A BIFURCATION FOR A GENERALIZED BURGERS' EQUATION IN DIMENSION ONE

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Abstract. We consider the generalized Burgers' equation

$$\begin{cases} \partial_t u = \partial_x^2 u - u\partial_x u + u^p - \lambda u & \text{in } \overline{\Omega} \text{ for } t > 0, \\ \mathcal{B}(u) = 0 & \text{on } \partial\Omega \text{ for } t > 0, \\ u(\cdot, 0) = \varphi \geq 0 & \text{in } \overline{\Omega}, \end{cases}$$

with $p > 1$, $\lambda \in \mathbb{R}$, Ω a subdomain of $\mathbb{R}$, and where $\mathcal{B}(u) = 0$ denotes some boundary conditions. First, using some phase plane arguments, we study the existence of stationary solutions under the Dirichlet or the Neumann boundary conditions and prove a bifurcation depending on the parameter λ. Then, we compare positive solutions of the parabolic equation with appropriate stationary solutions to prove that global existence can occur when $\mathcal{B}(u) = 0$ stands for the Dirichlet, the Neumann or the dissipative dynamical boundary conditions $\sigma\partial_t u + \partial_\nu u = 0$. Finally, for many boundary conditions, global existence and blow up phenomena for solutions of the nonlinear parabolic problem in an unbounded domain Ω are investigated by using some standard super-solutions and some weighted L^1-norms.

1. **Introduction.** Let Ω be a domain of the real line $\mathbb{R}$, not necessarily bounded. Let p be a real number with $p > 1$, $\lambda \in \mathbb{R}$ and φ a non-negative continuous function in $\overline{\Omega}$. Consider the following nonlinear parabolic problem

$$\begin{cases} \partial_t u = \partial_x^2 u - u\partial_x u + u^p - \lambda u & \text{in } \overline{\Omega} \text{ for } t > 0, \\ \mathcal{B}(u) = 0 & \text{on } \partial\Omega \text{ for } t > 0, \\ u(\cdot, 0) = \varphi & \text{in } \overline{\Omega}, \end{cases} \tag{1}$$

where $\mathcal{B}(u) = 0$ stands for the Dirichlet boundary conditions ($u = 0$), the Neumann boundary conditions ($\partial_\nu u = 0$) or the dynamical boundary conditions ($\sigma\partial_t u + \partial_\nu u = 0$ with σ a non-negative smooth function). For the local existence of the positive solutions of this problem, we refer to von Below and Mailly's results [6] and references therein, [2], [4] and [7] . In the first section, we study the stationary equation

$$u'' - uu' + u|u|^{p-1} - \lambda u = 0 \tag{2}$$

stemming from Problem (1). We aim to prove the existence of positive and sign-changing solutions using phase plane arguments and dealing with the first order

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system

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} v \\ uv - u|u|^{p-1} + \lambda u \end{pmatrix}. \tag{3}$$

We prove a bifurcation in the phase plane of this system, depending on the parameters λ and p, which influences the resolution of Equation (2) under the Dirichlet, the Neumann and the mixed boundary conditions. Then in a second section, using the comparison method from [3], we deduce from the solutions of the stationary Equation (2) some regular super-solutions for the Problem (1). Dealing with these super-solutions and with the blow-up results from [6], we investigate global existence and blow-up phenomena for the Problem (1) for different values of λ and p, and for the Dirichlet, the Neumann and the dynamical boundary conditions. We also examine both phenomena in unbounded domains: we obtain global existence results with the comparison method and using some well-known super-solutions (we mean explicit functions) for the Dirichlet, the Neumann and the dynamical boundary conditions. The blowing-up concerns the regular solutions of Problem (1) satisfying some growth order at infinity and some boundary conditions such that

- $\partial_\nu u = 0$ (Neumann b.c.),
- $\partial_\nu u = g(u)$ with g a polynomial of degree 2 (nonlinear b.c.).

We use some weighted L^1-norms: our technique is to prove the blowing-up of the solution by proving the blowing-up of appropriate L^1-norms.

Before starting, let us define the kind of solution we look for:

Definition 1.1. A function u is called a solution (or regular solution) of Equation (2) in Ω if u is of class $\mathcal{C}^2(\Omega)$ and satisfies the equation in the classical sense.
A function u is called a solution (or regular solution) of Problem (1) in Ω if u is of class $\mathcal{C}(\overline{\Omega} \times [0,T)) \cap \mathcal{C}^{2,1}(\overline{\Omega} \times (0,T))$ and satisfies the equations of Problem (1) in the classical sense in $\Omega \times [0,T)$ where $T \in (0,\infty]$ denotes the maximal existence time of the solution u.

2. **Stationary equation.** In this section, we study the existence of positive and sign-changing solutions of Equation (2) using a phase plane method. Unless otherwise stated, we suppose $p \in (1,\infty)$. For the theory of phase planes (nature of equilibrium, regularity, behaviour and uniqueness of trajectories), we refer to H.Amann's book [1]. Here we consider the system $(u',v')^t = F(u,v)$ with a $\mathcal{C}^1(\mathbb{R}^2,\mathbb{R}^2)$ function F given by $F(u,v) = (v, uv - u|u|^{p-1} + \lambda u)^t$, thus uniqueness and regularity ($\mathcal{C}^1$)of the solutions (u,v) come from the standard ODE's theorems. With $v = u'$, we deduce that u is of class $\mathcal{C}^2$. First, we can note that System (3) has three equilibrium points if $\lambda > 0$: $(0,0)$, $(\lambda^{\frac{1}{p-1}},0)$ and $(-\lambda^{\frac{1}{p-1}},0)$. Using Hartman-Grobman's linearization theorem (see Reference [1]), we can state that $(0,0)$ is a saddle point, $(\lambda^{\frac{1}{p-1}},0)$ is an unstable and repulsive vortex (if $1-4(p-1)\lambda^{\frac{p-3}{p-1}} < 0$), an unstable node (if $1-4(p-1)\lambda^{\frac{p-3}{p-1}} \geq 0$, which degenerates when $1-4(p-1)\lambda^{\frac{p-3}{p-1}} = 0$). And $(-\lambda^{\frac{1}{p-1}},0)$ is a stable and attractive vortex (for $1-4(p-1)\lambda^{\frac{p-3}{p-1}} < 0$), a stable node (for $1-4(p-1)\lambda^{\frac{p-3}{p-1}} \geq 0$ with degeneracy when equality occurs). If $\lambda \leq 0$, then $(0,0)$ is the only equilibrium point of System (3). We will prove later that $(0,0)$ is a center.

2.1. **Case $\lambda > 0$.** Let λ be a positive real number and $p > 1$. We want to study the phase plane of the System (3). First we prove a lemma on the symmetry of the trajectories:

Lemma 2.1. *The support of the trajectories of the System (3) are symmetric with respect to the ordinates axis.*

Proof. Let (u, v) denote a solution of the System (3) in $(-a, a)$ for some $a \in (0, \infty]$, and define

$$\begin{cases} w(x) &= -u(-x) \\ z(x) &= v(-x) \end{cases} \quad \text{for all } x \in (-a, a).$$

A simple calculus of the derivatives implies

$$w'(x) = u'(-x) = v(-x) = z(x),$$

and

$$\begin{aligned} z'(x) &= -u(-x) \\ &= -v'(-x) \\ &= -\Big[u(-x)v(-x) - u(-x)|u(-x)|^{p-1} + \lambda u(-x)\Big] \\ &= w(x)z(x) - w(x)|w(x)|^{p-1} + \lambda w(x). \end{aligned}$$

Then (w, z) is also a trajectory of the System (3), and it is symmetric to (u, v) with respect to the ordinates axis. □

Thus, we can reduce our phase plane analysis to the half plane $\mathbb{R}^+ \times \mathbb{R}$. In order to draw the phase plane of the System (3), we write the ordinate v as a function depending on the abscissa u: $v = f(u)$. We do not know the function f, but we can deduce its variations and convexity using the equations (3). For the variations, when $v \neq 0$, we have

$$\frac{dv}{du} = \frac{uv - u|u|^{p-1} + \lambda u}{v} = \frac{u}{v}\Big(v - |u|^{p-1} + \lambda\Big), \tag{4}$$

in particular, it vanishes along the axis $\{u = 0\}$ and along the curve $\{v = |u|^{p-1} - \lambda\}$. For $u < \lambda^{\frac{1}{p-1}}$, we have

$$\left.\frac{dv}{du}\right|_{v=0} = \infty$$

whereas for $u > \lambda^{\frac{1}{p-1}}$

$$\left.\frac{dv}{du}\right|_{v=0} = -\infty.$$

Then we have $\frac{dv}{du} > 0$ in the sets $\{u > 0, v > 0, v > |u|^{p-1} - \lambda\}$ and $\{u > 0, v < 0, v < |u|^{p-1} - \lambda\}$. On the other hand, $\frac{dv}{du} < 0$ in the sets $\{u > 0, v < 0, v > |u|^{p-1} - \lambda\}$ and $\{u > 0, v > 0, v < |u|^{p-1} - \lambda\}$. Next, we compute the convexity of the function f and we obtain

$$\frac{d^2v}{du^2} = 1 + \frac{1}{v^2}\left[(\lambda - p|u|^{p-1})v - u(\lambda - |u|^{p-1})\frac{dv}{du}\right]. \tag{5}$$

We have $\frac{d^2v}{du^2} < 0$ in $\{u > 0, v > 0, v < |u|^{p-1} - \lambda\}$ and $\frac{d^2v}{du^2} > 0$ in $\{u > 0, v < 0, v < |u|^{p-1} - \lambda\}$. Since

$$\frac{d^2v}{du^2}\bigg|_{u=0} = 1 + \frac{\lambda}{v} \text{ and } \frac{d^2v}{du^2}\bigg|_{v=|u|^{p-1}-\lambda} = (1-p)\frac{|u|^{p-1}}{v},$$

the convexity is sign-changing in $\{u > 0, v > |u|^{p-1} - \lambda\}$. These arguments are sufficient to know the profile of the trajectories in the half plane $\{v < |u|^{p-1} - \lambda\}$. We do not need to know how the trajectories behave in $\{u > 0, v < 0, v > |u|^{p-1} - \lambda\}$ to solve Equation (2). In $\{u > 0, v > 0, v > |u|^{p-1} - \lambda\}$, things are different: unbounded trajectories can appear (see §2.3). To ensure the occurrence of bounded trajectories, we need an additional hypothesis:

$$p \geq 3. \tag{6}$$

Lemma 2.2. *Under hypothesis (6), all the trajectories of the System (3) are bounded in* $A = \{u > 0, v > 0, v > |u|^{p-1} - \lambda\}$.

