a primitive ℓ th root of unity.

(i) Case $\ell \mid n'$. In this case $S = \emptyset$, $[M : M'] = \ell$, and:

$$\begin{aligned} N(1 + \xi \zeta) &= \prod_{\lambda=0}^{\ell-1} (1 + \xi^{1+\lambda n'} \zeta) = \prod_{\lambda=0}^{\ell-1} (1 + \xi \psi^\lambda \zeta) \\ &= 1 + \xi^\ell \zeta^\ell = 1 + \xi' \zeta^\ell = (1 + \xi' \zeta)^{s_\ell}. \end{aligned}$$

Then $N(\eta_1) = (1 + \xi' \zeta)^{s_\ell e_\omega} N(\zeta)^{-\frac{1}{2}} \sim (1 + \xi' \zeta)^{\ell e_\omega} \zeta^{-\frac{1}{2}\ell} = (\eta'_1)^\ell$ since $s_\ell e_\omega \equiv \ell e_\omega \pmod{p}$.

(ii) Case $\ell \nmid n'$. In this case $S = \{\ell\}$ and:

$$N(1 + \xi \zeta) = \prod_{\lambda=0, \lambda \neq \lambda_0}^{\ell-1} (1 + \xi^{1+\lambda n'} \zeta),$$

where λ_0 is the unique value modulo ℓ such that $1 + \lambda_0 n' \equiv 0 \pmod{\ell}$, giving from the computation in (i):

$$N(1 + \xi \zeta) = \frac{1 + \xi^\ell \zeta^\ell}{1 + \xi^{1+\lambda_0 n'} \zeta} = \frac{(1 + \xi' \zeta)^{s_\ell}}{1 + (\xi')^\mu \zeta},$$

where $1 + \lambda_0 n' = \mu \ell$, so that $\mu \equiv \ell^{-1} \pmod{n'}$. Thus:

$$\begin{aligned} N(1 + \xi \zeta) &= \frac{(1 + \xi' \zeta)^{s_\ell}}{1 + (\xi')^{\ell-1} \zeta} = \frac{(1 + \xi' \zeta)^{s_\ell}}{1 + (\xi')^{t'_\ell-1} \zeta} \\ &= \left(\frac{1 + \xi' \zeta}{1 + (\xi')^{t'_\ell-1} (\zeta)^{s_\ell^{-1}}} \right)^{s_\ell} = \left(\frac{1 + \xi' \zeta}{1 + (\xi' \zeta)^{\sigma'_\ell-1}} \right)^{s_\ell}, \end{aligned}$$

where $\sigma'_\ell \in \text{Gal}(M'/\mathbb{Q})$ is the Artin automorphism defined by $\sigma'_\ell(\theta) = \theta^\ell$ for any pn' th root of unity θ ; thus, since $\sigma'_\ell = s_\ell t'_\ell$, this yields:

$$N(1 + \xi \zeta)^{e_\omega} \sim \left(\frac{1 + \xi' \zeta}{1 + (\xi' \zeta)^{\sigma'_\ell-1}} \right)^{\ell e_\omega} = (1 + \xi' \zeta)^{\ell(1-\sigma'_\ell-1)e_\omega};$$

from $\sigma'_\ell^{-1} e_\omega = s_\ell^{-1} t'_\ell^{-1} e_\omega \equiv \ell^{-1} t'_\ell^{-1} e_\omega \pmod{p}$, we get $N(1 + \xi \zeta)^{e_\omega} \sim (1 + \xi' \zeta)^{\ell(1-\ell^{-1} t'_\ell^{-1}) e_\omega}$. Finally, since in this case $[M : M'] = \ell - 1$ and $N(\zeta) = \zeta^{\ell-1} = \zeta^{\ell(1-\ell^{-1} t'_\ell^{-1})}$, we get:

$$N(\eta_1) \sim (\eta'_1)^{\ell(1-\ell^{-1} t'_\ell^{-1})}$$

and the proposition follows. \square

If for instance Λ' is invertible modulo p , with inverse Ω' , then $\eta'_1 \sim N(\eta_1)^{\Omega'}$ and, over L , we can see the abelian extension $F'_{n'}$ (compositum of the conjugates of the $F'_{\xi'}$ over L') as a subfield of F_n , in which case, for suitable primes q and q' , the properties of the Frobenii studied in this paper can be compared to give strengthened conditions.

9. Analysis of the case $p = 3$ versus $p \neq 3$

In this section we suppose $p = 3$ and consider the solutions of the equation associated to SFLT (see Remark 1) which are, for $p > 3$, a logical obstruction to the relevance of general statements similar to Theorem 2 and to the property of ρ -law of decomposition of Theorem 4. We intend to explain why this obstruction actually exists for $p = 3$ but a priori not for $p > 3$ when we suppose that the set of nontrivial solutions is nonempty.

The main differences between the cases $p = 3$ and $p > 3$, are that there are infinitely many solutions for the case $p = 3$, contrary to the case $p > 3$, even if we have not proved this fact (which was known for Fermat's equation before Wiles proof), and that we will exhibit a group of automorphisms, acting on the set of solutions for $p = 3$, which creates some exceptional relations of compatibility with density theorems. So we conjecture that this fact does not exist for $p > 3$.

9.1. Analysis of the case $p = 3$ for the principle of Theorem 2.

Recall Theorem 1 in that case, for the choice of a prime $q \neq p$. We have $\eta_1 = (1 + \xi \zeta)^{e_\omega} \zeta^{-\frac{1}{2}}$, with $\zeta = j$ and $e_\omega = s - 1$, where $\xi \equiv \frac{v}{u} \pmod{\mathfrak{q}_\xi}$ is supposed of order $n \not\equiv 0 \pmod{3}$ for a nontrivial solution (u, v) , g.c.d. $(u, v) = 1$, of the SFLT equation.

Recall that if we put $\rho := \frac{v}{u}$ (distinct from 0 and ± 1) we may have $u \equiv 0 \pmod{3}$ in which case ρ is not defined modulo 3, but is always defined as a rational. Put $L = \mathbb{Q}(\mu_n)$ and $M = LK$:

(i) First case. Since $uv(u+v) \not\equiv 0 \pmod{3}$, we get $u \equiv v \equiv \pm 1 \pmod{3}$, so that $\left(\frac{\eta_1}{\Omega}\right)_M = j^{\frac{1}{2} \frac{v-u}{v+u} \kappa} = 1$ for any $\Omega | \mathfrak{q}_\xi$ in M .

