

# COHOMOLOGICAL INVARIANTS OF CENTRAL SIMPLE ALGEBRAS OF DEGREE 4

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ABSTRACT. In this paper, we prove a result of Rost, which describes the cohomological invariants of central simple algebras of degree 4 with values in  $H^*(-, \mu_2)$  when the base field contains a square root of  $-1$ .

## INTRODUCTION

In [5], Rost, Serre and Tignol defined a cohomological invariant of central simple algebras of degree 4 with values in  $H^4(-, \mu_2)$  when the base field contains a square root of  $-1$ . On the other hand, taking the Brauer class of the tensor square of a central simple algebra of degree 4 yields a cohomological invariant with values in  $H^2(-, \mu_2)$ . In this paper, we prove a result of Rost (unpublished), which asserts that these two invariants are essentially the only ones (see Section 4 for a more precise statement).

This paper is organized as follows. After proving some preliminary results in Section 1, we determine the cohomological invariants of multicyclic algebras in Section 2. Section 3 is devoted to the construction of a generic central simple algebra of degree 4. Finally, in Section 4, we prove that a cohomological invariant which vanishes on cyclic algebras and biquaternion algebras is identically zero (which is another result due to Rost). As a corollary, we obtain a complete description of cohomological invariants of central simple algebras of degree 4.

The proofs of all the results of this paper rely heavily on the use of valuations and residue maps. We let the reader refer to [2] for the basic definitions and results on these topics.

We are grateful to Alexander Merkurjev for providing us his private lecture notes on Rost's theorem.

## 1. PRELIMINARIES

Let  $F$  be a field of characteristic different from 2. We will denote by  $\mathbf{Fields}_F$  and  $\mathbf{Alg}_F$  the category of field extensions of  $F$  and the category of commutative  $F$ -algebras respectively.

The category of sets will be denoted by  $\mathbf{Sets}$ , and by  $\mathbf{Rings}$  the category of commutative rings.

For any field extension  $K/F$ , we will denote by  $H^*(K)$  the cohomology ring of  $K$  with coefficients in  $\mu_2$ . We then get a functor  $H^* : \mathbf{Fields}_F \rightarrow \mathbf{Rings}$ .

If  $K/F$  is a field extension, then  $H^*(K)$  carries a natural structure of a  $H^*(F)$ -module, given by the external law

$$\begin{aligned} H^*(F) \times H^*(K) &\longrightarrow H^*(K) \\ (a, \xi) &\longmapsto a \cdot \xi = \text{Res}_{K/F}(a) \cup \xi. \end{aligned}$$

We start with some results on the cohomology of rational extensions.

Let  $n \geq 1$  be an integer, let  $t_1, \dots, t_n$  be algebraically independent indeterminates over  $k$ , and let  $K_n = k(t_1, \dots, t_n)$ . Let us denote by  $\mathcal{P}_n$  the set of subsets of  $\{1, \dots, n\}$ . If  $I = \{i_1, \dots, i_r\} \in \mathcal{P}_n$ , we set

$$(\mathbf{t}_I) = (t_{i_1}) \cup \dots \cup (t_{i_r}).$$

Notice that this definition does not depend on the numbering of the elements of  $I$ , since the cup-product in  $H^*(K)$  is commutative.

If  $I = \emptyset$ , then  $(\mathbf{t}_I) = 1$  (and thus  $a \cdot (\mathbf{t}_I) = \text{Res}_{K/k}(a)$  in this case).

**Lemma 1.1.** *The cohomology classes  $(\mathbf{t}_I), I \in \mathcal{P}_n$  are linearly independent over  $H^*(F)$ .*

*Proof.* We proceed by induction on  $n$ . Assume that  $n = 1$ , and let  $a_0, a_1 \in H^*(F)$  such that  $a_0 \cdot 1 + a_1 \cdot (t_1) = 0$ . Taking residues with respect to the valuation  $v_{t_1}$  shows that  $a_1 = 0$ . Then multiplying by  $(-t_1)$  and taking residues shows that  $a_0 = 0$ .

Assume now that the result is proved for  $n \geq 1$ , and let us prove it for  $n + 1$  indeterminates. Let  $a_I \in H^*(F), I \in \mathcal{P}_{n+1}$  such that  $\sum_I a_I \cdot (\mathbf{t}_I) = 0$ . Then we

have

$$\sum_J a_J \cdot (\mathbf{t}_J) + \sum_J a_{J \cup \{n+1\}} \cdot (\mathbf{t}_J) \cup (t_{n+1}) = 0,$$

where  $J$  describes  $\mathcal{P}_n$ . Reasoning as in the case  $n = 1$ , we get

$$\sum_J a_J \cdot (\mathbf{t}_J) = \sum_J a_{J \cup \{n+1\}} \cdot (\mathbf{t}_J) = 0,$$

and by induction we have  $a_J = a_{J \cup \{n+1\}} = 0$  for all  $J \in \mathcal{P}_n$ , that is  $a_I = 0$  for all  $I \in \mathcal{P}_{n+1}$ . This concludes the proof.  $\square$

**Proposition 1.2.** *Let  $\xi \in H^*(K_n)$  be a cohomology class which is unramified at every discrete  $k$ -valuation  $v \neq v_{t_i}, i = 1, \dots, n$ . Then there exist unique cohomology classes  $a_I \in H^*(F)$  such that  $\xi = \sum_I a_I \cdot (\mathbf{t}_I)$ .*

*Proof.* We prove it by induction on  $n$ . Assume first that  $n = 1$ , and let  $a_1 \in H^*(F)$  be the residue of  $\xi$  with respect to the  $t_1$ -adic valuation. Then  $\xi - \text{Res}_{K/F}(a_1) \cup (t_1)$  is unramified at every discrete  $F$ -valuation  $v \neq v_{t_1}$  by assumption on  $\xi$  and because  $a_1$  is constant. By choice of  $a_1$ , it is also unramified at  $v_{t_1}$ , so  $\xi - \text{Res}_{K_1/F}(a_1) \cup (t_1)$  is a constant class, so we may finally write

$$\xi = \text{Res}_{K_1/F}(a_0) + \text{Res}_{K_1/F}(a_1) \cup (t_1) = a_0 \cdot 1 + a_1 \cdot (t_1).$$

Now assume that the result is proved for  $n \geq 1$ , and let  $\xi \in H^*(K_{n+1})$  be a cohomology class which is unramified at every discrete  $F$ -valuation  $v \neq v_{t_i}, i = 1, \dots, n + 1$ . In particular, it is unramified at every discrete  $K_n$ -valuation  $v \neq v_{t_{n+1}}$ .

