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par

Piotr Przemysław Karwasz

Self-adjoint (a, b)-modules and hermitian forms

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Membres du jury :

Daniel Barlet	Directeur de Thèse	Professeur, Nancy I
Ridha Belgrade	Examinateur	Maître de Conférences, Nancy I
Antoine Douai	Rapporteur	Maître de Conférences, Nice
Claude Sabbah	Président et Rapporteur	Professeur, École Polytechnique
Willem Veys	Rapporteur	Professeur, Leuven

Institut Élie Cartan Nancy, Laboratoire de Mathématiques, B.P. 239, 54506 Vandœuvre-les-Nancy Cedex

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Piotr P. Karwasz

Introduction (français)

Dans cette thèse nous présenterons un travail relatif à la théorie des (a, b)-modules. Nous nous intéresserons en particulier à trois problèmes liés à la dualité des (a, b)-modules que nous traiterons dans les chapitres 2, 3 et 4 de ce manuscrit.

Vu la relative nouveauté du sujet, en particulier en ce qui concerne la dualité des (a, b)-modules, le premier chapitre sera dédié à un rappel des principaux concepts et résultats de cette théorie. Nous nous concentrerons plus particulièrement sur les suites de Jordan-Hölder et les invariants de ces suites.

Dans le chapitre suivant, en se basant sur la définition de dual d'un (a, b)-module, on exposera le concept de module adjoint et de forme hermitienne. Dans notre analyse des formes hermitiennes nous serons amenés à définir la notion de (a, b)-module indécomposable et à montrer l'analogue du théorème de Krull-Schmidt dans la théorie des modules sur un anneau commutatif. On montrera par la suite l'existence de formes ou bien hermitiennes ou anti-hermitiennes sur les modules indécomposables auto-adjoints et on donnera un exemple non trivial de rang 4 admettant uniquement une forme anti-hermitienne.

Suivra un chapitre dédié aux suites de Jordan-Hölder de (a, b)-modules auto-adjoints. L'intérêt se portera en particulier sur les suites de Jordan-Hölder dites elles aussi auto-adjointes et on en montrera l'existence, pour tout (a,b)-module régulier auto-adjoint.

En guise de conclusion on appliquera les résultats obtenus aux (a, b)-modules associés à une hypersurface à singularité isolée, c'est-à-dire au complété formel de son module de Brieskorn. On montrera que le symétrisé de l'isomorphisme donné par R. Belgrade dans [Bel01] satisfait aux axiomes donnés par K. Saito dans la présentation de ses "higher residue pairings".

0.1 Les (a, b)-modules

Un (a, b)-module est une structure algébrique obtenue par la donnée d'un $\mathbb{C}[[b]]$-module E libre de rang fini sur l'anneau des séries formelles dans une variable b et d'une application $\mathbb{C}$-linéaire a de E dans lui-même qui satisfont à la relation :

$$ab - ba = b^2.$$

Introduits par D. Barlet (cf. [Bar93]), ils apparaissent comme l'abstraction algébrique du réseau de Brieskorn d'une fonction à singularité isolée (cf. [Bri70]). En effet étant donnée une fonction holomorphe à singularité isolée $f\colon \mathbb{C}^{n+1} \to \mathbb{C}$ et à son réseau de Brieskorn :

$$D := \frac{\Omega_0^{n+1}}{\mathrm{df} \wedge \mathrm{d}\,\Omega_0^{n-1}},$$

où Ω_0^i désigne l'ensemble des germes de i-formes holomorphes à l'origine, on peut associer un (a, b)-module en définissant deux opérations sur D :

$$a[\omega] = [f\omega]$$
$$b[\omega] = [\mathrm{df} \wedge v]$$

où $[\cdot]$ représente la classe modulo $\mathrm{df} \wedge \mathrm{d}\,\Omega_0^{n-1}$, $\omega \in \Omega_0^{n+1}$ et $\mathrm{d}\,v = \omega$. Le complété du réseau D pour la topologie b-adique est un (a, b)-module, comme il est démontré dans [Bar93].

Dans le premier chapitre on rappellera les notions fondamentales de la théorie des (a, b)-modules et en particulier le concept de régularité, celui de (a, b)-module simple et de sous-(a, b)-module normal.

Les suites de Jordan-Hölder auront une importance particulière dans le contexte de cette thèse. Une suite de Jordan-Hölder d'un (a, b)-module régulier est par définition une suite de sous-(a, b)-modules normaux F_i de E, tels que :

$$0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_n = E,$$

où tous les quotients F_i/F_{i-1} sont des (a, b)-modules simples.

0.2 Dualité et formes hermitiennes

Dans la première partie du deuxième chapitre on clarifie le concept de dual d'un (a, b)-module E. Deux notions similaires existent au même temps dans la littérature : le concept de dual introduit par D. Barlet dans [Bar97] et celui de δ-dual introduit par R. Belgrade dans [Bel01].

On choisit la définition de D. Barlet de dual E^* d'un (a, b)-module E, qui a l'avantage de ne pas changer la $\mathbb{C}[[b]]$-structure naturelle sur E^*. Le dual est ainsi le $\mathbb{C}[[b]]$-module $\mathrm{Hom}_{\mathbb{C}[[b]]}(E, E_0)$, où E_0 est le (a, b)-module élémentaire de rang 1 engendré par un élément e_0 qui vérifie $ae_0 = 0$, et la a-structure sur E^* est donnée par

$$(a\varphi)(x) = a(\varphi(x)) - \varphi(ax)$$

pour tout $\varphi \in E^*$ et $x \in E$.

La notion de δ-dual de Belgrade sera réduite à la notion de dual à l'aide d'un foncteur de conjugaison $\breve{}$, qui échange les signes des actions de a et b ($a \to -a$ et $b \to -b$), et du concept de produit tensoriel de deux (a, b)-modules E et F, qui factorise toute application $\mathbb{C}[[b]]$-bilinéaire

$$\Phi\colon E \times F \to G,$$

dans un troisième (a, b)-module G satisfaisant

$$a\left(\Phi(x, y)\right) = \Phi(ax, y) + \Phi(x, ay),$$

où $x \in E$ et $y \in F$. On montrera ainsi que le δ-dual d'un (a, b)-module E au sens de [Bel01] coïncide avec le produit tensoriel $\breve{E}^* \otimes E_\delta$, où E_δ est le (a, b)-module élémentaire de paramètre $\delta \in \mathbb{C}$ (cf. [Bar93]). Par analogie avec la théorie des espaces vectoriels sur $\mathbb{C}$, on appelle adjoint le module $\breve{E}^*$.

Dans [Bel01] R. Belgrade montre que le (a, b)-module de Brieskorn E associé à une fonction de $\mathbb{C}^{n+1}$ dans $\mathbb{C}$ à singularité isolée admet un isomorphisme avec son $(n + 1)$-dual. Pour mieux comprendre le lien entre cet isomorphisme et les "higher residue pairings" de K. Saito (cf. [Sai83]), en particulier la propriété de symétrie de ces "pairings", on étudie dans la deuxième partie du chapitre 2 les (a, b)-modules réguliers qui admettent un isomorphisme auto-adjoint.

En se ramenant au cas des (a, b)-modules indécomposables, c'est-à-dire qui ne peuvent s'écrire comme somme directe de (a, b)-modules de rang inférieur, on montre que tout (a, b)-module régulier indécomposable isomorphe à son adjoint admet ou bien un isomorphisme hermitien, ou bien un isomorphisme anti-hermitien. On montre à l'aide d'un exemple non trivial de rang 4, que le deuxième cas est bien possible et que tous les isomorphismes de cet (a, b)-module avec son adjoint sont anti-hermitiens.

0.3 Suites de Jordan-Hölder auto-adjointes

Il est mis en évidence dans [Bar93] que les quotients F_i/F_{i-1} d'une suite de Jordan-Hölder d'un (a, b)-module régulier E

$$0 = F_0 \subsetneq \cdots \subsetneq F_n = E$$

ne sont pas nécessairement isomorphes pour deux suites différentes, même en tenant compte des permutations possibles.

Dans le troisième chapitre on se restreint aux (a,b)-modules réguliers auto-adjoints E de rang n et on montre que sous ces hypothèses il existe une suite de Jordan-Hölder

$$0 = F_0 \subsetneq \cdots \subsetneq F_n = E,$$

avec les propriétés suivantes de symétrie :

(i) Les quotients “centraux” F_{n-i}/F_i sont tous auto-adjoints pour $0 \leqslant i \leqslant [n/2]$.

(ii) Pour chaque i le quotient simple F_i/F_{i-1} est isomorphe à l'adjoint de F_{n-i+1}/F_{n-i}.

0.4 Higher residue pairings

Dans le dernier chapitre on regarde de plus près la relation existante entre l'isomorphisme avec le δ-dual de Belgrade et la version axiomatique des “higher residue pairings” de K. Saito. En particulier, à partir d'un (a,b)-module E associé à une fonction $\mathbb{C}^{n+1} \to \mathbb{C}$ à singularité isolée et de l'isomorphisme

$$\Delta \colon E \to \breve{E}^* \otimes_{(a,b)} E_{n+1}$$

donné par R. Belgrade, on obtient un nouvel isomorphisme auto-adjoint $\left(\Delta + \breve{\Delta}^*\right)/2$ tel que la suite d'applications $\mathbb{C}$-bilinéaires induite

$$\Delta_k \colon E \times E \to \mathbb{C}$$

vérifie les quatre axiomes des “higher residue pairings”.

La question si les Δ_k qui respectent ces axiomes sont uniques et coïncident ainsi avec les “higher residue pairings” reste ouverte. Néanmoins, le premier terme Δ_0 coïncide avec le résidu de Grothendieck.

Chapter 1

Theory of (a, b)-modules

Since the theory of (a, b)-module is rather new, this chapter is intended as a short introduction to this theory, whose object of study is an algebraic generalisation of the Brieskorn modules, introduces by E. Brieskorn in [Bri70]. The first section will recall the definitions of the (a, b)-module structure, introduced by D. Barlet in [Bar93] and the principal subtypes of this object. The following section will be devoted to defining Jordan-Hölder decompositions and recall a version of the Jordan-Hölder theorem, originally showed by C. Jordan and O. Hölder for groups, which is applicable to (a, b)-modules. We refer to [Bar93] and [Bar97] for a reference on the subject.

1.1 The (a, b)-modules

1.1.1 General (a, b)-modules

Definition 1.1. *Let $\mathbb{C}[[b]]$ denote the ring of formal series in the variable b. An (a, b)-module is an algebraic structure composed by a free $\mathbb{C}[[b]]$-module E of finite rank and a $\mathbb{C}$-linear application $a : E \to E$ that satisfies the commutation relation*

$$ab - ba = b^2, \tag{1.1}$$

where $b : E \to E$ is the multiplication by the element $b \in \mathbb{C}[[b]]$.

Given an (a, b)-module E, we will refer to its $\mathbb{C}[[b]]$-module structure as the **b-structure**, whereas the linear map a will be referred to as the **a-structure**. In the spirit of the category theory we will moreover define an $(\mathbf{a}, \mathbf{b})$**-morphism** as an application

$$\varphi : E \to F$$

between two (a, b)-modules E and F, which is a morphism of the underlying $\mathbb{C}[[b]]$-modules and respects the a-structure:

$$\varphi(ax) = a\varphi(x),$$

for any element $x \in E$. We will call φ an **isomorphism** (resp. **endomorphism**) of (a, b)-modules if it is bijective (resp. $E = F$).

An alternative definition of (a, b)-module was given in [Bar97]. Let

$$\tilde{A} = \left\{ \sum_{0}^{\infty} Q_p(a) b^p, \quad \text{with} \quad Q_p \in \mathbb{C}[a] \right\},$$

the $\mathbb{C}[a]$-module of formal series in b with coefficients in $\mathbb{C}[a]$ and define a multiplication that satisfies $ab - ba = b^2$ and which is continuous for the b-adic topology. It is a ring that contains $\mathbb{C}[[b]]$ as a subring and we can define (a, b)-modules as:

Definition 1.2 (Alternative definition)**.** *Let $\mathbb{C}[[b]] \subset \tilde{A}$ be the ring given above. An (a, b)-module E is a left $\tilde{A}$-module which is free and of finite rank, when considered as an $\mathbb{C}[[b]]$-module.*

While definition 1.2 puts (a, b)-modules in a more general context of the theory of modules over a non-commutative ring, we will prefer the definition 1.1 for the clarity of proofs that follows from it.

As a first property of (a, b)-modules we remark that from equation 1.1 we can directly derive by induction on n

$$ab^n - b^n a = nb^{n+1}, \qquad n \in \mathbb{N}, \tag{1.2}$$

which shows us that $a(b^n E) \subset b^n E$, for all $n \in \mathbb{N}$. This shows the continuity of a for the b-adic topology on E. The continuity of the map a gives us another formulation for (1.2),

$$aS(b) = S(b)a + S'(b)b^2, \tag{1.3}$$

with $S(b) \in \mathbb{C}[[b]]$ and $S'(b)$ the formal derivative, which has the advantage of a much more concise form.

Remark 1.3. Given an (a, b)-module E and a $\mathbb{C}[[b]]$-basis $\{v_i\}$, $1 \leqslant i \leqslant n$, we can deduce from (1.3) that the values of a are uniquely determined by its values $a(v_i)$ on the basis. On the other hand for any choice $\epsilon_i \in E$ of elements, we can use (1.3) to define an application a, such that

$$a(v_i) = \epsilon_i$$

and a satisfies the properties of an a-structure.

We will give now some examples of the (a, b)-module structure:

Example 1.4. Let $E = \mathbb{C}[[z]]$ be the ring of formal series in the variable z. We endow it with a $\mathbb{C}[[b]]$-structure given by the formal integration:

$$bS(z) = \int_0^z S(t)dt,$$

where $S(z) \in \mathbb{C}[[z]]$. The a-structure is defined as the multiplication by the element z:

$$aS(z) = zS(z).$$

If we take $e := 1$ as a basis of E, it follows from remark 1.3 that E can be identified as the only (a, b)-module of rank 1 that satisfies:

$$ae = be.$$

Example 1.5 (Brieskorn lattice)**.** The standard source of (a, b)-modules is the theory of complex hypersurfaces with isolated singularity and particularly the Brieskorn lattices defined in [Bri70]. Let

$$f : \mathbb{C}^{n+1} \to \mathbb{C}, \quad n \in \mathbb{N},$$

be a holomorphic function with an isolated singularity at the origin and suppose $f(0) = 0$. The Brieskorn module associated to this singularity is the vector space

$$D := \frac{\Omega_0^{n+1}}{\mathrm{df} \wedge \mathrm{d}\,\Omega_0^{n-1}},$$

where Ω_0^i are germs of holomorphic i-forms at the origin. We can define on this space two operations a and b that verify the relation of (a, b)-modules.

Let $[\omega]$ be a germ of an $(n + 1)$-form at the origin, then we define

$$[a\omega] = [f\omega].$$

Since ω is a germ of an holomorphic form, we can find by Poincaré's lemma a v such that $\omega = \mathrm{d}\,v$. Then we define

$$[b\omega] = [\mathrm{df} \wedge v].$$

This definition doesn't depend on the choice of v. In fact if we choose another element w such that $\omega = \mathrm{d}\,w$, v and w will differ by an element $\mathrm{d}\,\alpha$ with $\alpha \in \Omega_0^{n-1}$ and therefore:

$$\mathrm{df} \wedge v = \mathrm{df} \wedge w + \mathrm{df} \wedge \mathrm{d}\,\alpha,$$

which implies that $\mathrm{df} \wedge v$ and $\mathrm{df} \wedge w$ are in the same class modulo $\mathrm{df}\,\mathrm{d}\,\Omega_0^{n-1}$.

We can easily verify that

$$[ab\omega] = [a\,(\mathrm{df} \wedge v)] = [f\,(\mathrm{df} \wedge v)]$$
$$[b(a+b)\omega] = [b\,(f\omega + \mathrm{df} \wedge v)] = [b\,\mathrm{d}\,(fv)] = [f\,\mathrm{df} \wedge v]$$

and hence $ab = ba + b^2$. Moreover b is injective on the Brieskorn module. Consider in fact the complex:

$$0 \to \Omega_0^0 \to^{\wedge\,\mathrm{df}} \cdots \to^{\wedge\,\mathrm{df}} \Omega_0^{n+1} \to 0.$$

It is acyclic in every degree except $(n+1)$, because f has an isolated singularity. Suppose that we have $\omega = \mathrm{d}\,v$ and $b[\omega] = 0$. This translates into

$$\mathrm{df} \wedge v = \mathrm{df} \wedge \mathrm{d}\,u, \quad u \in \Omega_0^{n-1} \qquad \text{or, equivalently} \qquad \mathrm{df} \wedge (v - \mathrm{d}\,u) = 0.$$

But the complex above is acyclic in degree n, therefore we obtain

$$v - \mathrm{d}\,u = \mathrm{df} \wedge \alpha, \quad \alpha \in \Omega_0^{n-1},$$

and by differentiating both sides we obtain

$$\mathrm{d}\,v = -\,\mathrm{df} \wedge \mathrm{d}\,\alpha,$$

which shows us that $[\omega] = 0$.

