DOI : 10.1007/s00186-010-0341-2



# The restricted core of games on distributive lattices: how to share benefits in a hierarchy

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June 25, 2010

**Abstract**

Finding a solution concept is one of the central problems in cooperative game theory, and the notion of core is the most popular solution concept since it is based on some rationality condition. In many real situations, not all possible coalitions can form, so that classical TU-games cannot be used. An interesting case is when possible coalitions are defined through a partial ordering of the players (or hierarchy). Then feasible coalitions correspond to teams of players, that is, one or several players with all their subordinates. In these situations, the core in its usual formulation may be unbounded, making its use difficult in practice. We propose a new notion of core, called the restricted core, which imposes efficiency of the allocation at each level of the hierarchy, is always bounded, and answers the problem of sharing benefits in a hierarchy. We show that the core we defined has properties very close to the classical case, with respect to marginal vectors, the Weber set, and balancedness.

**Keywords:** cooperative game, feasible coalition, core, hierarchy
**JEL Classification**: C71

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# 1 Introduction

In cooperative game theory, a central topic is to define a rational way for distributing the total outcome among players (solution concept of this game). For transferable utility (TU) games, there exist two well-known solutions: the Shapley value [27], and the core [19]. The first one is defined by a set of rationality axioms: linearity, null player axiom, symmetry, and efficiency, and it is applicable to any game. The second one avoids the formation of subcoalitions of the grand coalition, in the sense that any subcoalition will receive at least the amount it can achieve by itself[1]. It may happen that no such solution exists. Classical results show under which conditions the core is nonempty, and give the structure of the core when the game is convex [28, 25].

In the classical setting of TU-games, any coalition $S \subseteq N$ can form, and each player can participate or not participate to the game. Mathematically speaking, this amounts to define the characteristic function of a game as a real-valued function $v$ on the Boolean lattice $2^N$, and vanishing at the empty set. More general definitions allowing a better modelling of reality have been proposed. We may distinguish between games having a restricted set of feasible coalitions (which may induce in some cases a hierarchy among players), and games permitting a more complex mechanism of participation. In the first category, we find games with precedence constraints [13], games on matroids, convex geometries and other combinatorial structures [3, 5], games on regular set systems [31], games on augmenting systems [4], games on permission structures [10], games on communication graphs [26, 29, 30] (see a comparative survey of all these structures in [20]) ; in the second category, we find multichoice games of Hsiao and Raghavan [24], fuzzy games [7], and games on product of distributive lattices [21]. In many cases, the characteristic function of such general games can be considered as a real-valued function defined over a (often distributive) lattice.

In this paper, we propose a definition for the core of TU-games whose characteristic function is $v : L \to \mathbb{R}$, where $L$ is a distributive lattice. There are two main reasons for focusing on this kind of game. The first one appears clearly from the previous discussion, since many of the above examples are related to lattices, or even their internal structure are exactly distributive lattices. The second reason is that a distributive lattice, by Birkhoff's theorem, is generated by defining a partial order on the set of players. This is in fact exactly the framework considered by Faigle and Kern [13], since precedence constraints among players are nothing else than that. A partial order on players can be interpreted in several ways, according to the application context, but there is one which is self-evident: it defines a hierarchy on players, in the sense that $j \leq i$ means that $j$ is a subordinate of $i$. Moreover, the lattice generated by this order is composed by all possible downsets on $N$, where a downset is a subset of $N$ where all subordinates of players in the downset are present. Therefore, the lattice can be interpreted as the set of all possible teams compatible with the hierarchy. This clearly applies for example to companies, or any other structured entity producing some benefit. In this context, defining the core of games on such structures amounts to define a way of sharing the total benefit $v(N)$ achieved among the members, in a way that fully respects the achievement of each team.

As we will show, the study of the core appears to be more complex than in the clas-

[1]In this paper, we consider games as *profit games*, hence the core is seen as a rational way to share benefits. We may consider *cost games* as well, reversing inequalities accordingly.

sical case, although similar results still hold. A first fact is that the core, defined as in the classical way, is still a polyhedron but possibly unbounded. This is not surprising, considering the general results obtained by Derks and Reijnierse [11], about the boundedness of the core for games defined on a set of feasible coalitions. However, this negative result prevents us to use the core in its original definition as a way of sharing benefits, for monetary amounts should remain bounded. The only way to get out of this situation is to impose further constraints on the core, that is, to add new inequalities or equalities in its definition, so as to get it bounded. An obvious way to do it is to impose the nonnegativity of the payoffs for players. Then we obtain what is generally called the positive core, introduced by Faigle [12]. Since we have no special reason to impose nonnegativity[2], we have to find another way to impose constraints, which should reflect some rationality. One of the main achievements of this paper is precisely to solve this issue, by adding equality constraints playing the rôle of efficiency, at each level of the hierarchy. The new definition of the core we obtain is called the restricted core, and it is always bounded. Moreover, it has a clear interpretation in our context of sharing benefits. The second achievement is that we prove that the restricted core has properties very similar to the core of classical games: in particular, the inclusion of the restricted core into the (restricted) Weber set always holds, and equality holds when the game is convex.

There are in the literature other works dealing with hierarchies, in particular by Demange [9], and van den Brink et al. [30]. The latter is more related to communication graphs and deals with the selectope, while the former consider a rather different definition of a team, where all subordinates need not be present. In particular, any singleton is a team, which we do not think meaningful in our context. We discuss these related works at the end of the paper.

The paper is organized as follows. We begin by introducing the basic definitions for games on distributive lattices and partially ordered sets (posets) in Section 2. Then Sections 3, 4 present the basic definitions for games on distributive lattices and the core. In the next sections 5, 6, 7, we study their properties. We indicate in Section 8 how to apply our results to the case of product lattices, encompassing the case of multichoice games. In Section 9, we give a brief account on related works in the literature.

## 2 Posets, distributive lattices and levels

This section briefly recalls the necessary material on finite posets and lattices (see, e.g., Davey and Priestley [8] for details). A *partially ordered set* $(P, \leq)$, or *poset* for short, is a set $P$ endowed with a partial order $\leq$ (reflexive, antisymmetric and transitive). As usual, the asymmetric part of $\leq$ is denoted by $<$. For $x, y \in (P, \leq)$ (if no ambiguity occurs, we may write simply $P$), we write $x \prec y$ and say that $x$ is *covered* by $y$ if $x < y$ and there is no $z \in P$ such that $x < z < y$. An element $x \in P$ is minimal if there is no $y \in P$ such that $y < x$.