By the case  $n = 1$ , there exist  $b_0, b_1 \in H^*(K_n)$  such that  $\xi = b_0 \cdot 1 + b_1 \cdot (t_{n+1})$ . Let  $v$  be a discrete  $F$ -valuation on  $K_n, v \neq v_{t_i}, i = 1, \dots, n$ , and let  $w$  the  $F$ -valuation on  $K_{n+1}$  extending  $v$  such that  $w(t_{n+1}) = 0$ . Since  $\xi$  is unramified at  $v$ , taking residues shows that  $\text{Res}_{\kappa(w)/\kappa(v)}(r_v(b_1)) = 0$ . Now  $\kappa(w) = \kappa(v)(t_{n+1})$ , so  $\text{Res}_{\kappa(w)/\kappa(v)}$  is injective and  $r_v(b_1) = 0$ . Moreover,  $\xi \cup (-t_{n+1})$  is also unramified at  $v$ , and reasoning as before shows that  $r_v(b_0) = 0$ . Hence  $b_0$  and  $b_1$  are unramified at any discrete  $F$ -valuation  $v$  on  $K_n, v \neq v_{t_i}, i = 1, \dots, n$ . Now use the induction hypothesis to conclude that  $\xi = \sum_I a_I \cdot (\mathbf{t}_I)$  for some  $a_I \in H^*(F)$ .

The uniqueness of the classes  $a_I$  comes from the previous lemma. This completes the proof.  $\square$

**Definition 1.3.** Let  $\mathbf{F} : \mathbf{Fields}_F \rightarrow \mathbf{Sets}$  be a covariant functor. A *cohomological invariant of  $\mathbf{F}$*  over  $F$  is a natural transformation  $\mathbf{F} \rightarrow H^*$  of functors  $\mathbf{Fields}_F \rightarrow \mathbf{Sets}$ .

Cohomological invariants of  $\mathbf{F}$  form an  $H^*(F)$ -module, that we will denote by  $\text{Inv}(\mathbf{F}, H^*)$ .

We will need in the sequel the notion of a classifying pair for a functor  $\mathbf{F}$ .

**Definition 1.4.** Let  $\mathbf{F} : \mathbf{Alg}_F \rightarrow \mathbf{Sets}$  be a covariant functor. Let  $R$  be an  $F$ -algebra, and let  $a \in \mathbf{F}(R)$ . We say that the pair  $(R, a)$  is classifying for  $\mathbf{F}$ , if the following conditions hold:

- (1) The ring  $R$  is a noetherian domain;
- (2) For every field extension  $L/F$ , and every  $a' \in \mathbf{F}(L)$ , there exists a maximal ideal  $\mathfrak{m}$  of  $R$  and a morphism of  $F$ -algebras  $f : R \rightarrow L$  such that  $\ker(f) \supset \mathfrak{m}$  and  $\mathbf{F}(f)$  maps  $a$  onto  $a'$ .

The Specialization Theorem (cf. [2, Theorem 12.2]) then immediately gives:

**Lemma 1.5.** Let  $\mathbf{F} : \mathbf{Alg}_F \rightarrow \mathbf{Sets}$  be a subfunctor of  $H^1(-, G)$ , where  $G$  is an algebraic group over  $F$ . Assume that  $(R, a)$  is classifying for  $\mathbf{F}$ , and let  $\alpha, \beta \in \text{Inv}(\mathbf{F}, H^*)$ . Let  $K$  be the quotient field of  $R$ . If  $\alpha(a_K) = \beta(a_K)$ , then  $\alpha = \beta$ .

## 2. INVARIANTS OF MULTICYCLIC ALGEBRAS

For all integers  $r, n \geq 1$ , we will denote by  $\mathbf{CSA}_n : \mathbf{Alg}_F \rightarrow \mathbf{Sets}$  the functor of isomorphism classes of Azumaya algebras of rank  $n$ .

Let  $R$  be a commutative ring. Assume that  $n \nmid \text{char}(R)$  and that  $R$  contains a primitive  $n$ -th root of 1, that we will denote by  $\zeta_n$ . For all  $u, v \in R^\times$ , the  $R$ -algebra  $\{u, v\}_{n,R}$  generated by two elements  $e$  and  $f$  subject to the relations

$$e^n = u, f^n = v, fe = \zeta_n ef$$

is an Azumaya  $R$ -algebra of rank  $n$ , called a symbol algebra.

We will denote by  $\mathbf{MS}_{n,r}$  the subfunctor of  $\mathbf{CSA}_{nr}$  of isomorphism classes of tensor products of  $r$  symbols algebras of degree  $n$ .

We will denote by 1 the constant cohomological invariant of  $\mathbf{CSA}_n$  (where  $\mathbf{CSA}_n$  is now viewed as a functor from  $\mathbf{Fields}_F$  to  $\mathbf{Sets}$ ). If  $n = 2m$ , then for every field

extension  $K/F$  and every central simple  $K$ -algebra  $A$  of exponent dividing  $n$ , the class  $m[A]$  is killed by 2, and therefore defines a cohomology class of  $H^2(K)$  via the usual isomorphism. This defines a cohomological invariant of  $\mathbf{CSA}_n$ , as well as a cohomological invariant of  $\mathbf{MS}_{n,r}$  for all  $r \geq 1$ , that we will denote by  $f_m$  in both cases. In particular, we have

$$f_m(\{a_1, b_1\}_{n,L} \otimes_F \cdots \otimes_F \{a_r, b_r\}_{n,L}) = (a_1) \cup (b_1) + \dots + (a_r) \cup (b_r)$$

for every field extension  $L/F$  and all  $a_i, b_i \in L^\times$ .

If  $A$  is a central simple  $F$ -algebra, the trace form of  $A$  is the quadratic form

$$\begin{aligned} A &\longrightarrow F \\ q_A: a &\longmapsto \mathrm{Trd}_A(a^2). \end{aligned}$$

**Lemma 2.1.** *Assume that  $n \nmid \mathrm{char}(F)$  and that  $\mu_n \in F$ . Assume also that  $n = 2m$ .*

*For all  $u, v \in F^\times$ , the trace form of  $\{u, v\}_{n,F}$  is Witt-equivalent to*

$$\langle n, nu, nv, (-1)^m nuv \rangle.$$

*Proof.* One may check that  $\mathrm{Trd}_A(e^i f^j) = 0$  if  $(i, j) \neq (0, 0)$ . It easily follows that the subspaces

$$F \cdot 1, F \cdot e^m, F \cdot f^m, F \cdot e^m f^m \text{ and } F \cdot e^i f^j \oplus F \cdot e^{n-i} f^{m-j},$$

where  $0 \leq i \leq j \leq m$ ,  $(i, j) \neq (0, 0), (0, m), (m, 0), (m, m)$  are mutually orthogonal. Moreover, the  $2m^2 - 2$  planes above are hyperbolic since  $e^i f^j$  is isotropic. The result then follows from the fact that the reduced traces of  $1, e^m, f^m$  and  $e^m f^m$  are respectively  $n, nu, nv$  and  $(-1)^m nuv$ . This concludes the proof.  $\square$

Assume that  $n \nmid \mathrm{char}(F)$  and that  $\mu_n \in F$ . Moreover, assume that  $-1 \in F^{\times 2}$  (this condition is automatically satisfied if  $n \equiv 0[4]$ ).

If  $A = \{u_1, v_1\}_{n,F} \otimes_F \cdots \otimes_F \{u_r, v_r\}_{n,F}$ , it follows from the previous lemma that  $q_A$  is Witt-equivalent to  $\langle n^r \rangle \langle \langle u_1, v_1, \dots, u_r, v_r \rangle \rangle$ . Therefore,  $q_A \in I^{2r}F$ , and we have

$$e_{2r}(q_A) = (u_1) \cup (v_1) \cup \cdots \cup (u_r) \cup (v_r).$$

This defines an element of  $\mathrm{Inv}(\mathbf{MS}_{n,r}, H^*)$  that we denote by  $e_{2r}$ .