The b-adic completion E of D is a free $\mathbb{C}[[b]]$-module of finite rank. The injectivity of b gives us an injection of D into its completion E and so we can extend by continuity the definition of a and b to the whole E and give it at the same time the structure of an (a, b)-module. The details of this construction can be found in [Bar93].

Many efforts in the theory of (a, b)-modules are put towards the characterisation of the (a, b)-modules that are associated to the Brieskorn lattice of a singularity via the construction in the example above. As it was proven in [Bar93], we can restrict our search to the regular (a, b)-modules defined in the following subsection.

1.1.2 Regularity of (a, b)-modules

As shown in [Bar93], we can associate to every meromorphic differential system in one variable z, which has a simple pole at $z = 0$, an (a, b)-module that satisfies the property in the following definition:

Definition 1.6. *An (a, b)-module E is called a **simple-pole** (a, b)-module if $aE \subset bE$.*

This justifies the use of the term "simple-pole".

Remark 1.7. Note that on a simple-pole (a, b)-module the application $b^{-1}a$ is well defined. We will use this fact in the future.

The structure of simple-pole (a, b)-modules was studied thoroughly in the article [Bar93], where we can find a complete classification. However, not all Brieskorn lattices give simple-pole (a, b)-modules and in order to be able to study those which are not, we need to define a bigger class: regular (a, b)-modules, which we will do in the following definitions.

Definition 1.8. *A sub-$\mathbb{C}[[b]]$-module F of an (a, b)-module E is called a **sub-$(\mathbf{a}, \mathbf{b})$-module** if it is stable for the action of a,*

$$a(F) \subset F.$$

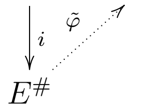
Definition 1.9. *An (a, b)-module E is called **regular** if it is a sub-(a, b)-module of a simple-pole (a, b)-module.*

For a regular (a, b)-module E we are interested in the "smallest" simple-pole (a, b)-module F that contains E as a sub-(a, b)-module. In order to compare two (a, b)-modules containing E we will stick to the algebraic convention of identifying an (a, b)-module E with its (a, b)-isomorphic image into another (a, b)-module and we will therefore give the following definition:

Definition 1.10. *Let E be a regular (a, b)-module and $E^{\#}$ a simple-pole (a, b)-module with an injective (a, b)-morphism $i : E \to E^{\#}$. We say that $E^{\#}$ is the **saturate** of the regular (a, b)-module E, if and only if it satisfies the following universal property: for every simple-pole (a, b)-module F and any morphism of (a, b)-modules*

$$\varphi : E \to F,$$

there exist an unique morphism $\tilde{\varphi} : E^{\#} \to F$, that makes the following diagram commutative:



To show the existence of the saturate, consider the following construction. Let E be an (a, b)-module and define a b-torsion-free $\mathbb{C}[[b]]$-module $E[b^{-1}]$ as

$$E[b^{-1}] = E \otimes_{\mathbb{C}[[b]]} \mathbb{C}[[b]][b^{-1}].$$

We can extend to $E[b^{-1}]$ the action of a on E by using the formula

$$a\left(v \otimes b^{-p}\right) = a(v) \otimes b^{-p} - pv \otimes b^{-p+1}. \tag{1.4}$$

The operation a on $E[b^{-1}]$ is well defined, since

$$\begin{aligned} a\left(bv \otimes b^{-p}\right) &= \left(ba(v) + b^2 v\right) \otimes b^{-p} - pbv \otimes b^{-p+1} \\ &= a(v) \otimes b^{-p+1} + (1-p)v \otimes b^{-p+2} \\ &= a\left(v \otimes b^{-p+1}\right) \end{aligned}$$

and satisfies $ab - ba = b^2$ as well. Hence $E[b^{-1}]$ satisfies all requirements for an (a, b)-module except that free and of finite rank is replaced by b-torsion-free. Moreover given two (a, b)-modules E and F and a morphism:

$$\varphi : E \to F,$$

there is only one way to extend it to a b-linear map between $E\left[b^{-1}\right]$ and $F\left[b^{-1}\right]$:

$$\begin{aligned} \tilde{\varphi} : E\left[b^{-1}\right] &\to F\left[b^{-1}\right] \\ v \otimes b^{-n} &\mapsto \varphi(v) \otimes b^{-n}. \end{aligned} \tag{1.5}$$

It is easy to verify using formula 1.4 that the resulting map $\tilde{\varphi}$ is also a-linear. We can now state the following proposition:

Proposition 1.11. *Let E be an (a, b)-module then the following are equivalent:*

(i) E is regular.

(ii) The sub-$\mathbb{C}[[b]]$-module of $E[b^{-1}]$:

$$E^{\#} = \sum_{k=0}^{\infty} \left(b^{-1}a\right)^k E$$

is of finite rank. In this case $E^{\#}$ is a simple-pole (a, b)-module for the induced structure and satisfies the universal property of the saturate.

Proof. The module $E^{\#}$ is clearly stable for $b^{-1}a$ and therefore stable for the morphism $a = b\left(b^{-1}a\right)$. If it has finite rank it is a simple-pole (a, b)-module and therefore E is regular.

Suppose E is regular and φ be an injective morphism into a simple-pole module F. Using formula 1.5 we can extend φ to an unique map

$$\tilde{\varphi} : E\left[b^{-1}\right] \to F\left[b^{-1}\right].$$

We remark that φ is injective if and only if $\tilde{\varphi}$ is. Moreover, since $(b^{-1}a)\,F \subset F$ we have

$$\tilde{\varphi}\left(\left(b^{-1}a\right)^k E\right) = \left(b^{-1}a\right)^k \tilde{\varphi}\,(E) \subset F,$$

the image by $\tilde{\varphi}$ of $E^{\#}$ is contained in F. Therefore the $E^{\#}$ is a sub-$\mathbb{C}[[b]]$-module of a finite type $\mathbb{C}[[b]]$-module and hence of finite rank. At the same time we can verify the universal property of $E^{\#}$ with the same procedure. □

The construction of the saturate $E^{\#}$ gives us as a corollary:

Corollary 1.12. *Given a regular (a,b)-module E and its saturate $E^{\#}$ the dimension over $\mathbb{C}$ of $E^{\#}/E$ is finite.*

Proof. The saturate $E^{\#} \subset E\,[b^{-1}]$ has finite rank, so it must be contained in $b^{-n}E$ for a certain $n \in \mathbb{N}$. Therefore the dimension over $\mathbb{C}$ of $E^{\#}/E$ is bounded by the dimension of $b^{-n}E/E$, which is $n \cdot \mathrm{rg}(E)$. □

Example 1.13. The smallest example of regular (a,b)-module which is not a simple-pole (a,b)-module occurs in rank two and is generated by two elements e_1 and e_2 with a defined as follows:

$$\begin{aligned} ae_1 &= \lambda b e_1 \\ ae_2 &= (\mu - 1) b e_2 + e_1 \end{aligned}$$

with λ and $\mu \in \mathbb{C}$. This (a,b)-module is contained in an (a,b)-module which is simple-pole and generated by $x = b^{-1}e_1$ and $y = (\lambda - \mu)e_2 - x$. It's easy to verify that:

$$\begin{aligned} ax &= (\lambda - 1) b x \\ ay &= (\mu - 1) b y \end{aligned}$$

which is trivially a simple pole (a,b)-module.

Example 1.14 (Non regular (a,b)-module)**.** D. Barlet shows in [Bar93] (example 2.1) that the (a,b)-module of rank two generated by two elements e_1 and e_2 satistfying

$$ae_1 = e_2$$
$$ae_2 = be_1$$

is not regular.

1.2 Jordan-Hölder composition series

In the theory of groups and modules we encounter composition series as a way to transform complex objects into simpler ones: simple groups and modules. The main result is given by the Jordan-Hölder theorem which states that any composition series is equivalent: up to a permutation the simple quotients are the same. We will introduce in this section the equivalent of composition series in the theory of (a, b)-modules. While many results are similar to those of group and module theory, there is no unicity of the quotients. Once again a complete reference can be found in [Bar93].

1.2.1 Normality

In the previous section we introduced the concept of sub-(a, b)-module. However, since in the general case quotients of an (a, b)-module E by a sub-(a, b)-module F are not (a, b)-modules, we will introduce normal (a, b)-modules and the basic properties they satisfy.

Definition 1.15. *A sub-(a, b)-module F of an (a, b)-module E is called* ***normal*** *if the $\mathbb{C}[[b]]$-module E/F is free. Equivalently if $bF = F \cap bE$.*

Clearly, since sub-(a, b)-modules are closed for the action of a, a induces a $\mathbb{C}$-linear application $\tilde{a}$ on the quotient E/F which satisfies $ab - ba = b^2$, so the definition of normality guarantees us that E/F has an induced structure of (a, b)-module.

Remark 1.16. Note that only $F \cap bE \subset bF$ has to be proved, since the reverse is always true. In the case of an (a, b)-module F of rank 1 generated by an element $x \in E$, the normality condition is equivalent to $x \notin bE$.

When the rank is arbitrary, we can prove the following:

Lemma 1.17. *Let E be an (a, b)-module and F a sub-module of E. Then F is normal if and only if the application*

$$\varphi : F/bF \to E/bE$$

induced by the inclusion of F in E is injective.

Proof. The kernel of φ is $F \cap bE$, so φ is injective if and only if we have $bF = F \cap bE$. □

The previous lemma, while quite obvious has two useful corollaries that we will use to test normality and compare normal (a, b)-modules.

Corollary 1.18. *Let F be a sub-(a,b)-module of an (a,b)-module E and $\{v_i\}$ be a $\mathbb{C}[[b]]$-basis of F. Then F is normal if and only if:*

$$\sum_i \alpha_i v_i \in bE, \qquad \alpha_i \in \mathbb{C}$$

implies $\alpha_i = 0$, $\forall i$.

Proof. It is enough to remark that the $\sum_i \alpha_i v_i$ are a set of representatives of the classes of F/bF. Hence by 1.17 F is normal if the only element belonging to bE is 0. □

Corollary 1.19. *Let E be an (a,b)-module and $F \subseteq G$ two normal sub-(a,b)-modules. Then they are equal if and only if they have the same rank.*

Proof. If $F = G$ their rank is equal.

On the other hand since F is normal in E, it is in particular normal in G. We remark that the dimensions of F/bF and G/bG are respectively the rank of F and G. Hence from lemma 1.17 we find out that $\varphi : F/bF \to G/bG$ is bijective, so F contains a $\mathbb{C}[[b]]$-basis of G and is therefore equal to G. □

Note that a normal sub-(a,b)-module F of a simple-pole module E is still simple-pole: we have $aF \subset bE \cap F = bF$. On the other hand any sub-(a,b)-module F of a regular (a,b)-module E is regular: they are both included in the saturate of E.

Similarly quotients of regular (resp. simple-pole) (a,b)-modules by a normal sub-(a,b)-module are regular (resp. simple-pole). A easy proof can be found in [Bar93] lemma 2.3.

1.2.2 Simple (a,b)-modules

Let us introduce the second fundamental part of our decomposition. As in the case of other algebraic categories, the basic blocks to build a Jordan-Hölder sequences are simple (a,b)-modules given by the following definition:

Definition 1.20. *An (a,b)-module E is called simple if and only if its only normal sub-(a,b)-modules are 0 and E.*

Example 1.21. We note $\mathbf{E}_\lambda$ with $\lambda \in \mathbb{C}$ the (a,b)-module of rank one generated by one element e_λ which satisfies $ae_\lambda = \lambda b e_\lambda$.

Since it is of rank 1 it follows directly from corollary 1.19 that E_λ is simple. Moreover it is a simple-pole (a,b)-module, hence regular. In fact these are the only regular simple (a,b)-modules, as we will show in the following subsection.

We will call the module E_λ the **elementary** (a,b)-module of parameter λ.

Example 1.22 (Simple (a, b)-module of rank 2)**.** The example of non regular (a, b)-module given at the end of subsection 1.1.2 is simple. Let, in fact, e_1 and e_2 be its generators that satisfy

$$\begin{aligned} ae_1 &= e_2 \\ ae_2 &= be_1. \end{aligned}$$

Since it has rank 2 in order to show that it's simple, it is enough to show that it doesn't contain any normal sub-(a, b)-module of rank 1. Let proceed by contradiction and suppose there is a normal sub-(a, b)-module F of rank 1 generated by an element $f \in F$. By eventually multiplying f by an invertible element of $\mathbb{C}[[b]]$, we can assume it is of the form $e_1 + S(b)e_2$ or $S(b)e_1 + e_2$, with $S(b) \in \mathbb{C}[[b]]$.

In the first case we have to solve the equation:

$$a\left(e_1 + S(b)e_2\right) = e_2 + S(b)be_1 + S'(b)b^2e_2 = T(b)\left(e_1 + S(b)e_2\right),$$

for an unknown $T(b) \in \mathbb{C}[[b]]$. Clearly we must have $T(b) = S(b)b$ looking at the coefficients of e_1. Therefore by identifying the coefficients of e_2 on the right and left side of the equation, we must have:

$$1 + S'(b)b^2 = S^2(b)b,$$

which is impossible, since the right side lacks a constant term.

In a similar manner we proceed for the case $f = S(b)e_1 + e_2$. The equation

$$a\left(S(b)e_1 + e_2\right) = S(b)e_2 + S'(b)b^2e_1 + be_1 = T(b)\left(S(b)e_1 + e_2\right),$$

gives us $T(b) = S(b)$ and we deduce another equation for $S(b)$:

$$b + S'(b)b^2 = S^2(b),$$

which does not have any solutions: if we write $S(b) = \sum_i s_i b^i$ we obtain that the constant term s_0^2 of $S^2(b)$ must be zero, hence the term $2s_0s_1b$ of degree 1 must be zero too, while the left side has a term of degree 1 equal to b.

1.2.3 Composition series

In this subsection we will expose the theory of Jordan-Hölder composition series in the context of (a, b)-modules. We will restrain ourselves to regular (a, b)-modules, for which D. Barlet proved the existence of the composition series and gave some invariance properties ([Bar93]). The general case, on the other hand, remains mainly unexplored.

Definition 1.23. *Let E be an (a,b)-module $n \in \mathbb{N}$ and $\{F_i\}$ a sequence of normal sub-(a,b)-modules of E, for $0 \leqslant i \leqslant n$ such that:*

$$0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_n = E.$$

The sequence $\{F_i\}$ is called a Jordan-Hölder composition series of E if and only if all quotients F_i/F_{i-1}, for $1 \leqslant i \leqslant n$, are simple (a,b)-modules.

Since (a,b)-modules have finite rank, we are always assured of the existence of a Jordan-Hölder composition series. We have indeed the following:

Proposition 1.24. *Let E be an (a,b)-module, then it admits a composition series:*

$$0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_r = E,$$

where $r \in \mathbb{N}$ and F_i are normal sub-(a,b)-modules of E and F_i/F_{i-1} are simple for all i.

Proof. We will prove this result by induction.

It follows for corollary 1.19 that every (a,b)-module of rank 1 must be simple.

Let E be an (a,b)-module of rank $n \in \mathbb{N}$ and assume the result true for all (a,b)-modules of rank less than n. Then we can have two cases: either E is simple and we have nothing to prove, or E has a normal non trivial sub-(a,b)-module F. Corollary 1.19 guarantees us that the rank of F is strictly less than n.

From our induction step follows that F and E/F admit Jordan-Hölder composition series,

$$0 = G_0 \subsetneq G_1 \subsetneq \cdots \subsetneq G_s = F$$

with G_i normal in F and

$$0 = H_0 \subsetneq H_1 \subsetneq \cdots \subsetneq H_t = E/F.$$

with H_i normal in E/F. Using lemma 1.17 we see that the natural application $\varphi_i : G_i/bG_i \to F/bF$ is injective for each i and so is the natural application $\vartheta : F/bF \to E/bE$. The composition $\vartheta \circ \varphi_i$ is the natural application between G_i/bG_i and E/bE and it is injective. Hence the modules G_i are also normal in E.

Let $\pi : E \to E/F$ be the projection of E onto E/F and let us define $H_i' = \pi^{-1}(H_i)$. The sub-(a,b)-modules H_i' are normal sub-(a,b)-modules of E. We can therefore combine the two sequences into a composition series of E:

$$0 = G_0 \subsetneq \cdots \subsetneq G_s = F = H_0' \subsetneq \cdots \subsetneq H_t' = E$$

□

Remark 1.25. If E is regular, then the sub-(a, b)-modules and quotients of the composition series are also regular.