Proof. Let $v_0 > 0$ and consider (u, v) the solution of the System (3) with initial data $(u(0), v(0)) = (0, v_0)$. The calculus of the variations (see Equation (4)) ensures that $(u(t), v(t)) \in A$ for small $t > 0$. We prove that there exist $0 < \tau < \infty$ such that $v(\tau) = |u(\tau)|^{p-1} - \lambda$. It means that (u, v) is bounded in A. Since (u, v) belongs to A, we have

$$\frac{dv}{du} = u + \frac{\lambda u}{v} - \frac{u|u|^{p-1}}{v} \leq u + \frac{\lambda u}{v}.$$

Then $\frac{dv}{du} \geq 0$ in A implies $v > v_0$ as long as $(u, v) \in A$, and we obtain

$$\frac{dv}{du} \leq u\Big(1 + \frac{\lambda}{v_0}\Big).$$

Integration gives

$$v \leq \frac{1}{2}\Big(1 + \frac{\lambda}{v_0}\Big)u^2 + v_0.$$

If $p > 3$, the intersection $\{v = |u|^{p-1} - \lambda\} \cap \{v = \frac{1}{2}\Big(1 + \frac{\lambda}{v_0}\Big)u^2 + v_0\}$ is non-empty for all $v_0 > 0$. If $p = 3$, we need to choose v_0 sufficiently big such that

$$\frac{1}{2}\Big(1 + \frac{\lambda}{v_0}\Big) < 1.$$

Then, the trajectory (u, v) belongs to the compact

$$\{u \geq 0, v \geq |u|^{p-1} - \lambda, v \leq \frac{1}{2}\Big(1 + \frac{\lambda}{v_0}\Big)u^2 + v_0\},$$

and, using $\frac{dv}{du} \geq 0$, we know that there exist $0 < \tau < \infty$ such that $v(\tau) = |u(\tau)|^{p-1} - \lambda$. This argument proves that each solution of the System (3) with initial data $(u(0), v(0)) = (0, v_0)$ is bounded in A if v_0 is big enough. Thanks to uniqueness of solution, it also proves the result for all the solutions initiated in A. □

Then, we complete this phase plane analysis by proving the existence of periodic trajectories.

Lemma 2.3. *Assume that hypothesis (6) is fulfilled. Then, there exists periodic trajectories of the System (3).*

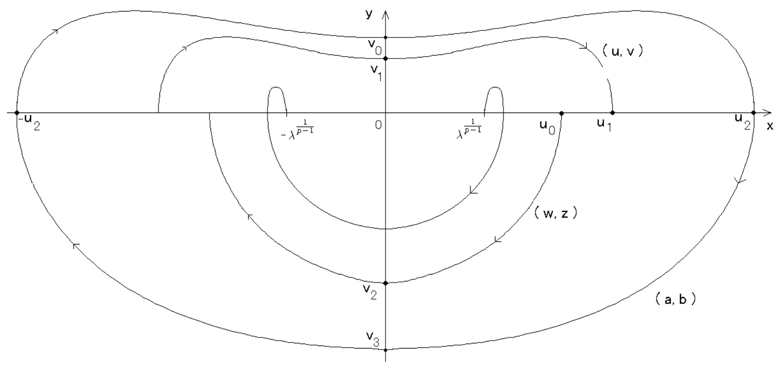

FIGURE 1. Phase plane for $p \geq 3$ and $\lambda > 0$.

Proof. Thanks to the symmetry (see Lemma 2.1), we just need to prove that for some initial data belonging to $\{0\} \times (0, \infty)$, there exists a trajectory which attains a point belonging to $\{0\} \times (-\infty, 0)$. First, consider a trajectory (u, v) initiated at $(0, v_1)$ with $v_1 > 0$. According to hypothesis (6), we know that (u, v) is bounded, and using its variations and its convexity (Equations (4) and (5)), we can deduce that (u, v) attains the $x-$axis at a point $(u_1, 0)$ with $u_1 > \lambda^{\frac{1}{p-1}}$ (see Figure 1). Then, using the reverse system

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} -v \\ -uv + u|u|^{p-1} - \lambda u \end{pmatrix},$$

and one of its trajectories initiated at $(0, v_2)$ with $v_2 < -\lambda)$ (trajectories of reverse system and of System (3) have same support), one can note that for $u_0 > \lambda^{\frac{1}{p-1}}$, there exists a trajectory (w, z) of (3) with $w(0) = u_0$ and $z(0) = 0$ (see Figure 1). Finally, let us consider the trajectory (a, b) of System (3) containing the point $(u_2, 0)$, where $u_2 > \max\{u_0, u_1\}$. Thanks to the uniqueness of the solutions, and using the information on the variations and the convexity, we deduce that there exist two real numbers $s < t$ such that $a(s) = a(t) = 0$, $b(s) = v_0$ and $b(t) = v_3$ (see Figure 1). Thus, the trajectory (a, b) is the periodic trajectory we look for. $\square$

Now, analysing the phase plane of the System (3), we deduce the following results concerning the Equation (2).

Theorem 2.4. *Assume hypothesis (6) and $\lambda > 0$. For each boundary conditions*

- $u(-\alpha) = u(\alpha) = 0$ *(Dirichlet b.c.)* ,
- $u'(-\alpha) = u'(\alpha) = 0$ *(Neumann b.c.)* ,
- $u(-\alpha) = u'(\alpha) = 0$ *(mixed-1 b.c.)*,
- $u'(-\alpha) = u(\alpha) = 0$ *(mixed-2 b.c.)*,

there exists a positive solution of the Equation (2)

$$u'' - uu' + u|u|^{p-1} - \lambda u = 0 \text{ in } (-\alpha, \alpha) \text{ for some } \alpha > 0.$$

Proof. We use the phase plane of System (3), see Figure 1. Consider the trajectory (a, b) between the points

- $(0, v_0)$ and $(0, v_3)$: we obtain the Dirichlet solution,
- $(0, v_0)$ and $(u_2, 0)$: we obtain the mixed-1 solution,
- $(u_2, 0)$ and $(0, v_3)$: we obtain the mixed-2 solution.

For the Neumann solution, consider $0 < \mu_0 < \lambda^{\frac{1}{p-1}}$ and the trajectory (μ, ν) of System (3) initiated at $(\mu_0, 0)$. Since (μ, ν) can not cross the trajectory (u, v) (see Figure 1), it must cross the $x-$axis at $(\mu_1, 0)$ with $\lambda^{\frac{1}{p-1}} < \mu_1 < u_1$. Thus, the abscissa of this trajectory is the Neumann solution we look for. Finally, the length (2α) of the existence interval is governed by the time needed by the trajectory to go from its initial data to its "final data". □

Theorem 2.5. *Assume hypothesis (6) and $\lambda > 0$. For some $\alpha > 0$, there exists a periodic sign-changing solution of the Equation (2)*

$$u'' - uu' + u|u|^{p-1} - \lambda u = 0 \text{ in } \mathbb{R}.$$

Proof. We just need to choose one of the periodic trajectories of the System (3) built in Lemma 2.3. □

Remark 1. Using the periodic solutions in the previous theorem, and restricting them to some suitable subintervals (non-trivial), we can build four sign-changing solutions satisfying the four boundary conditions: Dirichlet, Neumann, mixed-1 and mixed-2 (see Theorem 2.4).

Now, suppose that hypothesis (6) is not achieved. Then, we do not know if the solutions are bounded in $\{v > |u|^{p-1} - \lambda\}$: we will see in §2.3 that unbounded solutions appear. But in $\{v < |u|^{p-1} - \lambda\}$, the behaviour of the trajectories do not change.

Theorem 2.6. *Let $\lambda > 0$. For some $\alpha > 0$, there exists a positive solution of the Equation (2)*

$$u'' - uu' + u|u|^{p-1} - \lambda u = 0 \text{ in } (-\alpha, \alpha)$$

with the mixed boundary conditions $u'(-\alpha) = u(\alpha) = 0$. In addition, if

$$1 - 4(p-1)\lambda^{\frac{p-3}{p-1}} < 0, \tag{7}$$

then there exists a positive solution of the Equation (2) under the Neumann boundary conditions.

Proof. The first part of the statement comes from Theorem 2.4, the solution with mixed-2 boundary conditions is located in $\{v < |u|^{p-1} - \lambda\}$. The other part stems from Equation (7): in this case, the equilibrium $(\lambda^{\frac{1}{p-1}}, 0)$ is an unstable vortex. If we consider $u_0 > 0$ such that $|\lambda^{\frac{1}{p-1}} - u_0|$ is sufficiently small, the trajectory (u, v) of the System (3), with $u(0) = u_0$ and $v(0) = 0$, whirls around $(\lambda^{\frac{1}{p-1}}, 0)$. Thus, there exists $\tau > 0$ such that $v(\tau) = 0$ and $u(t) > 0$ for all $t \in [0, \tau]$. □

Without hypothesis (6), we can not construct positive solutions anymore for the Dirichlet, Neumann or mixed-1 boundary conditions. If we do not impose the positivity, we obtain this result:

Theorem 2.7. *Let $\lambda > 0$. For each boundary conditions*

- $u'(-\alpha) = u'(\alpha) = 0$ *(Neumann b.c.)* ,
- $u(-\alpha) = u'(\alpha) = 0$ *(mixed-1 b.c.)*,

there exists a solution of the Equation (2)

$$u'' - uu' + u|u|^{p-1} - \lambda u = 0 \text{ in } (-\alpha, \alpha) \text{ for some } \alpha > 0.$$

Proof. As we mentioned before, we consider the part $\{v < |u|^{p-1} - \lambda\}$ of the phase plane of the System (3) (see Figure 1). For the Neumann solution, we consider the trajectory (a, b) between $(u_2, 0)$ and $(-u_2, 0)$. For the mixed−1 solution, we can also consider the trajectory (a, b), but only between $(0, v_3)$ and $(-u_2, 0)$. □

Remark 2. The Neumann solution built above is sign changing, whereas the mixed−1 solution is negative.

