Weak solutions of backward stochastic differential equations with continuous generator

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Abstract

We prove the existence of a weak solution to a backward stochastic differential equation (BSDE)

$$Y_t = \xi + \int_t^T f(s, X_s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dW_s$$

in a finite-dimensional space, where $f(t, x, y, z)$ is affine with respect to z, and satisfies a sublinear growth condition and a continuity condition This solution takes the form of a triplet (Y, Z, L) of processes defined on an extended probability space and satisfying

$$Y_t = \xi + \int_t^T f(s, X_s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dW_s - (L_T - L_t)$$

where L is a continuous martingale which is orthogonal to any W. The solution is constructed as a solution measure, with the help of Young measures theory.

Keywords: Backward stochastic differential equation, weak solution, martingale solution, Young measure.

1 Introduction

Aim of the paper Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathrm{P})$ be a complete probability space, where $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration of a standard Brownian motion $W = (W_t)_{t\in[0,T]}$ on $\mathbb{R}^m$.

In this paper, we prove the existence of a weak solution (in the classical sense, i.e. defined on an extended probability space) to the equation

$$Y_t = \xi + \int_t^T f(s, X_s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dW_s - (L_T - L_t) \tag{1}$$

where where $f(t,x,y,z)$ is affine with respect to z, and satisfies a sublinear growth condition and a continuity condition, W is an $\mathbb{R}^m$-valued standard Brownian motion, Y and Z and L are unknown processes, Y and L take their values in $\mathbb{R}^d$, Z takes its values in the space $\mathbb{L}$ of linear mappings from $\mathbb{R}^m$ to $\mathbb{R}^d$, $\xi \in \mathrm{L}^2_{\mathbb{R}^d}$ is the terminal condition, and L is a continuous martingale orthogonal to any Brownian martingale, with $L_0 = 0$. The process $X = (X_t)_{0\leq t\leq T}$ is $(\mathcal{F}_t)$-adapted and continuous with values in a separable metric space $\mathbb{M}$. This process represents the random part of the generator f and plays a very small rôle in our construction. The space $\mathbb{M}$ can be for example some space of trajectories, and X_t can be for example the history until time t of some process ζ, i.e. $X_t = (\zeta_{s\wedge t})_{0\leq s\leq T}$.

Such a weak solution to (1) in the classical sense, i.e. defined on an extended probability space, can be considered as a generalized weak solution to the more classical equation

$$Y_t = \xi + \int_t^T f(s, X_s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dW_s. \tag{2}$$

Historical comments Existence and uniqueness of the solution (Y,Z) to a nonlinear BSDE of the form

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dW_s$$

have been proved in the seminal paper [22] by E. Pardoux and S. Peng, in the case when the generator $f(t,y,z)$ is Lispschitz with respect to (y,z) and satisfies a linear growth condition $\|f(t,y,z)\| \leq C(1 + \|y\| + \|z\|)$. In [20], J.P. Lepeltier and J. San Martin proved in the one dimensional case the existence of a solution when f is continuous and satisfies the same linear growth condition.

Equations of the form (2), whith f depending on some other process X, appear in forward-backward stochastic differential equations (FBSDE's), where X is a solution of a (forward) stochastic differential equation

As in the case of stochastic differential equations, one might expect that BSDE's with continuous generator always admit at least a *weak solution*, that is, a solution defined on a different probability space (generally larger than the space on which W is defined). A work in this direction but for forward-backward stochastic differential equations (FBSDE's) is that of K. Bahlali, B. Mezerdi, M. N'zi and Y. Ouknine [3], where the original probability is changed using Girsanov's theorem. Let us also mention the works on weak solutions to FBSDE's by Antonelli and Ma [2], and Delarue and Guatteri [12], where the change of probability space comes from the construction of the forward component.

Weak solutions where the filtration is enlarged have been studied by R. Buckdhahn, H.J. Engelbert and A. Rascanu in [10] (see also [8, 9]), using pseudopaths and the Meyer-Zheng topology. Pseudopaths were invented by Dellacherie and Meyer [13], actually they are Young measures on the Skorohod space $\mathbb{D}$ (see Subsection 3.2 for the definition of Young measures). Young measures on $\mathbb{D}$ have been re-invented by Pellaumail [23] under the name of rules, to construct weak solutions of SDE's. In the present paper, we also construct a weak solution with the help of Young measures on suitable spaces of trajectories. Note that the result of Buckdhahn, Engelbert and Rascanu [10, Theorem 4.6] is more general than ours in the sense that f in [10] depends functionally on Y, more precisely, their generator $f(t,x,y)$ is defined on $[0,T]\times\mathbb{D}\times\mathbb{D}$). Furthermore, in [10], W is only supposed to be a càdlàg martingale. On the other hand, it is assumed in [10] that f is bounded and does not depend on Z (but possibly on the martingale W). In the present paper, f satisfies only a linear growth condition, but the main novelty (and the main difficulty) is that f depends (linearly) on Z. As our final setup is not Brownian, the process Z we construct is not directly obtained by the martingale representation theorem, but as a limit of processes $Z^{(n)}$ which are obtained from the martingale representation theorem.

The existence of the orthogonal component L in our work comes from the fact that our approximating sequence $(Z^{(n)})$ does not converge in L^2, actually it converges to Z only in the weak topology of L^2, thus the stochastic integrals $\int_0^t Z^{(n)}\,dW_s$ do not need to converge to $\int_0^t Z\,dW_s$ in L^2, neither in distribution. Let us mention here the work of Ma, Zhang and Zheng [21], on the much more intricate problem of existence and uniqueness of weak solutions (in the classical sense) for forward-backward stochastic differential equations. Among other results, they prove existence of weak solutions with different methods and hypothesis (in particular the generator is assumed to be uniformly continuous in the space variables) which ensure that the approximating sequence $Z^{(n)}$ constructed in their paper converges in L^2 to Z.

Organization of the paper Definitions and hypothesis are provided in Section 2, along with remarks on pathwise uniqueness and existence of strong solutions. Section 3 is devoted to the construction of a weak solution: First, in Subsection 3.1, we construct a sequence $(Y^{(n)}, Z^{(n)})$ of strong solutions to approximating $BSDEs$ using a Tonelli type scheme, and we prove uniform boudedness in L^2 of these solutions. Finally, in Subsection 3.2, we obtain the solution by passing to the limit, using Young measures.

2 Definitions, notations and hypothesis

For any separable metric space $\mathbb{E}$, we denote by $C_{\mathbb{E}}[0,T]$ the space of continuous mappings on $[0,T]$ with values in $\mathbb{E}$. Similarly, for any $q \geq 1$, if $\mathbb{E}$ is a Banach space, and if $(\Sigma, \mathcal{G}, Q)$ is a measure space, we denote by $L^q_{\mathbb{E}}(\Sigma)$ the Banach space of measurable mappings $\varphi : \Sigma \to \mathbb{E}$ such that $\|\varphi\|^q_{L^q_{\mathbb{E}}} := \int_0^T \|\varphi(s)\|^q \, d\,Q(s) < +\infty$.

In the sequel, we are given a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$, the filtration $(\mathcal{F}_t)$ is the natural filtration of an $\mathbb{R}^m$-valued Brownian motion W, augmented with the P-negligible sets. We are also given an $\mathbb{R}^d$-valued random variable $\xi \in L^2_{\mathbb{R}^d}(\Omega, \mathcal{F}, P)$ (the terminal condition). The space of linear mappings from $\mathbb{R}^m$ to $\mathbb{R}^d$ is denoted by $\mathbb{L}$. We denote by $\mathbb{M}$ a separable metric space and by X a given $(\mathcal{F}_t)$-adapted $\mathbb{M}$-valued continuous process. Finally we are given a mapping $f : [0,T] \times \mathbb{M} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ which satifies the following growth and continuity conditions (H_1) and (H_2):

(H_1) There exists a constant $C_f \geq 0$ such that $\forall (t,x,y,z) \in [0,T] \times \mathbb{M} \times \mathbb{R}^d \times \mathbb{L}$, $\|f(t,x,y,z)\| \leq C_f(1 + \|z\|)$.

(H_2) (i) $f(t,x,y,z)$ is continuous with respect to (x,y) and affine with respect to z,

(ii) for all $x \in C_{\mathbb{M}}[0,T]$, $y \in C_{\mathbb{R}^d}[0,T]$, $z \in L^2_{\mathbb{L}}[0,T]$, and $t \in [0,T]$,

$$\lim_{n\to\infty} \int_t^T \left(f(s,x(s),y(s),z(s+1/n)) - f(s,x(s),y(s),z(s))\right) ds = 0 \text{ a.e.}$$

where v is extended to $[T, T+1]$ by $v(t) = 0$ for $t > T$.