We introduced in 1.21 a class of simple regular (a, b)-modules of rank 1. The following proposition, for which we give a condensed proof (cf. [Bar93]), shows us that they are the only regular simple (a, b)-modules.

Proposition 1.26. *Let E be a regular (a, b)-module, then it contains a normal simple sub-(a, b)-module of rank 1 of the form E_λ.*

Proof. We will reduce ourselves to the case of a simple-pole (a, b)-module. Using the result from lemma 1.12 we find a simple-pole (a, b)-module $E^\#$ that contains E and such that $E^\#/E$ is of finite dimension over $\mathbb{C}$.

Suppose now that there is a normal sub-(a, b)-module F of $E^\#$ of rank 1 generated by an element e and let $\lambda \in \mathbb{C}$ such that $ae = \lambda be$ (since F must be simple-pole). Since $E^\#/E$ is of finite dimension there is an $n \in \mathbb{N}$ such that $b^n e \in E$.

Let n_0 be the minimal n such that $b_0^n e \in E$ We assert that the sub-(a, b)-module F' of E generated by $b^{n_0}x$ is normal and isomorphic to $E_{\lambda+n_0}$ as an (a, b)-module.

In fact since $ae = \lambda be$, by using 1.2 we obtain $a.(b^{n_0})e = (\lambda + n_0)b.b^{n_0}e$, which shows that F' is isomorphic to $E_{\lambda+n_0}$.

On the other side F' is normal in E: if $n_0 > 0$ the minimality of this number guarantees us that $b^{n_0}e$ does not belong to bE (otherwise $b^{n_0-1}e \in E$) and the corollary 1.18 gives us the normality of F'.

If, however, $n_0 = 0$ we have by hypothesis that $F' = F$ is normal in $E^\#$, hence normal in E: if $E^\#/F$ is free, so is its sub-$\mathbb{C}[[b]]$-module E/F.

We can therefore reduce ourselves to show the proposition for a **simple-pole** (a, b)-module E, which we will do in the following lemma. □

Lemma 1.27 ([Bar93])**.** *Let E be a simple-pole (a, b)-module and f the $\mathbb{C}$-morphism induced by $b^{-1}a$ on E/bE. For each $\lambda \in \mathbb{C}$ eigenvalue of f let:*

$$\lambda_{min} = \min \{\lambda + j, \quad j \in \mathbb{Z} | \lambda + j \in \mathrm{Spec}(f)\}$$

then E has an element x that satisfies $ax = \lambda_{min} bx$ and the sub-(a, b)-module F generated by x is normal.

Proof. We'll build recursively a sequence of $x_i \in E$, $i \in \mathbb{N}$, such that

$$\left(b^{-1}a - \lambda_{min}\right) \left(\sum_{k=0}^{n} b^k x_k \right) \in b^{n+1}E \qquad \forall n \in \mathbb{N}$$

and since E is complete for the b-adic topology and a is continous the series

$$x = \sum_{k=0}^{\infty} b^k x_k$$

will converge and it will satisfy:

$$\left(b^{-1}a - \lambda_{min}\right) x \in \bigcap_n b^n E = 0$$

and therefore $(b^{-1}a - \lambda_{min})x = 0$ or $ax = \lambda_{min} bx$ as we wanted

Since by hypothesis $\lambda_{min} \in \mathrm{Spec}(f)$ there exists an x_0 that satisfies

$$(b^{-1}a - \lambda_{min})x_0 \in bE$$

.

Suppose now that for an $n \geqslant 0$ we have $n+1$ elements x_k, for i ranging between 0 and n, such that:

$$\left(b^{-1}a - \lambda_{min}\right) \left(\sum_{k=0}^{n} b^k x_k \right) = b^{n+1} y$$

for a certain $y \in E$. We are looking for an x_{n+1} such that

$$\left(b^{-1}a - \lambda_{min}\right) b^{n+1} x_{n+1} - b^{n+1} y \in b^{n+2} E$$

and by using formula 1.2 we obtain:

$$b^{n+1} \left(\left(b^{-1}a - \lambda_{min} + n + 1\right) x_{n+1} - y \right) \in b^{n+2} E,$$

the injectivity of b allowing us to write:

$$\left(b^{-1}a - (\lambda_{min} - n - 1)\right) x_{n+1} - y \in bE.$$

By the minimality property of λ_{min} we know that

$$\left(b^{-1}a - (\lambda_{min} - n - 1)\right)$$

is bijective on E/bE, we can therefore find the y we were looking for.

We find in this way an $x = \sum_{k=0}^{\infty} b^k x_k$ that satisfies $ax = \lambda_{min} bx$. It must also verify $x \notin bE$, otherwise let k be an integer such that $b^{-k}x \in E$, but $b^{-k} \notin bE$. This element is not 0 in E/bE and satisfies

$$a\left(b^{-k}x\right) = \left(\lambda_{min} - k\right) b \left(b^{-k}x\right),$$

which contradicts the minimality of λ_{min}. According to the corollary 1.18 this implies that the sub-(a,b)-module F generated by x is normal in E. □

We can now give the classification of all regular simple (a, b)-modules.

Corollary 1.28. *All regular simple (a, b)-modules are of rank 1 and are isomorphic to one of the modules E_λ.*

Proof. Apply proposition 1.26 to a simple regular (a, b)-module E. It follows that E must be of rank 1 and contain an (a, b)-module of the form E_λ. By corollary 1.19 we must have $E = E_\lambda$. □

Given the important role that the elements e verifying $ae = \lambda be$ play in the theory of (a, b)-modules, we will use from now on the following definition:

Definition 1.29. *Let E be an (a, b)-module and $x \in E$. We call x an monomial of E of type $(\lambda, 0)$ with $\lambda \in \mathbb{C}$ if and only if x satisfies:*

$$ax = \lambda bx.$$

1.2.4 Non unicity of composition series

As we already anticipated before, if E is a regular (a, b)-module and:

$$0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_n = E$$

the quotients are not unique and may in fact depend upon the decomposition. The following example will enlighten the situation:

Example 1.30. We recall the regular (a, b)-module E given in example 1.13. It satisfies:

$$\begin{aligned} ae_1 &= \lambda be_1 \\ ae_2 &= (\mu - 1)\, be_2 + e_1 \end{aligned}$$

for a basis $\{e_1, e_2\}$ and two complex numbers λ and μ. Let define:

$$e_1' = e_1 + (\mu - \lambda)\, be_2.$$

The couple $\{e_1', e_2\}$ is still a basis of E and it satisfies:

$$\begin{aligned} ae_1' &= \lambda be_1 + (\mu - \lambda)\left((\mu - 1)\, b^2 e_2 + be_1 + b^2 e_2\right) \\ &= (\lambda + \mu - \lambda)\, be_1 + \mu\, (\mu - \lambda)\, b^2 e_2 = \mu b e_1' \\ ae_2 &= (\mu - 1)\, be_2 + e_1' + (\lambda - \mu)\, be_2 = (\lambda - 1)\, be_2 + e_1' \end{aligned}$$

Thus we have two different Jordan-Hölder composition series. One given by the basis $\{e_1, e_2\}$:

$$0 \subsetneq F \subsetneq E,$$

with F being the module generated by e_1. The two quotients are respectively: $F \simeq E_\lambda$ and $E/F \simeq E_{\mu-1}$. The other decomposition is given by the other basis:

$$0 \subsetneq F' \subsetneq E,$$

where F' is generated by e'_1 and the quotients are: $F' \simeq E_\mu$ and $E/F' \simeq E_{\lambda-1}$.

Now if we choose $\lambda \neq \mu$, the quotients of the two Jordan-Hölder composition are in four different isomorphism classes..

Even if the unicity is not guaranteed, there are however some properties that don't depend upon the Jordan-Hölder sequence choosen. The following theorem is due to D. Barlet:

Theorem 1.31 ([Bar93]). *Let E be a regular (a, b)-module and $0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_n = E$ a Jordan-Hölder decomposition whose quotients are respectively $F_i/F_{i-1} \simeq E_{\lambda_i}$ for $1 \leqslant i \leqslant n$ and the $\lambda_i \in \mathbb{C}$. Then the following does not depend upon the decomposition choosen:*

(i) The complex number $\sum_{i=1}^n \lambda_i$.

(ii) The polynomial $\prod_{i=1}^n (z - \exp(2i\pi\lambda_i))$ (or the number of λ_i in each class of $\mathbb{C}$ modulo $\mathbb{Z}$ is constant).

Chapter 2

Duality and hermitian forms

In this chapter we will introduce the concept of duality for an (a, b)-module E. This notion was introduced by D. Barlet in [Bar97] and further developed by R. Belgrade in [Bel01]. The first section will be devoted to the comparison of the definitions of dual (a, b)-module given by D. Barlet and R. Belgrade. We will retain here the definition of D. Barlet, while we'll reduce R. Belgrade's definition of δ-dual to the concept of adjoint (a, b)-module and tensor product. We will moreover introduce hermitian forms on (a, b)-module and relate them to isomorphisms of an (a, b)-module with its adjoint. The second and last section will deal with the existence of hermitian non degenerate forms on (a, b)-modules. For this purpose we will introduce the concept of indecomposable modules and give the decomposition of an (a, b)-module into its indecomposable parts.

2.1 Duality

2.1.1 Barlet's definition

Let E and F be two (a, b)-modules. As defined by D. Barlet in [Bar97], the $\mathbb{C}[[b]]$-module $\mathrm{Hom}_{\mathbb{C}[[b]]}(E, F)$ of $\mathbb{C}[[b]]$-linear maps from E to F has a natural structure of (a, b)-module provided by an operator Λ that satisfies

$$(\Lambda\varphi)(x) = a_F(\varphi(x)) - \varphi(a_E x), \tag{2.1}$$

where $\varphi \in \mathrm{Hom}_{\mathbb{C}[[b]]}(E, F)$, x is an element of E and a_E and a_F are the a-structures of E and F respectively. We will designate this (a, b)-module with the notation $\mathrm{Hom}_{(a,b)}(E, F)$. For notation's sake we will denote a_E, a_F and Λ all by the letter a and to avoid the confusion that such a notation could pose we should read the expression

$$a \cdot \varphi(x)$$

as $(\Lambda\varphi)(x)$, whereas the expression $a_E(\varphi(x))$ will keep the conventional notation

$$a\varphi(x).$$

We will therefore rewrite the equation 2.1 as:

$$a \cdot \varphi(x) = a\varphi(x) - \varphi(ax).$$

Remark 2.1. Note that whereas $\mathrm{Hom}_{\mathbb{C}[[b]]}(E, F)$ is a free $\mathbb{C}[[b]]$-modules of finite rank, it was proven in theorem 1 bis of [Bar97] that the vector space $\mathrm{Hom}_{\tilde{A}}(E, F)$ of (a, b)-morphisms between E and F is of finite $\mathbb{C}$-dimension and therefore can not have a structure of (a, b)-module. No confusion can therefore arise when talking about the (a, b)-module $\mathrm{Hom}_{(a,b)}(E, F)$.

By choosing E_0 for the codomain of the morphisms, we can give the following definition:

Definition 2.2 (Barlet)**.** *Let E be an (a, b)-module and E_0 the elementary (a, b)-module of parameter* 0*, then we call the module*

$$\mathrm{Hom}_{(a,b)}(E, E_0)$$

*the **dual** (a, b)**-module** of E and note it by E^*.*

Remark 2.3. When considering only the b-structure of E, the $\mathbb{C}[[b]]$-module E^* corresponds exactly to the definition of dual of a $\mathbb{C}[[b]]$-module, since $E_0 = \mathbb{C}[[b]]e_0$, with $ae_0 = 0$.

Remark 2.4. The dual of the elementary (a, b)-module E_λ is isomorphic to $E_{-\lambda}$. In fact, if we consider the basis $\{e_\lambda\}$ of E_λ and $\{e^*_\lambda\}$ the dual basis of $(E_\lambda)^*$ such that

$$e^*_\lambda(e_\lambda) = e_0,$$

where e_0 is the generator of E_0, we just have to check the action of a on e^*_λ to obtain the result.

We have therefore:

$$a \cdot e^*_\lambda(e_\lambda) = ae^*_\lambda(e_\lambda) - e^*_\lambda(ae_\lambda) = ae_0 - e^*_\lambda(\lambda be_\lambda) = -\lambda be_0 = -\lambda be^*_\lambda(e_\lambda)$$

and we have so verified the equality

$$a \cdot e^*_\lambda = -\lambda b \cdot e^*_\lambda$$

on the $\mathbb{C}[[b]]$-basis $\{e^*_\lambda\}$ and we can conclude.

While the example above shows us that an (a,b)-module is not in general isomorphic to its dual, there exists always an isomorphism with its bidual $E^{**} = (E^*)^*$. In fact:

Proposition 2.5. *Let E be an (a,b)-module and $E^{**} = (E^*)^*$ its bidual then the natural application*

$$\begin{aligned} \mathrm{can} : E &\to E^{**} \\ v &\mapsto (\hat{v} : \varphi \mapsto \varphi(v))\,, \end{aligned}$$

where $v \in E$ and $\varphi \in E^*$, is an isomorphism.

Proof. Setwise the application 'can' is the same as the canonical application between E considered as $\mathbb{C}[[b]]$-module and its $\mathbb{C}[[b]]$-module dual. Therefore it is an isomorphism of $\mathbb{C}[[b]]$-modules.

We have just to verify that it preserves the a-structure. We will denote the action of a on E^{**} by $a \cdot \cdot$, which satisfies:

$$a \cdot \cdot \hat{v}(\varphi) = a\hat{v}(\varphi) - \hat{v}(a \cdot \varphi),$$

for each $\hat{v} \in E^{**}$ and $\varphi \in E^*$. With this notation we have

$$\begin{aligned} (a \cdot \cdot \,\mathrm{can}(v))\,(\varphi) &= a\,\mathrm{can}(v)(\varphi) - \mathrm{can}(v)(a \cdot \varphi) = \\ &= a\varphi(v) - a \cdot \varphi(v) = a\varphi(v) - a\varphi(v) + \varphi(av) = \mathrm{can}(av)(\varphi), \end{aligned}$$

for $\varphi \in E^*$ and $v \in E$. □

Another important property of dual (a,b)-modules is that the duality functor that associates an (a,b)-module E to its dual and every morphism $\varphi : E \to F$ to the morphism

$$\begin{aligned} \varphi^* : F^* &\to E^* \\ \psi &\mapsto \varphi \circ \psi \qquad \psi \in F^* \end{aligned}$$

is exact. We have in fact the following result:

Proposition 2.6. *Let E, F and G be (a,b)-modules and suppose that there exist an exact sequence of (a,b)-modules:*

$$0 \to F \xrightarrow{i} E \xrightarrow{\pi} G \to 0,$$

then the following sequence is also exact:

$$0 \to G^* \xrightarrow{\pi^*} E^* \xrightarrow{i^*} F^* \to 0,$$

where i^* (resp. π^*) send an element φ into $\varphi \circ i$ (resp. $\varphi \circ \pi$).

Proof. The dual sequence is exact as complex of $\mathbb{C}[[b]]$-modules and $\circ i$ and $\circ\varphi$ are a-linear. $\square$

By relating on the definition 1.2 of (a,b)-module, we can describe the dual of an (a,b)-module E in an alternative way.

Let E be an (a,b)-module and $\tilde{A}$ be the non commutative ring introduced in subsection 1.1.1. Then we can consider the object

$$\mathrm{Ext}^1_{\tilde{A}}\left(E,\tilde{A}\right),$$

which we can see at the same time as a right module over the ring $\tilde{A}$ or equivalently as a left module over the opposite ring $\tilde{A}^{op}$. Consider the morphism of rings defined by

$$\begin{aligned} \vartheta : \tilde{A} &\to \tilde{A}^{op} \\ 1 &\mapsto 1 \\ b &\mapsto b \\ a &\mapsto -a \\ x\cdot_{\tilde{A}} y &\mapsto \vartheta(x)\cdot_{\tilde{A}^{op}}\vartheta(y) = \vartheta(y)\cdot_{\tilde{A}}\vartheta(x), \end{aligned}$$

for x and $y \in \tilde{A}$ and $\cdot_{\tilde{A}}$ and $\cdot_{\tilde{A}^{op}}$ the multiplications of the rings $\tilde{A}$ and $\tilde{A}^{op}$ respectively. The morphism ϑ is the identity on the commutative subring generated by b and it maps a to $-a$. It is easy to verify that this is in fact an isomorphism.