Remark 3. In the general case, we can not build any solution with the Dirichlet boundary conditions using our phase plane method. Indeed, we will give a criterion in Theorem 2.18 concerning nonexistence of the Dirichlet solution.

Concerning the solutions in infinite interval, we can state:

Theorem 2.8. *Let $\lambda > 0$. Then the Equation (2)*

$$u'' - uu' + u|u|^{p-1} - \lambda u = 0$$

admits

- *a positive solution u in $(-\infty, 0]$ satisfying $u'(-\infty) = u'(0) = 0$ (Neumann).*
- *a positive solution v in $(-\infty, 0]$ satisfying $v'(-\infty) = v(0) = 0$ (mixed−2).*
- *a sign-changing solution w in $\mathbb{R}$ satisfying $w'(-\infty) = w'(\infty) = 0$ (Neumann).*
- *a negative solution u in $[0, \infty)$ satisfying $z(0) = z'(\infty) = 0$ (mixed−1).*

Proof. Consider $\mu_0 > 0$ with $\mu_0 > \lambda^{\frac{1}{p-1}}$ and with $|\lambda^{\frac{1}{p-1}} - \mu_0|$ small enough such that there exists a trajectory (μ, ν) of the System (3) satisfying

$$\mu(-\infty) = \lambda^{\frac{1}{p-1}},\ \nu(-\infty) = 0 \ \text{ and } \ \mu(0) = \mu_0,\ \nu(0) = 0.$$

Since $(\lambda^{\frac{1}{p-1}}, 0)$ is repulsive, the existence of (μ, ν) is clear. Hence, $u = \mu$ in $(-\infty, 0]$ is suitable for the first statement. Then, the trajectory (μ, ν) can be continued in the part $\{u > 0, v < 0\}$ using the information on its behaviour (see Equations (4) and (5)) until (μ, ν) attains the ordinate axis. Denote $t_1 > 0$ the time such that $\mu(t_1) = 0$ and $\nu(t_1) < 0$. We obtain the second statement setting $v(t) = \mu(t + t_1)$ for all $t \in (-\infty, 0]$. Finally, these results and the symmetry of the trajectories (see Lemma 2.1) imply the third and the fourth statements with the following definitions:

$$w(t) = \begin{cases} v(t) & \forall \quad t \leq 0 \\ -v(-t) & \forall \quad t > 0 \end{cases} \ \text{ and } \ z(t) = -v(-t) \text{ for all } t \geq 0.$$

□

2.2. **Case $\lambda \leq 0$.** First note that the System (3) has only one equilibrium point $(0, 0)$. As in the previous case, we can reduce our phase plane analysis to the half-plane $\mathbb{R}^+ \times \mathbb{R}$ since Lemma 2.1. Again, we obtain some information on the variations of the trajectories of the System (3) using Equation (4). We have $\frac{dv}{du} = 0$ along the curves $\{u = 0\}$ and $\{v = |u|^{p-1} - \lambda\}$. For $u > 0$

$$\left.\frac{dv}{du}\right|_{v=0} = -\infty$$

whereas for $u < 0$

$$\left.\frac{dv}{du}\right|_{v=0} = +\infty.$$

Then, we have $\frac{dv}{du} \geq 0$ in $\{u > 0, v < 0\} \cup \{v \geq |u|^{p-1} - \lambda\}$ and $\frac{dv}{du} \leq 0$ in $\{u > 0, v > 0, v \leq |u|^{p-1} - \lambda\}$. In addition, thanks to Equation (5), we know that

$\frac{d^2v}{du^2} \le 0$ in $\{u > 0, v > 0, v \le |u|^{p-1} - \lambda\}$, $\frac{d^2v}{du^2} \ge 0$ in $\{u > 0, v < 0\}$ while it is sign-changing in $\{u > 0, v \ge |u|^{p-1} - \lambda\}$. In this last part of the plane, we use the following lemma, similar to Lemma 2.2:

Lemma 2.9. *Let $\lambda \le 0$ and (u, v) be a trajectory of the System (3) with initial data $(0, v_0)$. If $v_0 > -\lambda$ satisfies*

$$\begin{cases} v_0 > -\lambda & \text{if } p \ge 3 \ , \\ v_0 \le -\lambda + (p-1)^{\frac{p-1}{3-p}} - \frac{1}{2}(p-1)^{\frac{2}{3-p}} & \text{if } p < 3 \ , \end{cases} \tag{8}$$

then the trajectory (u, v) is bounded in $A = \{u > 0, v \ge |u|^{p-1} - \lambda\}$.

Proof. The calculus of the variations (see Equation (4)) ensures that $(u(t), v(t)) \in A$ for small $t > 0$. We prove that there exists $0 < \tau < \infty$ such that $v(\tau) = |u(\tau)|^{p-1} - \lambda$. It means that (u, v) is bounded in A. Since (u, v) belongs to A and thanks to $\lambda \le 0$, we have

$$0 \le \frac{dv}{du} = u + \frac{\lambda u}{v} - \frac{u|u|^{p-1}}{v} \le u.$$

Then, integration between 0 and u gives

$$v \le \frac{1}{2}u^2 + v_0.$$

Hypothesis (8) implies that $\{u > 0, v = |u|^{p-1} - \lambda\} \cap \{u > 0, v = \frac{1}{2}u^2 + v_0\}$ is not empty. Thus, the trajectory (u, v) belongs to the compact

$$\{u \ge 0, v \ge |u|^{p-1} - \lambda, v \le \frac{1}{2}u^2 + v_0\}.$$

Using $\frac{dv}{du} \ge 0$, we know that there exist $\tau > 0$ such that $v(\tau) = |u(\tau)|^{p-1} - \lambda$. □

Now, the phase plane of the System (3) can be drawn, see Figure 2.

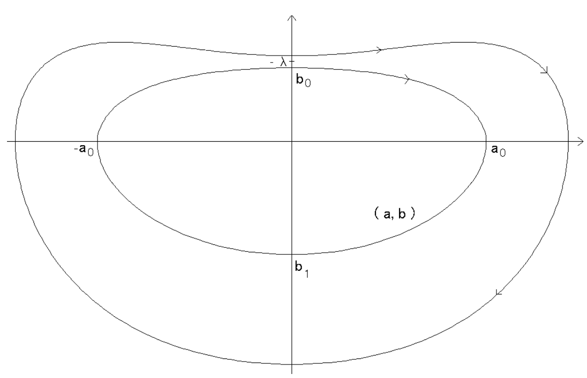

FIGURE 2. Phase plane for $\lambda \le 0$.

Corollary 1. *The equilibrium point $(0, 0)$ is a center for the System (3).*

Now, we use this information on the trajectories of the System (3) to obtain some results concerning the solutions of Equation (2).

Theorem 2.10. *Let $\lambda \le 0$. For each boundary conditions*

- $u(-\alpha) = u(\alpha) = 0$ *(Dirichlet b.c.)* ,
- $u(-\alpha) = u'(\alpha) = 0$ *(mixed−1 b.c.),*

- $u'(-\alpha) = u(\alpha) = 0$ *(mixed−2 b.c.),*

there exists a positive solution of the Equation (2)

$$u'' - uu' + u|u|^{p-1} - \lambda u = 0 \text{ in } (-\alpha, \alpha) \text{ for some } \alpha > 0.$$

Proof. We use the phase plane of System (3), see Figure 2. Consider the trajectory (a, b) between the points

- $(0, b_0)$ and $(0, b_1)$: we obtain the Dirichlet solution,
- $(0, b_0)$ and $(a_0, 0)$: we obtain the mixed−1 solution,
- $(a_0, 0)$ and $(0, b_1)$: we obtain the mixed−2 solution.

□

Theorem 2.11. *Let $\lambda \leq 0$. For all $\alpha > 0$, the Equation (2)*

$$u'' - uu' + u|u|^{p-1} - \lambda u = 0 \text{ in } (-\alpha, \alpha)$$

admits no positive solution under the Neumann boundary conditions.

Proof. Ab absurbo, suppose that there exists u a positive solution of (2) under the Neumann boundary conditions, and denote $v = u'$. Then the curve (u, v) is a trajectory of the System (3) located in $\mathbb{R}^+ \times \mathbb{R}$ with initial data on the axis $\{v = 0\}$. Then Equations (4) and (5) prove that (u, v) can not cross the axis $\{v = 0\}$ once again without going into $\mathbb{R}^- \times \mathbb{R}$. A contradiction with the positivity of u. □

Theorem 2.12. *Let $\lambda \leq 0$. For some $\alpha > 0$, the Equation (2)*

$$u'' - uu' + u|u|^{p-1} - \lambda u = 0 \text{ in } (-\alpha, \alpha)$$

admits a sign-changing solution under the Neumann boundary conditions.

Proof. Using the phase plane of System (3) (see Figure 2), consider the trajectory (a, b) between the points $(a_0, 0)$ and $(-a_0, 0)$. □

To conclude this section, let us give this result concerning the periodic solutions:

Theorem 2.13. *Let $\lambda \leq 0$. For some $\alpha > 0$, there exists a sign-changing periodic solution of the Equation (2)*

$$u'' - uu' + u|u|^{p-1} - \lambda u = 0 \text{ in } \mathbb{R}.$$

Proof. As in Lemma 2.3, we can build periodic trajectories of (3) using the symmetry (Lemma 2.1). □

2.3. **Unbounded solutions.** In the above paragraphs, we proved that all the trajectories of the System (3) are bounded for $p \geq 3$, but if $1 < p < 3$ we do not have a general answer: for example, we obtain some bounded trajectories when $\lambda \leq 0$ (see Lemma 2.9), but with our method, we do not have (yet) any result when $\lambda > 0$. In this paragraph, we show that there exists unbounded trajectories for every $\lambda \in \mathbb{R}$ and for all $p \in (1, 3)$. We start with a trajectory (u, v) with an initial data $(0, v_0)$.