A *chain* from $x$ to $y$ in $P$ is any sequence $x, x_1, \ldots, x_p, y$ of elements of $P$ such that $x < x_1 < \cdots < x_p < y$. The chain is *maximal* if no other chain from $x$ to $y$ contains it, i.e., if $x \prec x_1 \prec \cdots \prec x_p \prec y$. The *length* of a chain is its number of elements minus 1.

[2] It could happen that a player may induce some loss when participating to certain teams. Such a player should be penalized when sharing the benefit, therefore payoffs could be negative.

The *height* of $x \in P$, denoted by $h(x)$, is the length of a longest chain from a minimal element to $x$. The height of $P$, denoted by $h(P)$, is then the maximum of $h(x)$ taken over the elements of $P$. Interestingly, the height function induces a partition of $P$ into *levels*: level number $i$, denoted by $Q_i$, is the set of elements of height $i-1$. Note that $Q_1$ is the set of minimal elements of $P$, $Q_2$ is the set of minimal elements of $P \setminus Q_1$, etc. Also, any two distinct elements $x, y$ in a level set are incomparable, i.e., neither $x < y$ nor $x > y$ holds.

A particularly well-known class of posets are lattices. A lattice $(L, \leq)$ is a poset having the following property: for any $x, y \in L$, their supremum and infimum, denoted by $x \vee y$ and $x \wedge y$, exist in $L$. When a lattice is finite, it has a greatest element $\top = \bigvee_{x \in L} x$ (top element), and a smallest element $\bot = \bigwedge_{x \in L} x$ (bottom element). If $\vee, \wedge$ obey distributivity, then $L$ is said to be *distributive*. A nice property of distributive lattices is that all maximal chains from $\bot$ to $\top$ have same length $h(L)$.

We relate now lattices and posets. Consider a poset $(P, \leq)$ and some $Q \subseteq P$. Then $Q$ is a *downset* of $P$ if $x \in Q$ and $y \leq x$ imply $y \in Q$. Any element $x \in P$ generates a downset, defined by $\downarrow x := \{y \in P \mid y \leq x\}$. Let us denote by $\mathcal{O}(P)$ the set of all downsets of $P$ and remark that $\emptyset \in \mathcal{O}(P)$. It is not difficult to show that the poset $(\mathcal{O}(P), \subseteq)$ is a collection of subsets of $P$, containing $P$ and $\emptyset$, and is closed under union and intersection. More precisely, it is a distributive lattice of height $n$, and the fundamental result of Birkhoff [6] says that in the finite case, any distributive lattice can be obtained as the set of downsets of some poset.

Finally, consider again the levels in $P$ induced by the height function, and let us assume that there are $q$ levels. Observe that $Q_1$ (the set of minimal elements) is a downset, therefore it is an element of the lattice $\mathcal{O}(P)$. Similarly, by construction $Q_1 \cup Q_2$ is also a downset, and so are $\bigcup_{j=1}^{i} Q_j$, for $i = 1, \dots, q$. In other words, the levels sets in $P$ induce particular elements in the lattice, which we will denote by $\top_1, \dots, \top_q$:

$$\top_i := \bigcup_{j=1}^{i} Q_j.$$

Note that $\top_q = P$, the top element of $\mathcal{O}(P)$, and that $\top_1, \dots, \top_q$ form a chain in $\mathcal{O}(P)$. We put $\top_N := \{\top_1, \dots, \top_q\}$.

The following example illustrates these various definitions.

**Example 1.** We consider the poset $P$ and its corresponding distributive lattice $\mathcal{O}(P)$ on Fig. 1. Then

$$Q_1 = \{1, 4, 5\}, Q_2 = \{2\}, Q_3 = \{3\}.$$

$$\top_1 = 145, \top_2 = 1245, \top_3 = 12345.$$

## 3 Games on distributive lattices

As said in the introduction, there are two main applications of games defined on distributive lattices, namely to model restriction on the set of feasible coalitions, or to allow for each player several possible (partially ordered) actions for participation to the game. Our development will follow the first stream, and so is close to the framework of Faigle and

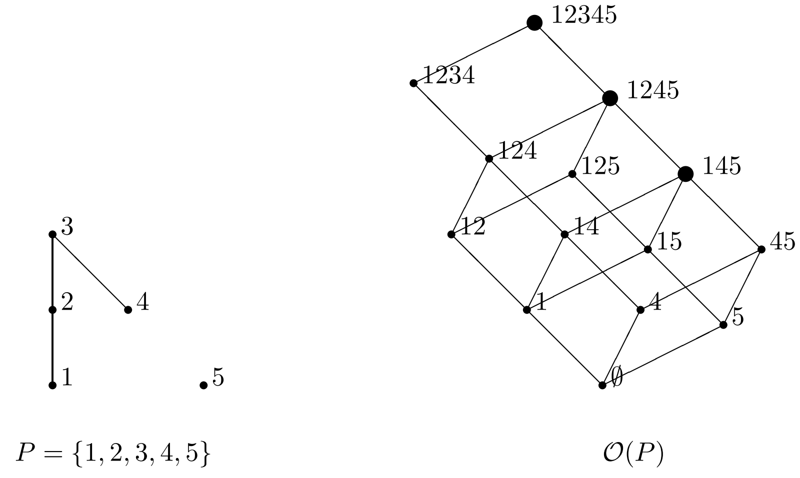



Figure 1: Example of poset $P$ (left) and the corresponding lattice $\mathcal{O}(P)$ (right)

Kern [13]. We will comment briefly the second one, which is developed in [21], in Section 8, where we will indicate how our results can be straightforwardly applied to this case.

In the rest of the paper, $N = \{1, \dots, n\}$ denotes the set of players, which we suppose to be endowed with a partial order $\leq$. The relation $i \leq j$, with $i, j \in N$, indicates that player $i$ is below player $j$, or a subordinate of $j$ (this is called *precedence constraint* by Faigle and Kern [13]). Hence, the relation $\leq$ describes a *hierarchy* among players. Practically, this means that, if $j$ participates to the game, all subordinates of $j$ must also participate to it. Therefore, a coalition $S \subseteq N$ is *feasible* if $j \in S$ and $i \leq j$ implies that $j \in S$. This has three important consequences, which can be drawn from Section 2:

(i) The set of feasible coalitions is precisely the set of all downsets of $(N, \leq)$, denoted by $\mathcal{O}(N)$.

(ii) The set of feasible coalitions is a distributive lattice of height $n$.

(iii) The set of feasible coalitions is closed under union and intersection, and contains $\emptyset$ and $N$.

**Definition 1.** Let $L := \mathcal{O}(N)$ be the collection of all feasible coalitions (all downsets of $(N, \leq)$). A *game on the distributive lattice* $L$ is a real-valued function $v : L \to \mathbb{R}$ such that $v(\emptyset) = 0$.

We make some noteworthy remarks for positioning our paper.