Then we can define the dual (a,b)-module as follows:

Definition 2.7 (Alternative definition)**.** *Let E be an (a,b)-module, then the dual (a,b)-module is the left $\tilde{A}^{op}$-module*

$$E^* = \mathrm{Ext}^1_{\tilde{A}}\left(E,\tilde{A}\right),$$

with the structure as left $\tilde{A}$-module given by ϑ:

$$\begin{aligned} \tilde{A}\times E^* &\to E^* \\ (x,v) &\mapsto \vartheta(x)v \end{aligned}$$

2.1.2 Belgrade's definition

In [Bel01] R. Belgrade gives a different definition of δ-dual (a,b)-module, for δ in $\mathbb{C}$. We shall show that Belgrade's 0-dual is in fact the dual of the conjugate as defined in this subsection, while the general δ-dual can be expressed in terms of the 0-dual and the tensor product, as we will show in the following subsection. Let us begin with Belgrade's definition:

Definition 2.8 (Belgrade). *Let E be an (a,b)-module. We call δ-dual of E, $\delta \in \mathbb{C}$, the $\mathbb{C}[[b]]$-module*

$$\mathrm{Hom}_{\mathbb{C}[[b]]}(E, E_\delta)$$

endowed with an (a,b)-structure defined as:

$$\begin{aligned}[a\cdot\varphi](x) &= \varphi(ax) - a\varphi(x)\\ [b\cdot\varphi](x) &= -b\varphi(x) = \varphi(-bx),\end{aligned}$$

with $\varphi \in \mathrm{Hom}_{\mathbb{C}[[b]]}(E, E_\delta)$ and $x \in E$.

In order to conciliate the two definitions, we will introduce the concepts of conjugate and adjoint (a,b)-modules.

As in the case of the complex field $\mathbb{C}$, the ring of formal series $\mathbb{C}[[b]]$ also admits a rather natural involution

$$\begin{aligned}\breve{} : \mathbb{C}[[b]] &\to \mathbb{C}[[b]]\\ S(b) &\mapsto \breve{S}(b) = S(-b),\end{aligned}$$

where $S(b) \in \mathbb{C}[[b]]$. This remark allows us to define the conjugate of an (a,b)-module in the same way as one defines the conjugate of a complex vector space.

Definition 2.9. *Let E be an (a,b)-module. We call (a,b)-**conjugate** of E and note it $\breve{E}$ the complex vector space E itself, endowed with an a- and b-structure given by:*

$$\begin{aligned}a \cdot_{\breve{E}} v &= -a \cdot_E v\\ b \cdot_{\breve{E}} v &= -b \cdot_E v,\end{aligned}$$

where $\cdot_{\breve{E}}$ and $\cdot_E$ denote the (a,b)-structure of $\breve{E}$ and E respectively.

Since we change signs of both a and b, the formula $ab - ba = b^2$ is still verified.

Remark 2.10. The conjugate of an elementary module E_λ is isomorphic to the module itself. Given a basis e_λ, this isomorphism can be written:

$$\begin{aligned}\Phi : E_\lambda &\to \breve{E}_\lambda\\ S(b)e_\lambda &\mapsto S(-b)e_\lambda \qquad (2.2)\end{aligned}$$

for $S(b) \in \mathbb{C}[[b]]$.

Remark 2.11. An (a, b)-module is not necessarily isomorphic to its conjugate. We can take, for example, the (a, b)-module of rank 2 generated by two elements x and y that satisfy:

$$\begin{aligned} ax &= \lambda bx \\ ay &= \lambda by + (1 + \alpha b)\, x, \end{aligned}$$

where λ and $\alpha \in \mathbb{C}$ and $\alpha \neq 0$. Its conjugate satisfies

$$\begin{aligned} ax &= \lambda bx \\ ay &= \lambda by + (1 - \alpha b)\, x, \end{aligned}$$

and the classification of rank 2 regular (a, b)-modules, given in [Bar93] implies that the two modules are not isomorphic.

One can see immediately that the conjugate of the conjugate $\left(\breve{E}\right)\breve{}$ of an (a, b)-module E is the (a, b)-module itself.

On the other hand let E and F be (a, b)-modules and φ a morphism between E and F. Since $\varphi(-ax) = -a\varphi(x)$ and $\varphi(-bx) = -b\varphi(x)$, for all $x \in E$ the application φ is also a morphism between the conjugates $\breve{E}$ and $\breve{F}$. We call conjugation functor the functor that associates to every (a, b)-module its conjugate and to every morphism, the morphism itself. Its easy to see that such a functor is exact. By combining proposition 2.6 and the remark above we have:

Proposition 2.12. *Let E, F and G be (a, b)-modules and*

$$0 \to E \to F \to G \to 0$$

be an exact sequence, then the sequence obtained by applying the duality and conjugation functors to the complex above is also exact:

$$0 \to \breve{G}^* \to \breve{F}^* \to \breve{E}^* \to 0,$$

with $\breve{E}^*$, $\breve{F}^*$ and $\breve{G}^*$ the conjugates of the duals of E, F and G.

With the terminology given above the **Belgrade's definition** of **0-dual (a, b)-module** corresponds exactly to the definition of **conjugate of the dual of an (a, b)-module**. For notation's sake we will call **adjoint** of an (a, b)-module the conjugate of its dual and we will call adjoint functor the composition of the conjugation and duality functors. An (a, b)-module which is isomorphic to its adjoint will be called **self-adjoint**.

We will give an equivalent of the δ-dual for a generic $\delta \in \mathbb{C}$ in the next subsection after introducing the tensor product of (a, b)-modules.

2.1.3 Bilinear forms and tensor product

In order to define $\mathrm{Hom}_{(a,b)}(E,F)$ we used the equivalent object for its underlying b-structure. We will introduce in a similar manner the concept of bilinear map of (a,b)-modules:

Definition 2.13. *Let E, F and G be two (a,b)-modules. An (a,b)**-bilinear form** on $E \times F$ is a $\mathbb{C}[[b]]$-linear map Φ,*

$$\Phi : E \times F \to G,$$

that satisfies the following property:

$$a\Phi(x,y) = \Phi(ax,y) + \Phi(x,ay).$$

Remark 2.14. If Φ is an (a,b)-bilinear map on $E \times F$ with values in G and v is an element of E:

$$\Phi_v := \Phi(v,\cdot) : w \mapsto \Phi(v,w) \qquad w \in F$$

is not necessarily an (a,b)-morphism. However the map $\pi : v \mapsto \Phi_v$ is an (a,b)-morphism between E and $\mathrm{Hom}_{(a,b)}(F,G)$. We have in fact:

$$\pi(av)(x) = \Phi_{av}(x) = a\Phi_v(x) - \Phi_v(ax) = a \cdot \Phi_v(x) = a\pi(v).$$

Inherently linked to the concept of (a,b)-bilinear maps is that of tensor products, that allows a more practical manipulation of these objects.

Definition 2.15. *Let E and F be two (a,b)-modules. We call (a,b)-tensor product of E and F and write it as $E \otimes_{(a,b)} F$ the $\mathbb{C}[[b]]$-module*

$$E \otimes_{\mathbb{C}[[b]]} F$$

endowed with an a-structure defined as follows:

$$a\,(v \otimes w) = (av) \otimes w + v \otimes (aw)$$

for every $v \in E$ and $w \in F$.

The a-structure we gave on $E \otimes_{(a,b)} F$ is well defined. We have in fact:

$$a\,(bv \otimes w) = a(bv) \otimes w + bv \otimes a(w) = ba(v) \otimes w + b^2 v \otimes w + v \otimes ba(w) = \\ a(v) \otimes bw + v \otimes a(bw) = a\,(v \otimes bw)\,,$$

for each $v \in E$, $w \in F$ and it satisfies $ab - ba = b^2$:

$$\begin{aligned} a\,(bv \otimes w) - ba\,(v \otimes w) = ba(v) \otimes w + b^2 v \otimes w + bv \otimes a(w) \\ - ba(v) \otimes w - bv \otimes a(w) = b^2\,(v \otimes w)\,. \end{aligned}$$

We can easily verify that the tensor product defined in this way satisfies the usual universal property: there exists a bilinear map

$$\Phi : E \times F \to E \otimes_{(a,b)} F,$$

such that for every bilinear map Ψ on $E \times F$ with values in a third (a, b)-module G, there exists a unique (a, b)-morphism $\tilde{\Psi}$ from $E \otimes_{(a,b)} F$ into G that makes the following diagram commutative:

$$\begin{array}{ccc} E \times F & \xrightarrow{\Psi} & G \\ \Big\downarrow{\scriptstyle \Phi} & \nearrow{\scriptstyle \tilde{\Psi}} & \\ E \otimes_{(a,b)} F. & & \end{array}$$

We can take as Φ the natural application

$$\begin{aligned} \Phi : E \times F &\to E \otimes_{(a,b)} F \\ (v, w) &\mapsto v \otimes_{(a,b)} w \end{aligned}$$

and define $\tilde{\Psi}$ as:

$$\begin{aligned} \tilde{\Psi} : E \otimes_{(a,b)} F &\to G \\ v \otimes_{(a,b)} w &\mapsto \Psi(v, w) \end{aligned}$$

The unicity of $\tilde{\Psi}$ follows directly from the universal property of the tensor product of $\mathbb{C}[[b]]$-modules. We need only verify that the map is a-linear. We will do it on the generators $v \otimes_{(a,b)} w$ of $E \otimes_{(a,b)} F$, for $v \in E$ and $w \in F$:

$$\begin{aligned} \tilde{\Psi}\left(a(v \otimes_{(a,b)} w)\right) = \tilde{\Psi}\left((av) \otimes_{(a,b)} w + v \otimes_{(a,b)} (aw)\right) = \\ \Psi(av, w) + \Psi(v, aw) = a\Psi(v, w) = a\tilde{\Psi}(v \otimes_{(a,b)} w). \end{aligned}$$

In a similar manner, by using the properties of the tensor product of $\mathbb{C}[[b]]$-modules, we can derive the other properties of the equivalent object in the theory of (a, b)-modules.

Lemma 2.16. *Let E, F and G be three (a, b)-modules, then the tensor product satisfies the following properties:*

(i)

$$E \otimes_{(a,b)} F \simeq F \otimes_{(a,b)} E,$$

(ii)

$$\left(E \otimes_{(a,b)} F\right) \otimes_{(a,b)} G \simeq E \otimes_{(a,b)} \left(F \otimes_{(a,b)} G\right),$$

(iii)

$$\left(E \otimes_{(a,b)} F\right)^* \simeq E^* \otimes_{(a,b)} F^*,$$

(iv)

$$\left(E \otimes_{(a,b)} F\right)^{\smile} \simeq \breve{E} \otimes_{(a,b)} \breve{F},$$

(v) The (a,b)-morphism

$$\begin{aligned} \Phi : E &\to E \otimes_{(a,b)} E_0 \\ v &\mapsto v \otimes_{(a,b)} e_0 \end{aligned}$$

where e_0 is a generator of the elementary (a,b)-module E_0, is an isomorphism.

(vi) We have the following isomorphism of (a,b)-modules:

$$\begin{aligned} E^* \otimes_{(a,b)} F &\to \mathrm{Hom}_{(a,b)}\left(E, F \otimes_{(a,b)} E_0\right) \\ \varphi \otimes_{(a,b)} y &\mapsto \left(\Phi : x \mapsto y \otimes_{(a,b)} \varphi(x)\right), \end{aligned}$$

where $\varphi \in E^*$, $x \in E$ and $y \in F$.

From property (v) and (vi) we have $E^* \otimes_{(a,b)} F \simeq \mathrm{Hom}_{(a,b)}(E, F)$, which in turn allows us to find an alternative description of the δ-dual of an (a,b)-module. In fact from definition 2.8 follows that the δ-dual of an (a,b)-module is the module

$$\mathrm{Hom}_{(a,b)}\left(\breve{E}, E_\delta\right),$$

which in turn can be rewritten as $\breve{E}^* \otimes_{(a,b)} E_\delta$.

We will call an (a,b)-bilinear application on $E \times F$ with values in G, an (a,b)-bilinear form if $G = E_0$. In the rest of this chapter we will deal with the existence of nondegenerate hermitian forms on (a,b)-modules. We will need therefore the following definitions.

Definition 2.17. *Let E and F be two (a,b)-modules and Φ a bilinear form on $E \times F$. We say that Φ is* ***nondegenerate****, if the (a,b)-morphism $v \mapsto \Phi(v, \cdot)$ is an isomorphism of E with F^*.*

Definition 2.18. *Let E be an (a,b)-module. A* ***sesquilinear*** *form on E is a bilinear form on $E \times \breve{E}$.*

Remark 2.19. Since a nondegenerate sequilinear form on an (a,b)-module E induces an isomorphism of E with its adjoint $\breve{E}^*$ it follows that not all (a,b)-modules are self-adjoint (e.g. E_λ with $\lambda \neq 0$ is not) and not every (a,b)-module admits a nondegenerate sesquilinear form.

Consider now a sesquilinear form Φ on E. By applying to it the conjugate functor we obtain a bilinear map $\breve{\Phi}$ on $\breve{E} \times E$ with values in $\breve{E}_0$. If we fix an isomorphism of $\breve{E}_0$ with E_0, we can consider $\breve{\Phi}$ as a sequilinear form on $\breve{E}$. Under this assumption, we define (a,b)-hermitian and anti-(a,b)-hermitian forms as:

Definition 2.20. *Let E be an (a,b)-module. An (a,b)-sesquilinear form H on E is called (a,b)-**hermitian** (respectively **anti-**(a,b)**-hermitian**) if it satisfies:*

$$H(v,w) = \breve{H}(w,v),$$

$$\left(\textit{respectively } H(v,w) = -\breve{H}(w,v)\right).$$

where $v \in E$, $w \in \breve{E}$ and $\breve{H}$ is the sesquilinear form on $\breve{E}$ defined above.

We have already shown that in order to admit a nondegenerate sesquilinear form, an (a,b)-module must be self-adjoint. We will refine the concept of self-adjoint by defining:

Definition 2.21. *Let E be a self-adjoint (a,b)-module. We say that E is **hermitian** (resp. **anti-hermitian**), if it admits a nondegenerate hermitian (resp. anti-hermitian) form.*

Let E be an (a,b)-module endowed with a hermitian form and consider $\Phi : E \to \breve{E}^*$ to be the linear form associated to the hermitian form via the remark 2.14.

We can translate the hermitian property into the identity between Φ and its adjoint $\breve{\Phi}^* : E \to \breve{E}^*$. In fact while $\Phi(v)$, for $v \in E$ is the linear map:

$$\varphi : w \mapsto \Phi(v,w), \qquad w \in \breve{E},$$

the adjoint map $\breve{\Phi}^*$ sends an element $v \in E = E^{**}$ to the map:

$$\varphi : w \mapsto v\left(\breve{\Phi}(w,\cdot)\right) = \breve{\Phi}(w,v).$$

We will use this formulation extensively in the following section.

Note moreover that to give an isomorphism Φ from an (a,b)-module E and its δ-dual $\breve{E}^* \otimes_{(a,b)} E_\delta$ is equivalent to specifying an isomorphism between $E \otimes_{(a,b)} E_{-\delta/2}$ and

$$\breve{E}^* \otimes_{(a,b)} E_\delta \otimes_{(a,b)} E_{-\delta/2} \simeq \breve{E}^* \otimes_{(a,b)} E_{\delta/2}.$$

Since we have

$$\left(\widetilde{E \otimes_{(a,b)} E_{-\delta/2}}\right)^* \simeq \check{E}^* \otimes_{(a,b)} \check{E}^*_{-\delta/2} \simeq \check{E}^* \otimes_{(a,b)} E_{\delta/2},$$

we can identify an isomorphism of E with its δ-dual with an hermitian form on $E \otimes_{(a,b)} E_{-\delta/2}$.

2.2 Existence of hermitian forms

We will analyze in this section the existence of nondegenerate hermitian forms on regular (a,b)-modules, which will be necessarily self-adjoint as of remark 2.19. We will proceed in two steps: in the first two subsections we will reduce ourselves to a subclass of (a,b)-modules called indecomposable (a,b)-modules and show that every regular (a,b)-module can be decomposed into the direct sum of indecomposable ones and that this decomposition is unique.

In the last subsection we will show that a self-adjoint (a,b)-module which is indecomposable admits at least a hermitian or anti-hermitian form. The result is optimal since there are examples that admit only a hermitian or only an anti-hermitian form.

2.2.1 Indecomposable (a,b)-modules

Definition 2.22. *Let E be an (a,b)-module. We say that E is* ***indecomposable*** *if it cannot be written as direct sum $F \oplus G$ of non zero (a,b)-modules.*

Since whenever we decompose an (a,b)-module E into a direct sum of (a,b)-modules $E = F \oplus G$ the rank of the components is strictly less than the rank of E, by proceeding by induction for every (a,b)-module E we can find a decomposition into a sum of indecomposable (a,b)-modules:

$$E = \bigoplus_{i=1}^{r} F_i,$$

where $r \in \mathbb{N}$ and F_i are indecomposable sub-(a,b)-modules.