**Proposition 2.2.** *Let  $F$  be a field of characteristic different from 2, and let  $n = 2m$ .*

(1) *Assume that  $n \nmid \mathrm{char}(F)$ , and that  $\zeta_n \in F$ , where  $\zeta_n$  is a primitive  $n^{\mathrm{th}}$ -root of 1. Then  $\mathrm{Inv}(\mathbf{MS}_{n,1}, H^*)$  is a free  $H^*(F)$ -module with basis  $1, f_m$ .*

(2) *Assume moreover that  $-1 \in F^{\times 2}$ . Then for all  $r \geq 2$ ,  $\mathrm{Inv}(\mathbf{MS}_{n,r}, H^*)$  is a free  $H^*(F)$ -module with basis  $1, f_m, e_{2r}$ .*

*Proof.* We first prove (1). Let  $\alpha \in \mathrm{Inv}(\mathbf{MS}_{n,1}, H^*)$ . Let  $t_1, t_2$  be two indeterminates over  $F$ , and let  $K = F(t_1, t_2)$ . Set  $A = \{t_1, t_2\}_{n,K}$ , and let  $v \neq v_{t_i}, i = 1, 2$  be a discrete  $F$ -valuation. By assumption on  $v, t_1, t_2 \in \mathcal{O}_v^\times$ , and we have  $A \simeq \{t_1, t_2\}_{n, \mathcal{O}_v} \otimes_{\mathcal{O}_v} K$ . It follows from [2, Theorem 11.7] that the class  $\alpha_K(\{t_1, t_2\}_{n,K})$  is unramified at  $v$ . By Proposition 1.2, there exist  $a_0, a_1, a_2, a_3 \in H^*(F)$  such that

$$\alpha_K(\{t_1, t_2\}_{n,K}) = a_0 \cdot 1 + a_1 \cdot (t_1) + a_2 \cdot (t_2) + a_3 \cdot (t_1) \cup (t_2).$$

Now let  $v$  be the  $(t_1 - 1)$ -adic valuation on  $F(t_2)$ . Applying [2, Theorem 11.7] shows by specialization that

$$\alpha_K(\{1, t_2\}_{n, F(t_2)}) = a_0 \cdot 1 + a_2 \cdot (t_2),$$

that is

$$\alpha_K(M_n(F(t_2))) = a_0 \cdot 1 + a_2 \cdot (t_2).$$

Now  $M_n(F(t_2))$  is unramified at  $v_{t_2}$ , and therefore, so is  $\alpha_K(M_n(F(t_2)))$ . Taking residues then yields  $a_2 = 0$ . Similar arguments show that  $a_1 = 0$ , and thus

$$\alpha_K(\{t_1, t_2\}_{n, K}) = a_0 \cdot 1 + a_3 \cdot (t_1) \cup (t_2).$$

Let  $R = F[t_1, t_1^{-1}, t_2, t_2^{-1}]$ . It is clear that the pair  $(R, \{t_1, t_2\}_{n, R})$  is classifying for  $\mathbf{MS}_{n,1}$ . The previous equality shows that the invariants  $\alpha$  and  $a_0 \cdot 1 + a_3 \cdot f_m$  coincide on a classifying pair, hence they are equal by Lemma 1.5.

Moreover, if  $a_0 \cdot 1 + a_3 \cdot f_m = 0$ , then applying this equality to  $\{t_1, t_2\}_{n, K}$  shows that  $a_0 \cdot 1 + a_3 \cdot (t_1) \cup (t_2)$ . Lemma 1.1 then yields  $a_0 = a_3 = 0$ . This proves (1).

Assume now that  $-1 \in F^{\times 2}$ . The fact that  $1, f_m, e_{2r}$  are linearly independent over  $H^*(F)$  if  $r \geq 2$  easily comes from Lemma 1.1. Let us prove by induction on  $r$  that  $1, f_m$  and  $e_{2r}$  span  $\text{Inv}(\mathbf{MS}_{n,r}, H^*)$  for all  $r \geq 1$  and every base field  $F$ . For  $r = 1$ , this follows from (1). Assume that the result is proved for  $r \geq 1$ , and let  $\alpha \in \text{Inv}(\mathbf{MS}_{n,r+1}, H^*)$ . Let  $K/F$  be a field extension, and let  $A$  be a symbol algebra over  $K$ .

Any field extension  $L/K$  yields a field extension  $L/F$ . The maps

$$\begin{aligned} B &\longrightarrow \mathbf{MS}_{n,r}(L) \\ \beta_L: H^*(L) &\longmapsto \alpha_L(B \otimes_L A_L) \end{aligned}$$

fit together into a cohomological invariant of  $\mathbf{MS}_{n,r}$ . By induction, there exists  $\alpha_{i,K}(A) \in H^*(K)$  such that

$$\beta = \alpha_{0,K}(A) \cdot 1 + \alpha_{1,K}(A) \cdot f_m + \alpha_{2,K}(A) \cdot e_{2r},$$

where  $\alpha_{2,K}(A) = 0$  if  $r = 1$ .

By linear independence of  $1, f_m, e_{2r}$  (resp. of  $1, f_m$  if  $r = 1$ ), these classes are unique. It easily follows that  $\alpha_{i,K}(A)$  only depends on the isomorphism class of  $A$ . Moreover, it follows from the uniqueness of  $\alpha_{i,K}(A)$  and the fact that  $\alpha$  is a cohomological invariant that the maps  $\alpha_{i,K}$  fit together into a cohomological invariant  $\alpha_i \in \text{Inv}(\mathbf{MS}_{n,1}, H^*)$ .

By (1), we may write  $\alpha_i = a_i \cdot 1 + b_i \cdot f_m$ , for some  $a_i, b_i \in H^*(F)$  (with  $a_2 = b_2 = 0$  if  $r = 1$ ). Taking  $K = F(u_1, v_1, \dots, u_{r+1}, v_{r+1})$ , where  $u_i, v_i$  are indeterminates,  $B = \{u_1, v_1\}_{n,K} \otimes_K \cdots \otimes_K \{u_r, v_r\}_{n,K}$  and  $A = \{u_{r+1}, v_{r+1}\}_{n,K}$ , we get

$$\begin{aligned} \alpha_K(\{u_1, v_1\}_{n,K} \otimes_K \cdots \otimes_K \{u_{r+1}, v_{r+1}\}_{n,K}) &= a_0 \cdot 1 + b_0 \cdot (u_{r+1}) \cup (v_{r+1}) + \\ (a_1 \cdot 1 + b_1 \cdot (u_{r+1}) \cup (v_{r+1})) \cup \sum_{i=1}^r (u_i) \cup (v_i) &+ (a_2 \cdot 1 + b_2 \cdot (u_{r+1}) \cup (v_{r+1})) \cup \bigcup_{i=1}^r (u_i) \cup (v_i). \end{aligned}$$