*Remark* 1. (i) The classical definition of a TU-game is recovered when $(N, \leq)$ is an antichain (i.e., all elements are minimal), that is, when there is no hierarchy and all players are "on the same level". Then clearly no restriction on coalitions exist, and any $S \in 2^N$ is feasible.

(ii) The structure of feasible coalitions we address in this paper is not new: as we mentionned, the idea of generating feasible coalitions by a partial order on $N$ has been proposed by Faigle and Kern [13]. Also, sets of feasible coalitions closed under union and intersection, but without the restriction of being of height $n$, have been studied by Derks and Gilles [10], using permission structures. In Section 9, we give a brief description of permission structures, and compare them to our framework. One can find also studies on collections of coalitions closed under union and intersection in the work of Faigle [12] and Faigle and Kern [14]. We mention also that this structure has been studied in the field of combinatorial optimization (see, e.g., the monograph of Fujishige [16]).

In fact, many works dealing with restricted coalitions exist, with various structures for the set of feasible coalitions (see a detailed survey in [20]). Our aim is not to bring new results for the case of collections of coalitions closed under union and intersection, already well-known. It is to bring the notion of level, closely related to the notion of hierarchy, inducing some natural restrictions in the definition of the core.

(iii) Once we have discovered that our set of feasible coalitions is closed under union and intersection, we may forget about distributive lattices and work only with union and intersection closedness. However, this would hide the way this collection is obtained (as downsets of a partial order), and the notion of level, fundamental in this paper, would disappear.

We introduce the following property.

**Definition 2.** Let $v$ be a game on $\mathcal{O}(N)$. The game $v$ is *convex* if $v(S\cup T)+v(S\cap T)\geq v(S)+v(T)$, for all $S,T\in\mathcal{O}(N)$.

# 4 The core and the restricted core

We take the classical point of view for defining the core, that is, it is a set of pre-imputations satisfying some rationality condition, which prevent players to form sub-coalitions. A *pre-imputation* is a vector $\phi\in\mathbb{R}^n$ such that $\sum_{i=1}^n\phi_i=v(N)$, where $\phi_i$ is the amount of money given to player $i$. We use the usual shorthand $\phi(S):=\sum_{i\in S}\phi_i$ for any subset $S\subseteq N$.

## 4.1 The core

In the classical case, the rationality condition is $\phi(S)\geq v(S)$ for all coalitions $S$. Adapting it to our framework leads to the following definition.

**Definition 3.** The *core of a game $v$ on $\mathcal{O}(N)$* is defined by the following set.

$$\mathcal{C}(v):=\{\phi\in\mathbb{R}^n\mid\phi(N)=v(N)\text{ and }\phi(S)\geq v(S),\ \forall S\in\mathcal{O}(N)\}.$$

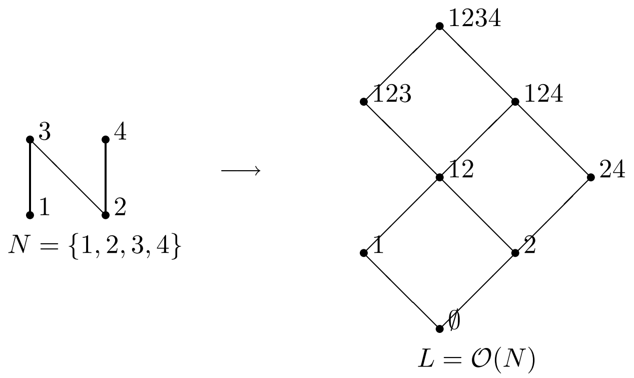
Clearly, the core is a closed convex polyhedron. Unlike the classical case where all coalitions are feasible, the core may be unbounded, i.e., it contains rays. We recall from the theory of polyhedra (see, e.g., Ziegler [32], Faigle et al. [15]) that, unless it contains a line in which case there is no vertices, a polyhedron is determined by its vertices (convex part) and rays (conic part). The structure of the polyhedron is determined by its *recession* *cone*, i.e., the set of inequalities where the right member is replaced by 0. In our case, this gives

$$\mathcal{C}(L) =: \{\phi \in \mathbb{R}^n \mid \phi(N) = 0 \text{ and } \phi(S) \geq 0, \forall S \in \mathcal{O}(N)\}.$$

The polyhedron has no line if and only if the recession cone is a pointed cone (and its rays are the rays of the polyhedron), and the polyhedron is bounded (polytope) if and only if the recession cone reduces to $\{0\}$.

Derks and Gilles [10] have shown that for our case (set of coalitions closed under union and intersection, of height $n$), the recession cone is always a pointed cone. Also, Derks and Reijnierse [11] provided necessary and sufficient conditions for the boundedness of the core of a game defined on a set of feasible coalitions, without special structure (set systems). See also the survey paper [20] for a general study of the core of games on set systems. The following example illustrates that the core can be unbounded.

**Example 2.** We consider the following poset $(N, \leq)$ (left) and its corresponding distributive lattice $\mathcal{O}(N)$ (right).




Let $v$ be a game on $\mathcal{O}(N)$. By definition of the core, any element $\phi$ of the core must satisfy:

$$\begin{aligned}
\phi_1 + \phi_2 + \phi_3 + \phi_4 &= v(\top) = v(1234) \\
\phi_1 &\geq v(1) \\
\phi_2 &\geq v(2) \\
\phi_1 + \phi_2 &\geq v(12) \\
\phi_2 + \phi_4 &\geq v(24) \\
\phi_1 + \phi_2 + \phi_3 &\geq v(123) \\
\phi_1 + \phi_2 + \phi_4 &\geq v(124).
\end{aligned}$$

Whenever $\phi_1, \phi_2$ are large enough, we can always find out some $\phi_3, \phi_4$ to satisfy all conditions, i.e., $\phi_1, \phi_2$ can be arbitrarily large. More precisely, the core contains 3 rays, which are $(0, 1, -1, 0)$, $(0, 1, 0, -1)$ and $(1, 0, -1, 0)$.

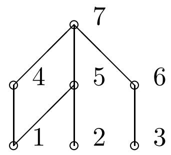
We denote the set of vertices of some convex set by $\text{Ext}(\cdot)$, and the convex hull of some set by $\text{co}(\cdot)$. We define the *convex part of the core* by $\mathcal{C}^F(v) := \text{co}(\text{Ext}(\mathcal{C}(v)))$.