(ii) Second case. We get $\left(\frac{\eta_1}{\Omega}\right)_M = j^{\pm \frac{1}{2} \kappa}$ for any $\Omega | \mathfrak{q}_\xi$ since $3 | uv$.

(iii) Special case. Then $\left(\frac{\eta_1}{\Omega}\right)_M = j^{\frac{1}{2} \frac{v+u}{3v} \kappa}$ for any $\Omega | \mathfrak{q}_\xi$ with $3 | v + u$; we have seen, at the end of Subsection 7.2, that $\frac{v+u}{3v}$ can take any value modulo 3.

From this, we see that the existence of q totally split in $H_L^- [3]/\mathbb{Q}$ for $L = \mathbb{Q}(\mu_{q-1})$, or at least $L = \mathbb{Q}(\mu_m)$ for a large $m | q - 1$, may be in contradiction with the existence of the solutions of the second and special cases when $\kappa \not\equiv 0 \pmod{3}$, i.e., 3 inert in \mathbb{Q}_1/\mathbb{Q} where $\mathbb{Q}_1 = \mathbb{Q}(\mu_9)^+$.

Definition 7. Consider the field $k(Y)$, where k is any field of characteristic distinct from 2 and 3, and the automorphism:

$$\begin{array}{ccc} T : k(Y) & \longrightarrow & k(Y) . \\ Y & \longmapsto & \frac{2Y-1}{Y+1} \end{array}$$

Let $F(Y) := (1 + Y \zeta)^{e_\omega} \zeta^{-\frac{1}{2}} \in K(Y)$ be the formal cyclotomic unit, yet defined in Subsection 7.3. \square

We intend to prove below various properties of compatibility, of this automorphism, with the method of cyclotomic units developed here.

Theorem 6. (i) *The automorphism T is of order 6 and we have the following orbit of Y :*

$$\begin{aligned} T(Y) &= \frac{2Y-1}{Y+1}; & T^2(Y) &= \frac{Y-1}{Y}; & T^3(Y) &= \frac{Y-2}{2Y-1}; \\ T^4(Y) &= \frac{-1}{Y-1}; & T^5(Y) &= \frac{-Y-1}{Y-2}; & T^6(Y) &= Y. \end{aligned}$$

(ii) *We have for $\zeta = j$ of order 3 and for $F(Y) = (1 + Yj)^{e_\omega} j$, the following formulas (equalities up to 3th powers in $K(Y)$):*

$$T(F(Y)) = (1 + Yj)^{e_\omega}; \quad T^2(F(Y)) = (1 + Yj)^{e_\omega} j^2; \quad T^3(F(Y)) = F(Y),$$

summarized by the identity $T^i(F(Y)) \sim F(Y) j^{\frac{1}{2}i}$, $0 \leq i < 3$.

Proof. We have:

$$T(F(Y)) = \left(1 + \frac{2Y-1}{Y+1}j\right)^{e_\omega} j = (Y+1 + (2Y-1)j)^{e_\omega} j = (1-j + (2j+1)Y)^{e_\omega} j;$$

since $2j+1 = j(1-j)$, we get finally $T(F(Y)) = (1-j)^{e_\omega} (1+Yj)^{e_\omega} j$; but $(1-j)^{e_\omega} = -j^2$, hence the result in this case. The other computations are obtained by induction. \square

We apply now the automorphism T to the solutions (u, v) in the following way. We put $T\left(\frac{v}{u}\right) =: \frac{V}{U}$ where (U, V) is defined up to the sign. We start for instance from the solution:

$$(u, v) = (-s^3 - t^3 + 3s^2t, -s^3 - t^3 + 3st^2)$$

(see Remark 1) to determine its orbit.

Theorem 7. *We obtain the following identities:*

$$\begin{aligned} T^0\left(\frac{v}{u}\right) &= \frac{v}{u} = \frac{-s^3 - t^3 + 3st^2}{-s^3 - t^3 + 3s^2t}, \\ T^1\left(\frac{v}{u}\right) &= \frac{2v-u}{v+u} = \frac{-s^3 - t^3 - 3s^2t + 6st^2}{-2s^3 - 2t^3 + 3s^2t + 3st^2}, \\ T^2\left(\frac{v}{u}\right) &= \frac{v-u}{v} = \frac{3s^2t - 3st^2}{s^3 + t^3 - 3st^2}, \\ T^3\left(\frac{v}{u}\right) &= \frac{v-2u}{2v-u} = \frac{-s^3 - t^3 + 6s^2t - 3st^2}{s^3 + t^3 + 3s^2t - 6st^2}, \\ T^4\left(\frac{v}{u}\right) &= \frac{-u}{v-u} = \frac{s^3 + t^3 - 3s^2t}{3st^2 - 3s^2t}, \\ T^5\left(\frac{v}{u}\right) &= \frac{-v-u}{v-2u} = \frac{2s^3 + 2t^3 - 3s^2t - 3st^2}{s^3 + t^3 - 6s^2t + 3st^2}, \end{aligned}$$

which leads to the six fundamental families of solutions of the SFLT equation for $p = 3$. \square

Remark 11. For $q \not\equiv 1 \pmod{3}$, $q \neq 2$, all the orbits in $\mathbb{F}_q \cup \{\infty\}$ have six elements (indeed, all the equations of the form $\frac{ay+b}{cy+d} = y$, deduced from the

rational fractions of Theorem 6, (i), reduce to $y^2 - y + 1$ which is irreducible over \mathbb{F}_q). We remark the orbit of 0 which is:

$$0 \rightarrow -1 \rightarrow \infty \rightarrow 2 \rightarrow 1 \rightarrow \frac{1}{2} \text{ in } \mathbb{F}_q \cup \{\infty\};$$

this is consistent with $|\mathbb{F}_q \cup \{\infty\}| = q + 1 \equiv 0 \pmod{6}$. \square

Let q be a prime number; to simplify we suppose $q \not\equiv 1 \pmod{3}$. Call $n_i | q - 1$ the orders modulo q of $T^i(\frac{v}{u})$, $0 \leq i < 6$, for any solution (u, v) .

As usual we put $\frac{v}{u} \equiv \xi \pmod{\mathfrak{q}_\xi = (q, u\xi - v)}$ and more generally:

$$T^i(\frac{v}{u}) =: \frac{v_i}{u_i} \equiv \xi_i \pmod{\mathfrak{q}_{\xi_i} = (q, u_i \xi_i - v_i)}, \quad 0 \leq i < 6,$$

where we recall that the pair $(\xi_i, \mathfrak{q}_{\xi_i})$ is defined up to conjugation, so that we can replace $(\xi_i, \mathfrak{q}_{\xi_i})$ by any conjugate $(\xi'_i, \mathfrak{q}_{\xi'_i})$.