Lemma 2.14. *Let $p \in (1, 3)$ and $\lambda \in \mathbb{R}$. Suppose that*

$$v_0 > 2\max\{-\lambda, 0\} + 2 \cdot 8^{\frac{p-1}{3-p}}. \tag{9}$$

Then the trajectory (u, v) is not bounded.

Proof. We will show that under hypothesis (9), the trajectory (u, v) always lies above the curve $\Big\{v = 2u^{p-1} + 2\max\{-\lambda, 0\}\Big\}$. Thus, using $\frac{dv}{du} \geq 0$ (Equation (4)), we obtain that (u, v) is not bounded. Ab absurdo, suppose that there exists $x_* > 0$ such that $u(x_*) = u_1 > 0$ and $v(x_*) = v_1 > 0$ satisfy

$$v_1 = 2u_1^{p-1} + 2\max\{-\lambda, 0\}, \tag{10}$$

and

$$v(x) > 2u(x)^{p-1} + 2\max\{-\lambda, 0\} \ \forall\ x \in [0, x_*).$$

Thus in $[0, x_*)$, we have

$$\frac{\lambda - u^{p-1}}{v} > -\frac{1}{2}. \tag{11}$$

On the other hand, Equation (4) gives

$$\frac{dv}{du} = u + u\frac{\lambda - u^{p-1}}{v},$$

and thanks to condition (11), we obtain

$$\frac{dv}{du} \geq \frac{1}{2}u \geq 0. \tag{12}$$

Then $v(u) \geq \frac{u^2}{4} + v_0$. Hence, for $u = u_1$, we have:

$$v_1 = v(u_1) \geq \frac{u_1^2}{4} + v_0,$$

and by definition (10) of u_1, we have

$$2u_1^{p-1} + 2\max\{-\lambda, 0\} \geq \frac{u_1^2}{4} + v_0.$$

Hypothesis (9) implies

$$-2 \cdot 8^{\frac{p-1}{3-p}} > \frac{u_1^2}{4} - 2u_1^{p-1}. \tag{13}$$

Meanwhile, if we study both cases $u_1 < 8^{\frac{1}{3-p}}$ and $u_1 > 8^{\frac{1}{3-p}}$, we remark that

$$\frac{u_1^2}{4} - 2u_1^{p-1} = \frac{u_1^{p-1}}{4}\Big(u_1^{3-p} - 8\Big) \geq -2 \cdot 8^{\frac{p-1}{3-p}}. \tag{14}$$

Equations (13) and (14) are not compatible. Thus, the trajectory (u, v) can not attain the curve $\Big\{v = 2u^{p-1} + 2\max\{-\lambda, 0\}\Big\}$. □

Concerning Equation (2), we obtain the following results:

Theorem 2.15. *Let $p \in (2, 3)$ and $\lambda \in \mathbb{R}$. For some $\alpha > 0$, there exists a positive and unbounded solution of the Equation (2)*

$$u'' - uu' + u|u|^{p-1} - \lambda u = 0 \ \textit{in}\ (-\alpha, \alpha).$$

satisfying

$$u(-\alpha) = 0 \ \textit{and}\ \lim_{x \to \alpha} u(x) = \infty.$$

Proof. The existence comes from the previous lemma. We just need to prove that the length of the existence interval is finite. Ab absurdo, suppose that there exists a positive and unbounded solution u of the Equation (2) in $[0,\infty)$. Let $b>0$ such that $u>|2\lambda|^{\frac{1}{p-1}}$ in $[b,\infty)$, and define $w(x,t)=u(x+t)$ for all $x\in[b,b+1]$ and for all $t\in[0,\infty)$. Thanks to the choice of b, we have

$$\partial_x^2 u - u\partial_x u + u^p - \lambda u \geq \partial_x^2 u - u\partial_x u + \frac{u^p}{2}$$

in $[b,b+1]\times[0,\infty)$. Because the solution u corresponds to a trajectory of the System (3) located in $\mathbb{R}\times\mathbb{R}^+$, we have $\partial_t w=\partial_x u>0$. Thus, w is super-solution of the following problem

$$\begin{cases} \partial_t v = \partial_x^2 v - v\partial_x v + \frac{1}{2}v^p & \text{in } [b,b+1]\times(0,\infty), \\ \partial_t v + \partial_\nu v = 0 & \text{on } \{\pm b\}\times(0,\infty), \\ v(\cdot,0) = |2\lambda|^{\frac{1}{p-1}} & \text{in } [b,b+1]. \end{cases}$$

By the comparison principle from [3], $w\geq v$ where v is the solution of the previous problem. But, according to [6], the solution v blows up in finite time. Since $w\geq v$, this contradicts the global existence of w. Thus, w can not exist on $[b,b+1]\times(0,\infty)$, and the solution u exists only in a finite interval. □

For $1<p\leq 2$, we do not have the blowing-up argument and we are not sure that the existence interval of the solution is finite.

Theorem 2.16. *Let $p\in(1,2]$ and $\lambda\in\mathbb{R}$. For some $\alpha\in(0,\infty]$, there exists a positive and unbounded solution of the Equation (2)*

$$u'' - uu' + u|u|^{p-1} - \lambda u = 0 \text{ in } (0,\alpha).$$

satisfying

$$u(-\alpha)=0 \text{ and } \lim_{x\to\alpha} u(x)=\infty.$$

With some assumption on the parameter λ, we can also build an unbounded trajectory (u,v) with an initial data $(u_0,0)$ belonging to the abscissa axis.

Lemma 2.17. *Let $p\in(1,3)$ and $\lambda\in\mathbb{R}^+$. Suppose that there exists $\beta>1$ such that*

$$\lambda > \max\left\{\frac{\beta-1}{2\beta}\left(\frac{2\beta^2}{\beta-1}\right)^{\frac{2}{3-p}},\ \beta\left(\frac{2\beta^2}{\beta-1}\right)^{\frac{p-1}{3-p}}\right\} \tag{15}$$

If

$$u_0 = \left(\frac{2\beta^2}{\beta-1}\right)^{\frac{1}{3-p}}, \tag{16}$$

then the trajectory (u,v) is not bounded.

Proof. We use the same method as in Lemma 2.14: we prove that, under hypotheses (15) and (16), the trajectory (u,v) always lies above the curve $\{v=\beta u^{p-1}-\lambda\}$. Ab absurdo, suppose that there exist $x_*>0$ such that $u(x_*)=u_1$ and $v_1=v(x_*)$ verify

$$v_1 = \beta u_1^{p-1} - \lambda, \tag{17}$$

and

$$v(x_*) > \beta u(x_*)^{p-1} - \lambda\ \forall\ 0<x<x_*.$$

Thus, in $[0, x_*)$, we have

$$\frac{\lambda - u^{p-1}}{v} \geq \frac{-1}{\beta}. \tag{18}$$

Equation (4) gives

$$\frac{dv}{du} = u + u\frac{\lambda - u^{p-1}}{v},$$

and condition (18) implies

$$\frac{dv}{du} \geq \frac{\beta - 1}{\beta}u \geq 0.$$

Integration between u_0 and u_1 leads to

$$v(u_1) \geq \frac{\beta - 1}{2\beta}(u_1^2 - u_0^2),$$

definition (17) gives

$$\beta u_1^{p-1} - \lambda \geq \frac{\beta - 1}{2\beta}(u_1^2 - u_0^2),$$

and we obtain

$$u_1^{p-1}\Big(1 - \frac{\beta - 1}{2\beta^2}u_1^{3-p}\Big) \geq \frac{1}{\beta}\Big(\lambda - u_0^2\frac{\beta - 1}{2\beta}\Big). \tag{19}$$

Since $u_0 < u_1$, Equations (15) and (16) imply

$$\lambda - u_0^2\frac{\beta - 1}{2\beta} > 0 \text{ and } 1 - \frac{\beta - 1}{2\beta^2}u_1^{3-p} < 0.$$

Hence, Equation (19) is a contradiction. □

Concerning Equation (2), and reasoning as in Theorem 2.15, we obtain the following result.

Theorem 2.18. *Let $p \in (1, 3)$ and $\lambda \in \mathbb{R}$ verifying Equation (15). For some $\alpha \in (0, \infty]$, there exists a positive and unbounded solution of the Equation (2)*

$$u'' - uu' + u|u|^{p-1} - \lambda u = 0 \text{ in } (0, \alpha).$$

satisfying

$$u'(0) = 0 \text{ and } \lim_{x \to \alpha} u(x) = \infty.$$

In addition, if $p \in (2, 3)$, then α is finite.

2.4. **Limiting case $p = 1$.** In this paragraph, we study the case where the exponent p attains the limit 1. Then, Equation (2) becomes

$$u'' - uu' + (1 - \lambda)u = 0 \text{ in } \mathbb{R},$$

and the System (3) is written

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} v \\ u(v + \lambda - 1) \end{pmatrix}. \tag{20}$$

For $\lambda \neq 1$, $(0, 0)$ is the only equilibrium point, while for $\lambda = 1$ the axis $\{v = 0\}$ is a continuum of equilibria. We begin with the case $\lambda = 1$. Here, we have $\frac{dv}{du} = u$, then

$$v(u) = \frac{1}{2}u^2 + c,$$

where c depends on the initial data. Thus, the phase plane is easily drawn, see Figure 3. Now, suppose $\lambda \neq 1$. One can compute the explicit trajectory

$$\begin{cases} u_e(x) = (1-\lambda)x \\ v_e(x) = (1-\lambda) \end{cases} \quad \forall\, x \in \mathbb{R}$$

Then, using the following equations

$$\frac{dv}{du} = u + \frac{u}{v}(\lambda - 1) \quad \text{and} \quad \frac{d^2v}{du^2} = 1 + + \frac{\lambda - 1}{v^2}\Big(v - u\frac{dv}{du}\Big)$$

we can draw the phase plane of the System (20), see Figure 3.