The fact that the core is unbounded makes its usage difficult, since once cannot deal with arbitrary large payoffs in practice. A simple remedy to this would be to impose that $\phi$ should be bounded from below by some quantity. For example, we may impose nonnegativity, and thus obtain what is called the *positive core* by Faigle [12]. However, as indicated in Footnote 2, we do not think appropriate to forbid negative payoffs. If the game is nonmonotone, there is some player inducing some loss by his participation. Therefore, this player should not be rewarded, rather he should be charged by some amount. In the sequel, we will provide a much less arbitrary and much better answer to this problem, both for mathematical properties (since we will see that we are able to keep many of the classical results on the core), and for the practical side, illustrated hereafter with an example of benefit sharing in a hierarchical structure, one of our main motivation.

## 4.2 How to share benefits in a hierarchical structure

The example we develop in this section will lead naturally to a new definition of the core.

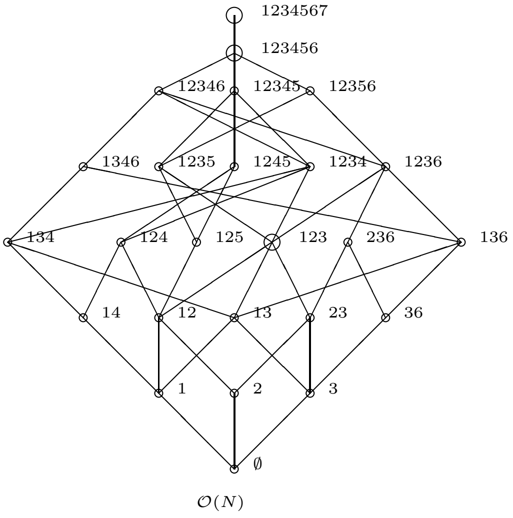
We consider for illustration purpose a company with 7 employees $N = \{1, 2, 3, 4, 5, 6, 7\}$, and we represent the hierarchy among employees by the partial order $\leq$ on $N$. To be enough general, we may even consider that one employee may have more than one direct superior (it could be the case if the employee participates to several projects or belongs to several teams). Hence the partial order is not necessarily a tree. The poset below depicts the hierarchy in $N$.




$$N = \{1, 2, 3, 4, 5, 6, 7 \mid 1 < 4 < 7, 2 < 5 < 7, 3 < 6 < 7 \text{ and } 1 < 5\}$$

We see that employee 1 has two direct superiors, namely 4 and 5.

As explained in Section 3, feasible coalitions are downsets of $(N, \leq)$. In this context, feasible coalitions correspond to *feasible teams* of the company, in the sense that the presence of an employee in a feasible team implies the presence of all employees below. It must be remarked that in general a feasible team in the above sense may be formed of several teams in the usual sense, which we may call *elementary teams* (that is, a boss and all employees below): in terms of poset terminology, this amounts to say that a downset is the union of principal ideals (see Section 2). For example, the feasible team 12356 is formed of the two elementary teams 125 and 36, with bosses 5 and 6. Note also that 3 itself is a team reduced to a singleton. We give below the distributive lattice of all teams ordered by inclusion.




Computing the levels $Q_k$ and top elements $\top_k$, we get

$$Q_1 = \{1, 2, 3\}, Q_2 = \{4, 5, 6\}, Q_3 = \{7\},$$
$$\top_1 = 123, \top_2 = 123456, \top_3 = 1234567 = N.$$

Level $Q_k$ corresponds to employees having the same rank[3] $k$ in the company, and $\top_k$ is the smallest feasible team containing all employees up to rank $k$. We call it the *principal team of rank k*.

At the end of each year, the total benefit (or a fixed proportion of it) has to be distributed among all employees as a bonus. We denote it by $v(N)$. For a given feasible team $S$, we denote by $v(S)$ the benefit achieved by $S$ (and only by $S$) which is brought to the company, and we denote by $\phi(S)$ the bonus or reward given to $S$. To achieve the sharing, we propose to perform a local sharing at each hierarchical level $Q_k$. More precisely:

- For hierarchical level $Q_k$, the amount to be shared among the employees of this level is $v(\top_k) - v(\top_{k-1})$, that is, roughly speaking, the difference between the benefit achieved by all employees up to level $k$, and the benefit achieved by all employees of level strictly lower than $k$. In a sense, this is the genuine contribution of level $k$.

- Inside a given level $Q_k$, the sharing is done freely, up to the condition that for each feasible team $S \in \mathcal{S}_k$, $\phi(S) \geq v(S)$. Otherwise, if for some $S$, $\phi(S) < v(S)$, then the team $S$ may split from $N$ and build a new independent company, because the benefit achieved by $S$ alone is more than that $S$ will receive.

Asuming there are $l$ hierarchical levels, this gives the linear system in $\phi$

$$\begin{aligned}\phi(Q_l) &= v(N) - v(\top_{l-1})\\ \phi(Q_{l-1}) &= v(\top_{l-1}) - v(\top_{l-2})\\ \vdots &= \vdots\\ \phi(Q_1) &= v(\top_1)\end{aligned}$$

[3]Mathematically speaking the same height, see Section 2.

and since $\top_k = \cup_{i=1}^k Q_i$, and the $Q_k$'s are pairwise disjoint, we deduce that $\phi(\top_k) = \sum_{i=1}^k \phi(Q_i) = v(\top_k)$ for $k = 1, \ldots, l$. Conversely, imposing $\phi(\top_k) = v(\top_k)$ for $k = 1, \ldots, l$ leads to the above system.

Applying this procedure to our example, we get

$$\begin{aligned} v(N) - v(123456) &\text{ is given to the group } \mathcal{Q}_3 = \{7\}, \\ v(123456) - v(123) &\text{ is given to the group } \mathcal{Q}_2 = \{4,5,6\}, \\ v(123) &\text{ is given to the group } \mathcal{Q}_1 = \{1,2,3\}. \end{aligned}$$

## 4.3 The restricted core

From the previous development, we are led to the following definition.

**Definition 4.** *The restricted core of a game $v$ on $\mathcal{O}(N)$* is defined by

$$\mathcal{RC}(v) := \{\phi \in \mathcal{C}(v) \mid \phi(\top_i) = v(\top_i), \forall \top_i \in \top_N\}.$$

Hence, the normalization condition is imposed at each level of the hierarchy. Note that the restricted core collapses to the classical one if the set of feasible coalitions is $2^N$. Indeed, in this case, $(N, \leq)$ is an antichain, so that there is only one level $Q_1 = N$, and $\top_1 = N$.