Condition (H_2) is satisfied if e.g. f is continuous in all variables, or if $f(t,x,y,z) = g(t,x,y) + h(x,y,z)$ with g and h continuous with respect to (x, y, z). For example, in the first case, we have

$$\begin{aligned}\int_t^T & f\left(s, x(s), y(s), z(s+1/n)\right) ds \\ &= \int_{t+1/n}^T f\left(s-1/n,\, x(s-1/n),\, y(s-1/n),\, z(s)\right) ds \\ &\sim \int_{t+1/n}^T f\left(s, x(s), y(s), z(s)\right) ds\end{aligned}$$

by continuity of f, x, and y.

Definition 2.1 A *strong solution* to (2) is an $\mathcal{F}_t$-adapted, $\mathbb{R}^d \times \mathbb{L}$- valued process (Y, Z) (defined on $\Omega \times [0, T]$) that satisfies (2).

Remark 2.2 Similarly, a strong solution to (1) should be a triplet (Y, Z, L) defined on $\Omega \times [0, T])$ satisfying (1), and such that L is a continuous martingale orthogonal to any $(\mathcal{F}_t)$-martingale (recall that $(\mathcal{F}_t)$ is the the natural filtration of W) and $L_0 = 0$, but this notion coincides that of a strong solution to (2), because then L would be an $(\mathcal{F}_t)$-martingale, hence $L = 0$.

We now define weak solutions as solutions defined on an extended probability space:

Definition 2.3 A *weak solution* to (2) is a stochastic basis $(\underline{\Omega}, \underline{\mathcal{F}}, (\underline{\mathcal{F}}_t)_t, \mu)$ along with a triplet (Y, Z, L) of processes defined on $\underline{\Omega}$ such that:

1 There exists a measurable space $(\Gamma, \mathcal{G})$, and a filtration $(\mathcal{G}_t)$ on $(\Gamma, \mathcal{G})$ such that

$$\underline{\Omega} = \Omega \times \Gamma, \quad \underline{\mathcal{F}} = \mathcal{F} \otimes \mathcal{G}, \quad \underline{\mathcal{F}}_t = \mathcal{F}_t \otimes \mathcal{G}_t \text{ for every } t,$$

and there exists a probability measure μ on $(\underline{\Omega}, \underline{\mathcal{F}})$ such that $\mu(A \times \Gamma) = \mathrm{P}(A)$ for every $A \in \mathcal{F}$.

Note that every random variable ζ defined on Ω can be identified to a random variable defined on $\underline{\Omega}$, by setting $\zeta(\omega, \gamma) = \zeta(\omega)$. Furthermore, $\mathcal{F}$ can be viewed as a sub-σ-algebra of $\underline{\mathcal{F}}$ by identifying each $A \in \mathcal{F}$ to the set $A \times \Gamma$. Similarly, each $\mathcal{F}_t$ can be considered as a sub-σ-algebra of $\underline{\mathcal{F}}_t$. We say that $(\underline{\Omega}, \underline{\mathcal{F}}, (\underline{\mathcal{F}}_t)_t, \mu)$ is an *extension* of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathrm{P})$.

2 The process $(W_t)_{0 \leq t \leq T}$ is a Brownian motion on $(\underline{\Omega}, \underline{\mathcal{F}}, (\underline{\mathcal{F}}_t)_t, \mu)$ (where $W(\omega, \gamma) := W(\omega)$ for all $(\omega, \gamma) \in \underline{\Omega}$),

3 The process Z is $\mathbb{L}$-valued, defined on Ω and $(\mathcal{F}_t)$-adapted, with $\mathrm{E}\int_0^T \|Z_s\|^2\ ds < \infty$, and the processes Y and L are $\mathbb{R}^d$-valued, defined on $\underline{\Omega}$ and $(\underline{\mathcal{F}}_t)$-adapted, L is a continuous martingale with $L_T \in \mathrm{L}^2_{\mathbb{R}^d}(\underline{\Omega})$ and $L_0 = 0$, and L is orthogonal to every $\mathbb{R}^d$-valued $(\mathcal{F}_t)$-martingale M, i.e. $M^{[i]}L^{[j]}$ is a martingale for all $i, j \in \{1, \ldots, d\}$, where $M^{[i]}$ and $L^{[j]}$ are respectively the ith and jth coordinates of M and L.

4 The BSDE (1) holds.

Remark 2.4 In our definition of a weak solution, Z is assumed to be defined on Ω, that is, $\int_0^t Z_s\, dW_s$ represents the $(\mathcal{F}_t)$-adapted component of the martingale $Y_t - Y_0 + \int_0^t f(s, X_s, Y_s, Z_s)\, ds$, i.e.

$$\int_0^t Z_s\, dW_s = \mathrm{E}^{\mathcal{F}_t}\left(\int_0^t Z_s\, dW_s + L_t\right)$$

where $\mathrm{E}^{\mathcal{F}_t}$ denotes the conditional expectation with respect to $\mathcal{F}_t$. Actually, it is easy to check (see the proof of Lemma 3.3) that (1) is equivalent to

$$Y_t = \mathrm{E}^{\underline{\mathcal{F}}_t}\left(\xi + \int_t^T f(s, X_s, Y_s, Z_s)\, ds\right) \tag{3}$$

$$\int_0^t Z_s\, dW_s + L_t = \mathrm{E}^{\underline{\mathcal{F}}_t}\left(\xi + \int_0^T f(s, X_s, Y_s, Z_s)\, ds\right) - \mathrm{E}\left(\xi + \int_0^T f(s, X_s, Y_s, Z_s)\, ds\right). \tag{4}$$

Remark 2.5 One easily sees that, under hypothesis (H_1) and(H_2), Equation (2) may have infinitely many *strong* solutions. For example, let $d = m = 1$, $\xi = 0$, and $f(s, x, y, z) = \sqrt{|y|}$. Then, for any $t_0 \in [0, T]$, we get a solution by setting $Z = 0$ and

$$Y_t = \begin{cases} \frac{1}{4}(t_0 - t)^2 & \text{if } 0 \le t \le t_0 \\ 0 & \text{if } t_0 \le t \le T. \end{cases}$$

Thus, in our setting, pathwise uniqueness does not necessarily hold.

T. G. Kurtz [19] has proved a very general version of the Yamada-Watanabe and Engelbert theorems on uniqueness and existence of strong solutions to stochastic equations, which includes SDEs, BSDEs and FBSDEs. His results are based on the convexity of the set of joint solution-measures. But, in our case, as Z is defined on Ω, this convexity does not hold. So Kurtz's theory does not apply directly to our problem, and the study of the relations between pathwise uniqueness and strong solutions remains to be done.

3 Construction of a weak solution

Theorem 3.1 *Assume that f satisfies hypotheses (H_1) and (H_2). Then Equation (2) admits a weak solution.*

Note that the counterexample given by Buckdahn and Engelbert in [8] does not fit in our framework, so we do not know any example of a BSDE of the form (2) under hypothesis (H_1) and (H_2) which has no strong solution.

3.1 Construction of an approximating sequence of solutions

In this subsection, we only assume that f is measurable and satisfies the growth condition (H_1). We construct approximating equations and solutions and prove some boundedness results of theses solutions through a series of Lemmas.

Approximating equations The proof of Lemma 3.3 will show that (2) amounts to the following equations (5) and (6):

$$Y_t = \mathrm{E}^{\mathcal{F}_t}\left(\xi + \int_t^T f(s, X_s, Y_s, Z_s)\, ds\right) \tag{5}$$

$$\int_0^t Z_s\, dW_s = \mathrm{E}^{\mathcal{F}_t}\left(\xi + \int_0^T f(s, X_s, Y_s, Z_s)\, ds\right) - \mathrm{E}\left(\xi + \int_0^T f(s, X_s, Y_s, Z_s)\, ds\right) \tag{6}$$

where $\mathrm{E}^{\mathcal{F}_t}$ denotes the conditional expectation with respect to $\mathcal{F}_t$. We need first some notations: in the sequel, f is extended by setting $f(t, x, y, z) = 0$ for $t > T$, and we denote $\widetilde{Z}_s^{(n)} = \mathrm{E}^{\mathcal{F}_s}\left(Z_{s+1/n}^{(n)}\right)$, with $Z_t^{(n)} = 0$ for $t > T$. We can now write the approximating equations for (5) and (6):

$$Y_t^{(n)} = \mathrm{E}^{\mathcal{F}_t}\left(\xi + \int_{t+1/n}^T f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds\right) \tag{7}$$

$$\int_0^t Z_s^{(n)}\, dW_s = \mathrm{E}^{\mathcal{F}_t}\left(\xi + \int_0^T f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds\right) - \mathrm{E}\left(\xi + \int_0^T f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds\right) \tag{8}$$

