

# Hermitian $(a, b)$ -modules and Saito's “higher residue pairings”

Piotr P. Karwasz\*

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## Abstract

Following the work of Daniel BARLET ([Bar97]) and Ridha BELGRADE ([Bel01]) the aim of this article is the study of the existence of  $(a, b)$ -hermitian forms on regular  $(a, b)$ -modules. We show that every regular  $(a, b)$ -module  $E$  with a non-degenerate bilinear form can be written in a unique way as a direct sum of  $(a, b)$ -modules  $E_i$  that admit either an  $(a, b)$ -hermitian or an  $(a, b)$ -anti-hermitian form or both; all three cases are equally possible with explicit examples.

As an application we extend the result in [Bel01] on the existence for all  $(a, b)$ -modules  $E$  associated with the Brieskorn module of a holomorphic function with an isolated singularity, of an  $(a, b)$ -bilinear non degenerate form on  $E$ . We show that with a small transformation Belgrade's form can be considered  $(a, b)$ -hermitian and that the result satisfies the axioms of Kyoji Saito's “higher residue pairings”.

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## 1 Introduction

In this article we will study the self-duality properties of  $(a, b)$ -modules and more precisely the conditions under which they admit a nondegenerate hermitian form. As such we wish to provide the reader with a short introduction to the theory of  $(a, b)$ -modules.

The  $(a, b)$ -modules were introduced by D. BARLET in [Bar93] as a formal completion of the Brieskorn module ([Bri70])

$$D := \frac{\Omega_0^{n+1}}{df \wedge d\Omega_0^{n-1}}$$

associated to a holomorphic function  $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$  with an isolated singularity at the origin, where we denote by  $\Omega_0^p$  the germs of holomorphic  $p$ -forms in 0.

We wish to recall briefly the basic results about  $(a, b)$ -modules and refer the reader to the articles [Bar93] and [Bar97] for further details.

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\*Institut Élie Cartan Nancy UMR 7502  
Nancy-Université, B.P. 70230, 54506 Vandœuvre-lès-Nancy, France  
E-mail: piotr.karwasz@iecn.u-nancy.fr

**Definition 1.** Let  $\mathbb{C}[[b]]$  be the ring of formal series in the variable  $b$ . An **(a, b)-module** is an algebraic structure composed by a free  $\mathbb{C}[[b]]$ -module  $E$  of finite rank and a  $\mathbb{C}$ -linear application  $a : E \rightarrow E$  that satisfies the commutation relation

$$ab - ba = b^2, \quad (1)$$

where  $b : E \rightarrow E$  is the multiplication by the element  $b \in \mathbb{C}[[b]]$ .

For a complex number  $\lambda \in \mathbb{C}$  and an  $(a, b)$ -module  $E$ , we call **monomial of type**  $(\lambda, 0)$ , an element  $x \in E$  that satisfies to the relation  $ax = \lambda bx$ . The simplest  $(a, b)$ -modules are those generated over  $\mathbb{C}[[b]]$  by a monomial  $e_\lambda$  of type  $(\lambda, 0)$ . These modules are called **elementary** and noted  $E_\lambda$ .

Given an  $(a, b)$ -module  $E$ , a sub- $\mathbb{C}[[b]]$ -module  $F$  of  $E$  closed to the multiplication by  $a$  is called sub- $(a, b)$ -modules. Since the quotient of an  $(a, b)$ -module  $E$  by a sub- $(a, b)$ -module  $F$  is not necessarily  $b$ -torsion free, a sub- $(a, b)$ -module  $F$  of  $E$  will be called **normal**, if  $E/F$  is free and hence have an induced  $(a, b)$ -module structure.

The  $(a, b)$ -modules associated to a Brieskorn module are all **regular**, i.e. they are sub- $(a, b)$ -modules an  $(a, b)$ -module  $E$  satisfying  $aE \subset bE$  (a **simple-pole**  $(a, b)$ -module). The composition series of regular  $(a, b)$ -module satisfy the following property:

**Proposition 2.** Let  $E$  be a regular  $(a, b)$ -module, then all its composition series are of the form:

$$0 = E_0 \subsetneq \cdots \subsetneq E_{n-1} \subsetneq E_n = E,$$

with  $E_i/E_{i-1}$  elementary  $(a, b)$ -modules  $E_\lambda$ .

As proven in [Bar93], the quotients of two composition series of an  $(a, b)$ -module  $E$  are not unique, even if we ignore consider the permutations of the quotients.

## 2 The $(a, b)$ -modules and their duality

The dual and bi-dual structure on  $(a, b)$ -modules were first introduced in [Bar97] and [Bel01] and then expanded in our thesis (cf. [Kar09]). We will therefore begin by giving a formal definition of the duality structures we work with.

In the spirit of the category theory we will define an **(a, b)-morphism** as an application  $\varphi : E \rightarrow F$  between two  $(a, b)$ -modules  $E$  and  $F$ , which is a morphism of the underlying  $\mathbb{C}[[b]]$ -modules and respects the  $a$ -structure  $\varphi(ax) = a\varphi(x)$ , for any element  $x \in E$ . We will call  $\varphi$  an **isomorphism** (resp. **endomorphism**) of  $(a, b)$ -modules if it is bijective (resp.  $E = F$ ).

### 2.1 $(a, b)$ -linear maps and dual $(a, b)$ -modules

Let  $E$  and  $F$  be two  $(a, b)$ -modules. As defined by D. Barlet in [Bar97], the  $\mathbb{C}[[b]]$ -module  $\text{Hom}_{\mathbb{C}[[b]]}(E, F)$  of  $\mathbb{C}[[b]]$ -linear maps from  $E$  to  $F$  has a natural structure of  $(a, b)$ -module provided by an operator  $\Lambda$  that satisfies

$$(\Lambda\varphi)(x) = a_F(\varphi(x)) - \varphi(a_E x), \quad (2)$$

where  $\varphi \in \text{Hom}_{\mathbb{C}[[b]]}(E, F)$ ,  $x$  is an element of  $E$  and  $a_E$  and  $a_F$  are the  $a$ -structures of  $E$  and  $F$  respectively. We use for this  $(a, b)$ -module the notation  $\text{Hom}_{(a,b)}(E, F)$ .

For notation's sake we will denote  $a_E$ ,  $a_F$  and  $\Lambda$  all by the letter  $a$  and to avoid the confusion that such a notation could pose we should read the expression

$$a \cdot \varphi(x)$$

as  $(\Lambda\varphi)(x)$ , whereas the expression  $a_E(\varphi(x))$  will keep the conventional notation

$$a\varphi(x).$$

We will therefore rewrite the equation 2 as:  $a \cdot \varphi(x) = a\varphi(x) - \varphi(ax)$ .

By choosing  $E_0$  for the codomain of the morphisms, we can give the following definition:

**Definition 3** (Barlet). *Let  $E$  be an  $(a, b)$ -module and  $E_0$  the elementary  $(a, b)$ -module of parameter 0, then we call the module*

$$\text{Hom}_{(a,b)}(E, E_0)$$

the **dual  $(a, b)$ -module** of  $E$  and note it by  $E^*$ .

*Remark 4.* When considering only the  $b$ -structure of  $E$ , the  $\mathbb{C}[[b]]$ -module  $E^*$  corresponds exactly to the definition of dual of a  $\mathbb{C}[[b]]$ -module, since  $E_0 = \mathbb{C}[[b]]e_0$ , with  $ae_0 = 0$ .

The duality functor  $*$  is exact (cf. [Bar97]).

## 2.2 Conjugate $(a, b)$ -module

In [Bel01] R. BELGRADE uses another definition of dual  $(a, b)$ -module which is not equivalent to the one of D. BARLET. In order to be able to express on concept in terms of the other the other, we will introduce an operation that exchanges the signs of both  $a$  and  $b$ , whose behaviour is similar to that of conjugation in the complex field.

As in the case of the complex field  $\mathbb{C}$ , the ring of formal series  $\mathbb{C}[[b]]$  also admits a rather natural involution

$$\begin{aligned} \check{\cdot} : \mathbb{C}[[b]] &\rightarrow \mathbb{C}[[b]] \\ S(b) &\mapsto \check{S}(b) = S(-b), \end{aligned}$$

where  $S(b) \in \mathbb{C}[[b]]$ . This remark allows us to define the conjugate of an  $(a, b)$ -module in the same way as one defines the conjugate of a complex vector space.

**Definition 5.** *Let  $E$  be an  $(a, b)$ -module. We call  **$(a, b)$ -conjugate** of  $E$  and note it  $\check{E}$  the set  $E$  itself, endowed with an  $a$ - and  $b$ -structure given by:*

$$\begin{aligned} a \cdot_{\check{E}} v &= -a \cdot_E v \\ b \cdot_{\check{E}} v &= -b \cdot_E v, \end{aligned}$$

where  $\cdot_{\check{E}}$  and  $\cdot_E$  denote the  $(a, b)$ -structure of  $\check{E}$  and  $E$  respectively.