We are interested in the question whether the isomorphism classes of the F_i are unique and do not depend upon the decomposition. We will need to this purpose an intermediary result:

Proposition 2.23. *Let E be a regular and indecomposable (a,b)-module. Then every endomorphism of E is either bijective or nilpotent.*

The proof of this proposition will need several steps beginning with a definition:

Definition 2.24. *Let E be a regular (a,b)-module and $\lambda \in \mathbb{C}$. We define:*

$$V_\lambda = \left\{ \sum F_i | F_i \subset E, F_i \simeq E_\lambda \right\},$$

the sum of all sub-(a,b)-modules of E isomorphic to E_λ.

The object V_λ is clearly a sub-(a,b)-module. We will use V_λ as an induction step in the proof of proposition 2.23, by choosing such a λ that V_λ is normal:

Proposition 2.25. *Let E be a regular (a,b)-module, $\lambda \in \mathbb{C}$ and:*

$$\lambda_{min} = \inf_j \{\lambda + j | j \in \mathbb{Z} \text{ and } \exists x \in E, ax = (\lambda + j)bx\}$$

be the minimal $\lambda + j$ such that E contains a monomial of type $(\lambda + j, 0)$.

Then $V_{\lambda_{min}}$ is a normal sub-(a,b)-module of E isomorphic as (a,b)-module to the direct sum of a finite number of copies of $E_{\lambda_{min}}$.

Proof. We will use two facts.

First, for every $W \simeq \bigoplus E_{\lambda_{min}}$ sub-(a,b)-module of E, W is normal in E. Let in fact $\{e_i\}$ be a basis of W with $1 \leqslant i \leqslant p$ the rank of W. Suppose by absurd that there exist some $x \in W$ which is in bE, but not in bW.

By eventually translating x by an element of bW, we can assume $x = \sum_{i=1}^p \alpha_i e_i$, $\alpha_i \in \mathbb{C}$. We can easily verify that

$$ax = \lambda_{min} bx$$

but now if $x = by$ we must have:

$$ay = (\lambda_{min} - 1)by,$$

and since $y \in E$ it contradicts the minimality of λ_{min}.

On the other hand we can show that $V_{\lambda_{min}}$ is a direct sum of $E_{\lambda_{min}}$. In fact let W be the largest (inclusionwise) direct sum of copies of $E_{\lambda_{min}}$ included in $V_{\lambda_{min}}$. We remark that since W is normal, for any sub-(a,b)-module F isomorphic to $E_{\lambda_{min}}$ only one of two cases is possible: either

$$W \cap F = \{0\} \text{ or } F \subset W.$$

If $W \cap F \neq \{0\}$, let e be the generator of F and $S(b)b^n e \in W$ with $S(0) \neq 0$, then $S(b)e \in W$ by normality and $e = S^{-1}(b)S(b)e \in W$. We have therefore $F \subset W$.

If W contains every sub-(a,b)-module module isomorphic to $E_{\lambda_{min}}$, then it is equal to $V_{\lambda min}$. Otherwise there is an F such that $W \cap F = \{0\}$, hence $W \oplus F$ is still in $V_{\lambda min}$, which contradicts the maximality of W. □

We will now use the sub-(a, b)-module $V_{\lambda_{min}}$ to show the following proposition

Proposition 2.26. *Let E be a regular (a, b)-module and let φ be a morphism between E and itself. Then φ is bijective if and only if φ is injective.*

Proof. To show that bijectivity follows from injectivity, we will proceed by induction on the rank of the module.

If E is of rank 1 the statement of the proof is satisfied: in fact E must be isomorphic to one of the E_λ and the only b-linear morphisms from a E_λ to itself that are also a-linear are those that send the generator e to αe, $\alpha \in \mathbb{C}$. They are all bijective for $\alpha \neq 0$.

Let now E be of rank $n > 1$. We can find, by lemma 1.27, a λ_{min} that satisfies the minimality property of the previous proposition. Hence the module $V_{\lambda_{min}}$ is normal and isomorphic to a direct sum of $E_{\lambda_{min}}$.

Let $\{e_i\}$ be a basis of $V_{\lambda_{min}}$ composed of monomials of type $(\lambda_{min}, 0)$ and let x another monomial of type $(\lambda_{min}, 0)$. We want to show that x is a linear combination of the elements of the basis, with coefficients in $\mathbb{C} \subset \mathbb{C}[[b]]$.

From the definition of $V_{\lambda_{min}}$ follows that $x \in V_{\lambda_{min}}$. Suppose now that $x = \sum_i S_i(b)e_i$ and let us apply a to both sides. We obtain:

$$ax = \sum_i \left(\lambda_{min} S_i(b) b e_i + S_i'(b) b^2 e_i\right) = \lambda_{min} b x + \sum_i S_i'(b) b^2 e_i$$

and since x is a monomial of type $(\lambda_{min}, 0)$, we must have $S'(b)_i = 0$ for all i and therefore

$$x = \sum_i S_i(0) e_i,$$

as we wanted.

Let $\varphi : E \to E$ be an injective endomorphism of E and $\{e_i\}$ a basis of $V_{\lambda_{min}}$. Every $\varphi(e_i)$ is a monomial of type $(\lambda_{min}, 0)$ and therefore is an element of $V_{\lambda_{min}}$. The restriction of φ to $V_{\lambda_{min}}$ is therefore an endomorphism of $V_{\lambda_{min}}$:

$$\varphi|_{V_{\lambda_{min}}} : V_{\lambda_{min}} \to V_{\lambda_{min}}.$$

Moreover since the coefficients of the $\varphi(e_i)$ in our base are complex constants, $\varphi|_{V_{\lambda_{min}}}$ behaves as a linear application between finite dimensional spaces: in particular if it is injective, it is also surjective.

In order to apply our induction hypothesis let us consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V_{\lambda_{min}} & \hookrightarrow & E & \twoheadrightarrow & E/V_{\lambda_{min}} & \longrightarrow & 0 \\
 & & \downarrow \varphi & & \downarrow \varphi & & \downarrow \tilde{\varphi} & & \\
0 & \longrightarrow & V_{\lambda_{min}} & \hookrightarrow & E & \twoheadrightarrow & E/V_{\lambda_{min}} & \longrightarrow & 0
\end{array}$$

where $\tilde{\varphi}$ is the (a, b)-linear morphism induced on the quotient. As we showed the first downward arrow is bijective.

The third arrow $\tilde{\varphi}$ is injective: suppose in fact that we have two different classes with representatives x and $y \in E$ that map to the same class modulo $V_{\lambda_{min}}$. Then $\varphi(x - y)$ is in $V_{\lambda_{min}}$. From the bijectivity of $\varphi|_{V_{\lambda_{min}}}$ we can find an element $v \in V_{\lambda_{min}}$ such that

$$\varphi(x - y) = \varphi(v)$$

which in turn implies $x - y = v$ by the injectivity of φ, which contradicts the fact that x and y are in distinct classes modulo $V_{\lambda_{min}}$.

Since the rank of $E/V_{\lambda_{min}}$ is strictly inferior to the rank of E, we can apply the induction hypothesis to show that $\tilde{\varphi}$ is also bijective.

It follows from a basic result of homological algebra that the second arrow is bijective if it is injective. □

We can now consider endomorphisms that are not necessarily injective. Once again the structure of (a, b)-modules does not differ essentially from that of finite vector spaces over $\mathbb{C}$:

Lemma 2.27. *Let E be a regular (a, b)-module and φ an endomorphism of E. Then E splits into the direct sum of two φ-stable sub-(a, b)-modules F and N, with φ bijective on F and nilpotent on N.*

Proof. Consider the sequence of normal sub-(a, b)-modules

$$K_n = \operatorname{Ker} \varphi^n, \quad n \in \mathbb{N}.$$

Since two normal sub-(a, b)-modules $F \subset G$ are equal if and only if they have the same rank, the sequence of K_n stabilizes beginning with a certain integer m: $K_m = K_{m+1}$.

On the other hand if we consider the sequence $I_n = \operatorname{Im} \varphi^n$, let us look at the restriction of φ to I_m:

$$\varphi|_{I_m} : I_m \to I_{m+1} \subset I_m.$$

This restriction is injective: if $y = \varphi^m(x) \in \operatorname{Ker} \varphi$, then $x \in K_{m+1} = K_m$. Hence $y = \varphi^m(x) = 0$. From the previous proposition we deduce that this restriction is in fact bijective, which means that $I_{m+1} = \varphi(I_m) = I_m$.

We can now take $F = I_m$ and $N = K_m$. They are clearly stable by φ. We will show that $E = F \oplus N$.

We have in fact $\operatorname{Ker} \varphi \cap F = \{0\}$, since the restriction of φ to I_m is injective. *À fortiori*, since $K \subset \operatorname{Ker} \varphi$ we have $F \cap N = \{0\}$.

Let's take an element $x \in E$. Since $I_m = I_{2m}$ we can find an element $y \in E$ such that $\varphi^m(x) = \varphi^{2m}(y)$ and call k the element $x - \varphi^m(y)$. Thus we can write x as a sum:

$$x = \varphi^m(y) + k$$

of an element $\varphi^m(y) \in I_m$ and an element $k \in K_m$, which implies that:

$$E = N \oplus F.$$

The restriction of φ to N is nilpotent, since $\varphi|_N^m = 0$, while we already showed that the restriction to $I_m = F$ is bijective. □

We have now all the elements necessary to prove proposition 2.23:

Proof. Let E be a regular indecomposable (a, b)-module and φ an endomorphism of E. Then by lemma 2.27 splits E into a sum

$$E = N \oplus F$$

of two (a, b)-modules, with φ nilpotent on N and bijective on F. But E is indecomposable, therefore either $N = 0$ and φ is bijective or $F = 0$ and φ is nilpotent. □

2.2.2 Krull-Schmidt theorem

This subsection will be devoted to the proof of a version of the Krull-Schmidt theorem for the theory of (a, b)-modules. The principal argument of the proof will be proposition 2.23 from the previous subsection.

Theorem 2.28 (Krull-Schmidt for (a, b)-modules)**.** *Suppose that we have two decompositions into direct sum of a regular (a, b)-module E:*

$$E = \bigoplus_{i=1}^{m} E_i$$

$$E = \bigoplus_{i=1}^{n} F_i$$

where $m, n \in \mathbb{N}$ and all E_i and F_i are indecomposable (a, b)-modules. Then $m = n$ and up to a reindexing of the modules E_i is isomorphic to F_i for all $1 \leqslant i \leqslant n$.

For the proof of this theorem we need a couple of lemmas:

Lemma 2.29. *Let E be a regular indecomposable (a, b)-module and φ an automorphism of E. Suppose moreover that $\varphi = \varphi_1 + \varphi_2$. Then at least one of φ_1, φ_2 is an isomorphism.*

Proof. Be applying φ^{-1} to both terms, we can assume without loss of generality that $\varphi = Id$ is the identity automorphism.

The two endomorphisms φ_1 and φ_2 commute. In fact:

$$\varphi_1\varphi_2 - \varphi_2\varphi_1 = \varphi_1(\varphi_1 + \varphi_2) - (\varphi_2 + \varphi_1)\varphi_1 = \varphi_1 - \varphi_1 = 0.$$

By lemma 2.23 the φ_i can be either nilpotent or isomorphisms. If they were both nilpotent, their sum would be nilpotent, which is absurd. Hence the result. □

Remark 2.30. By subsequently applying the previous lemma, we can extend the result to the sum of more than two endomorphisms.

Lemma 2.31. *Let E and F be indecomposable regular (a, b)-modules and $\alpha : E \to F$ and $\beta : F \to E$ two (a, b)-linear morphisms. Suppose that $\beta \circ \alpha$ is an isomorphism, then α and β are also isomorphisms.*

Proof. Let prove that $F = \operatorname{Im}\alpha \bigoplus \operatorname{Ker}\beta$. If $\alpha(x) \in \operatorname{Ker}\beta$, we have

$$\beta \circ \alpha(x) = 0,$$

hence $x = 0$ and therefore

$$\operatorname{Im}\alpha \cap \operatorname{Ker}\beta = \{0\}.$$

Consider now an element $x \in F$ and let

$$y = \alpha \circ (\beta \circ \alpha)^{-1} \circ \beta(x).$$

We have

$$\beta(x - y) = \beta(x) - \beta(y) = \beta(x) - (\beta \circ \alpha) \circ (\beta \circ \alpha)^{-1} \circ \beta(x) = \beta(x) - \beta(x) = 0.$$

We can thus write x as sum of an element y of $\operatorname{Im}\alpha$ and an element $x - y$ of $\operatorname{Ker}\beta$. This implies $F = \operatorname{Im}\alpha \bigoplus \operatorname{Ker}\beta$.

Now since $\beta \circ \alpha$ is injective, so must be α and $\operatorname{Im}\alpha$ can not be 0. But F is indecomposable therefore we must have $\operatorname{Im}\alpha = F$ and $\operatorname{Ker}\beta = 0$. It from proposition 2.26 that α is bijective and $\beta = (\beta \circ \alpha) \circ \alpha^{-1}$ must be also bijective. □

Proof of Krull-Schmidt theorem for (a, b)-modules. We will show this theorem by induction on m.

If $m = 1$, then E is indecomposable and we must have $n = 1$ and $E_1 \simeq F_1$.

In the general case consider the morphisms

$$q_i = \pi_i \circ p_1,$$

where the π_is are the projections on F_i and the p_js are the projections on E_j. Let consider the sum:

$$\sum_i p_1 \circ q_i = p_1 \circ \sum_i \pi_i \circ p_1 = p_1 \circ p_1 = p_1,$$

is the identity on the component E_1. By the lemma 2.2.2, there is an i such that $p_1 \circ q_i|_{E_1} : E_1 \to E_1$ is an isomorphism. Suppose, without loss of generality, it is $p_1 \circ q_1$, then by the lemma 2.31 $q_1|_{E_1} = \pi_1 : E_1 \to F_1$ is an isomorphism.

In order to apply the induction hypothesis, let note $G = \sum_{i=2}^m F_i$. We want to show that $E_1 \oplus G$ is equal to $E = F_1 \oplus G$. Since π_1 is an isomorphism of E_1 onto F_1 and its kernel is G we must have

$$E_1 \cap G = \{0\} :$$

if $x \in E_1 \cap G$, then $\pi_1(x) = 0$, but π_1 restricted to E_1 is injective, so $x = 0$. On the other hand every element of E can be written as $v + w$ with $v \in F_1$ and $w \in G$. If $y \in E_1$ is such that $\pi_1(y) = v$, then we have:

$$v + w = y + \pi_1(y) - y + w,$$

and $\pi_1(y) - y \in W$ by definition of π_1. We can then conclude that $E_1 \oplus G = E = E_1 \oplus \sum_{i=2}^m E_i$.

We have immediately $E/E_1 \simeq G \simeq \sum_{i=2}^m E_i$ and we can apply the induction hypothesis to G. □

We can now focus on finding self-adjoint isomorphisms of an (a, b)-module E with its adjoint $\breve{E}^*$. The Krull-Schmidt theorem will be useful to show the following decomposition:

Proposition 2.32. *Let E be a regular self-adjoint (a, b)-module. Then E is isomorphic to:*

$$E \simeq \bigoplus_{i=1}^r \left(F_i^{\oplus \alpha_i}\right) \oplus \bigoplus_{i=1}^s \left(G_i \oplus \breve{G}_i^*\right)^{\oplus \beta_i}$$

where r and s as well as the α_i and β_i are positive integers. The F_i are self-adjoint (a, b)-modules and the G_i are non self-adjoint (a, b)-modules. The isomorphism classes of the F_i, G_i and $\breve{G}_i^*$ are all disjoint.

Proof. Consider a decomposition of E into indecomposable (a, b)-modules

$$E = \sum_i E_i.$$

Since E is self-adjoint we have another decomposition given by

$$E \simeq \check{E}^* = \sum_i \check{E}_i^*.$$

The Krull-Schmidt theorem assures us that the factors are unique up to a permutation. So we can divide the E_i into two groups.

In the first group we find the self-adjoint components F_i with a certain multiplicity.

In the second one we find the non self-adjoint components G_i with the respective multiplicity. Since the two decompositions $\sum_i E_i$ and $\sum_i \check{E}_i^*$ must contains the same modules up to a permutation, the multiplicity of the G_i and the $\check{G}_i^*$ must be equal. □

Remark 2.33. From the proposition above we can immediately see that the non self-adjoint part of the decomposition always admits a hermitian nondegenerate form. In fact if we consider the module $G_i \oplus \check{G}_i^*$, a hermitian form can be given by:

$$\begin{aligned} \Phi : G_i \oplus \check{G}_i^* &\to (G_i \oplus \check{G}_i^*)^* = \check{G}_i^* \oplus G_i \\ (x, y) &\mapsto (y, x). \end{aligned}$$

If the multiplicity of a self-adjoint term F_i is pair, we fall into the same situation.