Proposition 3.2 *The system (7)-(8) admits a unique strong solution $(Y^{(n)}, Z^{(n)})$. Furthermore, $Y_t^{(n)} \in \mathrm{L}^2_{\mathbb{R}^d}(\Omega \times [0, T])$ and $Z^{(n)} \in \mathrm{L}^2_{\mathbb{L}}(\Omega \times [0, T])$.*

Proof Let $T_k = T - \frac{k}{n}$, $k = 0, \dots, \lceil nT \rceil$, where $\lceil nT \rceil$ is the integer part of nT. Observe first that for each k, (8) amounts on the interval $]T_{k+1}, T_k]$ to

$$\int_{T_{k+1}}^{t} Z_s^{(n)}\, dW_s = \mathrm{E}^{\mathcal{F}_t}\left(\xi + \int_{T_{k+1}}^{T} f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds\right) - \mathrm{E}^{\mathcal{F}_{T_{k+1}}}\left(\xi + \int_{T_{k+1}}^{T} f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds\right). \tag{9}$$

Now, the construction of $(Y^{(n)}, Z^{(n)})$ is easy by backward induction: For $T_1 \le t \le T = T_0$, we have $Y_t^{(n)} = \mathrm{E}^{\mathcal{F}_t}(\xi)$ and $(Z_t^{(n)})_{T_1 \le t \le T}$ is the unique adapted process such that $\mathrm{E} \int_{T_1}^{T} \left(Z_t^{(n)}\right)^2 ds < +\infty$ and

$$\int_{T_1}^{t} Z_s^{(n)}\, dW_s = \mathrm{E}^{\mathcal{F}_t}\left(\xi + \int_{T_1}^{T} f(s, X_s, Y_s^{(n)}, 0)\, ds\right) - \mathrm{E}^{\mathcal{F}_{T_1}}\left(\xi + \int_{T_1}^{T} f(s, X_s, Y_s^{(n)}, 0)\, ds\right).$$

Suppose $(Y^{(n)}, Z^{(n)})$ is defined on the time interval $]T_k, T]$, with $k < [nT]$, then $Y^{(n)}$ is defined in a unique way on $]T_{k+1}, T_k]$ by (7) and then $Z^{(n)}$ on the same interval by (9). Furthermore, we get by induction from (9) that $Z^{(n)} \in \mathrm{L}^2_{\mathbb{L}}(\Omega \times [0, T])$. Then, using this last result in (7), we deduce that $Y_t^{(n)} \in \mathrm{L}^2_{\mathbb{R}^d}(\Omega \times [0, T])$. □

The following result links (7) and (8) to an approximate version of (2):

Lemma 3.3 *Equations* (7) *and* (8) *are equivalent to*

$$Y_t^{(n)} = \xi + \int_t^T f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds - \int_t^T Z_s^{(n)}\, dW_s - U_t^{(n)} \tag{10}$$

with $Y^{(n)}$ *adapted and*

$$U_t^{(n)} = \mathrm{E}^{\mathcal{F}_t}\left(\int_t^{t+1/n} f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds\right).$$

Proof Assume (7) and (8). Denoting

$$M_t^{(n)} = \mathrm{E}^{\mathcal{F}_t}\left(\xi + \int_0^T f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds\right) = M_0^{(n)} + \int_0^t Z_s^{(n)}\, dW_s,$$

Equation (8) becomes

$$M_t^{(n)} = \mathrm{E}^{\mathcal{F}_t}\left(\xi + \int_{t+1/n}^T f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds\right) \\ + \int_0^t f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds + U_t^{(n)}$$

By (7), this yields

$$M_t^{(n)} = Y_t^{(n)} + \int_0^t f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds + U_t^{(n)},$$

that is,

$$\begin{aligned} Y_t^{(n)} &= M_t^{(n)} - \int_0^t f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds - U_t^{(n)} \\ &= M_0^{(n)} + \int_0^t Z_s^{(n)}\, dW_s - \int_0^t f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds - U_t^{(n)}. \end{aligned}$$

In particular,

$$Y_T^{(n)} = \xi = M_0^{(n)} + \int_0^T Z_s^{(n)}\, dW_s - \int_0^T f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds$$

thus

$$Y_t^{(n)} - Y_T^{(n)} = -\int_t^T Z_s^{(n)}\, dW_s + \int_t^T f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds - U_t^{(n)},$$

which proves (10).

Conversely, assume (10) and that $Y^{(n)}$ is adapted. Denote $V_t^{(n)} = \int_0^t Z_s^{(n)}\, dW_s$. We have

$$\begin{aligned} Y_t^{(n)} &= \mathrm{E}^{\mathcal{F}_t}\left(Y_t^{(n)}\right) \\ &= \mathrm{E}^{\mathcal{F}_t}\Big(\xi + \int_t^T f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds - \int_t^T Z_s^{(n)}\, dW_s \\ &\qquad - \int_t^{t+1/n} f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds\Big) \\ &= \mathrm{E}^{\mathcal{F}_t}\left(\xi + \int_{t+1/n}^T f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds\right) - \mathrm{E}^{\mathcal{F}_t}\left(V_T^{(n)} - V_t^{(n)}\right) \\ &= \mathrm{E}^{\mathcal{F}_t}\left(\xi + \int_{t+1/n}^T f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds\right), \end{aligned}$$

which proves (7).

Now, using (7) and (10), we have

$$\begin{aligned}
&\mathrm{E}^{\mathcal{F}_t}\left(\xi + \int_0^T f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds\right) \\
&\quad = \mathrm{E}^{\mathcal{F}_t}\left(\xi + \int_{t+1/n}^T f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds\right) \\
&\qquad\qquad + \int_0^t f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds + U_t^{(n)} \\
&\quad = Y_t^{(n)} + \int_0^t f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds + U_t^{(n)} \\
&\quad = \xi + \int_t^T f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds - \int_t^T Z_s^{(n)}\, dW_s - U_t^{(n)} \\
&\qquad + \int_0^t f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds + U_t^{(n)} \\
&\qquad\qquad = \xi + \int_0^T f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds - \int_t^T Z_s^{(n)}\, dW_s
\end{aligned}$$

In particular,

$$\begin{aligned}
&\mathrm{E}\left(\xi + \int_0^T f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds\right) \\
&\qquad\qquad = \xi + \int_0^T f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds - \int_0^T Z_s^{(n)}\, dW_s
\end{aligned}$$

thus

$$\begin{aligned}
&\int_0^t Z_s^{(n)}\, dW_s = \int_0^T Z_s^{(n)}\, dW_s - \int_t^T Z_s^{(n)}\, dW_s \\
&= \left(\xi + \int_0^T f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds - \mathrm{E}\left(\xi + \int_0^T f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds\right)\right) \\
&\quad - \left(\xi + \int_0^T f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds - \mathrm{E}^{\mathcal{F}_t}\left(\xi + \int_0^T f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds\right)\right) \\
&= \mathrm{E}^{\mathcal{F}_t}\left(\xi + \int_0^T f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds\right) - \mathrm{E}\left(\xi + \int_0^T f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds\right)
\end{aligned}$$

which proves (8). □

Boundedness results In the following part, we show some results that will be useful to prove that our approximating sequence of solutions $(Y^{(n)}, Z^{(n)})$ is tight.

Lemma 3.4 *Let*

$$\widetilde{Y}_t^{(n)} = Y_t^{(n)} + U_t^{(n)} = \xi + \int_t^T f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds - \int_t^T Z_s^{(n)}\, dW_s.$$

There exist constants $\mathfrak{a}, \mathfrak{b} > 0$ *such that, for all* t *such that* $0 \le t \le T$*,*

$$\mathrm{E} \int_t^T \left\| Z_s^{(n)} \right\|^2 ds \le \mathfrak{a}\, \mathrm{E} \int_t^T \left\| \widetilde{Y}_s^{(n)} \right\|^2 ds + \mathfrak{b}. \tag{11}$$

Proof Appling Itô's formula to the semi-martingale $\left\| \widetilde{Y}_t^{(n)} \right\|^2$, taking expectation in both sides and using the fact that $t' \mapsto \int_t^{t'} \left\| \widetilde{Y}_t^{(n)} \right\| \left\| Z_s^{(n)} \right\| dW_s$ is a martingale (thanks to Proposition 3.2), we get:

$$\mathrm{E} \left\| \widetilde{Y}_t^{(n)} \right\|^2 = \mathrm{E} \left\| \xi \right\|^2 + 2\, \mathrm{E} \int_t^T \widetilde{Y}_s^{(n)}.f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)}) ds - \mathrm{E} \int_t^T \left| Z_s^{(n)} \right|^2 ds$$

thus

$$\mathrm{E} \int_t^T \left\| Z_s^{(n)} \right\|^2 ds \le \mathrm{E} \left\| \xi \right\|^2 + 2\, \mathrm{E} \int_t^T \left\| \widetilde{Y}_s^{(n)} \right\| . \left\| f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)}) \right\| ds.$$