Since we change signs of both  $a$  and  $b$ , the formula  $ab - ba = b^2$  is still satisfied.

*Remark 6.* An  $(a, b)$ -module is not necessarily isomorphic to its conjugate. We can take, for example, the  $(a, b)$ -module of rank 2 generated by two elements  $x$  and  $y$  that satisfy:

$$\begin{aligned} ax &= \lambda bx \\ ay &= \lambda by + (1 + \alpha b)x, \end{aligned}$$

where  $\lambda$  and  $\alpha \in \mathbb{C}$  and  $\alpha \neq 0$ . Its conjugate satisfies

$$\begin{aligned} ax &= \lambda bx \\ ay &= \lambda by + (1 - \alpha b)x, \end{aligned}$$

and the classification of rank 2 regular  $(a, b)$ -modules, given in [Bar93] implies that the two modules are not isomorphic.

One can see immediately that for an  $(a, b)$ -module  $E$  the conjugate of the conjugate  $(\check{E})^\check{\phantom{E}}$  is the  $(a, b)$ -module itself.

On the other hand let  $E$  and  $F$  be  $(a, b)$ -modules and  $\varphi$  a morphism between  $E$  and  $F$ . Since  $\varphi(-ax) = -a\varphi(x)$  and  $\varphi(-bx) = -b\varphi(x)$ , for all  $x \in E$  the application  $\varphi$  is also a morphism between the conjugates  $\check{E}$  and  $\check{F}$ . We call conjugation functor the functor that associates to every  $(a, b)$ -module its conjugate and to every morphism, the morphism itself. Such a functor is exact.

For an  $(a, b)$ -module  $E$  we will be interested especially in a particular kind of conjugate, the conjugate of the dual, which we call **adjoint**  $(a, b)$ -module and note with  $\check{E}^*$ .

### 2.3 Bilinear forms and tensor product

In order to define  $\text{Hom}_{(a,b)}(E, F)$  we used the equivalent object for its underlying  $b$ -structure. We can proceed in a similar way to obtain the concept of  $(a, b)$ -bilinear maps:

**Definition 7.** Let  $E, F$  and  $G$  be two  $(a, b)$ -modules. An  $(a, b)$ -**bilinear map** on  $E \times F$  is a  $\mathbb{C}[[b]]$ -linear map  $\Phi$ ,

$$\Phi : E \times F \rightarrow G,$$

that satisfies the following property:

$$a\Phi(x, y) = \Phi(ax, y) + \Phi(x, ay).$$

*Remark 8.* If  $\Phi$  is an  $(a, b)$ -bilinear map on  $E \times F$  with values in  $G$  and  $v$  is an element of  $E$ :

$$\Phi_v := \Phi(v, \cdot) : w \mapsto \Phi(v, w) \quad w \in F$$

is not necessarily an  $(a, b)$ -morphism. However the map  $\pi : v \mapsto \Phi_v$  is an  $(a, b)$ -morphism between  $E$  and  $\text{Hom}_{(a,b)}(F, G)$ . We have in fact:

$$\pi(av)(x) = \Phi_{av}(x) = a\Phi_v(x) - \Phi_v(ax) = a \cdot \Phi_v(x) = a\pi(v).$$

Inherently linked to the concept of  $(a, b)$ -bilinear maps is that of tensor products, that allows a more practical manipulation of these objects.

**Definition 9.** Let  $E$  and  $F$  be two  $(a, b)$ -modules. We call  $(a, b)$ -tensor product of  $E$  and  $F$  and write it as  $E \otimes_{(a,b)} F$  the  $\mathbb{C}[[b]]$ -module

$$E \otimes_{\mathbb{C}[[b]]} F$$

endowed with an  $a$ -structure defined as follows:

$$a(v \otimes w) = (av) \otimes w + v \otimes (aw)$$

for every  $v \in E$  and  $w \in F$ .

The  $a$ -structure we gave on  $E \otimes_{(a,b)} F$  is well defined. We have in fact:

$$\begin{aligned} a(bv \otimes w) &= a(bv) \otimes w + bv \otimes a(w) = ba(v) \otimes w + b^2v \otimes w + v \otimes ba(w) = \\ &= a(v) \otimes bw + v \otimes a(bw) = a(v \otimes bw), \end{aligned}$$

for each  $v \in E$ ,  $w \in F$  and it satisfies  $ab - ba = b^2$ :

$$\begin{aligned} a(bv \otimes w) - ba(v \otimes w) &= ba(v) \otimes w + b^2v \otimes w + bv \otimes a(w) \\ &\quad - ba(v) \otimes w - bv \otimes a(w) = b^2(v \otimes w). \end{aligned}$$

We can easily verify that the tensor product defined satisfies the usual universal property: there exists a bilinear map

$$\Phi : E \times F \rightarrow E \otimes_{(a,b)} F,$$

such that for every bilinear map  $\Psi$  on  $E \times F$  with values in a third  $(a, b)$ -module  $G$ , there exists a unique  $(a, b)$ -morphism  $\tilde{\Psi}$  from  $E \otimes_{(a,b)} F$  into  $G$  that makes the following diagram commutative:

$$\begin{array}{ccc} E \times F & \xrightarrow{\Psi} & G \\ \downarrow \Phi & \searrow \tilde{\Psi} & \nearrow \\ E \otimes_{(a,b)} F & & \end{array}$$

We can take as  $\Phi$  the natural application

$$\begin{aligned} \Phi : E \times F &\rightarrow E \otimes_{(a,b)} F \\ (v, w) &\mapsto v \otimes_{(a,b)} w \end{aligned}$$

and define  $\tilde{\Psi}$  as:

$$\begin{aligned} \tilde{\Psi} : E \otimes_{(a,b)} F &\rightarrow G \\ v \otimes_{(a,b)} w &\mapsto \Psi(v, w) \end{aligned}$$

The unicity of  $\tilde{\Psi}$  follows directly from the universal property of the tensor product of  $\mathbb{C}[[b]]$ -modules. We need only to verify that the map is  $a$ -linear. We will do it on the generators  $v \otimes_{(a,b)} w$  of  $E \otimes_{(a,b)} F$ , for  $v \in E$  and  $w \in F$ :

$$\begin{aligned} \tilde{\Psi}(a(v \otimes_{(a,b)} w)) &= \tilde{\Psi}((av) \otimes_{(a,b)} w + v \otimes_{(a,b)} (aw)) = \\ &= \Psi(av, w) + \Psi(v, aw) = a\Psi(v, w) = a\tilde{\Psi}(v \otimes_{(a,b)} w). \end{aligned}$$

Basing ourselves on the properties of the tensor product of  $\mathbb{C}[[b]]$ -modules, we can derive in a similar manner the other properties of the equivalent object in the theory  $(a, b)$ -modules.

**Lemma 10.** *Let  $E$ ,  $F$  and  $G$  be three  $(a, b)$ -modules, then the tensor product verifies the following properties:*

- (i) 
$$E \otimes_{(a,b)} F \simeq F \otimes_{(a,b)} E,$$
- (ii) 
$$(E \otimes_{(a,b)} F) \otimes_{(a,b)} G \simeq E \otimes_{(a,b)} (F \otimes_{(a,b)} G),$$
- (iii) 
$$(E \otimes_{(a,b)} F)^* \simeq E^* \otimes_{(a,b)} F^*,$$
- (iv) 
$$(E \otimes_{(a,b)} F)^\vee \simeq \check{E} \otimes_{(a,b)} \check{F},$$
- (v) *The  $(a, b)$ -morphism*

$$\begin{aligned} \Phi : E &\rightarrow E \otimes_{(a,b)} E_0 \\ v &\mapsto v \otimes_{(a,b)} e_0 \end{aligned}$$

where  $e_0$  is a generator of the elementary  $(a, b)$ -module  $E_0$ , is an isomorphism.

(vi) *We have the following isomorphism of  $(a, b)$ -modules:*

$$\begin{aligned} E^* \otimes_{(a,b)} F &\rightarrow \text{Hom}_{(a,b)}(E, F \otimes_{(a,b)} E_0) \\ \varphi \otimes_{(a,b)} y &\mapsto (\Phi : x \mapsto y \otimes_{(a,b)} \varphi(x)), \end{aligned}$$

where  $\varphi \in E^*$ ,  $x \in E$  and  $y \in F$ .