The  $F$ -automorphism of  $K$  which exchanges  $u_1$  and  $u_{r+1}$ ,  $v_1$  and  $v_{r+1}$ , and leaves invariant the other indeterminates induces maps  $\mathbf{MS}_{n,r+1}(K) \longrightarrow \mathbf{MS}_{n,r+1}(K)$

and  $H^*(K) \longrightarrow H^*(K)$ . Notice that the first one maps the isomorphism class of  $\{u_1, v_1\}_{n,K} \otimes_K \cdots \otimes_K \{u_{r+1}, v_{r+1}\}_{n,K}$  onto itself. Since  $\alpha$  commutes with induced maps, we then get

$$\begin{aligned} \alpha_K(\{u_1, v_1\}_{n,K} \otimes_K \cdots \otimes_K \{u_{r+1}, v_{r+1}\}_{n,K}) &= a_0 \cdot 1 + b_0 \cdot (u_1) \cup (v_1) + \\ (a_1 \cdot 1 + b_1 \cdot (u_1) \cup (v_1)) \cup \sum_{i=2}^{r+1} (u_i) \cup (v_i) &+ (a_2 \cdot 1 + b_2 \cdot (u_1) \cup (v_1)) \cup \bigcup_{i=2}^{r+1} (u_i) \cup (v_i). \end{aligned}$$

Comparing with the previous equality and using Lemma 1.1, we get that  $b_0 = a_1$ , and  $b_1 = a_2 = 0$ . Therefore,  $\{u_1, v_1\}_{n,K} \otimes_K \cdots \otimes_K \{u_{r+1}, v_{r+1}\}_{n,K}$  has same image by  $\alpha$  and by  $a_0 \cdot 1 + a_1 \cdot f_m + b_2 \cdot e_{2(r+1)}$ . Now let  $R = F[u_i, u_i^{-1}, v_i, v_i^{-1}, i = 1, \dots, r+1]$ . The pair  $(R, \{u_1, v_1\}_{n,R} \otimes_R \cdots \otimes_R \{u_{r+1}, v_{r+1}\}_{n,R})$  is classifying for  $\mathbf{MS}_{n,r+1}$ . It follows from the previous considerations that the invariants  $\alpha$  and  $a_0 \cdot 1 + a_1 \cdot f_m + b_2 \cdot e_{2(r+1)}$  are equal. This finishes the proof by induction.  $\square$

### 3. CENTRAL SIMPLE ALGEBRAS OF DEGREE 4

We start this section by giving a parametrization of central simple algebras of degree 4.

Let  $R$  be a commutative ring such that  $2 \in R^\times$ . If  $a \in R^\times$ , we will denote by  $R[\sqrt{a}]$  the étale quadratic  $R$ -algebra  $R[X]/(X^2 - a)$ .

Let  $L$  be a biquadratic étale  $R$ -algebra, that is an  $R$ -algebra  $L$  generated by two elements  $\alpha, \beta$  subject to the relations

$$\alpha^2 = d, \beta^2 = d', \alpha\beta = \beta\alpha,$$

for some  $d, d' \in R^\times$ . Such an algebra will be denoted by  $F[\sqrt{d}, \sqrt{d'}]$ .

The group  $G$  of automorphisms of the  $R$ -algebra  $L = F[\sqrt{d}, \sqrt{d'}]$  is isomorphic to  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ , generated by the automorphisms  $\sigma$  and  $\tau$  which are uniquely determined by the following relations:

$$\sigma(\alpha) = \alpha, \sigma(\beta) = -\beta, \tau(\alpha) = -\alpha, \tau(\beta) = \beta.$$

Let  $a \in R[\sqrt{d}]^\times, b \in R[\sqrt{d'}]^\times, c \in R[\sqrt{dd'}]^\times$ , and set

$$n_a = N_{R[\sqrt{d}]/R}(a), n_b = N_{R[\sqrt{d'}]/R}(b), n_c = N_{R[\sqrt{dd'}]/R}(c).$$

Notice that  $n_a, n_b, n_c \in R^\times$  and that we have

$$n_a = a\tau(a), n_b = b\sigma(b), n_c = c\sigma(c) = c\tau(c).$$

Assume that  $n_c = n_a n_b$ , and let  $1, e_\sigma, e_\tau, e_{\sigma\tau}$  be the canonical basis of the  $L$ -vector space  $\text{Map}(G, L)$ . We define a product law on  $\text{Map}(G, L)$  by imposing the relations

$$\begin{aligned} e_\sigma^2 &= a, e_\tau^2 = b, e_{\sigma\tau}^2 = c, e_\sigma e_\tau = e_{\sigma\tau}, \\ e_\sigma \lambda &= \sigma(\lambda) e_\sigma, e_\tau \lambda = \tau(\lambda) e_\tau, \text{ for all } \lambda \in L, \end{aligned}$$

and extending by distributivity.

By definition, the elements  $e_\rho, \rho \in G$  are invertible with respect to this product law. Notice that the missing products  $e_\rho e_{\rho'}, \rho, \rho' \in G$  may be obtained using the

relations above, so that the product law on  $\text{Map}(G, L)$  is completely determined. For example, we have

$$e_\tau e_{\sigma\tau} = e_\sigma^{-1} e_\sigma e_\tau e_{\sigma\tau} = e_\sigma^{-1} (e_{\sigma\tau})^2 = a^{-1} e_\sigma c = a^{-1} \sigma(c) e_\sigma.$$

One can check that we obtain an Azumaya  $R$ -algebra of rank 4, that we will denote by  $(a, b, c, L/R)$ . Notice that if we set  $e = e_\sigma$  and  $f = e_\tau$ , then we have  $(a, b, c, L/R) = L \oplus Le \oplus Lf \oplus Lef$ , and

$$e^2 = a, f^2 = b, (ef)^2 = c, e\lambda = \sigma(\lambda)e, f\lambda = \tau(\lambda)f, \text{ for all } \lambda \in L.$$

Notice for later use that we have  $fe = a^{-1} \sigma(cb^{-1})ef$ , since

$$fe = e^{-1} (ef)^2 f^{-1} = a^{-1} cb^{-1} f = a^{-1} \sigma(cb^{-1})ef.$$

**Lemma 3.1.** *Let  $F$  be a field of characteristic different from 2. Then every central simple  $F$ -algebra of degree 4 is isomorphic to some  $(a, b, c, L/F)$ . Moreover, one may assume that  $a, b, c \notin F^\times$ .*

*Proof.* Assume first that  $A$  is a division  $F$ -algebra. By a theorem of Albert (see [1, Theorem 11.9] for example),  $A$  contains a biquadratic étale  $F$ -algebra  $L$ . By Skolem-Noether's theorem,  $\sigma$  and  $\tau$  extend to inner automorphisms  $\text{Int}(e)$  and  $\text{Int}(f)$ , for some  $e, f \in A^\times$ . This rewrites

$$e\lambda = \sigma(\lambda)e, f\lambda = \tau(\lambda)f, \text{ for all } \lambda \in L.$$

Since  $\text{Int}(e)|_L = \sigma$ , we have

$$\text{Int}(e^2)|_L = (\text{Int}(e)|_L)^2 = \sigma^2 = \text{Id}_L,$$

so  $a = e^2 \in C_A(L) = L$  (since  $L$  is a maximal subfield of  $A$ ). Since  $e \in A^\times$ , we have  $a \in L^\times$ . Similarly,  $b = f^2 \in L^\times$  and  $c = (ef)^2 \in L^\times$ .