From (H_1), this entails

$$\mathrm{E} \int_t^T \left\| Z_s^{(n)} \right\|^2 ds \le \mathrm{E} \left\| \xi \right\|^2 + 2C_f\, \mathrm{E} \int_t^T \left\| \widetilde{Y}_s^{(n)} \right\| (1 + \left\| \widetilde{Z}_s^{(n)} \right\|)\, ds.$$

Using that, for $a \ge 0, b \ge 0$, and $\lambda \ne 0$, we have $2ab \le a^2\lambda^2 + b^2/\lambda^2$, we get

$$\begin{aligned}
&\int_t^T \left\| \widetilde{Y}_s^{(n)} \right\| (1 + \left\| \widetilde{Z}_s^{(n)} \right\|)\, ds \\
&\qquad \le +2\lambda^2 \int_t^T \left\| \widetilde{Y}_s^{(n)} \right\|^2 ds + (T-t)/\lambda^2 + 1/\lambda^2 \int_t^T \left\| \widetilde{Z}_s^{(n)} \right\|^2 ds \\
&\qquad \le +2\lambda^2 \int_t^T \left\| \widetilde{Y}_s^{(n)} \right\|^2 ds + (T-t)/\lambda^2 + 1/\lambda^2 \int_t^T \left\| Z_s^{(n)} \right\|^2 ds
\end{aligned}$$

thus, taking $\lambda^2 > 2C_f$,

$$(1 - 2C_f/\lambda^2)\, \mathrm{E} \int_t^T \left\| Z_t^{(n)} \right\|^2 ds \le \mathrm{E} \left\| \xi \right\|^2 + 2C_f \left(T/\lambda^2 + \mathrm{E} \int_t^T \left\| \widetilde{Y}_t^{(n)} \right\|^2 ds \right)$$

which yields (11). □

Proposition 3.5 *Let* $\widetilde{Y}_t^{(n)} = Y_t^{(n)} + U_t^{(n)}$ *as in lemma 3.4. The families* $(\widetilde{Y}_t^{(n)})_{0\le t\le T,\, n\ge 1}$, $(Y_t^{(n)})_{0\le t\le T,\, n\ge 1}$ *and* $(U_t^{(n)})_{0\le t\le T,\, n\ge 1}$ *are bounded in* $\mathrm{L}^2_{\mathbb{R}^d}(\Omega)$.

Proof Let

$$\begin{aligned}
\widetilde{Y}_t^{(n)} &= Y_t^{(n)} + U_t^{(n)} \\
&= \mathrm{E}^{\mathcal{F}_t}\left(\xi + \int_{t+1/n}^T f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\right) + \mathrm{E}^{\mathcal{F}_t}\left(\int_t^{t+1/n} f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds\right) \\
&= \mathrm{E}^{\mathcal{F}_t}\left(\xi + \int_t^T f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds\right)
\end{aligned}$$

So, we have the following inequalities, where C denotes some constant which is not necessarily the same at each line but does not depend on n:

$$\begin{aligned}
\mathrm{E}\left\|\widetilde{Y}_t^{(n)}\right\|^2 &= \mathrm{E}\left\|\mathrm{E}^{\mathcal{F}_t}\left(\xi + \int_t^T f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds\right)\right\| \\
&\le C\,\mathrm{E}\left(\|\xi\|^2 + \int_t^T (1 + \left\|Z_s^{(n)}\right\|^2)\, ds\right) \\
&\le C\left(1 + \int_0^t \mathrm{E}\left\|\widetilde{Y}_s^{(n)}\right\|^2 ds\right).
\end{aligned}$$

The last inequality is a consequence of Lemma 3.4. Let $g(t) = \mathrm{E}\left\|\widetilde{Y}_{T-t}^{(n)}\right\|^2$. The preceding inequalities yield

$$g(t) \le C\left(1 + \int_0^t g(s)\, ds\right)$$

thus, by Gronwall Lemma,

$$g(t) \le C\left(1 + C\int_0^t e^{C(t-s)}\right) \le C\left(1 + C\int_0^T e^{C(T-s)}\right)$$

which proves that $(\widetilde{Y}_t^{(n)})_{0\leq t\leq T,\, n\geq 1}$ is bounded in $\mathrm{L}^2_{\mathbb{R}^d}(\Omega)$.
Now, we have

$$\begin{aligned}
\mathrm{E}\left\|Y_t^{(n)}\right\|^2 &= \mathrm{E}\left\|\mathrm{E}^{\mathcal{F}_t}\left(\xi + \int_{t+1/n}^T f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds\right)\right\| \\
&\leq C\,\mathrm{E}\left(\|\xi\|^2 + \int_{t+1/n}^T (1 + \left\|Z_s^{(n)}\right\|^2)\, ds\right) \\
&\leq C\left(1 + \int_0^t \mathrm{E}\left\|\widetilde{Y}_s^{(n)}\right\|^2 ds\right).
\end{aligned}$$

wich proves that $(Y_t^{(n)})_{0\leq t\leq T,\, n\geq 1}$ is bounded in $\mathrm{L}^2_{\mathbb{R}^d}(\Omega)$.

In the same way, we prove that $(U_t^{(n)})_{0\leq t\leq T,\, n\geq 1}$ is bounded in $\mathrm{L}^2_{\mathbb{R}^d}(\Omega)$.

□

Remark 3.6 Thanks to Proposition 3.5, we can improve Equation (11) in the following way: With the notations of Lemma 3.4, we have, for all t, t' such that $0 \leq t \leq t' \leq T$,

$$\mathrm{E}\int_t^{t'} \left\|Z_s^{(n)}\right\|^2 ds \leq \mathfrak{a}\,\mathrm{E}\int_t^{t'} \left\|\widetilde{Y}_s^{(n)}\right\|^2 ds + \mathfrak{b},$$

where the constants $\mathfrak{a}$ and $\mathfrak{b}$ do not depend on t and t'.

Indeed, by Proposition 3.5, (H_1), and (7), the family $(\widetilde{Y}_t^{(n)})_{n,t}$ is bounded in $\mathrm{L}^2_{\mathbb{R}^d}(\Omega)$ by some number $\mathfrak{M}$. We can now reproduce the reasoning of Lemma 3.4, replacing T by t' and $\mathrm{E}\,\|\xi\|^2$ by $\mathfrak{M}^2$.

Lemma 3.7 *Let $1 \leq q < 2$. There exists a constant $\mathfrak{C}_{q,2}$ such that, for all stopping times σ and τ with values in $[0, T]$,*

$$\mathrm{E}\left\|Y_\tau^{(n)} - Y_\sigma^{(n)}\right\|^q = \mathfrak{C}_{q,2}\left(\mathrm{E}\,|\tau - \sigma|\right)^{(2-q)/2} \tag{12}$$

Our proof of Lemma 3.7 relies on the following lemma, which will be used also in Subsection 3.2.

Lemma 3.8 *Let $1 \leq q < p$. There exists a constant $\alpha_{q,p}$ such that, for all $\mathcal{F}_T$-measurable random vectors $V \in \mathrm{L}^p_{\mathbb{R}^d}$ and for all stopping times σ and τ with values in $[0, T]$,*

$$\mathrm{E}\left\|\mathrm{E}^{\mathcal{F}_\tau}(V) - \mathrm{E}^{\mathcal{F}_\sigma}(V)\right\|^q = \alpha_{q,p}\left(\mathrm{E}\,|\tau - \sigma|\right)^{(p-q)/p} \|V\|^p_{\mathrm{L}^p_{\mathbb{R}^d}}. \tag{13}$$

Proof The result remains unchanged if we replace σ and τ by $\sigma \wedge \tau$ and $\sigma \vee \tau$, so we can assume without loss of generality that $\sigma \leq \tau$. By the martingale representation theorem, there exists an adapted process $H^{(V)}$, with $\mathrm{E} \int_0^T \left\| H_s^{(V)} \right\|^2 ds < \infty$, such that, for every $t \in [0, T]$,

$$M_t^{(V)} := \mathrm{E}^{\mathcal{F}_t}(V) - \mathrm{E}(V) = \int_0^t H_s^{(V)} \, dW_s.$$