*Remark 11.* In [Bel01], R. BELGRADE defines the concept of  $\delta$ -**dual** of an  $(a, b)$ -module  $E$ :

*Definition 12* (Belgrade). *Let  $E$  be an  $(a, b)$ -module and  $\delta \in \mathbb{C}$ , then we call the  $\delta$ -dual of  $E$  the set  $\text{Hom}_{\mathbb{C}[[b]]}(E, E_\delta)$  with the  $(a, b)$ -structure defined as follow:*

$$[a \cdot \varphi](x) = \varphi(ax) - a\varphi(x) \tag{3}$$

$$[b \cdot \varphi](x) = -b\varphi(x) = \varphi(-bx) \tag{4}$$

From property (v) and (vi) of the previous lemma we obtain the isomorphism  $E^* \otimes_{(a,b)} F \simeq \text{Hom}_{(a,b)}(E, F)$ , which in turn let us find an alternative description of the  $\delta$ -dual of an  $(a, b)$ -module. In fact from definition 12 it is easy to show that the  $\delta$ -dual of an  $(a, b)$ -module in Belgrade's terminology is the module

$$\text{Hom}_{(a,b)}(\check{E}, E_\delta),$$

which in turn can be rewritten as  $\check{E}^* \otimes_{(a,b)} E_\delta$ .

We will call an  $(a, b)$ -bilinear application on  $E \times F$  with values in  $G$ , an  $(a, b)$ -**bilinear form** if  $G = E_0$ . In the rest of this section we will deal with the existence of nondegenerate hermitian forms on  $(a, b)$ -modules. We will need therefore the following definitions.

**Definition 13.** Let  $E$  and  $F$  be two  $(a, b)$ -modules and  $\Phi$  a bilinear form on  $E \times F$ . We say that  $\Phi$  is **nondegenerate**, if the  $(a, b)$ -morphism  $v \mapsto \Phi(v, \cdot)$  is an isomorphism of  $E$  with  $F^*$ .

**Definition 14.** Let  $E$  be an  $(a, b)$ -module. A **sesquilinear** form on  $E$  is a bilinear form on  $E \times \check{E}$ .

*Remark 15.* Since a nondegenerate sesquilinear form on an  $(a, b)$ -module  $E$  induces an isomorphism to its adjoint  $\check{E}^*$  it follows that not all  $(a, b)$ -modules are self-adjoint (e.g.  $E_\lambda$  with  $\lambda \neq 0$  is not) not every  $(a, b)$ -module admits a nondegenerate sesquilinear form.

Consider now a sesquilinear form  $\Phi$  on  $E$ . By applying to it the conjugate functor we obtain a bilinear map  $\check{\Phi}$  on  $\check{E} \times E$  with values in  $\check{E}_0$ . If we fix an isomorphism of  $\check{E}_0$  with  $E_0$ , we can consider  $\check{\Phi}$  as a sesquilinear form on  $\check{E}$ . Under this assumption, we define  $(a, b)$ -hermitian and anti- $(a, b)$ -hermitian forms as:

**Definition 16.** Let  $E$  be an  $(a, b)$ -module. An  $(a, b)$ -sesquilinear form  $H$  on  $E$  is called  **$(a, b)$ -hermitian** (respectively **anti- $(a, b)$ -hermitian**) if it satisfies:

$$H(v, w) = \check{H}(w, v),$$

(respectively

$$H(v, w) = -\check{H}(w, v)).$$

where  $v \in E$ ,  $w \in \check{E}$  and  $\check{H}$  is the sesquilinear form on  $\check{E}$  defined above.

We have already shown that in order to admit a nondegenerate sesquilinear form, an  $(a, b)$ -module must be self-adjoint. We will refine the concept of self-adjoint by defining:

**Definition 17.** Let  $E$  be a self-adjoint  $(a, b)$ -module. We say that  $E$  is **hermitian** (resp. **anti-hermitian**), if it admits a nondegenerate hermitian (resp. anti-hermitian) form.

Let  $E$  be an  $(a, b)$ -module endowed with an hermitian form and let  $\Phi : E \rightarrow \check{E}^*$  be the linear form associated to the hermitian form via the remark 8.

We can translate the hermitian property into the identity between  $\Phi$  and its adjoint  $\check{\Phi}^* : E \rightarrow \check{E}^*$ . In fact while  $\Phi(v)$ , for  $v \in E$  is the linear map:

$$\varphi : w \mapsto \Phi(v, w), \quad w \in \check{E},$$

the adjoint map  $\check{\Phi}^*$  sends an element  $v \in E = E^{**}$  to the map:

$$\varphi : w \mapsto v(\check{\Phi}(w, \cdot)) = \check{\Phi}(w, v).$$

We will use this formulation extensively in the following section.

Note moreover that given an isomorphism  $\Phi$  from an  $(a, b)$ -module  $E$  and its  $\delta$ -dual  $\check{E}^* \otimes_{(a, b)} E_\delta$  is equivalent to specifying an isomorphism between  $E \otimes_{(a, b)} E_{-\delta/2}$  and

$$\check{E}^* \otimes_{(a, b)} E_\delta \otimes_{(a, b)} E_{-\delta/2} \simeq \check{E}^* \otimes_{(a, b)} E_{\delta/2}.$$

Since we have

$$(E \otimes_{(a, b)} \widetilde{E_{-\delta/2}})^* \simeq \check{E}^* \otimes_{(a, b)} \check{E}_{-\delta/2} \simeq \check{E}^* \otimes_{(a, b)} E_{\delta/2},$$

we can identify an isomorphism of  $E$  with its  $\delta$ -dual with an hermitian form on  $E \otimes_{(a, b)} E_{-\delta/2}$ .

### 3 Existence of hermitian forms

We will analyze in this section the existence of nondegenerate hermitian forms on regular  $(a, b)$ -modules. We will proceed in two steps: in the first two subsections we will reduce ourselves to a subclass of  $(a, b)$ -modules called indecomposable  $(a, b)$ -modules and show that every regular  $(a, b)$ -module can be decomposed into the direct sum of indecomposable ones and that this decomposition is unique (theorem 24).

In the last subsection we will show that a self-adjoint  $(a, b)$ -module which is indecomposable admits at least an hermitian or anti-hermitian form. The result is optimal since there are examples that admit only an hermitian or only an anti-hermitian form (theorem 30).

#### 3.1 Indecomposable $(a, b)$ -modules

**Definition 18.** *Let  $E$  be an  $(a, b)$ -module. We say that  $E$  is **indecomposable** if it cannot be written as direct sum  $F \oplus G$  of non zero  $(a, b)$ -modules.*

Whenever we decompose an  $(a, b)$ -module  $E$  into a direct sum of two  $(a, b)$ -modules  $E = F \oplus G$  the rank of the components is strictly less than the rank of  $E$ , hence by proceeding by induction for every  $(a, b)$ -module  $E$  we can find a decomposition into a sum of indecomposable  $(a, b)$ -modules:

$$E = \bigoplus_{i=1}^r F_i,$$

where  $r \in \mathbb{N}$  and  $F_i$  are indecomposable sub- $(a, b)$ -modules.

We are interested in the question whether the isomorphism classes of the  $F_i$  are unique and do not depend upon the decomposition. We will need to this purpose an introductory result:

**Proposition 19.** *Let  $E$  be a regular and indecomposable  $(a, b)$ -module. Then every endomorphism of  $E$  is either bijective or nilpotent.*

The proof of this proposition will need several steps beginning with a definition:

**Definition 20.** *Let  $E$  be a regular  $(a, b)$ -module and  $\lambda \in \mathbb{C}$ . We define*

$$V_\lambda = \left\{ \sum F_i \mid F_i \subset E, F_i \simeq E_\lambda \right\}$$

*to be the sum of all sub- $(a, b)$ -modules of  $E$  isomorphic to  $E_\lambda$ .*

The object  $V_\lambda$  is clearly a sub- $(a, b)$ -module. We will use  $V_\lambda$  as an induction step in the proof of proposition 19, by choosing a  $\lambda$  such that  $V_\lambda$  is normal:

**Proposition 21.** *Let  $E$  be a regular  $(a, b)$ -module,  $\lambda \in \mathbb{C}$  and:*

$$\lambda_{min} = \inf_j \{ \lambda + j \mid \exists x \in E, ax = (\lambda + j)bx \}$$

*be the minimal  $\lambda + j$  such that  $E$  contains a monome of type  $(\lambda + j, 0)$ .*

*Then  $V_{\lambda_{min}}$  is a normal sub- $(a, b)$ -module of  $E$  isomorphic as  $(a, b)$ -module to the direct sum of a finite number of copies of  $E_{\lambda_{min}}$ .*

*Proof.* We will use two facts.