Thus we have put $(u_0, v_0) := (u, v)$ and $\xi_0 := \xi$.

Consider for instance $T(\frac{v}{u}) = \frac{v_1}{u_1} \equiv \xi_1 \pmod{\mathfrak{q}_{\xi_1}}$ noting that $\frac{v}{u} \equiv \xi \pmod{\mathfrak{q}_\xi}$.

To compare the two congruences we can take a fixed prime ideal $\tilde{\mathfrak{q}} | q$ in $\tilde{L} := \mathbb{Q}(\mu_{q-1})$ such that $\tilde{\mathfrak{q}} | \mathfrak{q}_\xi$ and $\tilde{\mathfrak{q}} | \mathfrak{q}_{\xi_1}$ by suitable conjugation of $(\xi_1, \mathfrak{q}_{\xi_1})$, which gives the congruences $\frac{v}{u} \equiv \xi \pmod{\tilde{\mathfrak{q}}}$ and $\frac{v_1}{u_1} \equiv \xi_1 \pmod{\tilde{\mathfrak{q}}}$, hence $\xi_1 \equiv \frac{v_1}{u_1} = T(\frac{v}{u}) \equiv T(\xi) \pmod{\tilde{\mathfrak{q}}}$. More generally we can write for suitable choices of the ξ_i :

$$\xi_i \equiv T^i(\xi) \pmod{\tilde{\mathfrak{q}}}, \quad 0 \leq i < 6,$$

which yields, for the units η_1^i associated to the ξ_i (with $\eta_1^0 = \eta_1$):

$$\begin{aligned} \eta_1^i &:= (1 + \xi_i j)^{e_\omega j} \\ &\equiv (1 + T^i(\xi) j)^{e_\omega j} \\ &\equiv \eta_1 j^{\frac{1}{2}i} \pmod{\tilde{\mathfrak{Q}}}, \quad 0 \leq i < 3, \end{aligned}$$

(from Theorem 6, (ii)), for all $\tilde{\mathfrak{Q}}$ above $\tilde{\mathfrak{q}}$ in $\tilde{M} := \tilde{L}K$. Thus we have:

$$\left(\frac{\eta_1^i}{\tilde{\mathfrak{Q}}}\right)_{\tilde{M}} = \left(\frac{\eta_1}{\tilde{\mathfrak{Q}}}\right)_{\tilde{M}} \left(\frac{j^{\frac{1}{2}i}}{\tilde{\mathfrak{Q}}}\right)_{\tilde{M}} = \left(\frac{\eta_1}{\tilde{\mathfrak{Q}}}\right)_{\tilde{M}} j^{\frac{1}{2}i\kappa} \text{ for all } \tilde{\mathfrak{Q}} | \tilde{\mathfrak{q}}, \quad 0 \leq i < 3,$$

proving that the three symbols never coincide when $\kappa \not\equiv 0 \pmod{3}$.

These symbols are identical to the symbols $\left(\frac{\eta_1^i}{\mathfrak{Q}_{\xi_i}}\right)_{M^i}$, for all $\mathfrak{Q}_{\xi_i} | \mathfrak{q}_{\xi_i}$, $0 \leq i < 3$, where $M^i = L^i K$, with $L^i = \mathbb{Q}(\mu_{n_i})$.²¹

This proves that if for instance \mathfrak{q}_{ξ_0} splits in F_{ξ_0}/L^0 then \mathfrak{q}_{ξ_1} and \mathfrak{q}_{ξ_2} are inert in F_{ξ_1}/L^1 and F_{ξ_2}/L^2 , respectively; in other words, the three law of

²¹ The coherent choice of these ideals supposes that if $\tilde{\mathfrak{q}} = (q, \tilde{\xi} - \tilde{e})$ ($\tilde{\xi}$ of order $q-1$, $\tilde{e} \in \mathbb{Z}$ of order $q-1$ modulo q), $\frac{v_i}{u_i} \equiv \tilde{e}^{d_i} \pmod{q}$ (of order n_i modulo q), we must have chosen $\xi_i = \tilde{\xi}^{d_i}$ so that $\mathfrak{q}_{\xi_i} = (q, \xi_i - \tilde{e}^{d_i}) = (q, \tilde{\xi}^{d_i} - \tilde{e}^{d_i}) \equiv 0 \pmod{\tilde{\mathfrak{q}}}$, $0 \leq i < 3$.

ρ_i -decomposition, or the three symbols $\left[\frac{F_*/L^i}{\mathfrak{q}^*} \right]_{\rho_i, n_i}$ of Definition 7.2, yields to the three possibilities when $\kappa \not\equiv 0 \pmod{p}$.

So, since this phenomenon happens in $\tilde{L} = \mathbb{Q}(\mu_{q-1})$ (even if the fields L^i are distincts in general), statements like that of Theorem 6.1 are impossible for $p = 3$ since $\tilde{\mathfrak{q}}$ cannot be totally split in F_{q-1} .

This distribution of the three possible Frobenii, in the context of ρ_i -decompositions, must be compatible with the Čebotarev's theorem (see Subsection 9.2 for this aspect and Subsection 9.3 for some numerical evidence and especially Example 5).

Returning to the general case, it is necessary to see whether such a nontrivial automorphism T can exist for $p > 3$. If not, this will be a favorable argument for our purpose.

Theorem 8. Consider $\mathcal{M} := \mathbb{Z}_p \otimes_{\mathbb{Z}} K(Y)^\times$, as a multiplicative $\mathbb{Z}_p[g]$ -module, and the idempotent $\varepsilon_\omega := \sum_{s \in g} \omega^{-1}(s) s \in \mathbb{Z}_p[g]$ (see Definition 1, (i) and (ii)).

Then, for $p > 3$, there does not exist any automorphism T of $\mathbb{Q}(Y)$, distinct from the identity and the inversion $Y \mapsto Y^{-1}$, such that $T(1 + Y \zeta) := 1 + T(Y) \zeta$ be such that:

$$(1 + T(Y) \zeta)^{\varepsilon_\omega} = (1 + Y \zeta^\lambda)^{\varepsilon_\omega} \zeta^\mu,$$

up to a p th power in \mathcal{M} , for some $\lambda, \mu \in \mathbb{Z}$, $\lambda \not\equiv 0 \pmod{p}$.