Let k_q and K_q denote the universal constants in the Burholder-Davis-Gundy inequalities (see [18] for the multidimensional BDG inequalities), that is, for any continuous $\mathbb{R}^d$-valued martingale M and any stopping time ρ,

$$k_q \, \mathrm{E} \, \langle M \rangle_\rho^{q/2} \leq \mathrm{E} \left(M_\rho^* \right)^q \leq K_q \, \mathrm{E} \, \langle M \rangle_\rho^{q/2}, \tag{14}$$

where $M_t^* = \sup_{0 \leq s \leq t} \| M_s \|$. Applying BDG inequalities to the $(\mathcal{F}_{\sigma+t})_{t \geq 0}$-martingale $M_{\sigma+t}^{(V)}$, we get

$$\begin{aligned}
\mathrm{E} \left\| M_\tau^{(V)} - M_\sigma^{(V)} \right\|^q &= \mathrm{E} \left\| \int_\sigma^\tau H_s^{(V)} \, dW_s \right\|^q \\
&\leq K_q \, \mathrm{E} \int_\sigma^\tau \left\| H_s^{(V)} \right\|^q ds \\
&\leq K_q \left(\mathrm{E} \int_0^T \mathbb{1}_{[\sigma,\tau]}(s) \, ds \right)^{(p-q)/p} \left(\mathrm{E} \int_0^T \left\| H_s^{(V)} \right\|^p ds \right)^{q/p} \\
&\leq \frac{K_q}{(k_p)^{q/p}} \left(\mathrm{E} \left| \tau - \sigma \right| \right)^{(p-q)/p} \left(\mathrm{E} \left\| \int_0^T H_s^{(V)} \, dW_s \right\|^p \right)^{q/p} \\
&= \frac{K_q}{(k_p)^{q/p}} \left(\mathrm{E} \left| \tau - \sigma \right| \right)^{(p-q)/p} \left(\mathrm{E} \, \| V \|^p \right)^{q/p}
\end{aligned}$$

□

Proof of Lemma 3.7 As in the proof of Lemma 3.8, we can assume that $\sigma \leq \tau$. We have

$$\begin{aligned}
\mathrm{E} \left\| Y_\tau^{(n)} - Y_\sigma^{(n)} \right\|^q = \mathrm{E} \Bigg\| & \mathrm{E}^{\mathcal{F}_\tau} \left(\xi + \int_\tau^T f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)}) \, ds \right) \\
& - \mathrm{E}^{\mathcal{F}_\sigma} \left(\xi + \int_\sigma^T f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)}) \, ds \right) \Bigg\|^q \\
\leq A + B &
\end{aligned}$$

where, for some coefficient C_q which depends only on q,

$$
\begin{aligned}
A = C_q \,\mathrm{E} \Bigg\| \mathrm{E}^{\mathcal{F}_\tau} \left(\xi + \int_\tau^T f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds \right) \\
- \mathrm{E}^{\mathcal{F}_\sigma} \left(\xi + \int_\tau^T f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds \right) \Bigg\|^q, \\
B = C_q \,\mathrm{E} \Bigg\| \mathrm{E}^{\mathcal{F}_\sigma} \left(\xi + \int_\tau^T f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds \right) \\
- \mathrm{E}^{\mathcal{F}_\sigma} \left(\xi + \int_\sigma^T f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds \right) \Bigg\|^q.
\end{aligned}
$$

Now, by Lemma 3.4, Proposition 3.5 and the growth condition (H_1), the sequence $\left(\int_\tau^T f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds\right)$ is bounded in $\mathrm{L}^q_{\mathbb{R}^d}$ by some constant $\mathfrak{m}_q < +\infty$. From Lemma 3.8, we get

$$
A \le C_q \,\alpha_{q,2}\, \mathfrak{m}_q \left(\mathrm{E}\,|\tau - \sigma|\right)^{(2-q)/2} \tag{15}
$$

for some constant $\alpha_{q,2}$. On the other hand,we also have the following inequalities, where the coefficient C_q does not keep the same value from line to line:

$$
\begin{aligned}
B &= C_q \,\mathrm{E} \left\| \mathrm{E}^{\mathcal{F}_\sigma} \left(\int_\sigma^\tau f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds \right) \right\|^q \\
&\le C_q \,\mathrm{E} \left\| \int_\sigma^\tau f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds \right\|^q \\
&\le C_q \,\mathrm{E} \int_\sigma^\tau \left(1 + \left\| Z_s^{(n)} \right\|^q \right) ds \\
&= C_q \ \mathrm{E} \left(\int_0^T \mathbb{1}_{[\sigma,\tau]}(s) \left(1 + \left\| Z_s^{(n)} \right\|^q \right) ds \right) \\
&\le C_q \left(\mathrm{E}\,|\tau - \sigma|\right)^{(2-q)/2} \left(\mathrm{E} \int_0^T \left(1 + \left\| Z_s^{(n)} \right\|^2 \right) ds \right)^{q/2} \\
&\le C_q \left(\mathrm{E}\,|\tau - \sigma|\right)^{(2-q)/2} \mathfrak{M}_2^{q/2} \qquad (16)
\end{aligned}
$$

where $\mathfrak{M}_2 = \sup_n \mathrm{E} \int_0^T \left(1 + \left\| Z_s^{(n)} \right\|^2 \right) ds < +\infty$ by Lemma 3.4 and Proposition 3.5. Gathering (15) and (16) yields the result. □

Lemma 3.9 *Let $1 \le q < 2$. We have*

$$\lim_{n\to\infty} \sup_n \mathrm{E}\left(\sup_{0\le t\le T} \left\| U_t^{(n)} \right\|^q \right) = 0, \tag{17}$$

$$\lim_{\delta\to 0} \sup_n \sup_{\substack{\sigma,\tau\in\mathfrak{T}\\ 0\le|\tau-\sigma|\le\delta}} \mathrm{E}\left(\left\| U_\tau^{(n)} - U_\sigma^{(n)} \right\|^q \right) = 0 \tag{18}$$

where $\mathfrak{T}$ denotes the set of stopping times with values in $[0,T]$.

Proof Let $\mathfrak{M}_2 = \sup_n \mathrm{E} \int_0^T \left(1 + \left\| Z_s^{(n)} \right\|\right)^2 ds$. By Lemma 3.4 and Proposition 3.5, we have $\mathfrak{M}_2 < +\infty$. Using the growth condition (H_1), we get

$$\begin{aligned}
\mathrm{E}\left(\sup_{0\le t\le T} \left\| U_t^{(n)} \right\|^q \right) &\le C_f^q\, \mathrm{E}\left(\sup_{0\le t\le T} \mathrm{E}^{\mathcal{F}_t} \left(\int_t^{t+1/n} \left(1 + \left\| Z_s^{(n)} \right\|\right)^q ds \right)\right) \\
&\le C_f^q \left(\frac{1}{n}\right)^{(2-q)/2} \left(\mathrm{E} \int_0^T \left(1 + \left\| Z_s^{(n)} \right\|\right)^2 ds \right)^{q/2} \\
&\le C_f^q\, \mathfrak{M}_2 \left(\frac{1}{n}\right)^{(2-q)/2} \to 0 \text{ when } n \to 0,
\end{aligned}$$

which proves (17).

Let $\sigma, \tau \in \mathfrak{T}$. Denote temporarily

$$\begin{aligned}
\widetilde{U}_t^{(n)} &= \int_t^{t+1/n} f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds, \\
F_t^{(n)} &= f(t, X_t, Y_t^{(n)}, \widetilde{Z}_t^{(n)}).
\end{aligned}$$

Using Lemma 3.8 and proposition 3.5, we have, for some constant C,

$$\begin{aligned}
&\mathrm{E}\left(\left\|U_\tau^{(n)} - U_\sigma^{(n)}\right\|^q\right)\\
&\quad = \mathrm{E}\left(\left\|\mathrm{E}^{\mathcal{F}_\tau}\left(\widetilde{U}_\tau^{(n)}\right) - \mathrm{E}^{\mathcal{F}_\sigma}\left(\widetilde{U}_\sigma^{(n)}\right)\right\|^q\right)\\
&\quad \leq 2^{q-1}\,\mathrm{E}\Bigg(\left(\left\|\mathrm{E}^{\mathcal{F}_\tau}\left(\widetilde{U}_\tau^{(n)}\right) - \mathrm{E}^{\mathcal{F}_\sigma}\left(\widetilde{U}_\tau^{(n)}\right)\right\|^q\right)\\
&\qquad\qquad + \left(\left\|\mathrm{E}^{\mathcal{F}_\sigma}\left(\widetilde{U}_\tau^{(n)}\right) - \mathrm{E}^{\mathcal{F}_\sigma}\left(\widetilde{U}_\sigma^{(n)}\right)^q\right\|\right)\Bigg)\\
&\quad \leq 2^{q-1}\alpha_{q,2}\left(\mathrm{E}\left|\tau-\sigma\right|\right)^{(2-q)/2}\left\|\widetilde{U}_\tau^{(n)}\right\|^2_{\mathrm{L}^2_{\mathbb{R}^d}}\\
&\qquad\qquad + 2^{q-1}\,\mathrm{E}\left\|\int_\tau^{\tau+1/n} F_s^{(n)}\,ds - \int_\sigma^{\sigma+1/n} F_s^{(n)}\,ds\right\|^q\\
&\quad \leq C\left(\mathrm{E}\left|\tau-\sigma\right|\right)^{(2-q)/2}\mathrm{E}\int_0^T (1+\left\|Z_s^{(n)}\right\|)^2 ds + C\,\mathrm{E}\left\|\int_{A(\sigma,\tau)} F_s^{(n)}\,ds\right\|^2
\end{aligned}$$