First, for every  $W \simeq \bigoplus E_{\lambda_{min}}$  sub- $(a, b)$ -module of  $E$ ,  $W$  is normal in  $E$ . Let in fact  $\{e_i\}$  be a basis of  $W$  with  $1 \leq i \leq p$  the rank of  $W$ . Suppose by absurd that there exist some  $x \in W$  which is in  $bE$ , but not in  $bW$ .

By eventually translating  $x$  by an element of  $bW$ , we can assume  $x = \sum_{i=1}^p \alpha_i e_i$ ,  $\alpha_i \in \mathbb{C}$ . We can easily verify that  $ax = \lambda_{min}bx$  but now if  $x = by$  we must have  $ay = (\lambda_{min} - 1)by$ , and since  $y \in E$  it contradicts the minimality of  $\lambda_{min}$ .

On the other hand we can show that  $V_{\lambda_{min}}$  is a direct sum of  $E_{\lambda_{min}}$ . In fact let  $W$  be the largest (inclusionwise) direct sum of copies of  $E_{\lambda_{min}}$  included in  $V_{\lambda_{min}}$ . We remark that since  $W$  is normal, for any sub- $(a, b)$ -module  $F$  isomorphic to  $E_{\lambda_{min}}$  only one of two cases is possible: either

$$W \cap F = \{0\} \text{ or } F \subset W.$$

If  $W \cap F \neq \{0\}$ , let  $e$  be the generator of  $F$  and  $S(b)b^n e \in W$  with  $S(0) \neq 0$ , then  $S(b)e \in W$  by normality and  $e = S^{-1}(b)S(b)e \in W$ . We have therefore  $F \subset W$ .

If  $W$  contains every sub- $(a, b)$ -module isomorphic to  $E_{\lambda_{min}}$ , then it is equal to  $V_{\lambda_{min}}$ . Otherwise there is an  $F$  such that  $W \cap F = \{0\}$ , hence  $W \oplus F$  is still in  $V_{\lambda_{min}}$ , which contradicts the maximality of  $W$ .  $\square$

We will now use the sub- $(a, b)$ -module  $V_{\lambda_{min}}$  to show the following proposition

**Proposition 22.** *Let  $E$  be a regular  $(a, b)$ -module and  $\varphi$  an  $(a, b)$ -morphism between  $E$  and itself. Then  $\varphi$  is bijective if and only if  $\varphi$  is injective.*

*Proof.* To show that bijectivity follows from injectivity, we will proceed by induction on the rank of the module.

If  $E$  is of rank 1 the statement of the proof is satisfied: in fact  $E$  must be isomorphic to one of the  $E_\lambda$  and the only  $b$ -linear morphisms from a  $E_\lambda$  to itself that are also  $a$ -linear are those that send the generator  $e$  to  $\alpha e$ ,  $\alpha \in \mathbb{C}$ . They are all bijective for  $\alpha \neq 0$ .

Let now  $E$  be of rank  $n > 1$ . We can find a  $\lambda_{min}$  (cf. [Bar93]) that verifies the minimality property of the previous proposition. Hence the module  $V_{\lambda_{min}}$  is normal and isomorphic to a direct sum of copies of  $E_{\lambda_{min}}$ .

Let  $\{e_i\}$  be a basis of  $V_{\lambda_{min}}$  composed of monomials of type  $(\lambda_{min}, 0)$  and let  $x$  another monomial of type  $(\lambda_{min}, 0)$ . We want to show that  $x$  is a linear combination of the elements of the basis, with coefficients in  $\mathbb{C} \subset \mathbb{C}[[b]]$ .

From the definition of  $V_{\lambda_{min}}$  follows that  $x \in V_{\lambda_{min}}$ . Suppose now that  $x = \sum_i S_i(b)e_i$  and let us apply  $a$  to both sides. We obtain:

$$ax = \sum_i (\lambda_{min} S_i(b) b e_i + S'_i(b) b^2 e_i) = \lambda_{min} b x + \sum_i S'_i(b) b^2 e_i$$

and since  $x$  is a monomial of type  $(\lambda_{min}, 0)$ , we must have  $S'_i(b) = 0$  for all  $i$  and therefore

$$x = \sum_i S_i(0) e_i,$$

as we wanted.

Let  $\varphi : E \rightarrow E$  be an injective endomorphism of  $E$  and  $\{e_i\}$  a basis of  $V_{\lambda_{min}}$ . Every  $\varphi(e_i)$  is a monomial of type  $(\lambda_{min}, 0)$  and therefore is an element of  $V_{\lambda_{min}}$ . The restriction of  $\varphi$  to  $V_{\lambda_{min}}$  is therefore an endomorphism of  $V_{\lambda_{min}}$ :

$$\varphi|_{V_{\lambda_{min}}} : V_{\lambda_{min}} \rightarrow V_{\lambda_{min}}.$$

Moreover since the coefficients of the  $\varphi(e_i)$  in our base are complex constants,  $\varphi|_{V_{\lambda_{min}}}$  behaves as a linear application between finite dimensional spaces: in particular if it is injective, it is also surjective.

In order to apply our induction hypothesis let us consider the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & V_{\lambda_{min}} & \xrightarrow{\subset} & E & \twoheadrightarrow & E/V_{\lambda_{min}} \longrightarrow 0 \\ & & \downarrow \varphi & & \downarrow \varphi & & \downarrow \tilde{\varphi} \\ 0 & \longrightarrow & V_{\lambda_{min}} & \xrightarrow{\subset} & E & \twoheadrightarrow & E/V_{\lambda_{min}} \longrightarrow 0 \end{array}$$

where  $\tilde{\varphi}$  is the  $(a, b)$ -linear morphism induced on the quotient. As we showed the first downward arrow is bijective

The third arrow  $\tilde{\varphi}$  is injective: suppose in fact that we have two different classes with representatives  $x$  and  $y \in E$  that map to the same class modulo  $V_{\lambda_{min}}$ . Then  $\varphi(x - y)$  is in  $V_{\lambda_{min}}$ . From the bijectivity of  $\varphi|_{V_{\lambda_{min}}}$  we can find an element  $v \in V_{\lambda_{min}}$  such that  $\varphi(x - y) = \varphi(v)$  which in turn implies  $x - y = v$  by the injectivity of  $\varphi$ , which contradicts the fact that  $x$  and  $y$  are in distinct classes modulo  $V_{\lambda_{min}}$ .

Since the rank of  $E/V_{\lambda_{min}}$  is strictly inferior to the rank of  $E$ , we can apply the induction hypothesis to show that  $\tilde{\varphi}$  is also bijective.

It follows from a basic result of homological algebra that the second arrow is bijective if it is injective.  $\square$

We can now consider endomorphisms that are not necessarily injective. Once again the structure of  $(a, b)$ -modules does not differ essentially from that of finite vector spaces over  $\mathbb{C}$ :

**Lemma 23.** *Let  $E$  be a regular  $(a, b)$ -module and  $\varphi$  an endomorphism of  $E$ . Then  $E$  splits into the direct sum of two  $\varphi$ -stable sub- $(a, b)$ -modules  $F$  and  $N$ , with  $\varphi$  bijective on  $F$  and nilpotent on  $N$ .*

*Proof.* Consider the sequence of normal sub- $(a, b)$ -modules

$$K_n = \text{Ker } \varphi^n, \quad n \in \mathbb{N}.$$

Since two normal sub- $(a, b)$ -modules  $F \subset G$  are equal if and only if they have the same rank, the sequence of  $K_n$  stabilizes beginning with a certain integer  $m$ :  $K_m = K_{m+1}$ .

On the other hand if we consider the sequence  $I_n = \text{Im } \varphi^n$ , let us look at the restriction of  $\varphi$  to  $I_m$ :

$$\varphi|_{I_m} : I_m \rightarrow I_{m+1} \subset I_m.$$

This restriction is injective: if  $y = \varphi^m(x) \in \text{Ker } \varphi$ , then  $x \in K_{m+1}$  which is equal to  $K_m$ . Hence  $\varphi^m(x) = y = 0$ . From the previous proposition we deduce that this restriction is in fact bijective, which means that  $I_{m+1} = \varphi(I_m) = I_m$ .

We can now take  $F = I_m$  and  $N = K_m$ . They are clearly stable by  $\varphi$ . We will show that  $E = F \oplus N$ .

We have in fact  $\text{Ker } \varphi \cap F = \{0\}$ , since the restriction of  $\varphi$  to  $I_m$  is injective. *A fortiori*, since  $K \subset \text{Ker } \varphi$  we have  $F \cap N = \{0\}$ .