Proof. Suppose that such a nontrivial automorphism does exist and put $T(Y) = \frac{aY+b}{cY+d}$ with $a, b, c, d \in \mathbb{Q}$, $ad - bc \neq 0$. Note that the associated matrix:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is considered in $\text{Gl}_2(\mathbb{Q})/D$, where D is the subgroup of diagonal matrices $e I_2$, $e \in \mathbb{Q}^\times$, where I_2 is the unit matrix. In particular, T is of finite order if and only if there exists $n > 0$ such that $M^n = e I_2$. For instance, $M = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$ is such that $M^6 = -27 I_2$.

For simplicity we use a representative $e_\omega = \sum_{k=1}^{p-1} u_k s_k \in \mathbb{Z}[g]$ of ε_ω , with coefficients u_k such that $1 \leq u_k \leq p - 1$, and work in $K(Y)^\times / K(Y)^{\times p} \simeq \mathcal{M} / \mathcal{M}^p$ (so that e_ω is not here the usual representative; in particular, if $\Omega \in \mathbb{Q}(Y)^\times$ then $\Omega^{e_\omega} \in \mathbb{Q}(Y)^{\times p}$).

Then from the above identity we get the relation:

$$(cY + d + (aY + b) \zeta)^{e_\omega} = (1 + Y \zeta^\lambda)^{e_\omega} \zeta^\mu \cdot G(Y)^p, \quad G(Y) = \frac{A(Y)}{B(Y)} \in K(Y)^\times,$$

with $A, B \in K[Y]$, g.c.d. $(A, B) = 1$, hence the polynomial identity in $K[Y]$:

$$B(Y)^p (cY + d + (aY + b) \zeta)^{e_\omega} = A(Y)^p (1 + Y \zeta^\lambda)^{e_\omega} \zeta^\mu.$$

Since $(cY + d + (aY + b)\zeta)^{e_\omega}$ and $(1 + Y\zeta^\lambda)^{e_\omega}$ each have $p - 1$ distinct²² roots of orders of multiplicity u_k , with $1 \leq u_k \leq p - 1$, it is clear that $(cY + d + (aY + b)\zeta)^{e_\omega}$ and $(1 + Y\zeta^\lambda)^{e_\omega}$ each are prime to A and B , then have the same roots; hence there exists $\ell \not\equiv 0 \pmod{p}$ such that $\frac{d+b\zeta}{c+a\zeta} = \zeta^{-\lambda\ell} =: \zeta^\psi$, $\psi \not\equiv 0 \pmod{p}$. Then $\zeta^{\psi+1}a + \zeta^\psi c - \zeta b - d = 0$.

Since $p > 3$ we get $\psi \equiv -1$ or 1 modulo p , giving the solutions $(a, b, c, d) = (1, 0, 0, 1)$ (i.e., the identity), $(a, b, c, d) = (0, 1, 1, 0)$ (i.e., the inversion). \square

9.2. Analysis of the case $p = 3$ for the principle of Theorem 4.

We have now to explain why the phenomenon of ρ -law of decomposition (Theorem 4, i.e., $\left[\frac{F_\rho/L}{\mathfrak{q}_*}\right]_{\rho, n}$ independent of q in the sense of Remark 9) is indeed compatible for $p = 3$ but (conjecturally) not for $p > 3$.

The following analysis suggests a suitable property of repartition (in the meaning of Čebotarev density theorem) of the values of the Frobenii, due to the infiniteness of the set of solutions of the SFLT equation for $p = 3$ and the fact that this set is the union of six families (see Remark 1) having complementary properties for these values.

Let q be given such that $\kappa \not\equiv 0 \pmod{3}$. As usual, for the solutions $(u(s, t), v(s, t))$ of the SFLT equation, put $\rho := \frac{v}{u}$ and call ξ any primitive n th root of unity, where n is the order of ρ modulo q , n supposed prime to 3. Put $\eta_1 := (1 + \xi j)^{e_\omega} j^{-\frac{1}{2}} = (1 + \xi j)^{s-1} j^{-\frac{1}{2}}$, then $\mathfrak{q} := (q, u\xi - v)$, and denote by \mathfrak{Q} any prime ideal of $M = LK$ above \mathfrak{q} .

Of course, in this study n is not constant when the solution (u, v) varies, so that the statistical analysis cannot be done for a fixed field $L = \mathbb{Q}(\mu_n) \subseteq \mathbb{Q}(\mu_{q-1})$.

But up to this problem (probably not too tricky since the number of divisors n of $q - 1$ is finite and since it is probably better to work instead in $\mathbb{Q}(\mu_{q-1})$ for this statistical analysis), we have the following distribution of the possible cases, in a remarkable accordance with the definition of the solutions of the SFLT equation, that we summarize with the diagram of the compositum $L_1 F_\xi$ which is very simple for $p = 3$ (note that in the general case, $L_1 F_\xi / L$ contains $\frac{p^2-1}{p-1} = p + 1$ cyclic subextensions of degree p).

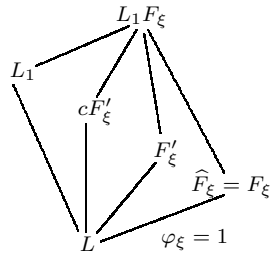
Indeed, for $p = 3$ the compositum $L_1 F_\xi$ contains L_1 , F_ξ , and two other cubic fields, F'_ξ and its conjugate cF'_ξ by the complex conjugation c (recall that F_ξ / L^+ is dihedral, L_1 / L^+ abelian, so that $L_1 F_\xi / L^+$ is Galois).

Moreover we will get \widehat{F}_ξ among the three extensions distinct from L_1 .

²² Indeed, for the roots $y_k := -\frac{d+b\zeta^k}{c+a\zeta^k}$, $1 \leq k \leq p - 1$, $y_k = y_{k'}$ is equivalent to the relation $(ad - bc)(\zeta^k - \zeta^{k'}) = 0$, hence the result. The other case is trivial.

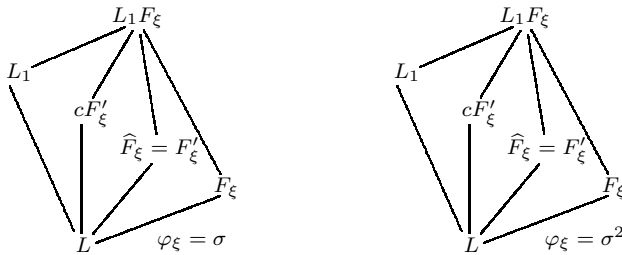
We denote by σ a fixed generator of $\text{Gal}(F_\xi/L)$ and call φ_ξ the Frobenius of \mathfrak{q} in F_ξ/L . We refer to Theorem 1 giving the symbol $\left(\frac{\eta_1}{\Omega}\right)_M$ for $p = 3$, where $\Omega \mid \mathfrak{q} = \mathfrak{q}_\xi$.