For the study of the restricted core, we may suppose without loss of generality that the game $v$ is monotone, i.e., $S \subseteq T$ implies $v(S) \leq v(T)$, and hence nonnegative. If this is not true, it suffices to define a vector $x \in \mathbb{R}^n$ so that for every $S, T \in L$ such that $S \subseteq T$ (in fact, one may restrict to pairs $S, T$ where $S$ is the immediate predecessor in $L$), it holds:

$$v(T) + x(T) \geq v(S) + x(S)$$

or equivalently

$$x(T \setminus S) \geq v(S) - v(T).$$

Observe that this set of inequalities has always a solution in $x$. Then we have

$$\mathcal{RC}(v) = \mathcal{RC}(v + x) - x := \{\phi - x \mid \phi \in \mathcal{RC}(v + x)\}.$$

Indeed, $\phi(S) \geq v(S) + x(S)$ is equivalent to $\underbrace{\phi(S) - x(S)}_{(\phi - x)(S)} \geq v(S)$, and similarly for equalities.

A first observation is that when the game is monotone, it is a subset of the positive core $\mathcal{C}^+(v) := \{\phi \in \mathbb{R}^n_+ \mid \phi(S) \geq v(S), \forall S \in L, \phi(N) = v(N)\}$.

**Theorem 1.** Assume $v$ on $L = \mathcal{O}(N)$ is monotone. Then $\mathcal{RC}(v) \subseteq \mathcal{C}^+(v)$.

*Proof.* Suppose that $\mathcal{RC} \neq \emptyset$, otherwise the result holds trivially. By definition of the positive core, it suffices to show that for any $\phi \in \mathcal{RC}(v)$, $\phi_i \geq 0$ for all $i \in N$.

If $i \in Q_1$, since $\{i\} \in \mathcal{O}(N)$, we have $\phi_i \geq v(\{i\}) \geq 0$ by monotonicity of $v$. Suppose then that $i \in Q_l$, $l > 1$. Since $\mathcal{RC}(v)$ is nonempty, the system

$$\begin{aligned} \phi(S) &\geq v(S), \quad S \in \mathcal{O}(N) \setminus \top_N \\ \phi(\top_k) &= v(\top_k), \quad \top_k \in \top_N \end{aligned}$$

has a solution, hence by the Farkas lemma, for any vector $y = (y_S)_{S \in L}$ such that $y_S \geq 0$ for all $S \in L \setminus \top_N$, and

$$\sum_{S \ni j} y_S = 0, \quad j \in N, \tag{1}$$

it holds $\sum_{S \in L} y_S v(S) \leq 0$. Let us choose such a vector $y^0$.

Consider now the following system for the above chosen $i \in N$:

$$\begin{aligned} \phi(S) &\geq v(S), \quad S \in \mathcal{O}(N) \setminus \top_N \\ \phi(\top_k) &= v(\top_k), \quad \top_k \in \top_N \\ -\phi_i &\geq \epsilon \end{aligned}$$

for some arbitrary small $\epsilon > 0$. Clearly, if this system is infeasible for any $\epsilon$, we must have $\phi_i \geq 0$. Hence let us build a vector $y' = \left[(y'_S)_{S \in L} \quad w\right]$ such that $y'_S \geq 0$ for $S \in L \setminus \top_N$, $w \geq 0$, satisfying the system

$$\sum_{S \ni j} y'_S = 0, \quad j \neq i \tag{2}$$

$$\sum_{S \ni i} y'_S - w = 0 \tag{3}$$

and $\sum_{S \in L} y_S v(S) + w\epsilon > 0$, which would imply infeasibility by the Farkas lemma. Remarking that $Q_1 \cup \cdots \cup Q_{j-1} \cup \{i\} = \top_{j-1} \cup \{i\}$ belongs to $\mathcal{O}(N)$, we build $y'$ from $y^0$ as follows:

$$y'_{\top_{j-1} \cup i} = y^0_{\top_{j-1} \cup i} + w, \quad y'_{\top_{j-1}} = y^0_{\top_{j-1}} - w, \quad y'_S = y^0_S \text{ for all other } S,$$

and verify it is adequate for our purpose. Since $w \geq 0$, the condition $y'_S \geq 0$ for $S \in L \setminus \top_N$ holds. Equation (3) becomes

$$y^0_{\top_{j-1} \cup i} + w + \sum_{S \ni i, S \neq \top_{j-1} \cup i} y^0_S - w = 0$$

which is true by (1). Clearly, equations in (2) with $j \not< i$ are identical to those of (1). Let us consider then (2) with $j < i$. Since any such equation contains both terms $S = \top_{j-1} \cup i$ and $S = \top_{j-1}$, it is clearly satisfied since identical to (1). Finally, we compute $\sum_{S \in L} y'_S v(S) + w\epsilon$:

$$\sum_{S \in L} y'_S v(S) + w\epsilon = \sum_{S \in L} y^0_S v(S) + w(v(\top_{j-1} \cup i) - v(\top_{j-1})) + w\epsilon.$$

The first term on the right hand side is a fixed nonpositive quantity, while the two other ones are nonnegative and proportional to $w$ (the second one is nonnegative by monotonicity of $v$). Then the expression can be made strictly positive by a sufficiently large $w$.

As a conclusion $\phi_i < 0$ cannot occur, for any $i \in N$. □

From this result, the fact that the positive core is bounded and the above remark on monotone games, it follows immediately:

**Corollary 1.** For any $v$ on $L = \mathcal{O}(N)$, $\mathcal{RC}(v)$ is a polytope (bounded polyhedron).

# 5 Balancedness

It is well known that the necessary and sufficient condition for the nonemptiness of the core of a game on $2^N$ is balancedness of the game. Now, this result can be directly adapted to our case, and in fact, to any structure of feasible coalitions.

**Definition 5.** (i) A collection $\mathcal{B}$ of elements of $\mathcal{O}(N) \setminus \{\emptyset\}$ is *balanced* if there exist positive coefficients $\mu(S)$, $S \in \mathcal{B}$, such that $\sum_{S:S\ni i} \mu(S) = 1$, for all $i \in N$.

(ii) A game $v$ on $\mathcal{O}(N)$ is *balanced* if for every balanced collection $\mathcal{B}$ of elements of $L \setminus \{\emptyset\}$ with coefficients $\mu(S)$, $S \in \mathcal{B}$, it holds

$$\sum_{S\in\mathcal{B}} \mu(S)v(S) \leq v(N).$$

**Proposition 1.** A game on $\mathcal{O}(N)$ has a nonempty core if and only if it is balanced.

We omit the proof of this result, since it is identical to the classical case. We mention that Faigle [12] has found other equivalent conditions for nonemptiness of the core in the general case of set systems. We turn now to the case of the restricted core.

Let $\mathcal{Q} = \{Q_1, \ldots, Q_q\}$ be the collection of levels of $N$ and $\top_N = \{\top_1, \ldots, \top_q\}$ be the collection of top elements of every level of $\mathcal{O}(N)$. Similarly, we introduce the notion of level-balancedness as follows.