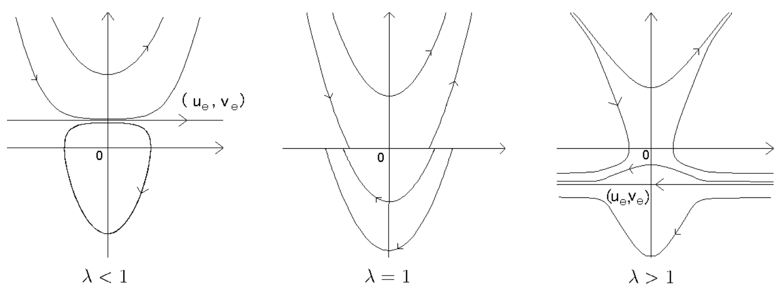

FIGURE 3. Phase planes for $p = 1$.

2.5. **Bifurcation.** According to the previous paragraphs, we can state that there exists a bifurcation of the phase plane of the System (3). First, we note that, for a fixed exponent p, the value of λ influences the phase plane of the System (3): for $\lambda > 0$, the System (3) admits three equilibrium points (a saddle point, an attractive equilibrium and a repulsive equilibrium). The distance between these equilibria goes to 0 when $\lambda \to 0$, and for $\lambda = 0$, they collapse and generate a unique center, which persists for all negative λ (see Figure 4).

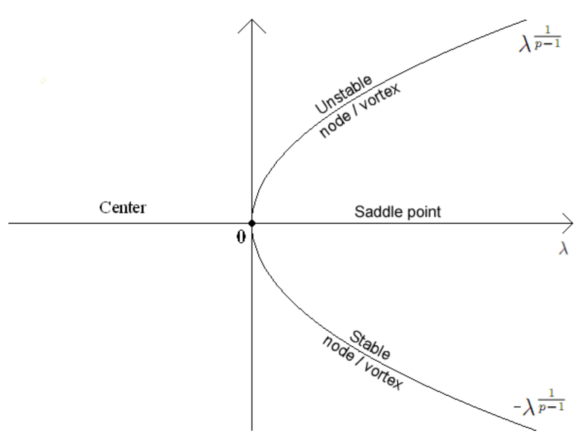

FIGURE 4. Abscissa of the equilibrium points of the System (3) depending on the parameter λ.

Now, for a fixed λ, the value of the exponent p has an important role. With λ, the value of p governs the type of the equilibrium points (node, improper node, vortex). The exponent p also establishes if all the trajectories of the System (3) are bounded ($p \geq 3$) or if there exists unbounded trajectories ($1 \leq p < 3$). Moreover, when p attains the limit 1, the critical value of λ changes from 0 (if $p > 1$) to 1 (for $p = 1$). The case $\lambda = 1$ is special because when $p \to 1$, the three equilibria of the System (3) (a saddle point, an attractive vortex and a repulsive vortex) generate a continuum of equilibria when p attains the limit 1 (see Figure 5).

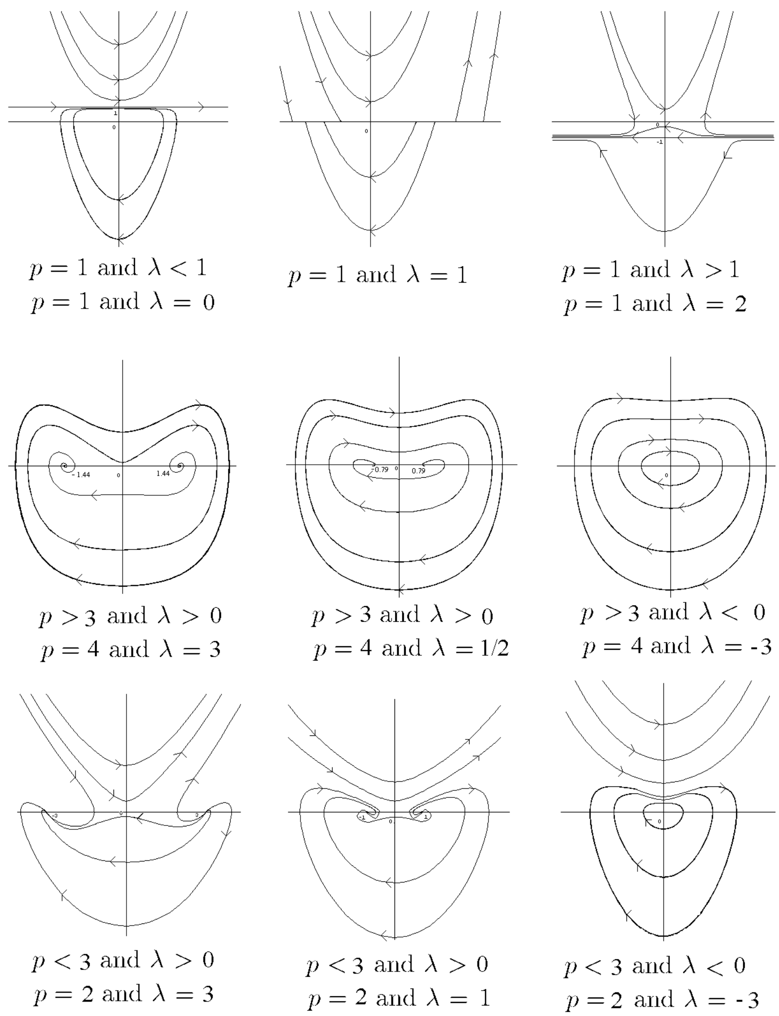

FIGURE 5. Phase planes of the System (3) with different parameters.

3. **Parabolic problem.** In this section, we study the positive solutions of the parabolic Problem (1) for many boundary conditions. First, we use the results concerning the stationary Equation (2) when the domain Ω is bounded. Then, we

consider the case of unbounded domains: we investigate global existence using the comparison method, and blow-up phenomenon thanks to a L^1-norm technique.

3.1. **Comparison.** We begin with the Dirichlet problem

$$\begin{cases} \partial_t u = \partial_x^2 u - u\partial_x u + u^p - \lambda u & \text{in } [-\alpha,\alpha]\times(0,\infty), \\ u = 0 & \text{on } \{\pm\alpha\}\times(0,\infty), \\ u(\cdot,0) = \varphi & \text{in } [-\alpha,\alpha], \end{cases} \quad (21)$$

where $\alpha > 0$, $p > 1$, $\lambda \in \mathbb{R}$ and $\varphi \in \mathcal{C}_0([-\alpha,\alpha])$ is non-negative. Thanks to the comparison principle [3] and with the results of the previous sections, we have:

Theorem 3.1. *Let $p > 1$ and $\lambda \in \mathbb{R}$. For some $\alpha > 0$, there exists a global positive solution*

$$u \in \mathcal{C}([-\alpha,\alpha]\times[0,\infty)) \cap \mathcal{C}^{2,1}([-\alpha,\alpha]\times(0,\infty))$$

of Problem (21) if the initial data $\varphi \in \mathcal{C}_0([-\alpha,\alpha])$ is sufficiently small.

Proof. If $p \geq 3$ and $\lambda > 0$, consider $\beta \in \mathcal{C}^2([-\alpha,\alpha])$ a solution of (2) with the Dirichlet boundary conditions (see Theorem 2.4). Suppose that φ is small enough: $\varphi \leq \beta$ in $[-\alpha,\alpha]$. Then, we obtain

$$\begin{cases} \partial_t \beta = 0 = \partial_x^2 \beta - \beta\partial_x \beta + \beta^p - \lambda\beta & \text{in } [-\alpha,\alpha]\times(0,\infty), \\ \beta = 0 & \text{on } \{\pm\alpha\}\times(0,\infty), \\ \beta(\cdot,0) \geq \varphi & \text{in } [-\alpha,\alpha]. \end{cases}$$

Thus, β is a non-negative upper solution of (21), and the constant 0 is a lower solution of (21). Using the comparison method from [3], we prove that there exists a solution u of (21) satisfying $0 \leq u \leq \beta$ for all $(x,t) \in [-\alpha,\alpha]\times(0,\infty)$. Thus, u is a global positive solution. If $1 < p < 3$ and $\lambda > 0$, then we just need to choose a positive solution β given in Theorem 2.6 (even if $\beta(\pm\alpha) > 0$). For $\lambda \leq 0$, we consider the Dirichlet solution given in Theorem 2.10. □

Now, we replace the Dirichlet boundary conditions by the dynamical boundary conditions. Consider the following problem

$$\begin{cases} \partial_t u = \partial_x^2 u - u\partial_x u + u^p - \lambda u & \text{in } [-\alpha,\alpha]\times(0,\infty), \\ \sigma\partial_t u + \partial_\nu u = 0 & \text{on } \{\pm\alpha\}\times(0,\infty), \\ u(\cdot,0) = \varphi & \text{in } [-\alpha,\alpha], \end{cases} \quad (22)$$

with $\alpha > 0$, $p > 1$, $\lambda \in \mathbb{R}$ and where $\varphi \in \mathcal{C}([-\alpha,\alpha])$ and $\sigma(\pm\alpha,\cdot) \in \mathcal{C}^1([0,\infty))$ are non-negative. We obtain two results, depending on the sign of λ.