where $A(\sigma,\tau)$ is the symmetric difference of the intervals $[\tau,\tau+1/n]$ and $[\sigma,\sigma+1/n]$, and has Lebesgue measure $|A(\sigma,\tau)| \leq 2((\tau-\sigma)\wedge(1/n))$. Thus

$$\begin{aligned}
\mathrm{E}\left(\left\|U_\tau^{(n)} - U_\sigma^{(n)}\right\|^q\right) &\leq C\left(\mathrm{E}\left|\tau-\sigma\right|\right)^{(2-q)/2}\mathrm{E}\int_0^T (1+\left\|Z_s^{(n)}\right\|)^2 ds\\
&\quad + C\left(\mathrm{E}\left|\tau-\sigma\right|^2\right)\mathrm{E}\int_0^T (1+\left\|Z_s^{(n)}\right\|)^2 ds,
\end{aligned}$$

which proves (18). □

Lemma 3.10 *The sequence* $(Y^{(n)})_{n\geq 1}$ *is tight in* $\mathrm{C}_{\mathbb{R}^d}[0,T]$.

Proof By a criterion of Aldous [1, 14], we only need to prove that

$$(A)\qquad \forall\epsilon>0,\ \exists R>0, \forall n\geq 1,\ \mathrm{P}\left(\sup_{0\leq t\leq T}\left\|Y_t^{(n)}\right\| \geq R\right)\leq\epsilon$$

$$(B)\quad \forall\epsilon>0, \forall\eta>0,\ \exists\delta>0 : \forall n\geq 1,\ \sup_{\substack{\sigma,\tau\in\mathfrak{T}\\ 0\leq|\tau-\sigma|\leq\delta}} \mathrm{P}\left(\left\|Y_\tau^{(n)} - Y_\sigma^{(n)}\right\| \geq \eta\right)\leq\epsilon$$

where $\mathfrak{T}$ denotes the set of stopping times with values in $[0,T]$. Property (B) is an immediate consequence of Lemma 3.7. Let us prove the following property (A'), which is stronger than (A):

$$(A') \qquad \sup_{n\geq 1} \mathrm{E}\left(\sup_{0\leq t\leq T} \left\| Y_t^{(n)} \right\|^2 \right) < +\infty.$$

Using (10), we get

$$\sup_{0\leq t\leq T} \left\| Y_t^{(n)} \right\|^2 \leq A_n + B_n + C_n$$

where

$$\begin{aligned} A_n &= 3 \sup_{0\leq t\leq T} \left\| \xi + \int_{t+1/n}^T f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds \right\|^2, \\ B_n &= 3 \sup_{0\leq t\leq T} \left\| \int_t^T Z_s^{(n)}\, dW_s \right\|^2, \\ C_n &= 3 \sup_{0\leq t\leq T} \left\| U_t^{(n)} \right\|^2. \end{aligned}$$

By Lemma 3.4 and Proposition 3.5, $(Z^{(n)})_{0\leq t\leq T,\, n\geq 1}$ is bounded in $\mathrm{L}^2_{\mathbb{L}}(\Omega \times [0,T])$, thus using the growth condition (H_1), we get

$$\sup_n \mathrm{E}\left(\sup_{0\leq t\leq T} \left(\|\xi\|^2 + C_f^2 \int_{t+1/n}^T \left(1 + \left\| Z_s^{(n)} \right\|\right)^2 ds \right) \right) < +\infty$$

which entails $\sup_n \mathrm{E}(A_n) < +\infty$. On the other hand, $V_t^{(n)} := \int_0^t Z_s^{(n)}\, dW_s$ is a martingale, so, using again Lemma 3.4 and Proposition 3.5,

$$\sup_n \mathrm{E}\,(B_n) \leq C \sup_n \mathrm{E} \left\| V_T^{(n)} \right\|^2 < +\infty.$$

Finally from (H_1), Lemma 3.4, and Proposition 3.5 we have $\sup_n C_n < +\infty$, and this proves (A'). □

Lemma 3.11 *The sequence* $(\int_\cdot^T Z_s^{(n)}\, dW_s)_{n\geq 1}$ *is tight in* $\mathrm{C}_{\mathbb{R}^d}[0,T]$.

Proof We use the criterion of Aldous we already used in the proof of Lemma 3.10. Let us denote $\Lambda_t^{(n)} = \int_t^T Z^{(n)}\,dW_s$ and recall that $\widetilde{Y}_t^{(n)} = Y_t^{(n)} + U_t^{(n)}$. By Lemma 3.4 and Proposition 3.5, we have

(19)

$$\sup_n \mathrm{E}\left(\sup_{0\le t\le T}\left\|\Lambda_t^{(n)}\right\|^2\right) \le \sup_n \mathrm{E}\left(\sup_{0\le t\le T}\int_t^T\left(\mathfrak{a}\left\|\widetilde{Y}_s^{(n)}\right\|^2+\mathfrak{b}\right)\right) < +\infty.$$

Now, let σ and τ be stopping times, with $0 \le \sigma \le \tau \le T$, and let $1 \le q < 2$. We deduce from (10), (H_1), (18), and Lemma 3.7, that

$$\lim_{\delta\to 0}\sup_{\substack{\sigma,\tau\in\mathfrak{T}\\ 0\le|\tau-\sigma|\le\delta}} \mathrm{E}\left(\left\|\Lambda_\tau^{(n)}-\Lambda_\sigma^{(n)}\right\|^q\right) = 0. \tag{20}$$

We conclude from (19) and (20), using Aldous criterion. □

Lemma 3.12 *The sequence* $(\int_{\cdot}^T f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\,ds)_{n\ge1}$ *is tight in* $\mathrm{C}_{\mathbb{R}^d}[0,T]$.

Proof This is an immediate consequence of (10), (17) and Lemma 3.11.

□

3.2 Construction of a limiting weak solution

This part of the proof of Theorem 3.1 follows the same lines as in [17], with some complications due to the processes $Z^{(n)}$. We now assume in the sequel that f satisfies the continuity assumptions (H_2).

Construction of Z By Proposition 3.5, the sequence $(Z^{(n)})$ is weakly sequentially compact in the space $\mathbb{G} = \mathrm{L}^2_{\mathbb{L}}(\Omega\times[0,T])$. Let us denote by $\mathbb{G}_\sigma$ the space $\mathbb{G}$ endowed with its weak topology. Considering if necessary a subsequence of $(Y^{(n)}, Z^{(n)})$, we can thus assume that there exists a random variable Z with values in $\mathbb{L}$ such that

$$(Z^{(n)}) \text{ converges to } Z \text{ in } \mathbb{G}_\sigma. \tag{21}$$

Let us denote by $\mathbb{H} = \mathrm{L}^2_{\mathbb{L}}([0,T])$, and by $\mathbb{H}_\sigma$ the space $\mathbb{H}$ endowed with its weak topology. By (21), we have in particular:

$$(Z^{(n)}) \text{ converges a.e. to } Z \text{ in } \mathbb{H}_\sigma. \tag{22}$$

Indeed, for any $A \in \mathcal{F}$ and any $g \in \mathbb{H}$, the function $\mathbb{1}_A \otimes g$ is in $\mathbb{G}$, thus we have

$$\lim_n \mathrm{E}\left(\mathbb{1}_A \int_0^T Z_s^{(n)} g(s)\, ds \right) = \mathrm{E}\left(\mathbb{1}_A \int_0^T Z_s g(s)\, ds \right).$$