The case of an unpair multiplicity of a self-adjoint component is far more interesting and we will study it in the next subsection.

2.2.3 Hermitian forms on regular and indecomposable (a, b)-modules

As already noted in the previous subsection, the situation of an indecomposable self-adjoint (a, b)-module concerning hermitian forms is not so simple as in the complex vector space case and the existence is not always guaranteed. We have in fact the following theorem:

Theorem 2.34. *Let E be a regular indecomposable self-adjoint (a, b)-module and $E \neq \{0\}$. Then it admits a hermitian nondegenerate form or an anti-hermitian one.*

Proof. Let $\Phi : E \to \check{E}^*$ be any isomorphism of E with its dual and pose $M = \Phi^{-1}\check{\Phi}^*$. Consider now the two endomorphisms of E given by:

$$Id + M$$

and

$$Id - M$$

they commute and can be either isomorphisms or nilpotent, since E is indecomposable. But if they were both nilpotent, their sum $2Id$ would be nilpotent too, which is absurd.

If $Id + M$ is an isomorphism, so is $S = \Phi + \check{\Phi}^*$, which is associated to a nondegenerate hermitian form. The bijectivity of $Id - M$ on the other hand gives us an isomorphism $A = \Phi - \check{\Phi}^*$, which comes from an anti-hermitian form. □

Example 2.35. Note that both cases of the previous theorem are equally possible.

The simplest example of a regular self-adjoint indecomposable (a, b)-module which admits a hermitian form is the elementary (a, b)-module E_0 with the isomorphism that sends the generator e to its adjoint $\check{e}^*$.

In order to obtain an anti-hermitian form, we can consider for a given $\lambda, \mu \in \mathbb{C}$ the (a, b)-module E of rank 4, generated by $\{e_1, e_2, e_3, e_4\}$ which satisfies:

$$\begin{aligned} ae_1 &= \lambda b e_1 \\ ae_2 &= \mu b e_2 + e_1 \\ ae_3 &= -\mu b e_3 + e_1 \\ ae_4 &= -\lambda b e_4 + e_2 - e_3 \end{aligned} \tag{2.3}$$

whose adjoint basis satisfies:

$$\begin{aligned} a \cdot \check{e}_4^* &= \lambda b \check{e}_4^* \\ a \cdot \check{e}_3^* &= \mu b \check{e}_3^* - \check{e}_4^* \\ a \cdot \check{e}_2^* &= -\mu b \check{e}_2^* + \check{e}_4^* \\ a \cdot \check{e}_1^* &= -\lambda b \check{e}_1^* + \check{e}_3^* + \check{e}_2^* \end{aligned}$$

It is now easy to check that the map given by:

$$\begin{aligned}\Phi : E &\to E^*\\ e_1 &\mapsto \check{e}_4^*\\ e_2 &\mapsto -\check{e}_3^*\\ e_3 &\mapsto \check{e}_2^*\\ e_4 &\mapsto -\check{e}_1^*\end{aligned}$$

is an isomorphism and is anti-self-adjoint.

If we choose λ and μ such that none of the numbers 2λ, 2μ and $\lambda \pm \mu$ is an integer, this module is indecomposable and does not admit a hermitian form.

We can show that the structure of this module is in fact very rigid and the only elements of the (a, b)-module that verify the equations 2.3 are the 4-tuplets $\{\alpha e_1, \alpha e_2, \alpha e_3, \alpha e_4\}$, for α ranging among the complex numbers: let x_1, x_2, x_3 and x_4 be four non zero elements that satisfy:

$$\begin{aligned}x_1 &= \lambda b x_1\\ x_2 &= \mu b x_2 + x_1\\ x_3 &= -\mu b x_3 + x_1\\ x_4 &= -\lambda b x_4 + x_2 - x_3\end{aligned}$$

Let $x_1 = Re_1 + Se_2 + Te_3 + Ue_4$, for R, S, T and $U \in \mathbb{C}[[b]]$, then the following equation is verified:

$$\begin{aligned}a(Re_1 + Se_2 + Te_3 + Ue_4) = \left(\lambda Rbe_1 + R'b^2e_1\right) +\\ \left(\mu Sbe_2 + Se_1 + S'b^2e_2\right) + \left(-\mu Tbe_3 + Te_1 + T'b^2e_3\right) +\\ \left(-\lambda Ube_4 + Ue_2 - Ue_3 + U'b^2e_4\right) = \left(\lambda Rb + R'b^2 + S + T\right) e_1 +\\ \left(\mu Sb + S'b^2 + U\right) e_2 + \left(-\mu Tb + T'b^2 - U\right) e_3 + \left(-\lambda Ub + U'b^2\right) e_4. \quad (2.4)\end{aligned}$$

Moreover the equation $\alpha S = bS'$, where $\alpha \in \mathbb{C}$ and $S \in \mathbb{C}[[b]]$ has non trivial solutions only if α is an integer.

By identifying the left and right term of the equation $ax_1 = \lambda b x_1$, we deduce $-2\lambda Ub + U'b^2 = 0$, hence $U = 0$ since $2\lambda \notin \mathbb{Z}$. In the same way we obtain

$$\begin{aligned}-\left(\lambda + \mu\right) Tb + T'b^2 = 0\\ -\left(\lambda - \mu\right) Sb + S'b^2 = 0\end{aligned}$$

which implies S and T equal to 0 too. Therefore we are left with $Rbe_1+R'b^2e_1$ which has as solution $R=\alpha$ for $\alpha\in\mathbb{C}$. We obtain in this way $x_1=\alpha e_1$.

In the same way we prove that if $x_2=\mu bx_2+x_1$ the only solution is $x_2=\alpha e_2$. If we let $x_2=Re_1+Se_2+Te_3+Ue_4$ we obtain in fact:

$$\begin{aligned}-(\lambda+\mu)\,Ub+U'b^2=0 &\quad \Rightarrow U=0\\ -2\mu Tb+T'b^2=0 &\quad \Rightarrow T=0\\ S'b^2=0 &\quad \Rightarrow S=\beta,\quad \beta\in\mathbb{C},\end{aligned}$$

and finally

$$\lambda Rb+R'b^2+\beta=\alpha,$$

which forces $\beta=\alpha$ when looking at the rank 0.

A similar proof gives us $x_3=\alpha e_3$ and $x_4=\alpha e_4$. We have showed therefore that the unique automorphisms of E are multiplications by a complex number. All the isomorphisms of E with its adjoint are therefore of the form $\alpha\Phi$ and are all anti-hermitian.

The same fact shows us that E is indecomposable. In fact if by absurd $E=F\oplus G$, E would possess at least another automorphism, e.g. the application which is the identity on F and $-Id$ on G.

Chapter 3

Self-adjoint composition series

Given the non unicity of the Jordan-Hölder composition series in the theory of (a, b)-modules, we are interested whether the particularities of certain (a, b)-modules can be transmitted to their composition series. This chapter will focus on the properties of Jordan-Hölder composition series of self-adjoint (a, b)-modules. In particular we will prove that a self-adjoint composition series alway exists for such (a, b)-modules.

3.1 Self-adjoint composition series

Consider a regular (a, b)-module E of rank $n \in \mathbb{N}$ and a Jordan-Hölder decomposition of itself:

$$0 = F_0 \subsetneq F_1 \subsetneq \ldots \subsetneq F_n = E.$$

with $F_j/F_{j-1} \simeq E_{\lambda_j}$, the elementary (a, b)-module of parameter λ_j. We say that the sequence is **self-adjoint** if $\lambda_{n-j+1} = -\lambda_j$ for all $1 \leqslant j \leqslant n$ and the (a, b)-module F_{n-j}/F_j is self-adjoint for all $0 \leqslant j \leqslant [n/2]$.

We shall prove the following theorem in the case of regular hermitian or anti-hermitian (a, b)-modules and we will extend it successively to all regular self-adjoint (a, b)-modules.

Theorem 3.1. *Let E be a regular hermitian or anti-hermitian (a, b)-module, then it has a Jordan-Hölder sequence*

$$0 = F_0 \subsetneq F_1 \subsetneq \ldots \subsetneq F_n = E$$

which is self-adjoint.

Before proving the theorem we shall introduce a couple of lemmas.

Lemma 3.2. *Let E be a regular hermitian (a,b)-module (respectively anti-hermitian) and $\Phi : E \to \breve{E}^*$ a self-adjoint (resp. anti-self-adjoint) isomorphism. If there exists F_1 normal sub-(a,b)-module isomorphic to E_λ such that $\Phi(F_1)(F_1) = 0$, then there exists a normal sub-(a,b)-module F_{n-1} of rang $n-1$ such that $E/F_{n-1} \simeq \breve{F_1}^*$ and F_{n-1}/F_1 is hermitian (resp. anti-hermitian).*

Proof. Let e_λ be the generator of F_1 and H the hermitian form associated to Φ by $H(x,y) = \Phi(y)(x)$ and consider

$$F_{n-1} := \{x \in E | H(e_\lambda, x) = 0\}.$$

We remark that the condition $H(e_\lambda, e_\lambda) = 0$ gives us $F_1 \subset F_{n-1}$ and F_{n-1} is normal, because it is the kernel of a morphism. Let's consider the following exact sequence:

$$0 \to F_1 \to E \to E/F_1 \to 0$$

by theorem 2.6 we can pass to the adjoint sequence:

$$0 \to (\overbrace{E/F_1})^* \to \breve{E}^* \xrightarrow{\pi} \breve{F}_1^* \to 0.$$

Let $K = \operatorname{Ker}\pi$, since π is the restriction morphism of forms on E to the sub-(a,b)-module F_1, we can describe K as follows:

$$K = \{\varphi \in \breve{E}^* | \varphi(F_1) = 0\}$$

since $(\overbrace{E/F_1})^*$ is isomorphic to K, we can consider it as the sub-(a,b)-module of $\breve{E}^*$ whose elements annihilate F_1.

Consider now the restriction of Φ to F_{n-1}.

$$\Phi : F_{n-1} \to \breve{E}^*.$$

Since by the definition of F_{n-1}, $\Phi(x)(e_\lambda) = 0$ for all $x \in F_{n-1}$, we obtain that $\Phi(F_{n-1}) \subset (\overbrace{E/F_1})^*$. On the other side since for all $\varphi \in (\overbrace{E/F_1})^*$ the element $y = \Phi^{-1}(\varphi)$ satisfies $\Phi(y)(e_\lambda) = 0$ we also have $(\overbrace{E/F_1})^* \subset \Phi(F_{n-1})$. Hence $\Phi(F_{n-1}) = (\overbrace{E/F_1})^*$ and since Φ is an (a,b)-linear isomorphism F_{n-1} is isomorphic to its image by Φ: $(\overbrace{E/F_1})^*$.

Let look at the following exact sequence:

$$0 \to (F_{n-1}/F_1) \to (E/F_1) \to (E/F_{n-1}) \to 0$$

and its adjoint sequence:

$$0 \to (\overbrace{E/F_{n-1}})^* \xrightarrow{i} (\overbrace{E/F_1})^* \xrightarrow{\pi} (\overbrace{F_{n-1}/F_1})^* \to 0.$$

π is the restriction application on the forms of $(\widecheck{E/F_1})^*$. $\operatorname{Ker}\pi$ is thus the forms of $(\widecheck{E/F_1})^*$ that annihilate $(\widecheck{F_{n-1}/F_1})^*$ or with the convention of the previous paragraph, the forms of $\check{E}^*$ that annihilate F_1 and F_{n-1}:

$$\operatorname{Ker}\pi = \{\varphi \in \check{E}^* \text{ s.t. } \varphi(F_{n-1}) = 0\}$$

since $F_1 \subset F_{n-1}$. We note that since Φ is self-adjoint (resp. anti-self-adjoint)

$$\Phi(e_\lambda)(F_{n-1}) = \pm\Phi(F_{n-1})(e_\lambda) = 0$$

and hence $\Phi(F_1) \subset \operatorname{Ker}\pi$. An easy calculation shows that $\operatorname{Ker}\pi$ is of rank 1. Since $\Phi(F_1)$ is normal, of rank 1 and included into $\operatorname{Ker}\pi$, they must be equal.

We obtain $(\widecheck{E/F_{n-1}})^* \simeq \operatorname{Ker}\pi \simeq F_1$. Now we know that Φ sends F_{n-1} onto $(E/F_1)^*$ and F_1 onto $\operatorname{Ker}\pi$, so starting with the following exact sequence:

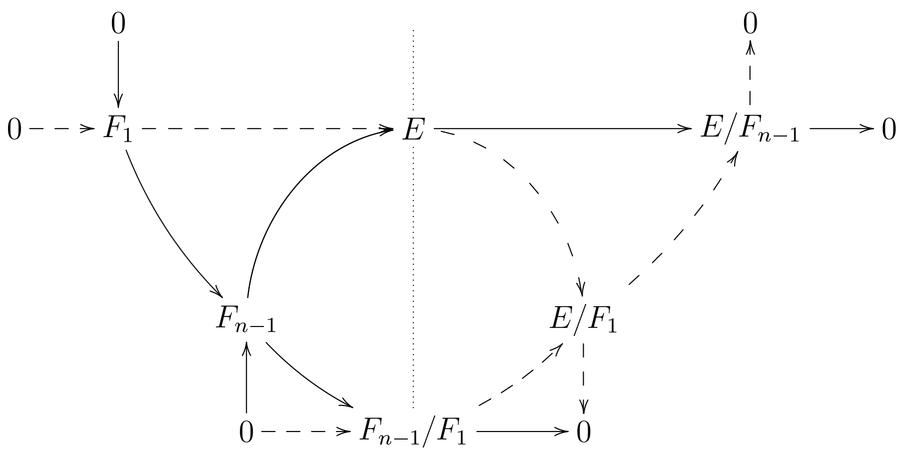
$$0 \to \operatorname{Ker}\pi \hookrightarrow (E/F_1)^* \xrightarrow{\pi} (F_{n-1}/F_1)^* \to 0$$

we can obtain another by substituting $\operatorname{Ker}\pi$ with F_1 and $(E/F_1)^*$ with F_{n-1}:

$$0 \to F_1 \to F_{n-1} \to (\widecheck{F_{n-1}/F_1})^* \to 0.$$

or in other terms $(\widecheck{F_{n-1}/F_1})^* \simeq (F_{n-1}/F_1)$. Note that the isomorphism is given by $x \to \Phi(x)|_{F_{n-1}}$ and is therefore hermitian.

Let resume the results with a graph of interwoven exact sequences:



Modules in symmetric positions with respect to the dotted line are each other's adjoints. □

Remark 3.3. If $ae_\lambda = \lambda be_\lambda$ and $2\lambda \notin \mathbb{N}$, then $H(e_\lambda, e_\lambda) = 0$. In fact $H(e_\lambda, e_\lambda) \in E_0$ must verify:

$$aH(e_\lambda, e_\lambda) = H(ae_\lambda, e_\lambda) + H(e_\lambda, -ae_\lambda) = 2\lambda bH(e_\lambda, e_\lambda)$$

which has non-trivial solutions in E_0 only if $2\lambda \in \mathbb{N}$.

Lemma 3.4. *If E is a regular hermitian (resp. anti-hermitian) (a, b)-module and there exists $\lambda \in \mathbb{C}$ such that E contains two distinct normal elementary sub-(a, b)-modules F and G of parameters $f = g = \lambda \bmod \mathbb{Z}$ then there exists $F_1 \subset F_{n-1}$ two normal sub-(a, b)-modules or rang 1 and $n-1$ respectively such that $(E/F_{n-1})^* \simeq F_1$ and F_{n-1}/F_1 is hermitian (resp. anti-hermitian).*

Proof. We will denote by H an hermitian (resp. anti-hermitian) form on E. Let e_f and e_g be generators of F and G and suppose without loss of generality that $f - g \geqslant 0$. We will show that there exist a normal elementary sub-(a, b)-module F_1 of E whose generator $e \in E$ satisfies $H(e, e) = 0$.

By the property 1.2 of (a, b)-modules we have $ab^{f-g}e_g = fb \cdot b^{f-g}e_g$. Let's pose $e_1 = b^{f-g}e_g$. Consider now the complex vector space:

$$V := \{\alpha e_f + \beta e_1 | \alpha, \beta \in \mathbb{C}\}$$

Note that every $v \in V$ satisfies $av = fbv$. The b-linearity of H and the definition of the action of a give us:

$$(a - 2fb)H(v, v) = 0$$

which has in E_0 only solutions of the form $\alpha b^{2f}e_0$, $\alpha \in \mathbb{C}$. There exists therefore an application B from $V \times V$ to $\mathbb{C}$ such that:

$$H(v, w) = B(v, w)b^{2f}e_0 \quad \forall v, w \in V$$

The bilinearity and symmetry of H imply that B is in fact a bilinear symmetric form on a 2 dimensional complex vector space, which entrains:

$$B(e_f + xe_1, e_f + xe_1) = a_0 + a_1x + a_2x^2$$

for some complex numbers a_i. This vector space has an isotropic vector $e \neq 0$ such that $B(e, e) = 0$, and therefore $H(e, e) = 0$.