Associativity of the product law of  $A$  implies that

$$\sigma(a) = a, \tau(b) = b, \sigma\tau(c) = c, n_c = n_a n_b.$$

Indeed, we have

$$e^3 = ee^2 = ea = \sigma(a)e = e^2e = ae,$$

and since  $e$  is invertible, we get  $\sigma(a) = a$ . Similar arguments show that  $\tau(b) = b$  and  $\sigma\tau(c) = c$ . We also have

$$\begin{aligned} be &= f^2e \\ &= f(fe) \\ &= fa^{-1}\sigma(cb^{-1})ef \\ &= \tau(a^{-1}\sigma(cb^{-1}))f(ef) \\ &= c\tau(a)^{-1}\sigma(b)^{-1}(fe)f \\ &= c\tau(a)^{-1}\sigma(b)^{-1}a^{-1}\sigma(cb^{-1})(ef)f \\ &= c\tau(a)^{-1}\sigma(b)^{-1}a^{-1}\sigma(c)\sigma(b)^{-1}eb \\ &= c\tau(a)^{-1}\sigma(b)^{-1}\sigma(c)a^{-1}\sigma(b)^{-1}\sigma(b)e \\ &= n_c n_a^{-1} \sigma(b)^{-1} e. \end{aligned}$$

Since  $e$  is invertible, we get  $b = n_c n_a^{-1} \sigma(b)^{-1}$ , that is  $n_c = n_a n_b$ . The proof of the linear independence of  $1, e, f, ef$  over  $L$  is straightforward and left to the reader. Hence  $A$  contains a subalgebra isomorphic to  $(a, b, c, L/F)$ , and therefore  $A \simeq (a, b, c, L/F)$  since they have same dimension over  $F$ .

Assume now that  $A$  is not a division algebra. Then  $A \simeq M_2(Q)$ , where  $Q = (d, d')$  is a (not necessarily division) quaternion  $F$ -algebra.

Let  $L = F[\sqrt{d}, \sqrt{d'}]$ , and let us consider the  $F$ -algebra  $(d, 1, d, L/F)$ , where  $\alpha^2 = d, \beta^2 = d'$  and  $\beta\alpha = \alpha\beta$ . In particular, we have

$$e^2 = d, f^2 = 1, (ef)^2 = d, e\alpha = \alpha e, e\beta = -\beta e, f\alpha = -\alpha f, f\beta = \beta f.$$

Notice that we also have  $fe = d\sigma(d^{-1})ef = ef$ .

Set  $i = e, j = \beta f, i' = f, j' = \alpha e$ . It is straightforward to check that the  $F$ -algebra  $B$  generated by  $i$  and  $j$  is isomorphic to  $Q$ , and that the  $F$ -algebra  $B'$  generated by  $i'$  and  $j'$  is isomorphic to  $(1, d^2) \simeq M_2(F)$ . Moreover one may check  $B'$  is contained in the centralizer of  $B$ . Since  $B'$  is a central simple  $F$ -algebra, we get

$$(d, 1, 1, L/F) \simeq B \otimes_F B' \simeq Q \otimes_F M_2(F) \simeq M_2(Q) \simeq A.$$

We now prove the last part. Write  $A \simeq (a, b, c, L/F)$ , for some biquadratic étale  $F$ -algebra  $L = F[\sqrt{d}, \sqrt{d'}]$ . Replacing  $f$  by  $(1 + \alpha + \beta)f$ , one may assume that  $b \notin F^\times$ . Indeed, if  $b \in F^\times$ , we then have

$$[(1 + \alpha + \beta)f]^2 = (1 + \alpha + \beta)(1 - \alpha + \beta)f^2 = b(1 + d' - d + 2\beta) \notin F^\times.$$

Write  $a = a_0 + a_1\alpha$ . Notice that, for all  $s \in F^\times$ , we have

$$\begin{aligned} [(s + \alpha + \beta)e]^2 &= (s^2 + d - d' + 2s\alpha)(a_0 + a_1\alpha) \\ &= (a_0(s^2 + d' - d) + 2sa_1d) + (a_1(s^2 + d - d') + 2a_0s)\alpha, \end{aligned}$$

and

$$[(s + \alpha + \beta)ef]^2 = (s + \alpha + \beta)(s - \alpha - \beta)c = c(s^2 - d - d' - 2s\alpha\beta).$$

Thus, after replacing  $e$  by  $(s + \alpha + \beta)e$  for a suitable  $s \in F^\times$ , one may also assume that  $a, c \notin F^\times$ . This concludes the proof.  $\square$

**Remark 3.2.** For any biquadratic  $F$ -algebra  $L$ , we have  $M_4(F) \simeq (1, 1, 1, L/F)$ .

Indeed, the subalgebra  $B$  generated by  $e$  and  $\beta$  is isomorphic to  $(1, d') \simeq M_2(F)$ , and its centralizer  $C$  is the subalgebra generated by  $f$  and  $\alpha e$ , which is isomorphic to  $(1, d^2) \simeq M_2(F)$ . Since  $B$  is a central simple  $F$ -algebra, we have

$$(1, 1, 1, L/F) \simeq B \otimes_F C \simeq M_2(F) \otimes_F M_2(F) \simeq M_4(F).$$

**Proposition 3.3.** *Let  $F$  be a field of characteristic different from 2, and let  $A = (a, b, c, L/F)$ . Then*

$$2[A] = [(d', n_c)] \in \text{Br}(F).$$

*Proof.* Recall from [4] that for every central simple  $F$ -algebra of degree  $n$ , we have

$$\det(q_A) = \det(q_{M_n(F)}), w_2(q_A) = w_2(q_{M_n(F)}) + \frac{n(n-1)}{2}[A].$$

Since  $q_A$  and  $q_{M_n(F)}$  have same dimension and determinant, the quadratic form  $q_A \perp -q_{M_n(F)}$  lies in  $I^2(F)$ , and the second equality may be rewritten as

$$c(q_A \perp -q_{M_n(F)}) + \frac{n(n-1)}{2}[A],$$

where  $c$  denotes the Clifford invariant. In particular, if  $n = 4$ , we have

$$c(q_A \perp -q_{M_n(F)}) = 6[A] = 2[A].$$

Let us introduce some notation. Let  $E$  be an  $F$ -algebra. For  $u \in E^\times$ , we denote by  $q_{E,u}$  the quadratic form

$$\begin{aligned} E &\longrightarrow F \\ q_{E,u}: x &\longmapsto \operatorname{Tr}_{E/F}(ux^2). \end{aligned}$$

If  $u = 1$ , we denote it by  $q_E$ .

Let  $\rho$  be an  $F$ -automorphism of  $L$  satisfying  $\rho^2 = \operatorname{Id}_E$ . If  $u \in E^\times$ ,  $\rho(u) = u$ , we denote by  $q_{E,\rho,u}$  the quadratic form

$$\begin{aligned} E &\longrightarrow F \\ q_{E,\rho,u}: x &\longmapsto \operatorname{Tr}_{E/F}(ux\rho(x)) \end{aligned}$$

If  $u = 1$ , we denote it by  $q_{E,\rho}$ .