Let's take an element  $x \in E$ . Since  $I_m = I_{2m}$  we can find an element  $y \in E$  such that  $\varphi^m(x) = \varphi^{2m}(y)$  and call  $k$  the element  $x - \varphi^m(y)$ . Thus we can write  $x$  as a sum:

$$x = \varphi^m(y) + k$$

of an element  $\varphi^m(y) \in I_m$  and an element  $k \in K_m$ , which implies that:

$$E = N \oplus F.$$

The restriction of  $\varphi$  to  $N$  is nilpotent, since  $\varphi|_N^m = 0$ , while we already showed that the restriction to  $I_m = F$  is bijective.  $\square$

We have now all the elements necessary to prove proposition 19:

*Proof.* Let  $E$  be a regular indecomposable  $(a, b)$ -module and  $\varphi$  an endomorphism of  $E$ . Then by lemma 23  $E$  splits into a sum

$$E = N \oplus F$$

of two  $(a, b)$ -modules, with  $\varphi$  nilpotent on  $N$  and bijective on  $F$ . But  $E$  is indecomposable, therefore either  $N = 0$  and  $\varphi$  is bijective or  $F = 0$  and  $\varphi$  is nilpotent.  $\square$

### 3.2 Krull-Schmidt theorem

This subsection will be devoted to the proof of a version of the Krull-Schmidt theorem for the theory of  $(a, b)$ -modules. The principal argument of the proof will be proposition 19 from the previous subsection.

**Theorem 24** (Krull-Schmidt for  $(a, b)$ -modules). *Suppose that we have two decompositions into direct sum of a regular  $(a, b)$ -module  $E$ :*

$$E = \bigoplus_{i=1}^m E_i$$

$$E = \bigoplus_{i=1}^n F_i$$

where  $m, n \in \mathbb{N}$  and all  $E_i$  and  $F_i$  are indecomposable  $(a, b)$ -modules. Then  $m = n$  and up to a reindexing of the modules  $E_i$  is isomorphic to  $G_i$  for all  $1 \leq i \leq n$ .

For the proof of this theorem we need a couple of lemmas:

**Lemma 25.** *Let  $E$  be a regular indecomposable  $(a, b)$ -module and  $\varphi$  an automorphism of  $E$ . Suppose moreover that  $\varphi = \varphi_1 + \varphi_2$ . Then at least one of  $\varphi_1, \varphi_2$  is an isomorphism.*

*Proof.* By applying  $\varphi^{-1}$  to both terms, we can assume without loss of generality that  $\varphi = Id$  is the identity automorphism.

The two endomorphisms  $\varphi_1$  and  $\varphi_2$  commute. In fact:

$$\varphi_1\varphi_2 - \varphi_2\varphi_1 = \varphi_1(\varphi_1 + \varphi_2) - (\varphi_2 + \varphi_1)\varphi_1 = \varphi_1 - \varphi_1 = 0.$$

By lemma 19 the  $\varphi_i$  can be either nilpotent or isomorphisms. If they were both nilpotent, their sum would be nilpotent, which is absurd. Hence the result.  $\square$

*Remark 26.* By subsequently applying the previous lemma, we can extend the result to the sum of more than two endomorphisms.

**Lemma 27.** *Let  $E$  and  $F$  be indecomposable regular  $(a, b)$ -modules and  $\alpha : E \rightarrow F$  and  $\beta : F \rightarrow E$  two  $(a, b)$ -linear morphisms. Suppose that  $\beta \circ \alpha$  is an isomorphism, then  $\alpha$  and  $\beta$  are also isomorphisms.*

*Proof.* Let prove that  $F = \text{Im } \alpha \oplus \text{Ker } \beta$ . If  $\alpha(x) \in \text{Ker } \beta$ , we have

$$\beta \circ \alpha(x) = 0,$$

hence  $x = 0$  and therefore

$$\text{Im } \alpha \cap \text{Ker } \beta = \{0\}.$$

Consider now an element  $x \in F$  and let

$$y = \alpha \circ (\beta \circ \alpha)^{-1} \circ \beta(x).$$

We have

$$\beta(x - y) = \beta(x) - \beta(y) = \beta(x) - (\beta \circ \alpha) \circ (\beta \circ \alpha)^{-1} \circ \beta(x) = \beta(x) - \beta(x) = 0.$$

We can thus write  $x$  as sum of an element  $y$  of  $\text{Im } \alpha$  and an element  $x - y$  of  $\text{Ker } \beta$ . This implies  $F = \text{Im } \alpha \oplus \text{Ker } \beta$ .

Now since  $\beta \circ \alpha$  is injective, so must be  $\alpha$  and  $\text{Im } \alpha$  can not be 0. But  $F$  is indecomposable therefore we must have  $\text{Im } \alpha = F$  and  $\text{Ker } \beta = 0$ . It follows that  $\alpha$  is bijective and  $\beta = (\beta \circ \alpha) \circ \alpha^{-1}$  must be also bijective.  $\square$

*Proof of Krull-Schmidt theorem for  $(a, b)$ -modules.* We will show this theorem by induction on  $m$ .

If  $m = 1$ , then  $E$  is indecomposable and we must have  $n = 1$  and  $E_1 \simeq F_1$ .

In the general case consider the morphisms

$$q_i = \pi_i \circ p_1,$$

where the  $\pi_i$ s are the projections on  $F_i$  and the  $p_j$ s are the projections on  $E_j$ . Let consider the sum:

$$\sum_i p_1 \circ q_i = p_1 \circ \sum_i \pi_i \circ p_1 = p_1 \circ p_1 = p_1,$$

is the identity on the component  $E_1$ . By the lemma 3.2, there is an  $i$  such that  $p_1 \circ q_i|_{E_1} : E_1 \rightarrow E_1$  is an isomorphism. Suppose, without loss of generality, it is  $p_1 \circ q_1$ , then by the lemma 27  $q_1|_{E_1} = \pi_1 : E_1 \rightarrow F_1$  is an isomorphism.

In order to apply the induction hypothesis, let note  $G = \sum_{i=2}^m F_i$ . We want to show that  $E_1 \oplus G$  is equal to  $E = F_1 \oplus G$ . Since  $\pi_1$  is an isomorphism of  $E_1$  onto  $F_1$  and its kernel is  $G$  we must have

$$E_1 \cap G = \{0\} :$$

if  $x \in E_1 \cap G$ , then  $\pi_1(x) = 0$ , but  $\pi_1$  restricted to  $E_1$  is injective, so  $x = 0$ . On the other hand every element of  $E$  can be written as  $v + w$  with  $v \in F_1$  and  $w \in G$ . If  $y \in E_1$  is such that  $\pi_1(y) = v$ , then we have:

$$v + w = y + \pi_1(y) - y + w,$$

and  $\pi_1(y) - y \in W$  by definition of  $\pi_1$ . We can then conclude that  $E = E_1 \oplus G = E_1 + \sum_{i=2}^m E_i$ .

We have immediately  $E/E_1 \simeq G \simeq \sum_{i=2}^m E_i$  and we can apply the induction hypothesis to  $G$ .  $\square$

We can now focus on finding hermitian isomorphisms of an  $(a, b)$ -module  $E$  with its adjoint  $\check{E}^*$ . The Krull-Schmidt theorem will be useful to show the following decomposition:

**Proposition 28.** *Let  $E$  be a regular self-adjoint  $(a, b)$ -module. Then  $E$  is isomorphic to:*

$$E \simeq \bigoplus_{i=1}^r (F_i^{\oplus \alpha_i}) \oplus \bigoplus_{i=1}^s (G_i \oplus \check{G}_i^*)^{\oplus \beta_i}$$

where  $r$  and  $s$  as well as the  $\alpha_i$  and  $\beta_i$  are positive integers. The  $F_i$  are self-adjoint  $(a, b)$ -modules and the  $G_i$  are non self-adjoint  $(a, b)$ -modules. The isomorphism classes of the  $F_i$ ,  $G_i$  and  $\check{G}_i^*$  are all disjoint.

*Proof.* Consider a decomposition of  $E$  into indecomposable  $(a, b)$ -modules

$$E = \sum_i E_i.$$

Since  $E$  is self-adjoint we have another decomposition given by

$$E \simeq \check{E}^* = \sum_i \check{E}_i^*.$$

The Krull-Schmidt theorem assures us that the factors are unique up to a permutation. So we can divide the  $E_i$  into two groups.

In the first group we find the self-adjoint components  $F_i$  with a certain multiplicity.

In the second one we find the non self-adjoint components  $G_i$  with the respective multiplicity. Since the two decompositions  $\sum_i E_i$  and  $\sum_i \check{E}_i^*$  must contain the same modules up to a permutation, the multiplicity of the  $G_i$  and the  $\check{G}_i^*$  must be equal.  $\square$

*Remark 29.* From the definition above we can immediately see that the non self-adjoint part of the decomposition always admits a hermitian nondegenerate form. In fact if we consider the module  $G_i \oplus \check{G}_i^*$ , a hermitian form can be given by:

$$\begin{aligned} \Phi : G_i \oplus \check{G}_i^* &\rightarrow (G_i \oplus \check{G}_i^*)^* = \check{G}_i^* \oplus G_i \\ (x, y) &\mapsto (y, x). \end{aligned}$$

If the multiplicity of a self-adjoint term  $F_i$  is pair, we fall into the same situation.