As A is arbitrary, we deduce that, almost everywhere,

$$\lim_n \int_0^T Z_s^{(n)} g(s)\, ds = \int_0^T Z_s\, g(s)\, ds.$$

Young measures Let us recall the definition of Young measures, see [27, 6] for more in-depth introductions to the topic. Let $\mathbb{E}$ be a separable metric space, or more generally a Suslin regular topological space (e.g. a Banach space endowed with its weak* topology). Let $\mathcal{B}(\mathbb{E})$ be the Borel σ-algebra of $\mathbb{E}$. A *Young measure* μ *with basis* P *on* $\mathbb{E}$ is a probability measure on $\Omega \times \mathbb{E}$, such that for any set $A \in \mathcal{F}$, $\mu(A \times \mathbb{E}) = \mathrm{P}(A)$. The space of Young measures with basis P is denoted by $\mathcal{Y}(\Omega, \mathcal{F}, \mathrm{P}; \mathbb{E})$. It is very useful to describe a Young measure μ by its *disintegration* (see [26]): for every $\mu \in \mathcal{Y}(\Omega, \mathcal{F}, \mathrm{P}; \mathbb{E})$, there exists a unique (up to equality P-a.e.) family $(\mu_\omega)_{\omega \in \Omega}$ of probabilities on $\mathbb{E}$ characterized, for any measurable nonnegative $\phi : \Omega \times \mathbb{E} \to \mathbb{R}$, by

$$\int_{\Omega \times \mathbb{E}} \phi d\mu = \int_\Omega \left(\int_\mathbb{E} \phi(\omega, \xi) \mu_\omega(d\xi)\, \mathrm{P}(dx) \right).$$

The space $\mathrm{L}^0(\Omega; \mathbb{E})$ of measurable functions from Ω to $\mathbb{E}$ is embedded in $\mathcal{Y}(\Omega, \mathcal{F}, \mathrm{P}; \mathbb{E})$ in the following way: we identify every $u \in \mathrm{L}^0(\Omega; \mathbb{E})$ with the unique Young measure μ whose support is the graph of u. We then have $\mu_\omega = \delta_{u(\omega)}$, where $\delta_{u(\omega)}$ denotes the Dirac mass at $u(\omega)$. The set $\mathcal{Y}(\Omega, \mathcal{F}, \mathrm{P}; \mathbb{E})$ is endowed with a metrizable topology, defined as follows: let $\mathrm{C}_\mathbb{E}[0, T]$ be the set of continuous bounded real valued functions defined on $\mathbb{E}$, then a sequence (μ^n) of Young measures converges to a Young measure μ if, for each $A \in \mathcal{F}$ and $f \in \mathrm{C}_\mathbb{E}[0, T]$, the sequence $(\mu^n(\mathbb{1}_A \otimes f))$ converges to $\mu(\mathbb{1}_A \otimes f)$ (where $\mathbb{1}_A$ is the indicator function of A and $(\mathbb{1}_A \otimes f)(\omega, \xi) = \mathbb{1}_A(\omega) f(\xi)$). In this case, we say that (μ^n) *converges stably* to μ (this terminology stems from Rényi [24]). Note that the restriction of the topology of stable convergence to $\mathrm{L}^0(\Omega; \mathbb{E})$ is the topology of convergence in probability, see [27].

The following technical lemma will be useful for limits of integrals of unbounded integrands with respect to Young measures.

Lemma 3.13 *Let $\mathbb{E}$ be a separable Banach space, and let $(K^{(n)})$ be a sequence of $\mathbb{E}$-valued continuous stochastic processes which converges in $\mathcal{Y}(\Omega, \mathcal{F}, \mathrm{P}; \mathrm{C}_{\mathbb{E}}[0,T])$ to a Young measure ν. Assume that $(K_t^{(n)})_{t\in[0,T],n\geq 1}$ is bounded in $\mathrm{L}^2_{\mathbb{E}}(\Omega)$. Let $\varphi:\ \Omega\times\mathrm{C}_{\mathbb{E}}[0,T]$ be measurable such that*

(i) $\varphi(\omega,.)$ is continuous for all $\omega\in\Omega$,

(ii) there exists a random variable $C\in\mathrm{L}^1_{\mathbb{R}}(\Omega)$ such that

$$|\varphi(\omega,x)|\leq C(\omega)\,(1+\|x\|)$$

for all $(\omega,x)\in\Omega\times\mathrm{C}_{\mathbb{E}}[0,T]$.

Then

$$\lim_{n\to\infty}\int_{\Omega\times\mathrm{C}_{\mathbb{E}}[0,T]}\varphi(\omega,K^{(n)}(\omega))\,d\,\mathrm{P}(\omega)=\int_{\Omega\times\mathrm{C}_{\mathbb{E}}[0,T]}\varphi\,d\nu.$$

Proof Each $K^{(n)}$ can be viewed as a $\mathrm{C}_{\mathbb{E}}[0,T]$-valued random variable. The sequence $(K^{(n)})$ is bounded in $\mathrm{L}^2_{\mathrm{C}_{\mathbb{E}}[0,T]}(\Omega)$, thus it is uniformly integrable. The conclusion follows from e.g. the equivalence 3 ⇔ 4 in [11, Proposition 2.4.1]. □

Construction of the extended probability space: the processes Y and V We now consider the space $\mathcal{Y}(\Omega,\mathcal{F},\mathrm{P};\mathrm{C}_{\mathbb{R}^d}[0,T]\times\mathrm{C}_{\mathbb{R}^d}[0,T])$, which we denote for simplicity by $\mathcal{Y}$. Let us denote $V_t^{(n)}=\int_0^t Z_s^{(n)}\,dW_s$. By Lemma 3.10 and Lemma 3.11, the sequence $(Y^{(n)},V^{(n)})$, seen as a sequence of random variables with values in $\mathrm{C}_{\mathbb{R}^d}[0,T]\times\mathrm{C}_{\mathbb{R}^d}[0,T]$, is tight. By Prohorov's sequential compactness criterion for Young measures [5, 11] we can extract a subsequence of $(Y^{(n)},V^{(n)})$ (for simplicity, we denote this extracted sequence by $(Y^{(n)},V^{(n)})$) which converges stably to some $\mu\in\mathcal{Y}$, that is, for every bounded continuous mapping $\Phi:\ \mathrm{C}_{\mathbb{R}^d}[0,T]\times\mathrm{C}_{\mathbb{R}^d}[0,T]\mapsto\mathbb{R}$ and for every $A\in\mathcal{F}$, we have

$$\lim_{n\to\infty}\int_A\Phi\left(Y^{(n)}(\omega),V^{(n)}(\omega)\right)\,d\,\mathrm{P}(\omega)=\int_\Omega\int_{\mathrm{C}_{\mathbb{R}^d}[0,T]\times\mathrm{C}_{\mathbb{R}^d}[0,T]}\mathbb{1}_A(\omega)\Phi(y,v)\,d\mu_\omega(y,v)\,d\,\mathrm{P}(\omega).\tag{23}$$

In particular, $(Y^{(n)},V^{(n)})$ converges in law to the image of μ by the canonical projection of $\Omega\times\mathrm{C}_{\mathbb{R}^d}[0,T]\times\mathrm{C}_{\mathbb{R}^d}[0,T]$ to $\mathrm{C}_{\mathbb{R}^d}[0,T]\times\mathrm{C}_{\mathbb{R}^d}[0,T]$.

Let us denote by $\mathcal{C}$ the Borel σ-algebra of $\mathrm{C}_{\mathbb{R}^d}[0,T]$ and, for each $t \in [0,T]$, let $\mathcal{C}_t$ be the sub-σ-algebra of $\mathcal{C}$ generated by $\mathrm{C}_{\mathbb{E}}[0,T]$. We define a stochastic basis $(\underline{\Omega}, \underline{\mathcal{F}}, (\underline{\mathcal{F}}_t)_t, \mu)$ by

$$\underline{\Omega} = \Omega \times \mathrm{C}_{\mathbb{R}^d}[0,T] \times \mathrm{C}_{\mathbb{R}^d}[0,T], \quad \underline{\mathcal{F}} = \mathcal{F} \otimes \mathcal{C} \otimes \mathcal{C}, \quad \underline{\mathcal{F}}_t = \mathcal{F}_t \otimes \mathcal{C}_t \otimes \mathcal{C}_t,$$

and we define a process (Y,V) on $\underline{\Omega}$ by

$$Y(\omega, y, v) = y, \quad V(\omega, y, v) = v.$$

Clearly, the law of (Y,V) is the projection of μ on $\mathrm{C}_{\mathbb{R}^d}[0,T] \times \mathrm{C}_{\mathbb{R}^d}[0,T]$, in particular $(Y^{(n)}, V^{(n)})$ converges in law to (Y,V). Furthermore, (Y,V) is $(\underline{\mathcal{F}}_t)$–adapted. Now, the random variables $(Y^{(n)}, V^{(n)})$ can be seen as random elements defined on $\underline{\Omega}$, using the notations

$$Y^{(n)}(\omega, y, v) := Y^{(n)}(\omega), \quad V^{(n)}(\omega, y, v) := V^{(n)}(\omega) \quad (n \geq 1).$$

Furthermore, $(Y^{(n)}, V^{(n)})$ is $(\underline{\mathcal{F}}_t)$–adapted for each n. Likewise, we set $W(\omega, y, v) = W(\omega)$.