By eventually dividing e by a certain power of b, operation that does not change the relation $H(e, e) = 0$, we can assume that $e \notin bE$, hence the module F_1 generated by e is normal.

We can now conclude by applying lemma 3.2 □

Lemma 3.5. *Let E be a regular (a,b)-module and:*

$$0 \subsetneq \ldots F_{i-1} \subsetneq F_i \subsetneq F_{i+1} \subsetneq \ldots E$$

be a Jordan-Hölder sequence with $F_i/F_{i-1} \simeq E_{\lambda_i}$ for all i and suppose there is a j such that $\lambda_{j+1} \neq \lambda_j \mod \mathbb{Z}$.

Then we can find another Jordan-Hölder sequence that differs only in the j-th term F'_j such that $F'_j/F_{j-1} \simeq E_{\lambda'_j}$ and $F_{j+1}/F'_j \simeq E_{\lambda'_{j+1}}$ with $\lambda_j = \lambda'_{j+1}$ mod $\mathbb{Z}$ and $\lambda_{j+1} = \lambda'_j$ mod $\mathbb{Z}$, i.e. we can permute the quotients up to an integer shift of the parameters.

Proof. Let consider $G := F_{j+1}/F_{j-1}$ and the canonical projection of E onto the quotient E/F_{j-1}, $\pi : E \to E/F_{j-1}$. G is a rank two module. Using the classification of regular (a,b)-modules of rank 2 given by D. Barlet in [Bar93] we see that the only two possibilities for G are:

$$G \simeq E_{\lambda_j} \oplus E_{\lambda_{j+1}}$$

in which case we take $F'_j = \pi^{-1}(E_{\lambda_{j+1}})$ or

$$G \simeq E_{\lambda_{j+1}+1,\lambda_j}$$

generated by y and t satisfying:

$$\begin{aligned} ay &= \lambda_j by \\ at &= \lambda_{j+1}bt + y \end{aligned}$$

that has also another set of generators: t and $x := y + (\lambda_{j+1} - \lambda_j + 1)bt$ which satisfy:

$$\begin{aligned} ax &= (\lambda_j + 1)bx \\ at &= (\lambda_j - 1)bt + x. \end{aligned}$$

In this case we take $F'_j = \pi^{-1}(< x >)$. □

Lemma 3.6. *Let λ be either 0 or $1/2$ and E be a regular hermitian (resp. anti-hermitian) (a,b)-module. Suppose that there is an unique normal elementary sub-(a,b)-module of parameter equal to λ modulo $\mathbb{Z}$ and suppose moreover that every Jordan-Hölder sequence contains at least 2 elementary quotients of parameter equal to λ modulo $\mathbb{Z}$.*

Then there exist $F_1 \subset \underbrace{F_{n-1}}$ two normal sub-(a,b)-modules of rang 1 and $n-1$ respectively such that $(E/F_{n-1})^* \simeq F_1$ and F_{n-1}/F_1 is hermitian (resp. anti-hermitian).

Proof. Let $F_1 \simeq E_\mu$ be the elementary sub-(a,b)-module of the hypothesis and $\{F_i\}$ a J-H sequence beginning with F_1 and such that E/F_{n-1} is of parameter μ' equal to λ mod $\mathbb{Z}$. We can find such a sequence by using repeatedly the previous lemma.

Consider the exact sequence:

$$0 \to F_{n-1} \to E \to (E/F_{n-1}) \to 0$$

and the dual sequence:

$$0 \to (\widecheck{E/F_{n-1}})^* \xrightarrow{i} \check{E}^* \xrightarrow{\pi} \check{F}^*_{n-1} \to 0.$$

The image of i is a normal elementary sub-(a,b)-module of $\check{E}^*$ of parameter equal to $-\lambda$ mod $\mathbb{Z}$ (since $(\widecheck{E/F_{n-1}})^* \simeq E_{-\mu'}$). But $\lambda = -\lambda$ mod $\mathbb{Z}$ and $E \simeq \check{E}^*$ so by the uniqueness of F_1 given in the hypothesis $\mathrm{Im}\left((\widecheck{E/F_{n-1}})^*\right) = \Phi(F_1)$, thus $(\widecheck{E/F_{n-1}})^* \simeq F_1$. By replacing $\check{E}^*$ by E and $\left(\widecheck{E/F_{n-1}}\right)^*$ by F_1 in the sequence we obtain:

$$0 \to F_1 \xrightarrow{i} E \to \check{F}^*_{n-1} \to 0$$

which is exact and i is the inclusion of sub-(a,b)-modules, so $\check{F}^*_{n-1} \simeq (E/F_1)$ or equivalently $F_{n-1} \simeq (\widecheck{E/F_1})^*$. Note that the first isomorphism is given by Φ^{-1}, while the second by the restriction of Φ.

Consider the following sequence and its adjoint:

$$0 \to F_{n-1}/F_1 \to E/F_1 \to E/F_{n-1} \to 0$$
$$0 \to (\widecheck{E/F_{n-1}})^* \to (\widecheck{E/F_1})^* \to (\widecheck{F_{n-1}/F_1})^* \to 0$$

by replacing $(\widecheck{E/F_{n-1}})^*$ and $(\widecheck{E/F_1})^*$ with F_1 and F_{n-1} we obtain:

$$0 \to F_1 \xrightarrow{\varphi} F_{n-1} \xrightarrow{\pi} (\widecheck{F_{n-1}/F_1})^* \to 0$$

for the uniqueness of F_1, φ can only be (up to multiplication by a complex number) the inclusion $F_1 \subset F_{n-1}$ and hence $(\widecheck{F_{n-1}/F_1})^* \simeq (F_{n-1}/F_1)$. Note that π is the restriction of Φ to F_{n-1}, so the isomorphism is self-adjoint (resp. anti-self-adjoint). □

We can now prove the theorem.

Proof of theorem 3.1. We will prove the theorem by induction on the rank of the (a,b)-module. For rank 0 and 1 the theorem is obvious.

Suppose we proved the theorem for every rank $< n$ and let's prove it for rank n. Let find $F_1 \subset F_{n-1}$ of rang 1 and $n-1$ such that $(\widetilde{E/F_{n-1}})^* \simeq F_1$ and F_{n-1}/F_1 is hermitian or anti-hermitian. We can have different cases which are exhaustive:

(i) There is G normal elementary sub-(a, b)-module of E of parameter λ not equal to 0 or 1/2 mod $\mathbb{Z}$. Then $\Phi(G)(G) = 0$ by remark 3.3 and we can apply lemma 3.2.

Only (a, b)-modules whose only normal elementary sub-(a, b)-modules have parameter $\lambda = 0$ or $\lambda = 1/2$ modulo $\mathbb{Z}$ remain after this point.

(ii) For $\lambda = 0$ or $\lambda = 1/2$ there are two distinct normal elementary sub-(a, b)-modules of parameter equal to λ mod $\mathbb{Z}$. We apply lemma 3.4.

After this point we can only have at most an unique normal elementary sub-(a, b)-module of parameter equal to 1/2 modulo $\mathbb{Z}$ and an unique normal elementary sub-(a, b)-module with an integer value of the parameter.

(iii) There is only one normal elementary sub-(a, b)-module of parameter equal to λ mod $\mathbb{Z}$, where $\lambda = 0$ or $1/2$, but two quotients of a J-H sequence are of parameter equal to λ mod $\mathbb{Z}$. We apply lemma 3.6.

Only modules of rank at most 2 (one for each possible value of λ) pass this far.

(iv) The rank of E is 2 and one quotient of a J-H sequence is equal to 0 mod $\mathbb{Z}$, the other equal to 1/2 mod $\mathbb{Z}$. By the classification of rank 2 modules this case is impossible. In fact with the notations of [Bar93]:

$$\begin{aligned}(\widetilde{E_\lambda \oplus E_\mu})^* &\simeq E_{-\lambda} \oplus E_{-\mu} \\ \check{E}^*_{\lambda,\mu} &\simeq E_{1-\lambda,1-\mu}\end{aligned}$$

so if $\lambda = 0$ mod $\mathbb{Z}$ and $\mu = 1/2$ mod $\mathbb{Z}$ the (a, b)-module is not self-adjoint.

By induction hypothesis F_{n-1}/F_1 has a J-H sequence that satisfies the hypothesis of the theorem and if we take the inverse image by the canonical morphism $F_{n-1} \to F_{n-1}/F_1$ and adding 0 and E we find a J-H sequence of E that satisfies the theorem. □

By considering the results of the previous chapter we can leave out the hermitian condition on the self-adjoint module. We have in fact the following:

Theorem 3.7. *Let E be a regular self-adjoint (a, b)-module. Then it admits a self-adjoint Jordan-Hölder composition series.*

Proof. By proposition 2.32 we can decompose E into

$$E = \bigoplus_{i=1}^{m} H_i$$

where m is an integer, while the H_i are either indecomposable self-adjoint or of the form $G \oplus \check{G}^*$, where G is indecomposable non self-adjoint (a, b)-module.

Each term of this sum admits a self-adjoint composition series. In fact if H_i is indecomposable self-adjoint, then it is hermitian or anti-hermitian by theorem 2.34. We can therefore apply the previous theorem 3.1.

On the other hand if H_i is the sum $G \oplus \check{G}^*$ of a module and its adjoint, we can easily find a self-adjoint Jordan-Hölder composition series. Take in fact any Jordan-Hölder series of G,

$$0 = G_0 \subsetneq \cdots \subsetneq G_n = G.$$

and consider the adjoint series

$$0 = (\widetilde{G/G_n})^* \subsetneq (\widetilde{G/G_{n-1}})^* \subsetneq \ldots (\widetilde{G/G_0})^* = \check{G}^*.$$

Then the following composition series of $G \oplus \check{G}^*$ is self-adjoint:

$$0 = G_0 \subsetneq G_1 \subsetneq \ldots G = G \oplus (\widetilde{G/G_n})^* \subsetneq G \oplus (\widetilde{G/G_{n-1}})^* \subsetneq \\ \cdots \subsetneq G \oplus (\widetilde{G/G_0})^* = G \oplus \check{G}^*.$$

We will now prove the theorem on induction on m. The case $m = 1$ was already proven.

Suppose now $m \geqslant 2$ and let $E' := H_1$ and $F := \sum_{i=2}^{m} H_i$. We have therefore $E = E' \oplus F$, and E' and F are both self-adjoint. By induction hypothesis we can find a self-adjoint composition series of E':

$$0 = E'_0 \subsetneq \cdots \subsetneq E'_r = E'$$

and of F:

$$0 = F_0 \subsetneq \cdots \subsetneq F_s = F.$$

Then the following composition series is self-adjoint:

$$0 = E'_0 \subsetneq E'_1 \subsetneq \cdots \subsetneq E'_{[r/2]} \subsetneq E'_{[r/2]} \oplus F_1 \subsetneq \cdots \subsetneq E'_{[r/2]} \oplus F_{[s/2]} [\cdots] \\ E'_{[(r+1)/2]} \oplus F_{[(s+1)/2]} \subsetneq E'_{[(r+1)/2]} \oplus F_{[(s+1)/2]+1} \subsetneq \cdots \subsetneq E'_{[(r+1)/2]} \oplus F \\ \subsetneq E'_{[(r+1)/2]+1} \oplus F \subsetneq \cdots \subsetneq E' \oplus F,$$

where depending on the parity of r and s, $[\cdots]$ stands for

(i) the $=$ sign if r and s are both even.

(ii) the $\subsetneq$ sign if one is even and the other odd.

(iii) the subsequence
$$\subsetneq E'_{[r/2]} \oplus F_{[(s+1)/2]} \subsetneq$$
This case needs a short verification. If r and s are odd, then the two central quotients of the series are isomorphic to $E'_{[(r+1)/2]}/E'_{[r/2]}$ and $F_{[(s+1)/2]}/F_{[s/2]}$. Since E'_i and F_i are self-adjoint series both quotients are self-adjoint (a, b)-modules of rank 1. They are therefore isomorphic to E_0.

□

Chapter 4

Higher-residue pairings

In the study of the Brieskorn lattice K. Saito introduced the concept of "higher residue pairings" (cf. [Sai83]), which can be defined using a set of axiomatic properties.

Using the theory of (a, b)-modules R. Belgrade showed the existence of a duality isomorphism between an (a, b)-module associated to a germ of a holomorphic function in $\mathbb{C}^{n+1}$ with an isolated singularity at the origin and its $(n + 1)$-dual. In this chapter we'll prove (as already noticed by R. Belgrade in [Bel01]) that the concept of "higher residue pairings" and self-adjoint (a, b)-module are linked.

In this chapter D will always denote the Brieskorn module associated to a holomorphic function in $\mathbb{C}^{n+1}$ with an isolated singularity, while E will denote its b-adic completion considered as an (a, b)-module.

4.1 Duality of geometric (a, b)-modules

The following theorem of R. Belgrade gives a relationship between E and its $(n + 1)$-dual.

Theorem 4.1 (Belgrade)**.** *Let E be the (a, b)-module associated to a germ of holomorphic function $f : \mathbb{C}^{n+1} \to \mathbb{C}$, then there is a natural isomorphism between E and its $(n + 1)$-dual:*

$$\Delta : \quad E \simeq \breve{E}^* \otimes_{(a,b)} E_{n+1}$$

We can obtain from this isomorphism a series $\Delta_k : E \times E \to \mathbb{C}$ of bilinear forms defined as follow:

$$[\Delta(y)](x) = (n + 1)! \sum_{k=0}^{+\infty} \Delta_k(x, y) b^k e_{n+1}$$

with x, $y \in E$. Notice that this definition differs from that of R. Belgrade by a factor of $(n+1)!$.

4.2 "Higher residue pairings" of K. Saito

K. Saito introduced in [Sai83] a series of pairings on the Brieskorn lattice D which are called "higher residue pairings":

$$K^{(k)}: \quad D \times D \to \mathbb{C} \qquad k \in \mathbb{N}$$

which are characterized by the following properties:

(i) $K^{(k)}(\omega_1, \omega_2) = K^{(k+1)}(b\omega_1, \omega_2) = -K^{(k+1)}(\omega_1, b\omega_2)$.

(ii) $K^{(k)}(a\omega_1, \omega_2) - K^{(k)}(\omega_1, a\omega_2) = (n+k)K^{(k-1)}(\omega_1, \omega_2)$.

(iii) $K^{(0)}$ satisfies

$$K^{(0)}(D, bD) = K^{(0)}(bD, D) = 0$$

and induces Grothendieck's residue on the quotient D/bD.

(iv) $K^{(k)}$ are $(-1)^k$-symmetric.

Remark 4.2. We notice that from the properties (i) and (iii) above we can deduce that $K^{(k)}(D, b^{k+1}D) = K^{(k)}(b^{k+1}D, D) = 0$, so we can consider the pairings $K^{(k)}$ as being defined on $D/b^{k+1}D$.

In the following section we'll show the following result:

Proposition 4.3. *The Δ_k verify the properties (i)–(iii) of the "higher residue pairings" of K. Saito.*

The prove will be performed by steps.

4.3 Proof of the proposition

4.3.1 Proof of (i)

We use the b-linearity of $\Delta(y)$ to obtain:

$$\sum_k (n+1)!\Delta_k(bx, y)b^k e_{n+1} = [\Delta(y)](bx) = b[\Delta(y)](x) =$$

$$\sum_k (n+1)!\Delta_k(x, y)b^{k+1} e_{n+1}$$

which gives us $\Delta_k(x,y) = \Delta_{k+1}(bx,y)$. And similarly by using the b-linearity of Δ and the definition of conjugate, we obtain:

$$\Delta(by)(x) = \left(b \cdot_{\check{E}^*} \Delta(y)\right)(x) = -b\Delta(y)(x),$$

and therefore

$$(n+1)! \sum_k \Delta_k(x,by) b^k e_{n+1} = \Delta(by)(x) = -b\Delta(y)(x) = (n+1)! \sum_k -\Delta_k(x,y) b^{k+1} e_{n+1}$$

which implies $\Delta_k(bx,y) = -\Delta_{k+1}(x,by)$.