(i) First case ($uv(u+v) \not\equiv 0 \pmod{3}$) corresponding to the relation $u + vj = j^2(s + tj)^3$. We have $\left(\frac{\eta_1}{\Omega}\right)_M = j^{\frac{1}{2} \frac{v-u}{v+u} \kappa} = 1$ since $u - v \equiv 0 \pmod{3}$, $\widehat{F}_\xi = F_\xi$, and the diagram:



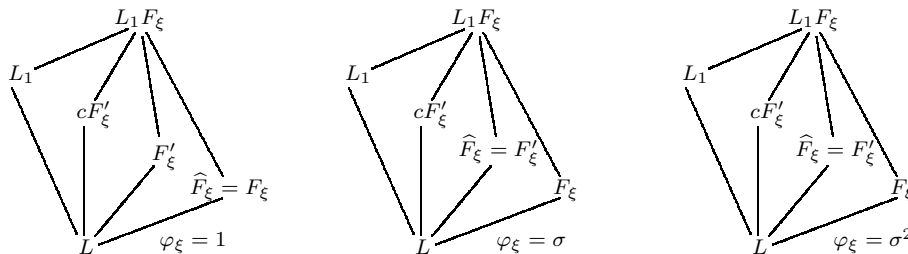
in which \mathfrak{q} is inert in F'_ξ/L , cF'_ξ/L , and L_1/L .

(ii) Second case ($uv \equiv 0 \pmod{3}$) corresponding to the two relations $u + vj = (s + tj)^3$ and $u + vj = j(s + tj)^3$. We have $\left(\frac{\eta_1}{\Omega}\right)_M = j^{\frac{1}{2} \frac{v-u}{v+u} \kappa} = j^{\pm \frac{1}{2} \kappa} = j$ or j^2 ; we get $\widehat{F}_\xi \neq F_\xi$, and the two equidistributed diagrams:



in which \mathfrak{q} is inert in F_ξ/L , cF'_ξ/L , and L_1/L .

(iii) Special case ($u+v \equiv 0 \pmod{3}$) corresponding to the three relations $u + vj = j^h(1-j)(s + tj)^3$, $0 \leq h < 3$. We have $\left(\frac{\eta_1}{\Omega}\right)_M = j^{\frac{1}{2} \frac{v+u}{3v} \kappa} = 1, j,$ or j^2 , and the three equidistributed diagrams:



in which the decomposition of \mathfrak{q} assembles all the above cases.

This suggests that the infiniteness of the solutions of the SFLT equation and their particular repartition into six families, is a necessary fact for the compatibility with Čebotarev's density theorem.

9.3. Numerical data for the case $p = 3$. We give some numerical experimentations, using [PARI], in the case $p = 3$, to highlight the above properties of this case.

We refer to Remark 1 for the six expressions of the solutions of the SFLT equation for $p = 3$; when we speak of “a solution (u, v) ”, we consider one of the six families $(u(s, t), v(s, t))$ with parameters s and t .

Proposition 4. *Let $n > 2$ be an integer not divisible by 3 and for any integers u, v with $\text{g.c.d.}(u, v) = 1$, let $\Phi_n(u, v) := \prod_{\xi' \text{ of order } n} (u \xi' - v)$.*

(i) *The set of primes $q \equiv -1 \pmod{3}$, $q \nmid n$, with $\kappa \not\equiv 0 \pmod{3}$, dividing at least one of the integers $\Phi_n(u, v)$, for a solution $(u(s, t), v(s, t))$ of the SFLT equation, is infinite when s, t vary in \mathbb{Z} with $\text{g.c.d.}(s, t) = 1$, $s + t \not\equiv 0 \pmod{3}$.*

Then there exist numbers of the form $\Phi_n(u, v)$ divisible by primes q as large as we need.

More precisely, the prime number q is solution if and only if for an $e \in \mathbb{Z}$, of order n modulo q , the polynomial $X^3 - 3e^{-1}X^2 - 3(1 - e^{-1})X + 1$ splits in $\mathbb{F}_q[X]$; the parameters (s, t) giving the solutions (u, v) such that $\frac{v}{u} \equiv e \pmod{q}$, are given via the three roots $\bar{\theta}_k \in \mathbb{F}_q$ of the polynomial, by the relation $s - t\theta_k \equiv 0 \pmod{q}$, $s, t \in \mathbb{Z}$ satisfying the above conditions, $k = 1, 2, 3$.

(ii) *The condition $q \mid \Phi_n(u, v)$ ($q \nmid n$), for a solution (u, v) of the SFLT equation, is equivalent to the ρ -splitting of q for $\widehat{\mathcal{F}}_n$ (i.e., it is equivalent to $\left[\frac{\widehat{F}_*/L}{\mathfrak{q}_*} \right]_{\rho, n} = 1$) for $\rho := \frac{v}{u}$.*

Proof. Let ξ of order n and let $L = \mathbb{Q}(\mu_n)$. Since $\text{g.c.d.}(s, t) = 1$, this yields immediately $\text{g.c.d.}(u, v) = 1$ for any solution, thus u and v are not divisible by any prime q dividing $\Phi_n(u, v)$ homogeneous of the form $u^{\phi(n)} \pm \dots \pm v^{\phi(n)}$ in coprime integers u and v .

From Lemma 2, $q \nmid n$ and $q \mid \Phi_n(u, v)$ is equivalent to the fact that $\frac{v}{u}$ is of order n modulo q , hence it is equivalent to the congruence $u \xi - v \equiv 0 \pmod{\mathfrak{q}}$, for a suitable and unique prime ideal $\mathfrak{q} \mid q$ in L ; then $\mathfrak{q} = (q, u \xi - v)$, which depends on (u, v) for ξ fixed, is one of the $\phi(n)$ prime ideals above q in L ; in the previous sections it was denoted \mathfrak{q}_ξ for given u, v .

We will prove that the condition $q \mid \Phi_n(u, v)$ ($q \nmid n$), for a solution of the SFLT equation, can be tested independently of the choice of the solution among the six possibilities, in the following sense.