Finally, if  $\alpha, \beta \in F^\times$ , we set  $\langle\langle\alpha, \beta\rangle\rangle = \langle 1, -\alpha \rangle \otimes \langle 1, -\beta \rangle$ .

Now assume that  $A = (a, b, u, L/F)$ . It is easy to check that we have

$$\operatorname{Tr}_A(x + ye + zf + tef) = \operatorname{Tr}_{L/F}(x), \text{ for all } x, y, z, t, \in L.$$

This implies that the subspaces  $L, Le, Lf$  and  $Le f$  are mutually orthogonal. It follows from the previous observation that we have

$$q_A \simeq q_L \perp q_{L,\sigma,a} \perp q_{L,\tau,b} \perp q_{L,\sigma\tau,c}.$$

Let  $\rho$  be an  $F$ -automorphism of  $L$ , and assume that  $\rho \neq \operatorname{Id}_L$ , so we may write  $L^{(\rho)} = F[\sqrt{\Delta}]$ , and  $L = L^{(\rho)}[\sqrt{\Delta'}]$ . Notice that  $ux\rho(x) \in L^{(\rho)}$ , so that we have  $\operatorname{Tr}_{L/F}(ux\rho(x)) = 2\operatorname{Tr}_{L^{(\rho)}/F}(ux\rho(x))$ .

Now if  $x = x_0 + x_1\alpha'$  where  $\alpha'^2 = \Delta'$  and  $x_i \in L^{(\rho)}$ , we have  $x\rho(x) = x_0^2 - \Delta'x_1^2$ , and therefore

$$q_{L,\rho,u} \simeq \langle 2, -2\Delta' \rangle \otimes q_{L^{(\rho)},u}.$$

Write  $u = u_0 + u_1\sqrt{\Delta}$ ,  $u_i \in F$ , and set  $n_u = u_0^2 - \Delta u_1^2$ . The representative matrix of the previous quadratic form in the basis  $1, \sqrt{\Delta}$  is

$$\begin{pmatrix} 2u_0 & 2u_1\Delta \\ 2u_1\Delta & 2u_0\Delta \end{pmatrix}.$$

If  $u_0 = 0$ , this 2-dimensional quadratic form is isotropic, hence hyperbolic. If  $u_0 \neq 0$ , this form then represents  $2u_0$  and has determinant  $4n_u\Delta$ , so it is isomorphic to  $\langle 2u_0 \rangle \otimes \langle 1, n_u\Delta \rangle$ . We then get

$$q_{L,\rho,u} \simeq \langle u_0 \rangle \otimes \langle\langle -n_u\Delta, \Delta' \rangle\rangle,$$

where this form has to be understood as the hyperbolic form of dimension 4 if  $u_0 = 0$ . Taking  $u = 1$ , we get

$$q_{L,\rho} \simeq \langle\langle -\Delta, \Delta' \rangle\rangle,$$

and thus

$$q_{L,\rho,u} \perp -q_{L,\rho} \simeq \langle 1, -\Delta' \rangle \otimes \langle u_0, u_0n_u\Delta, 1, \Delta \rangle.$$

Since  $c(\varphi \otimes \varphi') = [(disc(\varphi), disc(\varphi'))]$  for every even-dimensional forms  $\varphi, \varphi'$ , we get

$$c(q_{L,\rho,u} \perp -q_{L,\rho}) = [(\Delta', n_u)].$$

Remark 3.2 shows that  $q_A \perp -q_{M_4(F)}$  is Witt-equivalent to

$$(q_{L,\sigma,a} \perp -q_{L,\sigma}) \perp (q_{L,\tau,b} \perp -q_{L,\tau}) \perp (q_{L,\sigma\tau,c} \perp -q_{L,\sigma\tau}).$$

We then finally get

$$c(q_A \perp -q_{M_4(F)}) = [(d', n_a)] + [(d, n_b)] + [(d, n_c)].$$

Since the quaternion algebras  $(d, n_a)$  and  $(d', n_b)$  are split, we get

$$\begin{aligned} c(q_A \perp -q_{M_4(F)}) &= [(dd', n_a)] + [(dd', n_b)] + [(d, n_c)] \\ &= [(dd', n_a n_b)] + [(d, n_c)] \\ &= [(dd', n_c)] + [(d, n_c)] \\ &= [(d', n_c)]. \end{aligned}$$

This concludes the proof.  $\square$

We now define a classifying pair for  $\mathbf{CSA}_4$ . Let us consider the affine variety  $\mathcal{V} \subset \mathbb{A}^6$  defined by the equation

$$x^2 - uy^2 - vz^2 + uvt^2 + uv = 0.$$

This is a rational variety, with coordinate ring

$$F[\mathcal{V}] = F[X, Y, Z, T, U, V]/(X^2 - UY^2 - VZ^2 + UVT^2 + UV).$$

Let us denote by  $x, y, z, t, u, v$  the images of  $X, Y, Z, T, U, V$  in  $F[\mathcal{V}]$  respectively. Notice that  $u$  and  $v$  are algebraically independent over  $F$ . Indeed, we have a surjective  $F$ -algebra morphism

$$F[\mathcal{V}] \longrightarrow F[V, Z]$$

$$P(x, y, z, t, u, v) \longmapsto P(0, i, Z, i, VZ^2, V)$$

which maps  $u$  and  $v$  onto  $VZ^2$  and  $V$ . Since  $VZ^2$  and  $V$  are algebraically independent over  $F$ , so are  $u$  and  $v$ .

Now let us consider the open subset  $\mathcal{U}$  of  $\mathcal{V}$  defined by the equations

$$u \neq 0, v \neq 0, t \neq 0, y^2 - vt^2 \neq 0, z^2 - u \neq 0.$$

Then  $\mathcal{U}$  is also a rational variety, whose coordinate ring  $F[\mathcal{U}]$  is the localization of  $F[x, y, z, t, u, v]$  at  $u, v, t, y^2 - vt^2$  and  $z^2 - u$ . Set  $R_0 = F[\mathcal{U}]$ . Then  $R_0$  is a noetherian ring, whose quotient field  $F_0$  is a rational extension of  $F$ . Moreover,  $u, v \in R_0$  are algebraically independent over  $F$ . In particular, there exists a transcendence basis of  $F_0/F$  containing  $u$  and  $v$ .