4.3.2 Proof of (ii)

Since Δ is an isomorphism we have $\Delta(ay) = a \cdot_{\check{E}^*} [\Delta(y)]$ and:

$$(n+1)! \sum_k \Delta_k(x,ay) b^k e_{n+1} = \Delta(ay)(x) = a \cdot [\Delta(y)](x) =$$

$$= \Delta(y)(ax) - a[\Delta(y)(x)] = (n+1)! \sum_k \left(\Delta_k(ax,y) b^k e_{n+1} - \Delta_k(x,y) a b^k e_{n+1}\right)$$

The definition of (a,b)-module and E_{n+1} $(ae_{n+1} = (n+1)be_{n+1})$ gives the following relation

$$ab^k e_{n+1} = b^k a e_{n+1} + k b^{k+1} e_{n+1} = (n+k+1) b^{k+1} e_{n+1}$$

hence follows:

$$\Delta_k(ax,y) - \Delta_k(x,ay) = (n+k)\Delta_{k-1}(x,y)$$

4.3.3 Grothendieck's residue

We have to show now that the pairing Δ_0 induces Grothendieck's residue on $D/bD \simeq \Omega^{n+1}/df \wedge \Omega^n$.

Proof of (iv): From the definition of Δ_0 and the b-linearity of Δ it's easy to see that $\Delta_0(D,bD) = \Delta_0(bD,D) = 0$. We can hence consider Δ_0 as a pairing on D/bD.

Grothendieck's residue is defined as follows:

$$Res(g,h) := \lim_{\varepsilon_j \to 0, \forall j} \int_{|\partial f/\partial z_j| = \varepsilon_j} \frac{gh \, \mathrm{dz}}{\partial f/\partial z_1 \cdots \partial f/\partial z_{n+1}}$$

where $g, h \in \mathcal{O}$ and $\mathrm{dz} = \mathrm{dz}_1 \wedge \ldots \mathrm{dz}_{n+1}$.
The morphism Δ is defined as composed morphism of six (a, b)-modules morphism as showed by the following graph:

$$\begin{array}{ccccc} E & \xrightarrow{\alpha} & F_1 & \xrightarrow{\beta} & F_2 \\ & & & & \uparrow{\scriptstyle\gamma} \\ & & & & F_3 \\ & & & & \downarrow{\scriptstyle\delta} \\ \breve{E}_{n+1} & \xleftarrow[\zeta]{} & F_5 & \xleftarrow[\eta]{} & F_4 \end{array}$$

These morphisms pass to the quotient by the action of b in order to give a decomposition of the morphism Δ_0:

$$\begin{array}{ccccc} E/bE & \xrightarrow{\tilde{\alpha}} & F_1/bF_1 & \xrightarrow{\tilde{\beta}} & F_2/bF_2 \\ & & & & \uparrow{\scriptstyle\tilde{\gamma}} \\ & & & & F_3/bF_3 \\ & & & & \downarrow{\scriptstyle\tilde{\delta}} \\ \frac{\left(\breve{E}^* \otimes E_{n+1}\right)}{b\left(\breve{E}^* \otimes E_{n+1}\right)} & \xleftarrow[\tilde{\zeta}]{} & F_5/bF_5 & \xleftarrow[\tilde{\eta}]{} & F_4/bF_4 \end{array}$$

We have to verify that the image of $[g\,\mathrm{dz}]$ by Δ_0 is $Res(g, \cdot)$, where $g\,\mathrm{dz}$ is an element of Ω^{n+1}. We'll accomplish this in many steps using the decomposition above.

(i) **Step 1: E, F_1 and F_2.** We have the following isomorphisms:

$$\frac{F_1}{bF_1} \simeq \frac{\Omega^{n+1}}{df \wedge \Omega^n} \qquad \frac{F_2}{bF_2} \simeq \frac{\mathcal{D}b^{n+1}}{(\bar{\partial} - df\wedge)\mathcal{D}b^n},$$

the morphism $\tilde{\alpha}$ coincides with the identity on $\Omega^{n+1}/df \wedge \Omega^n$ and $\tilde{\beta}$ is induced by the inclusion $i : \Omega^{n+1} \mapsto \mathcal{D}b^{n+1}$. We deduce that $\tilde{\beta} \circ \tilde{\alpha}([g\,\mathrm{dz}]) = [i(g\,\mathrm{dz})]$. Let write $T \in \mathcal{D}b^{n+1,0}$ the current $i(g\,\mathrm{dz})$.

(ii) **Step 2: path between F_2 and F_3** By using the description of the lemma 3.4.2 of [Bel01] we see that:

$$\frac{F_3}{bF_3} = \frac{\mathrm{Ker}(\mathcal{D}b^{0,n+1} \xrightarrow{df\wedge} \mathcal{D}b^{1,n+1})}{\bar{\partial}\,\mathrm{Ker}(\mathcal{D}b^{0,n} \xrightarrow{df\wedge} \mathcal{D}b^{1,n})}$$

and the isomorphism $\tilde{\gamma}$ is induced by the inclusion $\mathcal{D}b^{0,n+1} \subset \mathcal{D}b^{n+1}$. In order to find $S := \tilde{\gamma}^{-1}(T)$ we have to solve the following system:

$$\begin{array}{rcl} T & = & df \wedge \alpha^{n,0} \\ \bar{\partial}\alpha^{n,0} & = & df \wedge \alpha^{n-1,1} \\ \dots & & \dots \\ \bar{\partial}\alpha^{1,n-1} & = & df \wedge \alpha^{0,n} \\ \bar{\partial}\alpha^{0,n} & = & S \end{array}$$

where the $\alpha^{p,q} \in \mathcal{D}b^{p,q}$. There is a solution to this system of equations since the complex $(\mathcal{D}b^{\bullet,q}; df\wedge)$ is acyclic in degree $\neq 0$ for all q in $0, \dots, n+1$ and the solution satisfies $[S] = [T]$ where $[\cdot]$ is the equivalence class in F_2/bF_2.

$$(\bar{\partial} - df\wedge) \sum_{k=0}^{n} \alpha^{k,n-k} = \bar{\partial}\alpha^{0,n} - df \wedge \alpha^{n,0} = S - T$$

We can compute this solution explicitly. Let be $(p,q) \in \mathbb{N}^2$ and $\varphi^{p,q}$ a C^∞ test form with compact support and of type (p,q). The action of T over $\varphi^{0,n+1}$ is given by:

$$< T, \phi^{0,n+1} >= \int \phi^{0,n+1} \wedge g \,\mathrm{dz}$$

then the following current satisfies $T = df \wedge \alpha^{n,0}$:

$$< \alpha^{n,0}, \phi^{1,n+1} >= \lim_{\varepsilon_1 \to 0} \int_{|\partial_1 f| \geqslant \epsilon_1} \frac{\phi^{1,n+1} \wedge g \,\mathrm{dz}_2 \wedge \dots \wedge \mathrm{dz}_{n+1}}{\partial_1 f}$$

in fact:

$$\begin{aligned} < df \wedge \alpha^{n,0}, \phi^{0,n+1} > &= \lim_{\varepsilon_1 \to 0} \int_{|\partial_1 f| \geqslant \epsilon_1} \frac{\phi^{0,n+1} \wedge df \wedge g \,\mathrm{dz}_2 \wedge \dots \wedge \mathrm{dz}_{n+1}}{\partial_1 f} \\ &= \int \phi^{0,n+1} \wedge g \,\mathrm{dz} \end{aligned}$$

and thanks to the Stokes' theorem:

$$\begin{aligned} < \bar{\partial}\alpha^{n,0}, \phi^{1,n} > &= - < \alpha^{n,0}, \bar{\partial}\phi^{1,n} > \\ &= \lim_{\varepsilon_1 \to 0} - \int_{|\partial_1 f| \geqslant \epsilon_1} \frac{\bar{\partial}\phi^{1,n} \wedge g \,\mathrm{dz}_2 \wedge \dots \wedge \mathrm{dz}_{n+1}}{\partial_1 f} \\ &= \lim_{\varepsilon_1 \to 0} \int_{|\partial_1 f| = \epsilon_1} \frac{\phi^{1,n} \wedge g \,\mathrm{dz}_2 \wedge \dots \wedge \mathrm{dz}_{n+1}}{\partial_1 f} \end{aligned}$$

We'll remark that the currents $\alpha_k^{n,0}$ defined above for $1 \leqslant k \leqslant n+1$ also verify $df \wedge \alpha_k^{n,0} = T$:

$$< \alpha_k^{n,0}, \phi^{1,n+1} >=$$

$$= \lim_{\varepsilon_k \to 0} \int_{|\partial_k f| \geqslant \epsilon_k} \frac{(-1)^{k+1}\phi^{1,n+1} \wedge g\, \mathrm{dz}_1 \wedge \ldots \widehat{\mathrm{dz}_k} \ldots \wedge \mathrm{dz}_{n+1}}{\partial_k f}$$

and that $[\bar{\partial}\alpha^{n,0}] = [\bar{\partial}\alpha_k^{n,0}]$ in F_2/bF_2: in fact $(\bar{\partial} - df\wedge)(\alpha^{n,0} - \alpha_k^{n,0}) = \bar{\partial}\alpha^{n,0} - \bar{\partial}\alpha_k^{n,0}$.
For all $k \in 0, \ldots, n$ and $1 \leqslant i_1 < \ldots < i_{k+1} \leqslant n+1$ let us define:

$$\alpha_{i_1,\ldots,i_{k+1}}^{n-k,k} = \frac{1}{(k+1)!} \lim_{\substack{\epsilon_{i_q} \to 0 \\ \forall 1 \leqslant q \leqslant k+1}} \int_{\substack{|\partial_{i_1} f| \geqslant \epsilon_{i_1} \\ |\partial_{i_q} f| = \epsilon_{i_q}}} \frac{(-1)^{(\sum_q i_q)+1} g \bigwedge_{l \neq i_1,\ldots,i_{k+1}} \mathrm{dz}_l}{\partial_{i_1} f \ldots \partial_{i_{k+1}} f}$$

and let $\alpha^{n-k,k} := \alpha_{1,2,\ldots,k+1}^{n-k,k}$.
A simple computation gives us:

$$\left\langle df \wedge \alpha_{i_1,\ldots,i_{k+1}}^{n-k,k}, \varphi^{k,n-k+1} \right\rangle = \left\langle \frac{1}{k+1} \sum_{q=1}^{k+1} \bar{\partial}\alpha_{i_1,\ldots,\widehat{i_q},\ldots,i_{k+1}}^{n-k+1,k-1}, \varphi^{k,n-k+1} \right\rangle$$

using this formula, we prove by induction on k that the class of the current $\alpha_{i_1,\ldots,i_{k+1}}^{n-k,k}$ doesn't depend upon the i_qs. This gives us

$$\left[df \wedge \alpha^{n-k,k}\right] = \left[\bar{\partial}\alpha^{n-k+1,k-1}\right].$$

In particular $\bar{\partial}\alpha^{0,n}$ acts upon the test function $\varphi^{n+1,0}$ in the following way:

$$< \bar{\partial}\alpha^{0,n}, \phi^{n+1,0} >= \frac{1}{(n+1)!} \lim_{\substack{\epsilon_k \to 0 \\ \forall k}} \int_{\substack{|\partial_k f| = \epsilon_k \\ \forall k}} \frac{\phi^{n+1,0} g}{\partial_1 f \ldots \partial_{n+1} f}$$

(iii) **Step 3 from F_3/bF_3 to $(D/bD)^*$**: let notice that S is a current of type $(0, n+1)$ with support in the origin.
We have the following isomorphisms:

$$\frac{F_4}{bF_4} \simeq \mathrm{Ker}\left(\mathcal{H}_0^{n+1}(X, \mathcal{O}) \xrightarrow{df\wedge} \mathcal{H}_0^{n+1}(X, \Omega^1)\right)$$

and the isomorphism between F_3/bF_3 and F_4/bF_4 is the natural one, and

$$\frac{F_5}{bF_5} \simeq \left(\frac{\Omega^{n+1}}{df \wedge \Omega^n}\right)^*$$

From steps (1)–(3) we deduce that Δ_0 induces Grothendieck's residue.

4.4 Property (iv)

Verifying that the Δ_k are $(-1)^k$-symmetric would follow closely the footsteps of K. Saito and R. Belgrade. We will follow another approach by proving instead that the isomorphism given by R. Belgrade can be easily transformed into one that satisfies the property.

Let $\Delta : E \to \breve{E}^* \otimes_{(a,b)} E_{n+1}$ be Belgrade's isomorphism As we already noted in chapter 2, the isomorphisms between E and $\breve{E}^* \otimes_{(a,b)} E_{n+1}$ are in bijection with the isomorphisms between $E \otimes_{(a,b)} E_{-(n+1)/2}$ and its adjoint, through the map that sends an isomorphism Φ to $\Phi \otimes_{(a,b)} Id_{E_{-(n+1)/2}}$.

By an easy calculation we can prove the following lemma:

Lemma 4.4. *Let $\Delta : E \to \breve{E}^* \otimes E_{n+1}$ be an isomorphism and*

$$\Delta(y)(x) = (n+1)! \sum_k \Delta_k(x,y) b^k e_{n+1}$$

for each x and $y \in E$. Then the Δ_k satisfy Saito's condition (iv) if and only if the isomorphism $\Delta \otimes_{(a,b)} Id_{E_{-(n+1)/2}}$ is self-adjoint.

Proof. $\Delta \otimes_{(a,b)} Id_{E_{-(n+1)/2}}$ is self-adjoint iff we have:

$$\Delta \otimes_{(a,b)} Id_{E_{-(n+1)/2}} \left(y \otimes e_{-(n+1)/2}\right) \left(x \otimes e_{-(n+1)/2}\right) = \sum_k S_k b^k e_0 \Leftrightarrow$$
$$\Delta \otimes_{(a,b)} Id_{E_{-(n+1)/2}} \left(x \otimes e_{-(n+1)/2}\right) \left(y \otimes e_{-(n+1)/2}\right) = \sum_k S_k (-b)^k e_0.$$

for all x and $y \in E$. On the other hand we have:

$$\Delta \otimes_{(a,b)} Id_{E_{-(n+1)/2}} \left(y \otimes e_{-(n+1)/2}\right) \left(x \otimes e_{-(n+1)/2}\right) = \sum_k S_k b^k e_0 \Leftrightarrow$$
$$\Delta(y)(x) = \sum_k S_k b^k e_{n+1}.$$

□

By combining the previous equivalence with the results of chapter 2, we can state the following theorem:

Theorem 4.5. *Let E be a regular (a,b)-module associated to a holomorphic function from $\mathbb{C}^{n+1}$ to $\mathbb{C}$ with an isolated singularity. Then there exists an isomorphism $\Phi : E \to \breve{E}^* \otimes_{(a,b)} E_{n+1}$ with*

$$\Phi(y)(x) = (n+1)! \sum_k \Phi_k(x,y) b^k e_{n+1},$$

for all x and y such that the sequence of $\mathbb{C}$-bilinear forms Φ_k satisfies all four properties of Saito's "higher residue pairings".

Proof. Let Δ be Belgrade's isomorphism and Δ_k defined as at the beginning of this chapter. Consider the isomorphism

$$\check{\Delta}^* \otimes_{(a,b)} Id_{E_{n+1}} : E \to \check{E}^* \otimes_{(a,b)} E_{n+1}$$

and let $\Phi = \left(\Delta + \check{\Delta}^* \otimes_{(a,b)} Id_{E_{n+1}}\right)/2$.

It is easy to see that the Φ_k satisfy properties (i) and (ii). Moreover since Δ_0 is symmetric (Grothendieck's residue) and $\check{\Delta}^* \otimes_{(a,b)} Id_{E_{n+1}}$ induces the transposed of Δ_0 on E/bE, we have

$$\Phi_0 = \left(\Delta_0 + {}^t\Delta_0\right)/2 = \Delta_0.$$

We have also

$$\left(\Phi \otimes_{(a,b)} \widetilde{Id_{E_{-(n+1)/2}}}\right)^* = \check{\Phi}^* \otimes_{(a,b)} Id_{E_{(n+1)/2}} = \Phi \otimes_{(a,b)} Id_{E_{-(n+1)/2}},$$

therefore the Φ_k satisfy Saito's property (iv).

We just have to show that $\Phi \otimes_{(a,b)} Id_{E_{-(n+1)/2}}$ is an isomorphism. Since there exists an isomorphism between $E \otimes_{(a,b)} E_{-(n+1)/2}$ and its adjoint, we can apply proposition 2.26 and reduce ourselves to prove the injectivity of $\Phi \otimes_{(a,b)} Id_{E_{-(n+1)/2}}$. But if $\Phi \otimes_{(a,b)} Id_{E_{-(n+1)/2}}$ were not injective Φ would induce a degenerate form on E/bE, which is absurd. □

The existence of a hermitian form on $E \otimes_{(a,b)} E_{-(n+1)/2}$ gives us an interesting restriction on the kind of (a,b)-module associated with Brieskorn lattices:

Corollary 4.6. *Let E be a regular (a,b)-module associated to a holomorphic function from $\mathbb{C}^{n+1}$ to $\mathbb{C}$ with an isolated singularity. Then $E \otimes_{(a,b)} E_{-(n+1)/2}$ is a hermitian (a,b)-module.*

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