Starting from a parametric solution (u, v) such that $u\xi - v \equiv 0 \pmod{\tilde{\mathfrak{q}}}$ for some $\tilde{\mathfrak{q}} \mid \mathfrak{q}_\xi$ in $\tilde{L} = \mathbb{Q}(\mu_{q-1})$, consider the solution (u', v') defined by:

$$\frac{v'}{u'} := T\left(\frac{v}{u}\right) = \frac{2v-u}{v+u}.$$

We have the congruence $u'\xi' - v' \equiv 0 \pmod{\tilde{\mathfrak{q}}}$ where ξ' is the unique $(q-1)$ th root of unity congruent to $T(\xi) = \frac{2\xi-1}{\xi+1}$ modulo $\tilde{\mathfrak{q}}$ (the order n' of ξ' divides $q-1$). Then we have:

$$\begin{aligned} u'\xi' - v' &\equiv (v+u)\frac{2\xi-1}{\xi+1} - (2v-u) \\ &\equiv \frac{1}{\xi+1}((v+u)(2\xi-1) - (2v-u)(\xi+1)) \\ &\equiv \frac{3}{\xi+1}(u\xi - v) \pmod{\tilde{\mathfrak{q}}}, \end{aligned}$$

proving the equivalence of the two congruences. Hence the result by induction on the powers of T . From Theorem 7, the six families of solutions give the congruences $u_i\xi_i - v_i \equiv 0 \pmod{\tilde{\mathfrak{q}}}$ for which $\frac{v_i}{u_i} := T^i\left(\frac{v}{u}\right)$, $\xi_i \equiv T^i(\xi) \pmod{\tilde{\mathfrak{q}}}$; each congruence reduces to a congruence modulo \mathfrak{q}_{ξ_i} in $L^i := \mathbb{Q}(\mu_{n_i})$, where $\mathfrak{q}_{\xi_i} = \tilde{\mathfrak{q}} \cap Z_{L^i}$ and n_i is the order of ξ_i (prime to 3 since $q \equiv -1 \pmod{3}$).

Warning: the orders n_i are divisors of $q-1$, not necessarily equal to n (see Example 5). But the conditions $q \nmid n_i$ and $q \mid \Phi_{n_i}(u, v)$, $0 \leq i < 6$, are equivalent to each other.

For instance, take the general solution of the second case $3 \mid v$; then we have to study the congruence $(s^3 + t^3 - 3st^2)\xi - 3st(s-t) \equiv 0 \pmod{\mathfrak{q}}$.

Put $\theta := \frac{s}{t}$, which yields to the congruence:

$$\theta^3 - 3\xi^{-1}\theta^2 - 3(1 - \xi^{-1})\theta + 1 \equiv 0 \pmod{\mathfrak{q}}.$$

Recall that for n fixed, the $\phi(n)$ ideals of L above q are the $(q, \xi - e)$, where $e \in \mathbb{Z}$, defined modulo q , is of order n in \mathbb{F}_q^\times ; so the congruence:

$$\theta^3 - 3\xi^{-1}\theta^2 - 3(1 - \xi^{-1})\theta + 1 \equiv 0 \pmod{\mathfrak{q} = (q, \xi - e)}$$

is equivalent to:

$$\theta^3 - 3e^{-1}\theta^2 - 3(1 - e^{-1})\theta + 1 \equiv 0 \pmod{q}$$

for the choice of $e \equiv \xi \pmod{\mathfrak{q}}$. Since the pair (ξ, \mathfrak{q}) is defined up to conjugation, we can select e of order n , which implies suitable ξ and \mathfrak{q} .

When q is solution, there exist infinitely many (u, v) such that $q \mid \Phi_n(u, v)$: for a root $\bar{\theta} \in \mathbb{F}_q$, $\theta \in \mathbb{Z}$, of the above congruence, the parameters (s, t) are obtained from the congruence $s \equiv \theta t \pmod{q}$ (see Example 8). At this step we have proved (i) under the existence of e such that the polynomial $X^3 - 3e^{-1}X^2 - 3(1 - e^{-1})X + 1$ splits in $\mathbb{F}_q[X]$.

The polynomial $X^3 - 3\xi^{-1}X^2 - 3(1 - \xi^{-1})X + 1$ defines the cyclic extension \widehat{F}_ξ : indeed, with $X = \xi^{-1}(Y + 1)$ one obtains the polynomial:

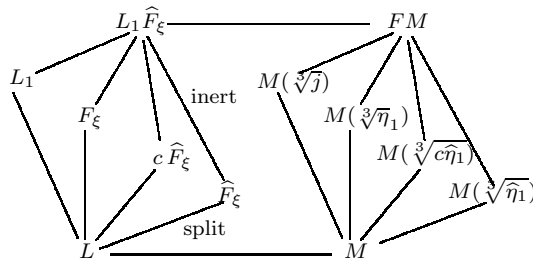
$$Y^3 - 3(\xi^2 - \xi + 1)Y - (2 - \xi)(\xi^2 - \xi + 1)$$

from the universal polynomial, irreducible of degree 3 over L (see Subsection 7.3), obtained from the cubic root of $(1 + \xi j)^{s+2} = \widehat{\eta}_1$ up to a 3th power.

Thus, the condition $q \mid \Phi_n(u, v)$ ($q \nmid n$) is equivalent to the ρ -splitting of q for $\widehat{\mathcal{F}}_n$, where $\rho := \frac{v}{u}$ (see Subsection 7.2) or to the ρ_i -splitting of q for $\widehat{\mathcal{F}}_{n_i}$ where $\rho_i := \frac{v_i}{u_i} = T^i(\frac{v}{u})$, and n_i is the order modulo q of ρ_i , $0 \leq i < 6$. This proves (ii).

From the Dirichlet–Čebotarev theorem, we get a precise result taking a nontrivial Frobenius in $L_1\widehat{F}_\xi/\widehat{F}_\xi$, and we obtain the prime ideal $\mathfrak{q}_\xi = (q, u\xi - v)$ where the (u, v) are obtained from the three roots $\bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3$ of the polynomial as explained above. We obtain infinitely many values of q with clearly a nonzero density. In other words, for $\kappa \not\equiv 0 \pmod{3}$ these primes q give again the splitting of \mathfrak{q}_ξ in \widehat{F}_ξ/L , hence its inertia in L_1/L , F_ξ/L , and in the fourth cubic subfield \widehat{F}'_ξ/L of the compositum $L_1\widehat{F}_\xi$ (note that $\widehat{F}'_\xi = c\widehat{F}_\xi$ and that $cF_\xi = F_\xi$ is dihedral over L^+). This makes clear the point (i) of the proposition. \square

In the case where (u, v) is for instance the general solution for the second case of the SFLT equation we get, with $\eta_1 = (1 + \xi j)^{e_\omega} j$, $\widehat{\eta}_1 := \eta_1 j^{\frac{1}{2}} = (1 + \xi j)^{e_\omega}$, and $c\widehat{\eta}_1 := \eta_1 j$, the following diagram:



There are six analogous diagrams over each field L^i .