Set  $w = u(y^2 - vt^2) \in R_0^\times$  and let  $L_0 = R_0[\sqrt{u}, \sqrt{uw}]$ . We now define three elements  $a_0, b_0, c_0 \in L_0^\times$  by

$$a_0 = z + \sqrt{u} \in R_0[\sqrt{u}]^\times, b_0 = \frac{y}{t} + \frac{\sqrt{uw}}{tu} \in R_0[\sqrt{uw}]^\times, c_0 = x + \sqrt{w} \in R_0[\sqrt{w}]^\times.$$

Let us check that  $n_{a_0} n_{b_0} = n_{c_0}$ . We have

$$n_{a_0} n_{b_0} = \frac{(z^2 - u)(y^2 u^2 - uw)}{t^2 u^2} = \frac{(z^2 - u)(y^2 u - w)}{t^2 u}.$$

Now  $y^2 u - w = uvt^2$ , and thus  $n_{a_0} n_{b_0} = (z^2 - u)v$ . Now  $z^2 v - uv = x^2 - uy^2 + uvt^2 = x^2 - w = n_{c_0}$ , and we are done. Hence we may consider the Azumaya algebra

$$\mathcal{A}_0 = (a_0, b_0, c_0, L_0/R_0).$$

**Lemma 3.4.** *The pair  $(R_0, \mathcal{A}_0)$  is a classifying pair for  $\mathbf{CSA}_4$ .*

*Proof.* Let  $E/F$  be a field extension, and let  $A$  be a central simple  $E$ -algebra of degree 4. By Lemma 3.1, we have  $A \simeq (a, b, c, L/E)$ , with  $a, b, c \notin E^\times$ . Write  $L = E[\sqrt{d}, \sqrt{d'}]$ , with generators  $\alpha, \beta$ . Write

$$a = \lambda_0 + \lambda_1\alpha, b = \mu_0 + \mu_1\beta, c = \gamma_0 + \gamma_1\alpha\beta.$$

We set

$$\bar{x} = \gamma_0, \bar{y} = \frac{\mu_0\gamma_1}{\lambda_1\mu_1}, \bar{z} = \lambda_0, \bar{t} = \frac{\gamma_1}{\lambda_1\mu_1}, \bar{u} = \lambda_1^2d, \bar{v} = \mu_0^2 - \mu_1^2d'.$$

Using the equality

$$(\lambda_0^2 - \lambda_1^2d)(\mu_0^2 - \mu_1^2d) = (\gamma_0^2 - \gamma_1^2d),$$

and the fact that  $\lambda_1, \mu_1, \gamma_1 \in F^\times$ , one may check that  $\mathbf{p} = (\bar{x}, \bar{y}, \bar{z}, \bar{t}, \bar{u}, \bar{v}) \in \mathcal{U}(E)$ . Then evaluation at  $\mathbf{p}$  yields an  $F$ -algebra morphism  $f : R_0 \rightarrow E$  such that

$$\mathcal{A}_0 \otimes_{R_0} E \simeq A.$$

Now it suffices to notice that  $\ker(f) = (x - \bar{x}, y - \bar{y}, z - \bar{z}, t - \bar{t}, u - \bar{u}, v - \bar{v})$  is a maximal ideal to conclude.  $\square$

#### 4. COHOMOLOGICAL INVARIANTS OF $\mathbf{CSA}_4$

We now prove the following theorem, due to Rost (unpublished):

**Theorem 4.1** (Rost). *Let  $F$  be a field of characteristic different from 2. Assume that  $-1 \in F^{\times 2}$ . Then the map*

$$\text{Inv}(\mathbf{CSA}_4, H^*) \longrightarrow \text{Inv}(\mathbf{MS}_{2,2}, H^*) \times \text{Inv}(\mathbf{MS}_{4,1}, H^*)$$

*is injective. In other words, a cohomological invariant which is zero on biquaternion algebras and cyclic algebras is identically zero.*

*Proof.* Let  $\alpha \in \text{Inv}(\mathbf{CSA}_4, H^*)$ . Let  $(R_0, \mathcal{A}_0)$  be the classifying pair of  $\mathbf{CSA}_4$  defined in the previous section, and let  $F_0$  be the quotient field of  $R_0$ . Finally, set

$$A_0 = \mathcal{A}_0 \otimes_{R_0} F_0 = (z + \sqrt{u}, \frac{y}{t} + \frac{\sqrt{uw}}{tu}, x + \sqrt{w}, F_0[\sqrt{u}, \sqrt{uw}]/F_0).$$

Let  $v$  be a discrete  $F$ -valuation on  $F_0$ . If  $v \neq v_\pi$ , with  $\pi = u, v, t, y^2 - vt^2, z^2 - u$ , then  $u, v, w, t \in \mathcal{O}_v^\times$  and  $z + \sqrt{u}, \frac{y}{t} + \frac{\sqrt{uw}}{tu}, x + \sqrt{w} \in \mathcal{O}_v[\sqrt{u}, \sqrt{uw}]^\times$ . We then get that

$$A_0 \simeq (z + \sqrt{u}, \frac{y}{t} + \frac{\sqrt{uw}}{tu}, x + \sqrt{w}, \mathcal{O}_v[\sqrt{u}, \sqrt{uw}]/\mathcal{O}_v) \otimes_{\mathcal{O}_v} F_0,$$

and therefore  $\alpha_{F_0}(A_0)$  is unramified at  $v$  by [2, Theorem 11.7].

Assume now that  $v = v_\pi$ , with  $\pi = t, z^2 - u$  or  $y^2 - vt^2$ . Then  $u, v \in \mathcal{O}_v^\times$ , and thus we have

$$(u) \cup (v) = j_v((\bar{u}) \cup (\bar{v})),$$

where  $j_v : H^2(\kappa(v)) \hookrightarrow H^2((F_0)_v)$  is the canonical injection. Now in all three cases, the quadratic form  $\langle 1, -\bar{u}, -\bar{v}, \bar{u}\bar{v} \rangle$  is isotropic over  $\kappa(v)$ . It is clear if  $\pi = z^2 - u$  or  $y^2 - vt^2$ , and follows from the equalities

$$0 = x^2 - uy^2 - vz^2 + uvt^2 + uv = x^2 - uy^2 - vz^2 + uv \pmod{t},$$

if  $\pi = t$ . Thus  $(\bar{u}) \cup (\bar{v}) = 0$ , and therefore  $(u) \cup (v) = 0$  as well. It follows that the quaternion algebra  $(u, v)_{F_0}$  is split over  $(F_0)_v$ . Hence, by Proposition 3.3,  $2[A_0]$  splits over  $(F_0)_v$ . By a theorem of Albert,  $A_0 \otimes_{F_0} (F_0)_v$  is a biquaternion algebra. By assumption, we get

$$\text{Res}_{(F_0)_v/F_0}(\alpha_{F_0}(A_0)) = \alpha_{(F_0)_v}(A_0 \otimes_{F_0} (F_0)_v) = 0.$$

Consequently,  $\partial_v(\alpha_{F_0}(A_0)) = 0$  and  $\alpha_{F_0}(A_0)$  is once again unramified at  $v$ .

Finally,  $\alpha_{F_0}(A_0)$  is unramified at every discrete  $F$ -valuation  $v \neq v_u, v_v$ . In particular, it is unramified at every  $F_1$ -valuation, where  $F_1 = F(u, v)$ . Since there exists a transcendence basis of  $F_0/F$  containing  $u$  and  $v$ , the field extension  $F_0/F_1$  is rational, and we deduce that  $\alpha_{F_0}(A_0) = \text{Res}_{F_0/F_1}(\beta)$ , for some  $\beta \in H^*(F_1)$ . Let  $v_1$  be a discrete  $F$ -valuation on  $F_0$ , which different from the  $u$ -adic and the  $v$ -adic valuations. Let us show that  $\beta$  is unramified at  $v_1$ .