**Definition 6.** (i) A collection $\mathcal{B}$ of elements of $\mathcal{O}(N) \setminus \{\emptyset\}$ is *level-balanced* if it exist positive coefficients $\mu(S)$, $S \in \mathcal{B}$, such that $\sum_{S:S\ni i} \mu(S) = q - k + 1$, for all $i \in Q_k, k = 1, \ldots, q$.

(ii) A game $v$ on $\mathcal{O}(N)$ is *level-balanced* if for every balanced collection $\mathcal{B}$ of elements of $\mathcal{O}(N) \setminus \{\emptyset\}$ with coefficients $\mu(S)$, $S \in \mathcal{B}$, it holds

$$\sum_{S\in\mathcal{B}} \mu(S)v(S) \leq \sum_{k=1}^{q} v(\top_k).$$

Let us come back to Example 2. The conditions for level-balancedness read

$$\sum_{S\ni 1} \mu(S) = \sum_{S\ni 2} \mu(S) = 2, \quad \sum_{S\ni 3} \mu(S) = \sum_{S\ni 4} \mu(S) = 1.$$

The sum for elements of lower height have a higher value since the more an element is in the bottom of the hierarchy, the more it is frequent in coalitions. Examples of level-balanced collections are

$$\mathcal{B} = \{123, 124\} \text{ with weights } 1,\ 1$$
$$\mathcal{B} = \{1234, 12, 1, 2\} \text{ with weights } 1,\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}.$$

**Proposition 2.** A game on $\mathcal{O}(N)$ has a nonempty restricted core if and only if it is level-balanced.

*Proof.* Proving nonemptiness of the restricted core of a game is equivalent to constructing a vector $\phi \in \mathbb{R}^n$ satisfying the following conditions:

$$\phi(\top_i) = v(\top_i), \forall \top_i \in \top_N \text{ and } \phi(S) \geq v(S), \forall S \in \mathcal{O}(N) \setminus \{\emptyset\}.$$

Consider the following linear program with the variables $\phi_j \in \mathbb{R}$, $j \in N$:

$$\min z = \sum_{\top_i \in \top_N} \phi(\top_i) = \sum_{i=1}^{q} \sum_{k=1}^{i} \phi(Q_k)$$

under

$$\sum_{j:j \in S} \phi_j \geq v(S), \forall S \in \mathcal{O}(N) \setminus \{\emptyset\}.$$

Its optimal value is $z = \sum_{\top_i \in \top_N} v(\top_i)$ if and only if the restricted core is nonempty.

Remarking that in the objective function the variable $\phi_i$ has coefficient $q - k + 1$ for $i \in Q_k$, its dual problem is

$$\max w = \sum_{S \in \mathcal{O}(N) \setminus \{\emptyset\}} \mu(S) v(S)$$

under

$$\sum_{S:S \ni j} \mu(S) = q - k + 1, \; \forall j \in Q_k, k = 1, \dots, q$$

$$\mu(S) \geq 0, \; \forall S \in \mathcal{O}(N) \setminus \{\emptyset\}.$$

By the duality theorem, it has the same optimal value $w = \sum_{\top_i \in \top_N} v(\top_i)$ if we can find out some $\mu$ satisfying all conditions. This is the desired result. □

# 6 The Weber set

We have mentionned in Section 2 that $\mathcal{O}(N)$ has all its maximal chains from $\bot$ to $\top$ of length $n$. The consequence is that they correspond to permutations on $N$ in the following way: let $C = \{S_0 := \emptyset \prec S_1 \prec \cdots \prec S_n := N\}$ be a maximal chain in $L := \mathcal{O}(N)$. Define the permutation $\pi$ on $N$ as $\pi(i) := S_i \setminus S_{i-1}$, $i = 1, \dots, n$. Hence we have $S_i = \{\pi(1), \pi(2), \dots, \pi(i)\}$.

It is easy to see that $\pi$ defines a linear extension of $\leq$ on $N$, and moreover, any linear extension of $\leq$ corresponds to such a permutation $\pi$. Indeed, $i < j$ implies that $\pi(i) > \pi(j)$ will never happen, for any permutation. Conversely, if $i_1, \dots, i_n$ is a linear extension, then $k < l$ implies that $i_k > i_l$ cannot happen. Hence $\{\{i_1\}, \{i_1, i_2\}, \dots, \{i_1, \dots, i_n\}\}$ is a chain of downsets, defining a permutation $\pi$ on $N$.

**Definition 7.** The *marginal worth vector* $\psi^C \in \mathbb{R}^n$ associated to $C$ and $v$ is defined by

$$\psi^C_j := v(S_i) - v(S_{i-1}), \quad \forall i \in N,$$

with $j = S_i \setminus S_{i-1}$.

The set of all marginal worth vectors $\psi^C$ for all maximal chains is denoted by $\mathcal{M}(v)$. We can easily get

$$\psi^C(S_i) := \sum_{k=1}^{i} \psi^C_{\pi(k)} = \sum_{k=1}^{i}(v(S_k) - v(S_{k-1})) = v(S_i), \forall S_i \in C.$$

**Definition 8.** The *Weber set* $\mathcal{W}(v)$ of $v$ is defined as the convex hull of all vectors in $\mathcal{M}(v)$:

$$\mathcal{W}(v) := \mathrm{co}(\mathcal{M}(v)).$$

**Theorem 2.** For any game $v$ on $\mathcal{O}(N)$, the convex part of the core is included in the Weber set, i.e, $\mathcal{C}^F(v) \subseteq \mathcal{W}(v)$.

This result has been shown independently by several authors (including us: see an earlier version of this paper as a technical report [23]). It can be found in Faigle and Kern [14, Section 5], and it is also mentionned in Derks and Gilles [10], where the result is shown for acyclic permission structures. Now, from Algaba et al. [1], it is known that acyclic permission structures exactly correspond to collections of feasible coalitions of the form $\mathcal{O}(N)$.

The next result shows that, like in the classical case, equality holds only for convex games.

**Theorem 3.** Let $v$ be a game on $\mathcal{O}(N)$. Then $v$ is convex if and only if $\mathrm{Ext}(\mathcal{C}(v)) = \mathcal{M}(v)$, i.e., $\mathcal{C}^F(v) = \mathcal{W}(v)$.

Again, this result has been shown several times (see our proof in [23]), the first one being Fujishige and Tomizawa [17] (also cited in [16]). Derks and Gilles have also shown this theorem in [10], for acyclic permission structures.

We investigate now if similar results hold for the restricted case.

# 7 The restricted Weber set

Let $C$ be a maximal chain from $\bot$ to $\top$ in $\mathcal{O}(N)$. We say that $C$ is a *restricted* maximal chain if $\top_1, \dots, \top_q$ belong to $C$. A marginal vector defined w.r.t a restricted maximal chain is called a *restricted marginal vector*. We denote by $\mathcal{RM}(v)$ the set of all restricted marginal vectors.