Remark 12. Let q be a prime number such that $\kappa \not\equiv 0 \pmod{3}$. Then for a divisor $m > 2$ of $q - 1$, there is not necessarily a solution $(u, v) = (s^3 + t^3 - 3st^2, 3st(s - t))$, $s, t \in \mathbb{Z}$, g.c.d. $(s, t) = 1$, $s + t \not\equiv 0 \pmod{3}$, such that the order n of $\frac{v}{u}$ modulo q is equal to m (see Example 6).

The cases $m \leq 2$ correspond, for the above solution, to the congruences:

$$s^3 + t^3 - 3st^2 \pm 3st(s - t) \equiv 0 \pmod{q},$$

equivalent to the splitting, modulo q , of $X^3 + 1 - 3X \pm 3X(X - 1)$. One verifies that these polynomials of $\mathbb{Q}[X]$ define the number field \mathbb{Q}_1 ; so, as

by assumption $\kappa \not\equiv 0 \pmod{3}$, we obtain that the orders 1 and 2 are never possible. The case $m > 2$ is less trivial. \square

Example 5. We illustrate Proposition 5 with the prime $q = 41$ and the solution $(u, v) = (139193, 76626)$ obtained with the parameters $(s, t) = (-11, 43)$; we note that for $e = \overline{22} \in \mathbb{Z}/41\mathbb{Z}$ the polynomial:

$$X^3 - 3e^{-1} X^2 - 3(1 - e^{-1}) X + 1$$

splits in $\mathbb{Z}/41\mathbb{Z}[X]$ into $(X - \overline{38})(X - \overline{31})(X - \overline{15})$ and we have chosen $\overline{\theta} = \overline{15}$ for which $s - 15t \equiv 0 \pmod{41}$. Using the automorphism T , we obtain the six steps:

$$\begin{aligned} T^0(e) = e &= \overline{22} \text{ of order } 40 \\ T^0\left(\frac{v}{u}\right) = \frac{v}{u} &= \frac{76626}{139193}, \text{ solution of the second case,} \\ T(e) = e_1 &= \overline{9} \text{ of order } 4 \\ T\left(\frac{v}{u}\right) = \frac{v_1}{u_1} &= \frac{14059}{215819}, \text{ solution of the special case,} \\ T^2(e) = e_2 &= \overline{14} \text{ of order } 8 \\ T^2\left(\frac{v}{u}\right) = \frac{v_2}{u_2} &= \frac{-62567}{76626}, \text{ solution of the second case,} \\ T^3(e) = e_3 &= \overline{10} \text{ of order } 5 \\ T^3\left(\frac{v}{u}\right) = \frac{v_3}{u_3} &= \frac{-201760}{14059}, \text{ solution of the special case,} \\ T^4(e) = e_4 &= \overline{39} \text{ of order } 20 \\ T^4\left(\frac{v}{u}\right) = \frac{v_4}{u_4} &= \frac{139193}{62567}, \text{ solution of the first case,} \\ T^5(e) = e_5 &= \overline{5} \text{ of order } 20 \\ T^5\left(\frac{v}{u}\right) = \frac{v_5}{u_5} &= \frac{215819}{201760}, \text{ solution of the special case.} \end{aligned}$$

As a consequence, we have:

$$\begin{aligned} \Phi_{40}(139193, 76626) &\equiv \Phi_4(215819, 14059) \equiv \Phi_8(76626, -62567) \equiv \\ \Phi_5(14059, -201760) &\equiv \Phi_{20}(62567, 139193) \equiv \Phi_{20}(201760, 215819) \equiv 0 \pmod{41}. \end{aligned}$$

We have obtained the set of orders $\{40, 4, 8, 5, 20\}$. For instance, this implies the inertia of $\mathfrak{q}_{\xi_{40}}$ in $F_{\xi_{40}}/\mathbb{Q}(\mu_{40})$ and that of \mathfrak{q}_{ξ_4} in $F_{\xi_4}/\mathbb{Q}(\mu_4)$, which illustrates the incompatibility with statements like Theorem 2 for $p = 3$. \square

Example 6. We have found the following numerical example to illustrate Remark 12, with $m = 5$ for which $L = \mathbb{Q}(\mu_5)$ is principal. Consider the prime $q = 48738631$ for which $q - 1 = 2 \cdot 3 \cdot 5 \cdot 163 \cdot 9967$ and $\kappa \not\equiv 0 \pmod{3}$.

Then $\mathfrak{q} = (\xi^2 + \xi^3 - 3 - 90(3\xi^2 + 5\xi + 3))\mathbb{Z}[\xi]$, where ξ is a primitive 5th root of unity, is a prime ideal above q .

Since $\xi^2 + \xi^3 - 3 \in L^+$, this ideal satisfies the relation $\mathfrak{q}^{1-c} = (\alpha) \mathbb{Z}[\xi]$, $\alpha \equiv 1 \pmod{9}$, which means that q totally splits in $H_L^- [3]/\mathbb{Q}$.

Concerning the solutions $(u, v) = (s^3 + t^3 - 3st^2, 3st(s-t))$, $s, t \in \mathbb{Z}$, g.c.d. $(s, t) = 1$, $s + t \not\equiv 0 \pmod{3}$, such that $\Phi_5(u, v) \equiv 0 \pmod{q}$, we try to find the smallest values of the order n of $\frac{v}{u}$ modulo q . It is clear that the value $n = 5$ is by construction impossible. There is also no solution for $n = 10$ since $\mathbb{Q}(\mu_{10}) = \mathbb{Q}(\mu_5) = L$ with q totally split in $H_L^- [3]/\mathbb{Q}$.