Theorem 3.2. *Let $p > 1$ and $\lambda > 0$. There exists a global positive solution*

$$u \in \mathcal{C}([-\alpha,\alpha]\times[0,\infty)) \cap \mathcal{C}^{2,1}([-\alpha,\alpha]\times(0,\infty))$$

of Problem (22)

- *for all $\alpha > 0$ if $\varphi \leq \lambda^{\frac{1}{p-1}}$.*
- *for some $\alpha > 0$ if $\varphi - \lambda^{\frac{1}{p-1}}$ is sign-changing and $\max\{\varphi - \lambda^{\frac{1}{p-1}}, 0\}$ is sufficiently close to 0.*
- *for no $\alpha > 0$ if $\varphi > \lambda^{\frac{1}{p-1}}$ and $p > 2$.*

Proof. For the first statement, we just need to note that the constant function $\lambda^{\frac{1}{p-1}}$ is a super-solution of (22) when $0 \leq \varphi \leq \lambda^{\frac{1}{p-1}}$. For the second statement, we consider two cases: when $p \geq 3$, we consider a positive solution w of Equation (2) under the Neumann boundary conditions, see Theorem 2.4. Choosing φ such that $0 \leq \varphi \leq w$, w becomes a non-negative super-solution of (22). If $1 < p < 3$, we consider a trajectory (μ, γ) of System (3) with $0 < \mu(0) < \lambda^{\frac{1}{p-1}}$ and $\gamma(0) = 0$. According to Equation (4), for a small $x_* > 0$, we have $\gamma(-x) < 0$ and $\gamma(x) > 0$ for all $x \in (0, x_*)$. Thus, μ satisfies $\partial_\nu \mu(-x_*) = -\gamma(-x_*) > 0$ and $\partial_\nu \mu(x_*) = \gamma(x_*) > 0$, and it is a super-solution of (22) when $0 \leq \varphi \leq \mu$ in $[-x_*, x_*]$.

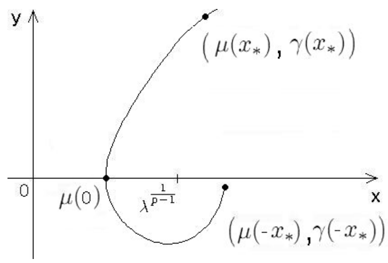

FIGURE 6. Trajectory (μ, γ).

Then, using these super-solutions and the comparison principle from [3], we prove first and second assertions. For the third statement, consider $c > 0$ such that

$$\varphi > c > \lambda^{\frac{1}{p-1}}.$$

The comparison principle from [3] implies that $u > c$, where u denote the solution of (22) with the initial data φ. Hence, there exists $d > 0$ such that

$$u^p - \lambda u \geq du^p \text{ for all } x \in [-\alpha, \alpha] \text{ and for all } t > 0.$$

Thus, u verifies

$$\begin{cases} \partial_t u \geq \partial_x^2 u - u\partial_x u + du^p & \text{in } [-\alpha, \alpha] \text{ for } t > 0, \\ \sigma\partial_t u + \partial_\nu u = 0 & \text{on } \{-\alpha,\ \alpha\} \text{ for } t > 0, \\ u(\cdot, 0) > c > 0 & \text{in } [-\alpha, \alpha]. \end{cases}$$

Then, blow-up results from [6] imply the blowing-up in finite time of u. □

Theorem 3.3. *Let $p > 2$ and $\lambda \leq 0$. For all $\alpha > 0$, the positive solution u of Problem (22) blows up in finite time if the initial data φ satisfies*

$$\varphi \geq 0,\ \ \varphi \not\equiv 0,\ \ \varphi \in \mathcal{C}([-\alpha, \alpha])$$

Proof. Since $\lambda \leq 0$, the function u verifies

$$\begin{cases} \partial_t u \geq \partial_x^2 u - u\partial_x u + u^p & \text{in } [-\alpha, \alpha] \text{ for } t > 0, \\ \sigma\partial_t u + \partial_\nu u = 0 & \text{on } \{-\alpha,\ \alpha\} \text{ for } t > 0, \\ u(\cdot, 0) > 0 & \text{in } [-\alpha, \alpha]. \end{cases}$$

Thanks to the blow-up results from [6] and [8], we know that u blows up in finite time. □

Remark 4. The Neumann boundary conditions are included here, with the special case $\sigma \equiv 0$.

3.2. **Global existence in unbounded domains.** We study the Problem (1) under the Dirichlet, the Neumann and the dynamical boundary conditions when Ω is an unbounded domain. Using some explicit super-solutions, we look for global existence in the three types of unbounded domains: $(-\infty, 0)$, $(0, \infty)$ and $\mathbb{R}$. We begin with the case $\lambda > 0$:

Theorem 3.4. *Let $p > 1$, $\lambda > 0$, $\varphi \in \mathcal{C}(\overline{\Omega})$ a non-negative function, and let Ω be any unbounded domain. Then, the Problem (1) admits a global positive solution if the initial data satisfies*

$$0 \leq \varphi \leq \lambda^{\frac{1}{p-1}},$$

and when $\mathcal{B}(u) = 0$ stands for the Dirichlet, the Neumann, the Robin ($\partial_\nu u + au = 0$ with $a \geq 0$) or the dynamical boundary conditions.

Proof. As in the proof of Theorem 3.2, we consider the constant function $v(x,t) = \lambda^{\frac{1}{p-1}}$ for all $(x,t) \in \Omega \times (0,\infty)$. Then, v satisfies Burgers' Equation. On the boundary, we have:

$$\begin{array}{lll} v & \geq 0 & \text{(Dirichlet)}. \\ \partial_\nu v & = 0 & \text{(Neumann)}. \\ \partial_\nu v + av & \geq 0 & \text{(Robin)}. \\ \sigma\partial_t v + \partial_\nu v & = 0 & \text{(Dynamical)}. \end{array}$$

The choice of φ implies $\varphi \leq v(\cdot, 0)$ in Ω. Thus, v is super-solution of (1) for the four boundary conditions above, and we conclude with the comparison method from [3]. □

If $\lambda \leq 0$, we must add some restrictions, and we obtain the following results.

Theorem 3.5. *Assume $\Omega = (0,\infty)$ and let $p \in (1,2]$, $\lambda \leq 0$ and $\varphi \in \mathcal{C}(\overline{\Omega})$ a non-negative function. Then, the Problem (1) admits a global positive solution if the initial data is bounded and when $\mathcal{B}(u) = 0$ stands for the Dirichlet boundary conditions or the dynamical boundary conditions with $\sigma > 0$ constant.*

Proof. We deal with the comparison principle [3] and the explicit function $v(x,t) = Ae^{\alpha x + (t+t_0)^2}$ defined in $\mathbb{R}^+ \times \mathbb{R}^+$. Computing the partial derivatives, we have

$$\begin{array}{rcl} \partial_t v(x,t) & = & 2(t+t_0)v. \\ \partial_x v(x,t) & = & \alpha v. \\ \partial_x^2 v(x,t) & = & \alpha^2 v. \end{array}$$

Choosing $t_0 \geq \frac{1}{2}\Big(\alpha^2 - \lambda\Big)$, we obtain

$$\partial_t v - \partial_x^2 v + v\partial_x v - v^p + \lambda v \geq v^2\Big(\alpha - v^{p-2}\Big).$$

Thanks to $p \leq 2$ and with $\alpha x + (t+t_0)^2 \geq 0$ in $\mathbb{R}^+ \times \mathbb{R}^+$, we have $v^{p-2} \leq A^{p-2}$. Choosing $A^{p-2} \leq \alpha$, we obtain $\partial_t v - \partial_x^2 v + v\partial_x v - v^p + \lambda v \geq 0$. Since $v \geq 0$, the case of the Dirichlet boundary conditions is trivial. Choosing $t_0 \geq \frac{\alpha}{2\sigma}$, the case of the dynamical boundary conditions is verified thanks to

$$\sigma\partial_t v + \partial_\nu v = v\Big(2\sigma(t+t_0) - \alpha\Big) \geq 0.$$

Finally, choosing $A \geq \sup_\Omega \varphi$, v is a super-solution of Problem (1) under the above boundary conditions. Thus, using the comparison method from [3], we prove that there exist a positive solution of Problem (1) bounded by v, and then, this solution must be global. □

Remark 5. In the previous proof, one can see that the dynamical boundary conditions are satisfied for a more general coefficient σ verifying

$$\sigma(x,t) \geq \frac{\alpha}{2(t+t_0)}.$$

And replacing the function v by $w(x,t) = Ae^{\alpha x + (t+t_0)^n}$, we can consider smaller coefficients $\sigma > 0$ with $\sigma(x,t) \underset{t\to\infty}{\sim} t^{-n+1}$.

Corollary 2. *Suppose $\Omega = (-\infty, 0)$ or $\Omega = \mathbb{R}$. Let $p = 2$, $\lambda \leq 0$ and $\varphi \in \mathcal{C}(\Omega)$. Then the Problem (1) admits a global positive solution if there exists $C > 0$ and $a > 0$ such that*

$$0 \leq \varphi(x) \leq Ce^{ax} \text{ in } \Omega$$

and when $\mathcal{B}(u) = 0$ stands for the Dirichlet, the Neumann or the dynamical boundary conditions with $\sigma \geq 0$.

Proof. As in the previous theorem, we consider $v(x,t) = Ae^{\alpha x + (t+t_0)^2}$. Thanks to $p = 2$ and with some appropriate constants A and α, we have

$$\begin{cases} \partial_t v - \partial_x^2 v + v\partial_x v - v^p + \lambda v \geq 0 & \text{in} \quad \Omega \times [0,\infty). \\ v(\cdot, 0) \geq \varphi & \text{in} \quad \Omega. \end{cases}$$

The case $\Omega = \mathbb{R}$ (no boundary) and the case of Dirichlet boundary conditions are trivial. For $\Omega = (-\infty, 0)$ (the boundary is $\{0\}$), we have $\partial_\nu v = \partial_x v = \alpha v > 0$ for $x = 0$. Thus, the Neumann boundary conditions and the dynamical boundary conditions with $\sigma \geq 0$ are verified. □

When $\lambda = 0$, $\Omega = (-\infty, 0)$ and $p > 3$, the Green function of the heat equation is a suitable super-solution for the Problem (1).