**Definition 9.** The *restricted Weber set* is defined as the convex hull of all restricted marginal worth vectors:

$$\mathcal{RW}(v) := \mathrm{co}(\mathcal{RM}(v)).$$

**Theorem 4.** For any game $v$ on $\mathcal{O}(N)$, the restricted core is included in the restricted Weber set, i.e., $\mathcal{RC}(v) \subseteq \mathcal{RW}(v)$.

*Proof.* We prove it by induction on the level number. If a poset $N$ has only one level, then the result coincides with the one of Theorem 2.

Suppose that the statement is true for all posets having at most $k$ levels. We assume now that the poset has $k+1$ levels. Let $v' := v_{|\top_k}$ be the restriction of $v$ to $\top_k$, that is, it is defined on the collection of coalitions $\mathcal{O}(N)_{|\top_k} := \{T \in \mathcal{O}(N) \mid T \subseteq \top_k\}$, and $v'(T) = v(T)$ for all $T \in \mathcal{O}(N)_{|\top_k}$. Note that $v'$ is a game on a distributive lattice with $k$ levels. Let $\phi \in \mathrm{Ext}(\mathcal{RC}(v))$. Clearly, $\phi_{|\top_k} \in \mathcal{RC}(v_{|\top_k}) = \mathcal{RC}(v')$. Then $\phi_{|\top_k} \in \mathrm{Ext}(\mathcal{RC}(v'))$. Indeed, if $\phi_{|\top_k} \notin \mathrm{Ext}(\mathcal{RC}(v'))$, then $\exists \phi^1 \neq \phi^2 \in \mathcal{RC}(v'), \exists \lambda \in (0,1)$ such that $\phi_{|\top_k} = \lambda\phi^1 + (1-\lambda)\phi^2$. Let $\phi' \in \mathbb{R}^n$ be defined by

$$\phi'^l_i := \begin{cases} \phi^l_i & \forall i \in \top_k, \\ \phi_i & \forall i \in \top \setminus \top_k \end{cases}$$

for $l = 1, 2$. Then $\phi = \lambda\phi'^1 + (1-\lambda)\phi'^2$, which contradicts $\phi \in \mathrm{Ext}(\mathcal{RC}(v))$.

Similarly, we define the game $v''$ on $N \setminus \top_k$, whose collection of feasible coalitions is $\mathcal{O}(N)_{N\setminus\top_k} := \{T \subseteq N \setminus \top_k \mid T \cup \top_k \in \mathcal{O}(N)\}$, by $v''(T) := v(T \cup \top_k) - v(\top_k)$ for such $T$'s. Observe that by definition of levels, $\mathcal{O}(N)_{|N\setminus\top_k} = \{T \subseteq N \setminus \top_k = Q_{k+1}\}$, therefore $v''$ is a game on the Boolean lattice $2^{Q_{k+1}}$ with one level. We have

$$\phi_{|N\setminus\top_k}(T) = \phi(T \cup \top_k) - \phi(\top_k) \geq v(T \cup \top_k) - v(\top_k) = v''(T), \forall T \subseteq \top_k,$$

hence $\phi_{|\top\setminus\top_k} \in \mathcal{RC}(v'')$. By the induction hypothesis, we can write

$$\phi_i = \begin{cases} \sum_{\psi^r \in \mathcal{RM}(v')} \alpha^r \psi^r_i, & \forall i \in \top_k \\ \sum_{\psi^s \in \mathcal{RM}(v'')} \beta^s \psi^s_i, & \forall i \in N \setminus \top_k, \end{cases}$$

with $\alpha^r \geq 0, \beta^s \geq 0$, $\sum_r \alpha^r = 1$, and $\sum_s \beta^s = 1$. Any $\psi^r \in \mathcal{RM}(v')$ corresponds to a restricted maximal chain $C^r$ in $\mathcal{O}(N)$ from $\emptyset$ to $\top_k$, and any $\psi^s \in \mathcal{RM}(v'')$ corresponds to a maximal chain $C^s := \{\emptyset =: S_0, S_1, \ldots, S_m = N \setminus \top_k\}$ in the Boolean lattice $2^{Q_{k+1}}$ from $\emptyset$ to $N \setminus \top_k$. Define the chain $(C^s)' := \{\top_k, S_1 \cup \top_k, \ldots, N\}$ obtained from $C^s$ by adding $\top_k$ to each coalition. Then $(C^s)'$ is restricted maximal chain in $\mathcal{O}(N)$ from $\top_k$ to $N$. Therefore the concatenation of $C^r$ and $(C^s)'$, denoted by $C$, is a restricted maximal chain in $\mathcal{O}(N)$ from $\emptyset$ to $N$, which induces the restricted marginal vector $\psi^{(r,s)}$ defined by:

$$\psi^{(r,s)}_i := \begin{cases} \psi^r_i, & \text{if } i \in \top_k \\ \psi^s_i, & \text{if } i \in N \setminus \top_k. \end{cases}$$

We have for any $i \in \top_k$

$$\phi_i = \sum_r \alpha^r \psi^r_i = \sum_r \alpha^r \psi^{(r,s)}_i = \sum_s \beta^s \big(\sum_r \alpha^r \psi^{(r,s)}_i\big) = \sum_s \sum_r \alpha^r \beta^s \psi^{(r,s)}_i$$

and similarly for $i \in N \setminus \top_k$

$$\phi_i = \sum_s \beta^s \psi^s_i = \sum_s \beta^s \psi^{(r,s)}_i = \sum_r \sum_s \alpha^r \beta^s \psi^{(r,s)}_i.$$

Therefore $\phi = \sum_r \sum_s \alpha^r \beta^s \psi^{(r,s)}$, i.e., $\phi \in \mathcal{RW}(v)$. This is the desired result. □

We examine the case of convex games.

**Theorem 5.** If a game $v$ on $\mathcal{O}(N)$ is convex, $\mathrm{Ext}(\mathcal{RC}(v)) = \mathcal{RM}(v)$, or equivalently $\mathcal{RC}(v) = \mathcal{RW}(v)$.

*Proof.* Consider a restricted maximal chain $C_r$ and its associated marginal worth vector $\psi^{C_r}$. We know by Theorem 3 that it is a vertex of the core $\mathcal{C}(v)$, and since $\psi^{C_r}$ coincide with $v$ on $C_r$, it has the property $\psi^{C_r}(x) = v(x)$, $\forall x \in \top_N$, hence it belongs to the restricted core and is a vertex of it. Therefore, we have established that $\mathcal{RM}(v) \subseteq \mathrm{Ext}(\mathcal{RC}(v))$.