The case of an unpair multiplicity of a self-adjoint component is far more interesting and we will study it in the next subsection.

### 3.3 Hermitian forms on indecomposable $(a, b)$ -modules

As already noted in the previous subsection, the situation of an indecomposable self-adjoint  $(a, b)$ -module concerning hermitian forms is far less regular and the existence is not always guaranteed. We have in fact the following theorem:

**Theorem 30.** *Let  $E$  be a regular indecomposable self-adjoint  $(a, b)$ -module and  $E \neq \{0\}$ . Then it admits a hermitian nondegenerate form or an anti-hermitian one.*

*Proof.* Let  $\Phi : E \rightarrow \check{E}^*$  be any isomorphism of  $E$  with its dual and pose  $M = \Phi^{-1}\check{\Phi}^*$ . Consider now the two endomorphisms of  $E$  given by:

$$Id + M$$

and

$$Id - M$$

they commute and can be either isomorphisms or nilpotent, since  $E$  is indecomposable. But if they were both nilpotent, their sum  $2Id$  would be nilpotent too, which is absurd.

If  $Id + M$  is an isomorphism, so is  $S = \Phi + \check{\Phi}^*$ , which is associated to a nondegenerate hermitian form. The bijectivity of  $Id - M$  on the other hand gives us an isomorphism  $A = \Phi - \check{\Phi}^*$ , which comes from an anti-hermitian form.  $\square$

Note that all the cases of the previous theorem are equally possible.

**Example 31.** The simplest example of a regular self-adjoint and indecomposable  $(a, b)$ -module which admits only a hermitian form is the elementary  $(a, b)$ -module  $E_0$  with the isomorphism that sends the generator  $e$  to its adjoint  $\check{e}^*$ .

**Example 32.** In order to obtain only an anti-hermitian form, we can consider for a given  $\lambda, \mu \in \mathbb{C}$  the  $(a, b)$ -module  $E$  of rank 4, generated by  $\{e_1, e_2, e_3, e_4\}$  which verifies:

$$\begin{aligned} ae_1 &= \lambda be_1 \\ ae_2 &= \mu be_2 + e_1 \\ ae_3 &= -\mu be_3 + e_1 \\ ae_4 &= -\lambda be_4 + e_2 - e_3 \end{aligned} \tag{5}$$

whose adjoint basis satisfies:

$$\begin{aligned} a \cdot \check{e}_4^* &= \lambda b \check{e}_4^* \\ a \cdot \check{e}_3^* &= \mu b \check{e}_3^* - \check{e}_4^* \\ a \cdot \check{e}_2^* &= -\mu b \check{e}_2^* + \check{e}_4^* \\ a \cdot \check{e}_1^* &= -\lambda b \check{e}_1^* + \check{e}_3^* + \check{e}_2^*. \end{aligned}$$

It is easy to show by calculation that the only isomorphism between  $E$  and  $\check{E}^*$  is, up to multiplication by a complex number, the one that sends  $e_1, e_2, e_3$  and  $e_4$  into  $\check{e}_4, -\check{e}_3, \check{e}_2$  and  $-\check{e}_1$  respectively.

This isomorphism is anti-hermitian and since there are no other isomorphisms  $E$  is also indecomposable.

**Example 33.** The regular  $(a, b)$ -module  $E_0 \oplus E_0$  admits both an hermitian and anti-hermitian form.

## 4 Duality of geometric $(a, b)$ -modules

In the study of the Brieskorn lattice  $K$ . Saito introduced the concept of “higher residue pairings” (cf. [Sai83]), which can be defined using a set of axiomatic properties.

Using the theory of  $(a, b)$ -modules R. Belgrade showed the existence of a duality isomorphism between an  $(a, b)$ -module associated to a germ of a holomorphic function in  $\mathbb{C}^{n+1}$  with an isolated singularity at the origin and its  $(n+1)$ -dual. In this section we’ll prove (as already noticed by R. Belgrade in [Bel01]) that the concept of “higher residue pairings” and self-adjoint  $(a, b)$ -module are linked.

In this section  $D$  will always denote the Brieskorn module associated to a holomorphic function in  $\mathbb{C}^{n+1}$  with an isolated singularity, while  $E$  will denote its  $b$ -adic completion considered as an  $(a, b)$ -module.

The following theorem of R. Belgrade gives a relationship between  $E$  and its  $(n+1)$ -dual.

**Theorem 34** (Belgrade). *Let  $E$  be the  $(a, b)$ -module associated to a germ of holomorphic function  $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ , then there is a natural isomorphism between  $E$  and its  $(n+1)$ -dual:*

$$\Delta : E \simeq \check{E}^* \otimes_{(a,b)} E_{n+1}$$

We can obtain from this isomorphism a series  $\Delta_k : E \times E \rightarrow \mathbb{C}$  of bilinear forms defined as follow:

$$[\Delta(y)](x) = (n+1)! \sum_{k=0}^{+\infty} \Delta_k(x, y) b^k e_{n+1}$$

with  $x, y \in E$ .

## 5 “Higher residue pairings” of K. Saito

K. Saito introduced in [Sai83] a series of pairings on the Brieskorn lattice  $D$  which are called “higher residue pairings”:

$$K^{(k)} : D \times D \rightarrow \mathbb{C} \quad k \in \mathbb{N}$$

which are characterized by the following properties:

- (i)  $K^{(k)}(\omega_1, \omega_2) = K^{(k+1)}(b\omega_1, \omega_2) = -K^{(k+1)}(\omega_1, b\omega_2)$ .

(ii)  $K^{(k)}(a\omega_1, \omega_2) - K^{(k)}(\omega_1, a\omega_2) = (n+k)K^{(k-1)}(\omega_1, \omega_2)$ .

(iii)  $K^{(0)}$  satisfies

$$K^{(0)}(D, bD) = K^{(0)}(bD, D) = 0$$

and induces Grothendieck's residue on the quotient  $D/bD$ .

(iv)  $K^{(k)}$  are  $(-1)^k$ -symmetric.

*Remark 35.* We notice that from the properties (i) and (iii) above we can deduce that  $K^{(k)}(D, b^{k+1}D) = K^{(k)}(b^{k+1}D, D) = 0$ , so we can consider the pairings  $K^{(k)}$  as being defined on  $D/b^{k+1}D$ .

In the following section we'll show the following result:

**Proposition 36.** *The  $\Delta_k$  verify the properties (i)–(iii) of the “higher residue pairings” of K. Saito.*

The prove will be performed by steps.

## 6 Proof of the proposition

### 6.1 Proof of (i)

We use the  $b$ -linearity of  $\Delta(y)$  to obtain:

$$\begin{aligned} \sum_k (n+1)! \Delta_k(bx, y) b^k e_{n+1} &= [\Delta(y)](bx) = b [\Delta(y)](x) = \\ &= \sum_k (n+1)! \Delta_k(x, y) b^{k+1} e_{n+1} \end{aligned}$$

which gives us  $\Delta_k(x, y) = \Delta_{k+1}(bx, y)$ . And similarly by using the  $b$ -linearity of  $\Delta$  and the adjoint morphism, we obtain:

$$\Delta(by)(x) = \check{\Delta}^*(x)(by) = -b\check{\Delta}^*(x)(y) = -b\Delta(y)(x),$$

and therefore

$$\begin{aligned} (n+1)! \sum_k \Delta_k(x, by) b^k e_{n+1} &= \Delta(by)(x) = -b\Delta(y)(x) = \\ &= (n+1)! \sum_k -\Delta_k(x, y) b^{k+1} e_{n+1}, \end{aligned}$$

which implies  $\Delta_k(bx, y) = -\Delta_{k+1}(x, by)$ .

### 6.2 Proof of (ii)

Since  $\Delta$  is an isomorphism we have  $\Delta(ay) = a \cdot \check{E}^*_{\otimes E_\delta} [\Delta(y)]$  and:

$$\begin{aligned} (n+1)! \sum_k \Delta_k(x, ay) b^k e_{n+1} &= \Delta(ay)(x) = a \cdot [\Delta(y)](x) = \\ &= \Delta(y)(ax) - a[\Delta(y)(x)] = (n+1)! \sum_k (\Delta_k(ax, y) b^k e_{n+1} - \Delta_k(x, y) a b^k e_{n+1}) \end{aligned}$$

The definition of  $(a, b)$ -module and  $E_{n+1}$  ( $ae_{n+1} = (n+1)be_{n+1}$ ) gives the following relation

$$ab^k e_{n+1} = b^k ae_{n+1} + kb^{k+1} e_{n+1} = (n+k+1)b^{k+1} e_{n+1}$$

hence follows:

$$\Delta_k(ax, y) - \Delta_k(x, ay) = (n+k)\Delta_{k-1}(x, y)$$

### 6.3 Grothendieck's residue

We have to show now that the pairing  $\Delta_0$  induces Grothendieck's residue on  $D/bD \simeq \Omega^{n+1}/df \wedge \Omega^n$ .