Theorem 3.6. *Assume $\Omega = (-\infty, 0)$, $p > 3$, $\lambda = 0$ and $\varphi \in \mathcal{C}(\Omega)$. Then the Problem (1) admits a global positive solution if the initial data φ is sufficiently small and when $\mathcal{B}(u) = 0$ stands for the Dirichlet, the Neumann or the dynamical boundary conditions with $\sigma \geq 0$ constant.*

Proof. Consider the function $v(x,t) = A(t+1)^{-\gamma} e^{\frac{-(x+y)^2}{4t+4}}$ defined in $\mathbb{R}^- \times \mathbb{R}^+$ with $A > 0$, $\gamma = \frac{1}{p-1}$ and $y = -2\sigma\gamma$. A simple calculation leads to

$$\partial_t v - \partial_x^2 v + v\partial_x v - v^p = \frac{v}{2(t+1)}\Big(-2\gamma + 1 - (x+y)v - v^{p-1}\Big).$$

By definition of γ and $p > 3$, we have $-2\gamma + 1 > 0$. Since $v^{p-1} \leq A^{p-1}$, and because $-(x+y) > 0$ for all $x \in \Omega$, we obtain $\partial_t v - \partial_x^2 v + v\partial_x v - v^p \geq 0$ by choosing A small enough. The case of the Dirichlet boundary conditions is clear because $v \geq 0$. For the dynamical boundary conditions and the Neumann boundary conditions ($\sigma \equiv 0$), we use the definition of y and we have

$$\sigma\partial_t v(0,t) + \partial_\nu v(0,t) \geq \frac{v(0,t)}{2(t+1)}\Big(-2\sigma\gamma - y\Big) \geq 0.$$

Thus, v is a super-solution of the Problem (1) as soon as we choose $0 \leq \varphi \leq v(\cdot, 0)$ in Ω.

□

3.3. **Blow up in unbounded domains.** Here, using some weighted L^1-norms, we examine blow-up phenomena for some solutions of Problem (1) in unbounded domains satisfying the Neumann, the Robin, and some nonlinear boundary conditions. We only consider regular solutions satisfying this standard growth order condition at infinity: for all $a > 0$ and for all $t > 0$

$$\lim_{|x|\to\infty} u(x,t)e^{-a|x|} = 0 \text{ and } \lim_{|x|\to\infty} |\partial_x u(x,t)|e^{-a|x|} = 0. \tag{23}$$

Unless otherwise stated, we always suppose $\Omega = (0,\infty)$. We begin with a lemma which gives a criterion for the blowing-up of the solution.

Lemma 3.7. *Let u be a solution of Problem (1) which satisfies the condition (23). If there exists $\alpha > 0$ such that*

$$N_\alpha(t) := \int_0^\infty u(x,t)e^{-\alpha x}\, dx$$

blows-up in finite time, then u also blows-up in finite time.

Proof. Consider $\alpha > 0$ such that N_α blows-up in finite time. Using the following inequality

$$N_\alpha(t) \le \int_0^\infty e^{-\alpha x/2}\, dx \cdot \sup_\Omega u(x,t)e^{-\frac{\alpha}{2}x} = \frac{2}{\alpha}\sup_\Omega u(x,t)e^{-\frac{\alpha}{2}x},$$

and because N_α blows up, we can deduce the blowing up in finite time of the function $u(x,t)e^{-\frac{\alpha}{2}x}$. Then, thanks to the growth order condition (23), the solution u must blow up too. □

We also need this technical lemma.

Lemma 3.8. *Let u be a solution of Problem (1) where the boundary conditions are the Neumann, the Robin, or some nonlinear boundary conditions $\partial_\nu u = g(u)$. Then, for all $\tau > 0$ there exists $c > 0$ such that*

$$u(0,t) \ge c \quad \textit{for all} \quad t \ge \tau\ .$$

Proof. Let u be the positive solution of Problem (1) with one of the above boundary conditions (denoted by $\mathcal{B}(u) = 0$), and with the initial data φ. Let v be the positive solution of the following problem

$$\begin{cases} \partial_t v = \partial_x^2 v - v\partial_x v + v^p - \lambda v & \text{in} \quad [0,1]\times[0,\infty), \\ \mathcal{B}(v) = 0 & \text{on} \quad \{0\}\times[0,\infty), \\ v = 0 & \text{on} \quad \{1\}\times[0,\infty), \\ v(\cdot,0) = \varphi_1 & \text{in} \quad [0,1], \end{cases}$$

where $\mathcal{B}(v) = 0$ denote the same boundary conditions as in $\mathcal{B}(u) = 0$, where $\varphi_1 \in \mathcal{C}^2([0,1])$ satisfies $\varphi_1(1) = 0$, $\partial_x^2\varphi_1 - \varphi_1\partial_x\varphi_1 + \varphi_1^p - \lambda\varphi_1 \ge 0$ and $0 \le \varphi_1 \le \varphi$ in $[0,1]$. We refer to [6] for the existence of v. Thanks to $u(\cdot,0) \ge v(\cdot,0)$ in $[0,1]$ and $u(1,t) \ge 0 = v(1,t)$ for all $t > 0$, the comparison principle from [3] implies

$$u(x,t) \ge v(x,t) \text{ for all } x \in [0,1] \text{ and } t > 0.$$

Then, the comparison principle and the maximum principle from [3] imply

$$\partial_t v(x,t) \ge 0 \text{ and } v(x,t) > 0.$$

for all $x \in [0,1]$ and $t > 0$, see Lemma 2.1 in [5]. Thus, for all $\tau > 0$, we obtain

$$u(0,t) \ge v(0,t) \ge v(0,\tau) > 0 \text{ for all } t \ge \tau.$$

Remark that, we have $v(0,\tau) \ge \varphi(0)$, and if $\varphi(0) > 0$, we can choose $c = \varphi(0)$. □

Theorem 3.9. *Let $\lambda < 0$ and $p \geq 2$. Then the Problem (1) admits no global positive solution when $\mathcal{B}(u) = 0$ stands for the Neumann boundary conditions.*

Proof. We aim to prove the existence of $\alpha > 0$ and $\beta > 0$ such that $N'_\alpha \geq \beta N_\alpha^p$ where

$$N_\alpha(t) := \int_0^\infty u(x,t)e^{-\alpha x}\, dx$$

Derivating the function N_α, we obtain

$$\begin{aligned} N'_\alpha(t) &= \int_0^\infty \partial_t u(x,t) e^{-\alpha x}\, dx \\ &= \int_0^\infty \Big(\partial_x^2 u(x,t)\Big) e^{-\alpha x}\, dx - \int_0^\infty \Big(u(x,t)\partial_x u(x,t)\Big) e^{-\alpha x}\, dx \\ &\quad + \int_0^\infty u^p(x,t)e^{-\alpha x}\, dx - \lambda \int_0^\infty u(x,t)e^{-\alpha x}\, dx. \end{aligned}$$

Using the growth order condition (23) and integrating by parts, we obtain

$$\int_0^\infty \Big(\partial_x^2 u(x,t)\Big)e^{-\alpha x}\, dx = \alpha^2 \int_0^\infty u(x,t)e^{-\alpha x}\, dx + \partial_\nu u(0,t) - \alpha u(0,t)$$

and

$$\int_0^\infty \Big(u(x,t)\partial_x u(x,t)\Big)e^{-\alpha x}\, dx = \frac{\alpha}{2}\int_0^\infty u^2(x,t)e^{-\alpha x}\, dx - \frac{u^2(0,t)}{2}.$$

Thus, we have

$$\begin{aligned} N'_\alpha(t) &= \int_0^\infty u(x,t)e^{-\alpha x}\Big(\alpha^2 - \frac{\alpha}{2}u(x,t) - \lambda + u^{p-1}(x,t)\Big)\, dx \\ &\quad + \partial_\nu u(0,t) - \alpha u(0,t) + \frac{u^2(0,t)}{2}. \end{aligned} \tag{24}$$

Thanks to Lemma 3.8, and considering u from a time $\tau > 0$, we can assume that

$$c := \min_{t>0} u(0,t) > 0\ .$$

Then, if α is small enough ($\alpha \leq c/2$), we have $-\alpha u(0,t) + \frac{u^2(0,t)}{2} \geq 0$. Then, the Neumann boundary conditions imply

$$N'_\alpha(t) \geq \int_0^\infty u(x,t)e^{-\alpha x}\Big(\alpha^2 - \frac{\alpha}{2}u(x,t) - \lambda + u^{p-1}(x,t)\Big)\, dx. \tag{25}$$

Shrinking α, we can suppose $\alpha \leq -2\lambda$ and $\alpha \leq 1$. When $u(x,t) \leq 1$, we have $-\lambda - \alpha u(x,t)/2 > 0$. On the other hand, if $u(x,t) \geq 1$, we have $u^{p-1}(x,t) - \alpha u(x,t)/2 \geq u^{p-1}(x,t)/2$. Hence, we obtain:

$$N'_\alpha(t) \geq \frac{1}{2}\int_0^\infty u^p(x,t)e^{-\alpha x}\, dx.$$

Hölder inequality

$$\int_0^\infty u(x,t)e^{-\alpha x}\, dx \leq \left(\int_0^\infty u^p(x,t)e^{-\alpha x}\, dx\right)^{\frac{1}{p}} \left(\int_0^\infty e^{-\alpha x}\, dx\right)^{\frac{p-1}{p}}$$

leads to $N'_\alpha(t) \geq \beta N_\alpha^p(t)$ with

$$\beta = \frac{1}{2}\left(\int_0^\infty e^{-\alpha x}\, dx\right)^{1-p}.$$

Finally, we prove the blowing-up of N_α in finite time. Integrating the differential inequality $N'_\alpha(t) \geq \beta N^p_\alpha(t)$ between 0 and $t > 0$, we obtain

$$\frac{1}{1-p}\Big(N_\alpha^{1-p}(t) - N_\alpha^{1-p}(0)\Big) = \int_{N_\alpha(0)}^{N_\alpha(t)} s^{-p}\ ds = \int_0^t \frac{N'_\alpha(t)}{N^p_\alpha(t)}\ dt \geq \beta t,$$

and

$$N_\alpha(t) \geq \Big(N_\alpha^{1-p}(0) - (p-1)\beta t\Big)^{\frac{-1}{p-1}}.$$

Since of $\frac{-1}{p-1} < 0$, the right hand side term blows up at $t = \frac{N_\alpha^{1-p}(0)}{(p-1)\beta} > 0$. We conclude with Lemma 3.7. □

Corollary 3. *Let $\lambda < 0$ and $p \geq 2$. Then the Problem (1) admits no global positive solution when $\mathcal{B}(u) = 0$ stands for the nonlinear boundary conditions $\partial_\nu u = g(u)$, where g is a function such that there exists $\delta > 0$ and $\varepsilon \leq 1/2$ satisfying*

$$g(\eta) \geq \delta\eta - \varepsilon\eta^2.$$

Proof. We follow the proof of Theorem 3.9. We just change the choice of α: let $\alpha > 0$ such that $\alpha \leq \delta$, and use the following minoration in Equation (24):

$$\begin{aligned}\partial_\nu u(0,t) - \alpha u(0,t) + \frac{1}{2}u^2(0,t) =& g(u) - \alpha u(0,t) + \frac{1}{2}u^2(0,t)\\ \geq& (\delta-\alpha)u(0,t) + (\frac{1}{2} - \varepsilon)u^2(0,t) \geq 0.\end{aligned}$$

Then, we return to Equation (25) and we can prove that there exists a $\beta > 0$ such that $N'_\alpha(t) \geq \beta N^p_\alpha(t)$ for $t \in (0,T)$. □

When $\lambda = 0$, the choice of α is too strict. Meanwhile, we obtain some blow-up results imposing some restrictions on the exponent p and on the initial data.