By Theorem 4, we know that the convex part of the restricted core is included into the restricted Weber set. Therefore the vertices of the two sets coincide. □

Remark that $\mathcal{RC}(v) = \mathcal{RW}(v)$ does not imply that $v$ is convex. Put differently, $\mathcal{RC}(v) = \mathcal{RW}(v)$ is not equivalent to $\mathcal{C}^F(v) = \mathcal{W}(v)$. This is shown by the following counterexample.

**Example 3.** Let $v$ be a game on $\mathcal{O}(N)$ with $N = \{1, 2, 3, 4, 5\} : 1 < 2 < 3, 4 < 5$. Consider $v$ satisfying $v(S) = \sum_{s \in S} s$ for any $S \neq \{12\}$ and $v(12) = 1$. We have $\mathcal{RC}(v) = \mathcal{RW}(v) = \{(1, 2, 3, 4, 5)\}$ but $v(12345) + v(12) = 16 < v(1245) + v(123) = 18$. Therefore $v$ is not convex.

To end this section, we come back to Example 2 and compute its restricted core. The four restricted maximal chains are

$$C_1 := \{\emptyset, 1, 12, 123, 1234\}, \quad C_2 := \{\emptyset, 1, 12, 124, 1234\}$$
$$C_3 := \{\emptyset, 2, 12, 123, 1234\}, \quad C_4 := \{\emptyset, 2, 12, 124, 1234\}.$$

Under convexity of $v$, the restricted core of $v$ is the convex hull of the four following vectors:

$$\phi^1 := (v(1), v(12) - v(1), v(123) - v(12), v(N) - v(123))$$
$$\phi^2 := (v(1), v(12) - v(1), v(N) - v(124), v(124) - v(12))$$
$$\phi^3 := (v(12) - v(2), v(2), v(123) - v(12), v(N) - v(123))$$
$$\phi^4 := (v(12) - v(2), v(2), v(N) - v(124), v(124) - v(12)).$$

In general, it is a 3-dimensional polytope with 4 vertices, hence a 3-dimensional simplex.

# 8 Games with a partially ordered set of actions

We give a brief indication about games where each player has at disposal a partially ordered set of (elementary) actions. This notion of game is described in [21]. Consider a set of players $N$, and for each $i \in N$, define $P_i$ the partially ordered set of possible actions of player $i$. A simple but useful example is to take the case of multichoice games [24]. Then the $P_i$'s are totally ordered sets $P_i := \{0 =: a_0, a_1, \dots, a_m\}$, where $a_0 < a_1 < \cdots < a_m$ indicate levels of participation.

We consider the distributive lattices $L_i := \mathcal{O}(P_i)$, $i \in N$. They represent all possible combinations of elementary actions, where if action $x$ is performed and $y \leq x$ in the poset of actions, then $y$ must be performed too. Considering all players together, a given *profile* of actions is an element of the product lattice $L := L_1 \times \cdots \times L_n$.

Since $L$ is again distributive, all previous definitions and results can be applied to $L$. In particular, the restricted core of $v$ is defined as the set of pre-imputations $\phi$ on $L$

such that $\phi$ dominates $v$ on $L$, and coincides with $v$ on each element of $L$ of the form $(\top_1^k, \top_2^k, \dots, \top_n^k)$, where $\top_i^k$ is the top element associated to the $k$-th level of $P_i$. When $L$ corresponds to a multichoice game, we recover the results shown previously by the authors in [22].

# 9 Related works

There is a substantial amount of research devoted to the core of games defined on a set of feasible coalitions (see a survey on this topic by the first author [20]), and it will be out of the scope of this paper to detail this. Our proposition solves the problem of unboundedness of the core, by imposing some additional normalization conditions. Up to our knowledge, there is no work taking the same philosophy.

On the other hand, the notion of hierarchy has received some attention by several authors, in particular by Gilles et al. [18] who propose permission structures, Demange [9], and van den Brink et al. [30].

A (conjunctive) permission structure is a mapping $\sigma : N \to 2^N$ such that $i \notin \sigma(i)$. The players in $\sigma(i)$ are the direct subordinates of $i$. "Conjunctive" means that a player $i$ has to get the permission to act of all his superiors. Consequently, an *autonomous* coalition contains all superiors of every member of the coalition, i.e., the set of autonomous coalitions generated by the permission structure $\sigma$ is

$$\mathcal{F}_\sigma = \{S \in 2^N \mid S \cap \sigma(N \setminus S) = \emptyset\}.$$

This collection is closed under union and intersection (and conversely, any collection of feasible coalitions closed under union and intersection corresponds to a permission structure). Clearly, our collection $\mathcal{O}(N)$ is closed under union and intersection, and so should correspond to a permission structure. However, the notions of team in our sense and of autonomous coalition are quite opposite, since a team must contain all subordinates of its members, and an autonomous coalition must contain all superiors. These are two different viewpoints of a hierarchy. A team $S$ is an entity able to perform some work giving rise to some profit $v(S)$. It is considered that the work cannot be achieved if one subordinates is missing. This view is suitable for projects, companies, etc. An autonomous coalition $T$ is able to achieve some work because they have the permission of all their superiors, this permission being represented simply by the presence of the superiors in the coalition. Therefore, $v(T)$ has not the meaning of some profit achieved by the coalition.

In the work od Demange, a hierarchy is the same as our partial order defined on $N$, up to the difference that a greatest element exists (called the principal), so that each player is a subordinate of the principal. Also, the notion of team differs: any singleton is a team, and if a team has at least two members, any two members have a superior in the team, and if $i$ is a superior of $j$, all intermediates between $i$ and $j$ must be present. Therefore, any "interval", i.e, a chain in the hierarchy, is a team. Again, this definition does not fit our idea of defining team as entities being able to produce something. Clearly, a single player, unless he has no subordinate, cannot produce something by himself. The same remark is valid for intervals.

The work of van den Brink et al. concerns oriented communication graphs. Most of the research on communication graphs do not consider orientation, since it is generally assumed that communication is in both directions. Defining an orientation implicitely defines some order among players, hence some hierarchy. The philosophy adopted in this work is that if player $i$ is higher in the hierarchy than $j$ (i.e., there is an oriented path from $i$ to $j$), the payoff (or cost) given to player $i$ should be higher than the one of player $j$. This is well suited to the well known water distribution problem of Ambec and Sprumont [2], also considered by van den Brink et al. [29]. However, it is not suited for our view, since assuming $S \supset S'$, it may be the case that $v(S) = v(S')$, which means that the superior(s) in $S \setminus S'$ do not really add some value to the team. Therefore, their payoff should be zero or very low.

## Acknowledgment

We are grateful to the two referees for pointing out many references to the literature related to this work, and for helping at improving the presentation.

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