**Proof of (iv):** From the definition of  $\Delta_0$  and the  $b$ -linearity of  $\Delta$  it's easy to see that  $\Delta_0(D, bD) = \Delta_0(bD, D) = 0$ . We can hence consider  $\Delta_0$  as a pairing on  $D/bD$ .

Grothendieck's residue is defined as follows:

$$Res(g, h) := \lim_{\varepsilon_j \rightarrow 0, \forall j} \int_{|\partial f / \partial z_j| = \varepsilon_j} \frac{gh dz}{\partial f / \partial z_1 \cdots \partial f / \partial z_{n+1}}$$

where  $g, h \in \mathcal{O}$  and  $dz = dz_1 \wedge \dots \wedge dz_{n+1}$ .

The morphism  $\Delta$  is defined as composed morphism of six  $(a, b)$ -modules morphism ([Bel01]) as showed by the following graph:

$$\begin{array}{ccccccc} E & \xrightarrow{\alpha} & F_1 & \xrightarrow{\beta} & F_2 & & \\ & & & & & \swarrow \gamma & \\ & & & & & & F_3 \\ & & & & & \searrow \delta & \\ & & & & & & \\ \check{E}_{n+1} & \xleftarrow{\zeta} & F_5 & \xleftarrow{\eta} & F_4 & & \end{array}$$

These morphisms pass to the quotient by the action of  $b$  in order to give a decomposition of the morphism  $\Delta_0$ :

$$\begin{array}{ccccccc} E/bE & \xrightarrow{\tilde{\alpha}} & F_1/bF_1 & \xrightarrow{\tilde{\beta}} & F_2/bF_2 & & \\ & & & & & \swarrow \tilde{\gamma} & \\ & & & & & & F_3/bF_3 \\ & & & & & \searrow \tilde{\delta} & \\ & & & & & & \\ \frac{(\check{E}^* \otimes E_{n+1})}{b(\check{E}^* \otimes E_{n+1})} & \xleftarrow{\tilde{\zeta}} & F_5/bF_5 & \xleftarrow{\tilde{\eta}} & F_4/bF_4 & & \end{array}$$

We have to verify that the image of  $[g dz]$  by  $\Delta_0$  is  $Res(g, \cdot)$ , where  $g dz$  is an element of  $\Omega^{n+1}$ . We'll accomplish this in many steps using the decomposition above.

(i) **Step 1:  $E, F_1$  and  $F_2$ .** We have the following isomorphisms:

$$\frac{F_1}{bF_1} \simeq \frac{\Omega^{n+1}}{df \wedge \Omega^n} \quad \frac{F_2}{bF_2} \simeq \frac{\mathcal{D}b^{n+1}}{(\bar{\partial} - df \wedge)\mathcal{D}b^n},$$

the morphism  $\tilde{\alpha}$  coincides with the identity on  $\Omega^{n+1}/df \wedge \Omega^n$  and  $\tilde{\beta}$  is induced by the inclusion  $i : \Omega^{n+1} \mapsto \mathcal{D}b^{n+1}$ . We deduce that  $\tilde{\beta} \circ \tilde{\alpha}([g dz]) = [i(g dz)]$ . Let write  $T \in \mathcal{D}b^{n+1,0}$  the current  $i(g dz)$ .

(ii) **Step 2: path between  $F_2$  and  $F_3$**  By using the description of the lemma 3.4.2 of [Bel01] we see that:

$$\frac{F_3}{bF_3} = \frac{\text{Ker}(\mathcal{D}b^{0,n+1} \xrightarrow{df \wedge} \mathcal{D}b^{1,n+1})}{\bar{\partial} \text{Ker}(\mathcal{D}b^{0,n} \xrightarrow{df \wedge} \mathcal{D}b^{1,n})}$$

and the isomorphism  $\tilde{\gamma}$  is induced by the inclusion  $\mathcal{D}b^{0,n+1} \subset \mathcal{D}b^{n+1}$ . In order to find  $S := \tilde{\gamma}^{-1}(T)$  we have to solve the following system:

$$\begin{aligned} T &= df \wedge \alpha^{n,0} \\ \bar{\partial} \alpha^{n,0} &= df \wedge \alpha^{n-1,1} \\ \dots &\dots \\ \bar{\partial} \alpha^{1,n-1} &= df \wedge \alpha^{0,n} \\ \bar{\partial} \alpha^{0,n} &= S \end{aligned}$$

where the  $\alpha^{p,q} \in \mathcal{D}b^{p,q}$ . There is a solution to this system of equations since the complex  $(\mathcal{D}b^{*,q}; df \wedge)$  is acyclic in degree  $\neq 0$  for all  $q$  in  $0, \dots, n+1$  and the solution verifies  $[S] = [T]$  where  $[\cdot]$  is the equivalence class in  $F_2/bF_2$ .

$$(\bar{\partial} - df \wedge) \sum_{k=0}^n \alpha^{k,n-k} = \bar{\partial} \alpha^{0,n} - df \wedge \alpha^{n,0} = S - T$$

We can compute this solution explicitly. Let be  $(p, q) \in \mathbb{N}^2$  and  $\varphi^{p,q}$  a  $C^\infty$  test form with compact support and of type  $(p, q)$ . The action of  $T$  over  $\varphi^{0,n+1}$  is given by:

$$\langle T, \phi^{0,n+1} \rangle = \int \phi^{0,n+1} \wedge g dz$$

then the following current verifies  $T = df \wedge \alpha^{n,0}$ :

$$\langle \alpha^{n,0}, \phi^{1,n+1} \rangle = \lim_{\varepsilon_1 \rightarrow 0} \int_{|\partial_1 f| \geq \varepsilon_1} \frac{\phi^{1,n+1} \wedge g dz_2 \wedge \dots \wedge dz_{n+1}}{\partial_1 f}$$

in fact:

$$\begin{aligned} \langle df \wedge \alpha^{n,0}, \phi^{0,n+1} \rangle &= \lim_{\varepsilon_1 \rightarrow 0} \int_{|\partial_1 f| \geq \varepsilon_1} \frac{\phi^{0,n+1} \wedge df \wedge g dz_2 \wedge \dots \wedge dz_{n+1}}{\partial_1 f} \\ &= \int \phi^{0,n+1} \wedge g dz \end{aligned}$$

and thanks to the Stokes' theorem:

$$\begin{aligned}
\langle \bar{\partial}\alpha^{n,0}, \phi^{1,n} \rangle &= - \langle \alpha^{n,0}, \bar{\partial}\phi^{1,n} \rangle \\
&= \lim_{\epsilon_1 \rightarrow 0} - \int_{|\partial_1 f| \geq \epsilon_1} \frac{\bar{\partial}\phi^{1,n} \wedge g \, dz_2 \wedge \dots \wedge dz_{n+1}}{\partial_1 f} \\
&= \lim_{\epsilon_1 \rightarrow 0} \int_{|\partial_1 f| = \epsilon_1} \frac{\phi^{1,n} \wedge g \, dz_2 \wedge \dots \wedge dz_{n+1}}{\partial_1 f}
\end{aligned}$$

We'll remark that the currents  $\alpha_k^{n,0}$  defined below for  $1 \leq k \leq n+1$  also satisfy  $df \wedge \alpha_k^{n,0} = T$ :

$$\begin{aligned}
\langle \alpha_k^{n,0}, \phi^{1,n+1} \rangle &= \\
&= \lim_{\epsilon_k \rightarrow 0} \int_{|\partial_k f| \geq \epsilon_k} \frac{(-1)^{k+1} \phi^{1,n+1} \wedge g \, dz_1 \wedge \dots \wedge \widehat{dz_k} \wedge \dots \wedge dz_{n+1}}{\partial_k f}
\end{aligned}$$

and that  $[\bar{\partial}\alpha^{n,0}] = [\bar{\partial}\alpha_k^{n,0}]$  in  $F_2/bF_2$ : in fact  $(\bar{\partial} - df \wedge)(\alpha^{n,0} - \alpha_k^{n,0}) = \bar{\partial}\alpha^{n,0} - \bar{\partial}\alpha_k^{n,0}$ .