Theorem 3.10. *Let $\lambda = 0$ and $1 < p \leq 3$. Then the Problem (1) admits no global positive solution when $\mathcal{B}(u) = 0$ stands for the Neumann boundary conditions.*

Proof. Return to the proof of Theorem 3.9. Under the Neumann boundary conditions and with $\lambda = 0$, Equation (24) becomes

$$N'_\alpha(t) = \int_0^\infty u(x,t)e^{-\alpha x}\Big(\alpha^2 - \frac{\alpha}{2}u(x,t) + u^{p-1}(x,t)\Big)\ dx - \alpha u(0,t) + \frac{u^2(0,t)}{2}.$$

Let $\beta \in (0,1)$ and put it into the previous equation:

$$\begin{aligned}N'_\alpha(t) =& \int_0^\infty u(x,t)e^{-\alpha x}\Big(\alpha^2 - \frac{\alpha}{2}u(x,t) + \beta u^{p-1}(x,t)\Big)\ dx\\ &- \alpha u(0,t) + \frac{u^2(0,t)}{2} + (1-\beta)\int_0^\infty u^p(x,t)e^{-\alpha x}\ dx.\end{aligned}$$

If $u \leq 2\alpha$, we have $\alpha^2 - \alpha u/2 \geq 0$, whereas $u > 2\alpha$ implies

$$-\frac{\alpha}{2}u + \beta u^{p-1} \geq u\Big(-\frac{\alpha}{2} + \beta(2\alpha)^{p-2}\Big).$$

It is non negative if

$$\beta\alpha^{p-3} \geq 2^{1-p}. \tag{26}$$

Thanks to $1 < p \leq 3$, Equation (26) is achieved by choosing $\alpha > 0$ sufficiently small and $\beta \in (0,1)$ depending on p. Thus, we obtain

$$N'_\alpha(t) \geq -\alpha u(0,t) + \frac{u^2(0,t)}{2} + (1-\beta)\int_0^\infty u^p(x,t)e^{-\alpha x}\ dx.$$

Then, we can suppose that $u(0,t) > c > 0$ for all $t > 0$ (see Lemma 3.8), and with $\alpha < c/2$ we have $-\alpha u(0,t) + \frac{u^2(0,t)}{2} > 0$. Hence

$$N'_\alpha(t) \geq (1-\beta)\int_0^\infty u^p(x,t)e^{-\alpha x}\ dx.$$

As in the proof of Theorem 3.9, we use Hölder inequality and we are led to $N'_\alpha \geq \delta N_\alpha^p$ with $\delta > 0$ depending on α, β and p. Hence, N_α blows-up in finite time, so does the solution u, see Lemma 3.7. □

Theorem 3.11. *Let $\lambda = 0$ and $p > 3$. Then the Problem (1) admits no global positive solution when $\mathcal{B}(u) = 0$ stands for the Neumann boundary conditions and if the initial data satisfies $\varphi(0) > 2^{\frac{1-p}{p-3}}$.*

Proof. The proof is similar to the previous one. Go back to Equation (26): since $p > 3$, we must choose α such that

$$\alpha \geq 2^{\frac{1-p}{p-3}}\beta^{\frac{-1}{p-3}}.$$

Under this condition, N_α satisfies the differential inequality

$$N'_\alpha(t) \geq -\alpha u(0,t) + \frac{u^2(0,t)}{2} + (1-\beta)\int_0^\infty u^p(x,t)e^{-\alpha x}\ dx.$$

Because α can not be too small, we must use the assumption $\varphi(0) > 2^{\frac{1-p}{p-3}}$. Using Lemma 3.8, we have

$$u(0,t) \geq \varphi(0) > 2^{\frac{1-p}{p-3}}\ \text{, for all } t > 0.$$

Thus, with β very close to 1 and with $\alpha = 2^{\frac{1-p}{p-3}}\beta^{\frac{-1}{p-3}}$, we obtain $-\alpha u(0,t) + \frac{u^2(0,t)}{2} \geq 0$. Hence, we have

$$N'_\alpha(t) \geq (1-\beta)\int_0^\infty u^p(x,t)e^{-\alpha x}\ dx.$$

We conclude with Hölder inequality and the blowing up of N_α. □

Corollary 4. *Let $\lambda = 0$ and $p > 3$. Then the Problem (1) admits no global positive solution when $\mathcal{B}(u) = 0$ stands for the Neumann boundary conditions and if the initial data satisfies*

$$\int_0^\infty \varphi(x)e^{-x}\ dx > \frac{1}{2}. \tag{27}$$

Proof. Return to the proof of Theorem 3.9. Under the Neumann boundary conditions and introducing β and $\delta \in (0,1)$ in Equation (24), we obtain

$$\begin{aligned} N'_\alpha(t) = &\int_0^\infty u(x,t)e^{-\alpha x}\Big(\delta\alpha^2 - \frac{\alpha}{2}u(x,t) + \beta u^{p-1}(x,t)\Big)\ dx \\ &- \alpha u(0,t) + \frac{u^2(0,t)}{2} + (1-\delta)\alpha^2 N_\alpha(t) + (1-\beta)\int_0^\infty u^p(x,t)e^{-\alpha x}\ dx. \end{aligned}$$

Studying both cases $u \geq 2\alpha\delta$ and $u \leq 2\alpha\delta$, we obtain $\delta\alpha^2 - \alpha u/2 + \beta u^{p-1} \geq 0$ if

$$\alpha = 2^{\frac{1-p}{p-3}}\beta^{\frac{-1}{p-3}}\delta^{\frac{2-p}{p-3}}.$$

Since of $u^2/2 - \alpha u \geq -\alpha^2/2$ and using Hölder inequality we have

$$N'_\alpha(t) \geq (1-\delta)\alpha^2 N_\alpha(t) + \gamma N_\alpha^p(t) - \frac{\alpha^2}{2}, \tag{28}$$

where $\gamma = (1-\beta)\Big(\int_0^\infty e^{-\alpha x}\, dx\Big)^{1-p} > 0$. First, consider this minoration

$$N_\alpha'(t) \geq (1-\delta)\alpha^2 N_\alpha(t) - \frac{\alpha^2}{2}.$$

Thus, N_α satisfies

$$N_\alpha(t) \geq \frac{1}{2(1-\delta)} + Ae^{(1-\delta)\alpha^2 t},\ A \in \mathbb{R}.$$

In particular, $N_\alpha(0) \geq (2-2\delta)^{-1} + A$. Choosing $\delta > 0$ close to 0 and with $\beta \in (0,1)$ close to 1, Hypothesis (27) implies $N_\alpha(0) > (2-2\delta)^{-1}$. Thus, A is positive and we obtain

$$(1-\delta)\alpha^2 N_\alpha(t) - \frac{\alpha^2}{2} \geq 0.$$

From Equation (28), we deduce

$$N_\alpha'(t) \geq \gamma N_\alpha^p(t).$$

Hence N_α blows-up, and the solution u blows up too, see Lemma 3.7. □

Finally, if $\Omega = (-\infty, 0)$, we must change the weight in N_α and we obtain this results concerning the nonlinear boundary conditions.

Theorem 3.12. *Let $\lambda \leq 0$ and $p \geq 2$. Then the Problem (1) admits no global positive solution when $\mathcal{B}(u) = 0$ stands for the nonlinear boundary conditions $\partial_\nu u = g(u)$, where g is a function such that there exists $c > 0$ and $d > 0$ satisfying*

$$g(\eta) \geq c\eta^2 + d\eta.$$

Proof. As in the case of $\Omega = (0,\infty)$, we use a weighted L^1-norm:

$$N_\alpha(t) = \int_{-\infty}^0 u(x,t)e^{\alpha x}\, dx\ , \quad \text{with } \alpha > 0.$$

We compute $N_\alpha'(t) = \int_{-\infty}^0 \partial_t u(x,t)e^{\alpha x}\, dx$, and using the equations of Problem (1), integration by parts leads to

$$N_\alpha'(t) = \int_{-\infty}^0 (\alpha^2 u + \alpha u^2 + u^p)e^{\alpha x}\, dx + \partial_x u(0,t) - \alpha u(0,t) - \frac{\alpha}{2}u^2(0,t).$$

Thanks to $\partial_\nu u(0,t) = \partial_x u(0,t)$ in $(-\infty,0)$, choosing $\alpha = \min\{2c,\ d\}$, we obtain

$$N_\alpha'(t) \geq \int_{-\infty}^0 (\alpha^2 u + \alpha u^2 + u^p)e^{\alpha x}\, dx \geq \int_{-\infty}^0 u^p e^{\alpha x}\, dx.$$

Hölder inequality leads to the differential equation $N_\alpha'(t) \geq \gamma N_\alpha^p(t)$ with $\gamma > 0$. Hence N_α and the solution u blow up in finite time. □

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