Let  $u, v, t_1, t_2, t_3$  be a transcendence basis of  $F_0/F$ , and let us extend  $v_1$  to a discrete  $F$ -valuation on  $F_0$  by setting

$$v_{F_1} = v_1, v(t_i) = 0, i = 1, 2, 3.$$

Then the corresponding ramification index is 1 and  $\kappa(v_1) = \kappa(v)(t_1, t_2, t_3)$ , so that  $\kappa(v)/\kappa(v_1)$  is a rational extension. Since  $\alpha_{F_0}(A_0)$  is unramified at  $v$ , we get

$$0 = \partial_v(\alpha_{F_0}(A_0)) = \text{Res}_{\kappa(v)/\kappa(v_1)}(\partial_{v_1}(\beta)).$$

Since  $\kappa(v)/\kappa(v_1)$  is rational, the map  $\text{Res}_{\kappa(v)/\kappa(v_1)}$  is injective, and we get that  $\beta$  is unramified at  $v_1$ . By Proposition 1.2, we have

$$\beta = a_0 \cdot 1 + a_1 \cdot (u) + a_2 \cdot (v) + a_3 \cdot (u) \cup (v), a_i \in H^*(F).$$

Hence we get

$$\alpha_{F_0}(A_0) = a_0 \cdot 1 + a_1 \cdot \text{Res}_{F_0/F_1}((u)) + a_2 \cdot \text{Res}_{F_0/F_1}((v)) + a_3 \cdot \text{Res}_{F_0/F_1}((u) \cup (v)).$$

Let us consider the  $(v-1)$ -adic valuation on  $F_0$ , and let  $F_2$  its residue field. Since  $A_0$  is unramified at this valuation, one may specialize  $A_0$  to a central simple  $F_2$ -algebra  $B$ . Then  $2[A_0]$  specializes to  $2[B]$ . But  $2[A_0] = [(u, v)]$  specializes to 0, and therefore  $B$  is a biquaternion algebra. Then  $\alpha_{F_2}(B) = 0$  by assumption, and therefore

$$a_0 \cdot 1 + a_1 \cdot \text{Res}_{F_2/F_1}((u)) = 0.$$

In other words,  $\text{Res}_{F_2/F_1}(a_0 \cdot 1 + a_1 \cdot (u)) = 0$ . Since  $F_1 \subset F_2 \subset F_0$ , and  $F_0/F_1$  is rational, the map  $\text{Res}_{F_2/F_0}$  is injective and thus

$$a_0 \cdot 1 + a_1 \cdot (u) = 0 \text{ in } H^*(F_1).$$

By Lemma 1.1, we get  $a_0 = a_1 = 0$ . Considering the  $u-1$ -adic valuation shows that  $a_2 = 0$  in a similar way, so that

$$\alpha_{F_0}(A_0) = a_3 \cdot \text{Res}_{F_0/F_1}((u) \cup (v)).$$

Since  $\text{Res}_{F_0/F_1}((u) \cup (v))$  corresponds to  $[(u, v)] = 2[A_0]$  in  $\text{Br}_2(F_0)$  via the usual isomorphism, the previous equality rewrites

$$\alpha_{F_0}(A_0) = a_3 \cdot f_{2, F_0}(A_0).$$

By Lemma 1.5, we get  $\alpha = a_3 \cdot f_2$ . By assumption on  $\alpha$ , we then have

$$0 = \alpha_{F_1}(\{u, v\}_{4, F_1}) = a_3 \cdot (u) \cup (v) \in H^*(F_1).$$

By Lemma 1.1, we get  $a_3 = 0$ , and thus  $\alpha = 0$ . This concludes the proof.  $\square$

We now describe the cohomological invariants of  $\mathbf{CSA}_4$ . Let  $F$  be a field of characteristic different from 2. Assume that  $-1 \in F^{\times 2}$ , and let  $K/F$  be a field extension. By [5], for every central simple  $K$ -algebra  $A$  of degree 4, there exists a 2-fold Pfister form  $q_2$  and a 4-fold Pfister form  $q_4$  such that

$$q_A \sim q_2 + q_4 \in W(K).$$

Taking the cohomology class  $e_4(q_4) \in H^4(K)$  yields a cohomological invariant of  $\text{Inv}(\mathbf{CSA}_4, H^*)$ . This invariant restricts to zero on cyclic algebras, and its restriction to  $\mathbf{MS}_{2,2}$  obviously coincide with the invariant  $e_4$  defined in Section 2. Therefore, we still denote this invariant by  $e_4$ .

**Corollary 4.2.** *Let  $F$  be a field of characteristic different from 2. Assume that  $-1 \in F^{\times 2}$ . Then  $\text{Inv}(\mathbf{CSA}_4, H^*)$  is a free  $H^*(F)$ -module with basis  $1, f_2, e_4$ .*

*Proof.* We first prove that  $1, f_2, e_4$  are linearly independent. Assume that we have

$$a_0 \cdot 1 + a_1 \cdot f_2 + a_2 \cdot e_4 = 0 \text{ for some } a_i \in H^*(F).$$

Since the restriction of  $e_4$  to  $\mathbf{MS}_{4,1}$  is zero, it follows that  $a_0 + a_1 \cdot f_2 = 0 \in \text{Inv}(\mathbf{MS}_{4,1}, H^*)$ . By Proposition 2.2 (1), we get  $a_0 = a_1 = 0$ . Thus, we get  $a_2 \cdot e_4 = 0 \in \text{Inv}(\mathbf{CSA}_4, H^*)$ . In particular,  $a_2 \cdot e_4 = 0 \in \text{Inv}(\mathbf{MS}_{2,2}, H^*)$ . By Proposition 2.2 (2), we get  $a_2 = 0$ .

We now prove that  $1, f_2, e_4$  span  $\text{Inv}(\mathbf{CSA}_4, H^*)$  as an  $H^*(F)$ -module. Let  $\alpha \in \text{Inv}(\mathbf{CSA}_4, H^*)$ . By Proposition 2.2, we have

$$\alpha|_{\mathbf{MS}_{2,2}} = a_0 \cdot 1 + a_1 \cdot f_2 + a_2 \cdot e_4,$$

for some  $a_i \in H^*(F)$ , and also

$$\alpha|_{\mathbf{MS}_{4,1}} = b_0 \cdot 1 + b_1 \cdot f_2,$$

for some  $b_i \in H^*(F)$ . Since  $M_4(F) \simeq \{1, 1, \}_{4,F} \simeq (1, 1) \otimes_F (1, 1)$ , we may apply  $M_4(F)$  to both equalities to get  $a_0 = b_0$ . Now we have

$$M_2((u, v)) \simeq (1, 1) \otimes_{F(u,v)} (u, v) \simeq \{u^2, v^2\}_{4,F(u,v)},$$

and applying  $M_2((u, v))$  to both equalities yields

$$a_1 \cdot (u) \cup (v) = b_1 \cdot (u) \cup (v) \in H^*(F(u, v)).$$

It follows from Lemma 1.1 that  $a_1 = b_1$ . Since  $e_4$  is zero on  $\mathbf{MS}_{4,1}$ , we conclude that  $\alpha$  and  $a_0 \cdot 1 + a_1 \cdot f_2 + a_2 \cdot e_4$  coincide on  $\mathbf{MS}_{4,1}$  and  $\mathbf{MS}_{2,2}$ . By the previous theorem, we get  $\alpha = a_0 \cdot 1 + a_1 \cdot f_2 + a_2 \cdot e_4$ , and this concludes the proof.  $\square$

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