We find the values:

$$\begin{aligned} n = 6 & \text{ for } (s, t) = (357, 42643), \\ n = 15 & \text{ for } (s, t) = (1531, 3232), \\ n = 163 & \text{ for } (s, t) = (143, 947), \\ n = 326 & \text{ for } (s, t) = (132, 883), \\ n = 489 & \text{ for } (s, t) = (79, 526), \\ n = 815 & \text{ for } (s, t) = (9, 971) \dots \end{aligned}$$

As we have seen, the orders $n = 1$ and 2 are impossible. \square

Example 7. In another point of view, in the following example we fix the solution $(u, v) = (19, 18)$ corresponding to $(s, t) = (3, 1)$ and we give the order n of $\frac{v}{u}$ modulo q for primes $q < 3 \cdot 10^6$ with $\kappa \not\equiv 0 \pmod{3}$, such that $n < q^{\frac{1}{3}}$ to limit the data.

q	n	q	n	q	n	q	n
79	3	137	4	751	5	17341	17
46663	11	49999	13	97373	44	225751	43
352771	55	419693	13	464549	47	536609	41
809359	22	816401	52	1037471	35	1115447	41
1167937	84	1252057	104	1403627	14	1529249	32
1995781	29	2040601	25	2743501	59	2912521	39

Example 8. Let $q = 113 = 1 + 2^4 \cdot 7$. In the following example we fix n and use a polynomial $X^3 - 3e^{-1}X^2 - 3(1 - e^{-1})X + 1$ which splits modulo 113; for $e = 83$, of order $n = 14$ modulo 113, its roots are $\overline{5}$, $\overline{28}$, and $\overline{46}$ modulo 113.

Recall that for ξ of order n and $e \in \mathbb{Z}$ defining the prime ideal $\mathfrak{q} = (q, \xi - e)$ above q , the solutions (s, t) giving $q \mid \Phi_n(u, v)$ for the corresponding solutions $(u, v) = (s^3 + t^3 - 3st^2, 3st(s-t))$, are defined for instance via the congruence $s - 5t \equiv 0 \pmod{113}$, g.c.d. $(s, t) = 1$, and $s + t \not\equiv 0 \pmod{3}$.

s	t	$\Phi_n(u, v)$
118	1	$113 \cdot 3557 \cdot 3942401 \cdot 744072113 \cdot 16254128953756891$
231	1	$113 \cdot 211 \cdot 239 \cdot 116929 \cdot 550757191489 \cdot 9432961248517529143$
457	1	$113 \cdot 8821 \cdot 18484859 \cdot 4489993033 \cdot 9077382763538364383220967$
123	2	$29 \cdot 43 \cdot 113 \cdot 3011 \cdot 11047 \cdot 1005000683 \cdot 8371388009051383$
128	3	$113 \cdot 385897 \cdot 8800908691961 \cdot 205376563933889209$
241	3	$29 \cdot 113 \cdot 3557 \cdot 26209 \cdot 136067 \cdot 2120693 \cdot 2348198329 \cdot 34945284137$
467	3	$113 \cdot 1451130199 \cdot 6673578443419738169458023356294472959$

133	4	113 · 421 · 43270571265013 · 74514155796456659333
138	5	113 · 2577267166287809480749101354040384043
251	5	113 · 547 · 2381 · 75688397 · 318274119451 · 4136563302302243
477	5	29 · 113 · 5503 · 26385694924317373 · 3324436493654921921540503
143	6	113 · 1847609 · 2588587173822250293234785701459

We observe a unique case where 113^2 divides $\Phi_n(u, v)$. □

Example 9. We consider the prime number $q = 401 = 1 + 2^4 \cdot 5^2$ and for all possible values of $\rho := \frac{v}{u}$ modulo q , for the general solution of the second case, we give the order of ρ modulo q . The resolution of $\frac{3st(s-t)}{s^3+t^3-3st^2} \equiv \rho \pmod{q}$ is of course equivalent to get the values ρ such that the polynomial $X^3 - 3\rho^{-1}X^2 - 3(1 - \rho^{-1})X + 1$ splits modulo q .

There are $133 = 7 \cdot 13$ distinct values of such ρ with the following repartition of the orders n : 53 for order 400; 28 for 200; 13 for 80; 12 for 100; 7 for 50 and 25; 4 for 40; 3 for 20; 2 for 10; 1 for 16, 8, 5, and 4. As we know, orders 1 and 2 cannot exist. These densities are in accordance with the expression $\frac{1}{3}\phi(n)$. □

The above computations for $p = 3$ suggest the following conjecture.

Conjecture 5. For all $m > 0$ and for all prime numbers $q \equiv 1 \pmod{m}$, $q \equiv 2$ or $5 \pmod{9}$, and q totally split in F_m/\mathbb{Q} , there exists a solution (u, v) of the SFLT equation for $p = 3$, for which the order of $\frac{v}{u}$ modulo q is $\geq m$. □

This conjecture (to be compared with Conjecture 2 for $p > 3$) is very reasonable since, in practice, the order of $\frac{v}{u}$ modulo q is often $q - 1$.

10. Conclusion

In Subsection 9.1, we have given a justification of the fact that Theorem 2 (or any weak form) cannot exist for $p = 3$.

For $p > 3$, if the number of solutions (u, v) of the SFLT equation is finite, for any bound N the number of primes q , such that the $\frac{v}{u}$ are of order modulo q less than N , is finite. So for primes q' for which we assume that all the prime ideals above q' in $L' = \mathbb{Q}(\mu_{m'})$, $m' | q' - 1$ large enough, totally split in $H_{L'}^-[p]/L'$, we get large values of q' , hence large values of the orders n' of the $\frac{v}{u}$ modulo q' , say $n' \gg N$. So, contrary to the case $p = 3$, the effectiveness of the statement of a weak form of Theorem 2 is more credible.

We have justified, in Subsection 9.2, why the case $p = 3$ is specific for the arithmetic of the fields $\mathbb{Q}(\mu_n)$ in relation with the abelian 3-ramification; which suggests that, for $p > 3$, a result like Theorem 4, on the constraints on the ρ -laws of decomposition of infinitely many primes q , gives a non trivial obstruction and is likely to lead to a proof of SFLT.

In other words, we can hope that for $p > 3$ any statistical analysis of the decomposition laws is legitimate.

To summarize it is not excluded that the two main principles of approach of the SFLT problem that we have developed in this paper may be successful for $p > 3$.

However, it should be noted that results like Theorem 2 are sufficient diophantine conditions, probably too strong, and that it would be better to return to the principle of laws of ρ -decomposition of infinitely many primes q for the canonical families \mathcal{F}_n (see Subsection 7.1, Theorem 4, and Conjecture 3); this aspect can be approached from an analytic point of view with the aim to show that such a constraint is impossible for $p > 3$.

In this direction, an interesting fact would be that the case $p = 3$ would have, in some sense, a reciprocal statement, namely that the infiniteness of the set of solutions of the SFLT equation and their particular repartition into six families, is in fact necessary for the Čebotarev's density theorem. Thus for $p > 3$, in the same spirit as the case $p = 3$, the set of nontrivial solutions (if nonempty) would be necessarily infinite with some structural properties in order to be compatible with the above principle, which seems clearly impossible for geometrical reasons (Theorem 8 for instance).

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