For all  $k \in 0, \dots, n$  and  $1 \leq i_1 < \dots < i_{k+1} \leq n+1$  let us define:

$$\alpha_{i_1, \dots, i_{k+1}}^{n-k,k} = \frac{1}{(k+1)!} \lim_{\substack{\epsilon_{i_q} \rightarrow 0 \\ \forall 1 \leq q \leq k+1}} \int_{\substack{|\partial_{i_1} f| \geq \epsilon_{i_1} \\ |\partial_{i_q} f| = \epsilon_{i_q}}} \frac{(-1)^{(\sum_q i_q)+1} g \wedge_{l \neq i_1, \dots, i_{k+1}} dz_l}{\partial_{i_1} f \dots \partial_{i_{k+1}} f}$$

and let  $\alpha^{n-k,k} := \alpha_{1,2,\dots,k+1}^{n-k,k}$ .

A simple computation gives us:

$$\left\langle df \wedge \alpha_{i_1, \dots, i_{k+1}}^{n-k,k}, \varphi^{k,n-k+1} \right\rangle = \left\langle \frac{1}{k+1} \sum_{q=1}^{k+1} \bar{\partial}\alpha_{i_1, \dots, \widehat{i_q}, \dots, i_{k+1}}^{n-k+1, k-1}, \varphi^{k,n-k+1} \right\rangle$$

using this formula, we can prove by induction on  $k$  that the class of the current  $\alpha_{i_1, \dots, i_{k+1}}^{n-k,k}$  doesn't depend upon the  $i_q$ s. This gives us

$$[df \wedge \alpha^{n-k,k}] = [\bar{\partial}\alpha^{n-k+1, k-1}].$$

In particular  $\bar{\partial}\alpha^{0,n}$  acts upon the test function  $\varphi^{n+1,0}$  in the following way:

$$\langle \bar{\partial}\alpha^{0,n}, \varphi^{n+1,0} \rangle = \frac{1}{(n+1)!} \lim_{\substack{\epsilon_k \rightarrow 0 \\ \forall k}} \int_{|\partial_k f| = \epsilon_k} \frac{\phi^{n+1,0} g}{\partial_1 f \dots \partial_{n+1} f}$$

- (iii) **Step 3 from  $F_3/bF_3$  to  $(D/bD)^*$** : let notice that  $S$  is a current of type  $(0, n+1)$  with support in the origin.

We have the following isomorphisms:

$$\frac{F_4}{bF_4} \simeq \text{Ker} \left( \mathcal{H}_0^{n+1}(X, \mathcal{O}) \xrightarrow{df \wedge} \mathcal{H}_0^{n+1}(X, \Omega^1) \right)$$

and the isomorphism between  $F_3/bF_3$  and  $F_4/bF_4$  is the natural one, and

$$\frac{F_5}{bF_5} \simeq \left( \frac{\Omega^{n+1}}{df \wedge \Omega^n} \right)^*$$

From steps 1–3 we deduce that  $\Delta_0$  induces Grothendieck's residue.

## 6.4 Property (iv)

We will that the isomorphism given by R. Belgrade can be easily transformed into one that verifies the property.

Let  $\Delta : E \rightarrow \check{E}^* \otimes_{(a,b)} E_{n+1}$  be Belgrade's isomorphism. By tensoring with  $E_{(n+1)/2}$  we can show that, the isomorphisms between  $E$  and  $\check{E}^* \otimes_{(a,b)} E_{n+1}$  are in bijection with the isomorphisms between  $E \otimes_{(a,b)} E_{-(n+1)/2}$  and its adjoint, through the map that sends an isomorphism  $\Phi$  to  $\Phi \otimes_{(a,b)} Id_{E_{-(n+1)/2}}$ .

By an easy calculation we can prove the following lemma:

**Lemma 37.** *Let  $\Delta : E \rightarrow \check{E}^* \otimes_{(a,b)} E_{n+1}$  be an isomorphism and*

$$\Delta(y)(x) = (n+1)! \sum_k \Delta_k(x, y) b^k e_{n+1}$$

for each  $x$  and  $y \in E$ . Then the  $\Delta_k$  satisfy Saito's condition (iv) if and only if the isomorphism  $\Delta \otimes_{(a,b)} Id_{E_{-(n+1)/2}}$  is hermitian.

*Proof.*  $\Delta \otimes_{(a,b)} Id_{E_{-(n+1)/2}}$  is self-adjoint iff we have:

$$\begin{aligned} \Delta \otimes_{(a,b)} Id_{E_{-(n+1)/2}} (y \otimes e_{-(n+1)/2}) (x \otimes e_{-(n+1)/2}) &= \sum_k S_k b^k e_0 \Leftrightarrow \\ \Delta \otimes_{(a,b)} Id_{E_{-(n+1)/2}} (x \otimes e_{-(n+1)/2}) (y \otimes e_{-(n+1)/2}) &= \sum_k S_k (-b)^k e_0. \end{aligned}$$

for all  $x$  and  $y \in E$ . On the other hand we have:

$$\begin{aligned} \Delta \otimes_{(a,b)} Id_{E_{-(n+1)/2}} (y \otimes e_{-(n+1)/2}) (x \otimes e_{-(n+1)/2}) &= \sum_k S_k b^k e_0 \Leftrightarrow \\ \Delta(y)(x) &= \sum_k S_k b^k e_{n+1}. \end{aligned}$$

□

By combining the previous equivalence with the results on the existence of hermitian forms, we can extend Belgrade's result:

**Theorem 38.** *Let  $E$  be a regular  $(a, b)$ -module associated to a holomorphic function from  $\mathbb{C}^{n+1}$  to  $\mathbb{C}$  with an isolated singularity. Then there exists an isomorphism  $\Phi : E \rightarrow \check{E}^* \otimes_{(a,b)} E_{n+1}$  with*

$$\Phi(y)(x) = (n+1)! \sum_k \Phi_k(x, y) b^k e_{n+1},$$

for all  $x$  and  $y$  such that the sequence of  $\mathbb{C}$ -bilinear forms  $\Phi_k$  satisfies all four properties of Saito's "higher residue pairings".

*Proof.* Let  $\Delta$  be Belgrade's isomorphism and  $\Delta_k$  defined as at the beginning of this section. Consider the isomorphism

$$\check{\Delta}^* \otimes_{(a,b)} Id_{E_{n+1}} : E \rightarrow \check{E}^* \otimes_{(a,b)} E_{n+1}$$

and let  $\Phi = (\Delta + \check{\Delta}^* \otimes_{(a,b)} Id_{E_{n+1}}) / 2$ .

It is easy to see that the  $\Phi_k$  satisfy properties (i) and (ii). Moreover since  $\Delta_0$  is symmetric (Grothendieck's residue) and  $\tilde{\Delta}^* \otimes_{(a,b)} Id_{E_{n+1}}$  induces the transposed of  $\Delta_0$  on  $E/bE$ , we have

$$\Phi_0 = (\Delta_0 + {}^t\Delta_0)/2 = \Delta_0.$$

We have also

$$\Phi \otimes_{(a,b)} Id_{E_{-(n+1)/2}} = \left( \widetilde{\Phi \otimes_{(a,b)} Id_{E_{-(n+1)}}} \right)^* = \check{\Phi}^* \otimes_{(a,b)} Id_{E_{(n+1)/2}} = \Phi \otimes_{(a,b)} Id_{E_{-(n+1)/2}},$$

therefore the  $\Phi_k$  satisfy Saito's property (iv).

We just have to show that  $\Phi \otimes_{(a,b)} Id_{E_{-(n+1)/2}}$  is an isomorphism. Since there exists an isomorphism between  $E \otimes_{(a,b)} E_{-(n+1)/2}$  and its adjoint  $\Delta \otimes_{(a,b)} Id_{E_{-(n+1)/2}}$ , we can apply proposition 22 and reduce ourselves to prove the injectivity of  $\Phi \otimes_{(a,b)} Id_{E_{-(n+1)/2}}$ . But if  $\Phi \otimes_{(a,b)} Id_{E_{-(n+1)/2}}$  were not injective  $\Phi$  would induce a degenerate form on  $E/bE$ , which is absurd.  $\square$

The existence of a hermitian form on  $E \otimes_{(a,b)} E_{-(n+1)/2}$  gives us an interesting restriction on the kind of  $(a, b)$ -module associated with Brieskorn lattices:

**Corollary 39.** *Let  $E$  be a regular  $(a, b)$ -module associated to a holomorphic function from  $\mathbb{C}^{n+1}$  to  $\mathbb{C}$  with an isolated singularity. Then  $E \otimes_{(a,b)} E_{-(n+1)/2}$  is a hermitian  $(a, b)$ -module.*

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