Lemma 3.14 *The process W is an $(\underline{\mathcal{F}}_t)$–Wiener process under the probability μ.*

Proof Clearly, W is $(\underline{\mathcal{F}}_t)$–adapted. By a result of Balder [4, 5], each subsequence of $(Y^{(n)}, V^{(n)})$ contains a further subsequence (Y^{n_k}, V^{n_k}) which K–converges to μ, that is, for each subsequence $(Y^{(n'_k)}, V^{(n'_k)})$ of $(Y^{(n_k)}, V^{(n_k)})$, we have

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} \delta_{(Y^{(n'_k)}(\omega), V^{(n'_k)}(\omega))} = \mu_\omega \text{ a.e.}$$

where $\delta_{(y,z)}$ denotes the Dirac measure on (y,v) and the limit is taken in the narrow convergence. This entails that, for every $A \in \mathcal{C}_t$, the mapping $\omega \mapsto \mu_\omega(A)$ is $\mathcal{F}_t$–measurable[1]. Then it is very easy to check that W has independent increments under μ which proves that W is again a Brownian motion on $(\underline{\Omega}, \underline{\mathcal{F}}, (\underline{\mathcal{F}}_t)_t, \mu)$. Indeed, let $t \in [0,T]$ and let $s > 0$ such that $t + s \in [0,T]$. Let us prove that, for any $A \in \underline{\mathcal{F}}_t$ and any Borel subset C of $\mathbb{R}^m$, we have

$$\mu\left(A \cap \{W_{t+s} - Wt \in C\}\right) = \mu(A)\, \mu\{W_{t+s} - W_t \in C\}. \tag{24}$$

[1]Note that, from [15, Lemma 2.17], this means that $(\underline{\Omega}, \underline{\mathcal{F}}, (\underline{\mathcal{F}}_t)_t, \mu)$ is a *very good extension* of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, r)$ in the sense of [15], that is, every martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathrm{P})$ remains a martingale on $(\underline{\Omega}, \underline{\mathcal{F}}, (\underline{\mathcal{F}}_t)_t, \mu)$. This condition is also called *compatinbility* in [19].

Let $B = \{\omega \in \Omega;\, W_{t+s}(\omega) - W_t(\omega) \in C\}$. We have

$$\begin{aligned}
\mu\left(A \cap (B \times \mathrm{C}_{\mathbb{R}^d}[0,T] \times \mathrm{C}_{\mathbb{R}^d}[0,T])\right) \\
&= \int_{\Omega\times \mathrm{C}_{\mathbb{R}^d}[0,T]\times \mathrm{C}_{\mathbb{R}^d}[0,T]} \mathbb{1}_A(\omega,y,v))\,\mathbb{1}_B(\omega)\,d\mu(\omega,y,v)\\
&= \int_\Omega \mu_\omega(\,\mathbb{1}_A(\omega,.))\,\mathbb{1}_B(\omega)\,d\,\mathrm{P}(\omega)\\
&= \int_\Omega \mu_\omega(\,\mathbb{1}_A(\omega,.))\,d\,\mathrm{P}(\omega)\ \mathrm{P}(B)\\
&= \mu(A)\,\mu(B \times \mathrm{C}_{\mathbb{R}^d}[0,T] \times \mathrm{C}_{\mathbb{R}^d}[0,T]),
\end{aligned}$$

which proves (24). Thus $W_{t+s} - W_t$ is independent of $\underline{\mathcal{F}}_t$. □

Properties of the processes Y and V

Lemma 3.15 *Let H and K be $\mathbb{R}^d$-valued random variables defined on $\underline{\Omega}$. Let $t \in [0,T]$. In order that H and K have the same conditional expectation with respect to $\underline{\mathcal{F}}_t$, it is sufficient that*

$$(25)\quad \begin{aligned}\int_\Omega \int_{\times \mathrm{C}_{\mathbb{R}^d}[0,T]\times \mathrm{C}_{\mathbb{R}^d}[0,T]} \mathbb{1}_A(\omega) g(y,v) H(\omega,y,v)\,d\mu_\omega(y,v)\,d\,\mathrm{P}(\omega)\\
= \int_\Omega \int_{\times \mathrm{C}_{\mathbb{R}^d}[0,T]\times \mathrm{C}_{\mathbb{R}^d}[0,T]} \mathbb{1}_A(\omega) g(y,v) K(\omega,y,v)\,d\mu_\omega(y,v)\,d\,\mathrm{P}(\omega)\end{aligned}$$

for every $A \in \mathcal{F}_t$ and every $\mathcal{C}_t$-mesurable bounded continuous function $g :$ $\mathrm{C}_{\mathbb{R}^d}[0,T] \to \mathbb{R}$.

Proof Let $\mathcal{C}$ be the set of functions of the form $\mathbb{1}_A \otimes g$, where $A \in \mathcal{F}_t$ and $g :\, \mathrm{C}_{\mathbb{R}^d}[0,T] \to \mathbb{R}$ is a $\mathcal{C}_t$-mesurable bounded continuous function. The set $\mathcal{C}$ is stable by multiplication of two functions and generates $\underline{\mathcal{F}}_t$. Assume that (25) holds for every $\mathbb{1}_A \otimes g \in \mathcal{C}$, and let $\mathcal{E}$ be the vector space of bounded $\underline{\mathcal{F}}_t$-measurable functions h defined on $\Omega \times \mathrm{C}_{\mathbb{R}^d}[0,t]$ such that

$$(26)\quad \begin{aligned}\int_\Omega \int_{\times \mathrm{C}_{\mathbb{R}^d}[0,T]\times \mathrm{C}_{\mathbb{R}^d}[0,T]} h(\omega,y,v) H(\omega,y,v)\,d\mu_\omega(y,v)\,d\,\mathrm{P}(\omega)\\
= \int_\Omega \int_{\times \mathrm{C}_{\mathbb{R}^d}[0,T]\times \mathrm{C}_{\mathbb{R}^d}[0,T]} h(\omega,y,v) K(\omega,y,v)\,d\mu_\omega(y,v)\,d\,\mathrm{P}(\omega).\end{aligned}$$

The space $\mathcal{E}$ contains $\mathcal{C}$. Furthermore, $\mathcal{E}$ contains the constant functions and is stable under monotone limits of uniformly bounded sequences. By the monotone class theorem (see [25, Appendix A0] and [13, Theorème 21, page 20]), $\mathcal{E}$ contains all bounded $\underline{\mathcal{F}}_t$-measurable functions. $\square$

Lemma 3.16 *The process V is a martingale with respect to $(\underline{\Omega}, \underline{\mathcal{F}}, (\underline{\mathcal{F}}_t)_t, \mu)$, and $V_0 = 0$ μ-a.e.*

Proof Let $t \in [0,T]$, and let $s \in [0, T-t]$. By Lemma 3.15, in order to prove that $\mathrm{E}^{\underline{\mathcal{F}}_t}(V_{t+s}) = V_t$, we only need to show that, for any $A \in \mathcal{F}_t$, and for any $\mathcal{C}_t$-mesurable bounded continuous function $g: \mathrm{C}_{\mathbb{R}^d}[0,T] \to \mathbb{R}$.

$$\mathrm{E}\left(\mathbb{1}_A \otimes g\, V_{t+s}\right) = \mathrm{E}\left(\mathbb{1}_A \otimes g\, V_t\right). \tag{27}$$

Let us denote $\pi_r(v) = v(r)$ for every $r \in [0,T]$ and every $v \in \mathrm{C}_{\mathbb{R}^d}[0,t]$. The mapping $\pi_r : \mathrm{C}_{\mathbb{R}^d}[0,t] \to \mathbb{R}^d$ is continuous. Using the definition of V, and Lemma 3.13 (with $\varphi(\omega, y, v) = \mathbb{1}_A(\omega) g(v) \left(\pi_{t+s}(v) - \pi_t(v)\right)$), and the fact that each $V^{(n)}$ is a martingale, we have

$$\begin{aligned}
&\mathrm{E}\left(\mathbb{1}_A \otimes g\ (V_{t+s} - V_t)\right)\\
&\quad = \int_\Omega \int_{\mathrm{C}_{\mathbb{R}^d}[0,T] \times \mathrm{C}_{\mathbb{R}^d}[0,T]} \mathbb{1}_A(\omega) g(v) \left(\pi_{t+s} - \pi_t\right)(v)\, d\mu_\omega(y,v)\ d\,\mathrm{P}(\omega)\\
&\quad = \lim_{n\to\infty} \int_\Omega \mathbb{1}_A(\omega) g(V^{(n)}(\omega)) \left(V^{(n)}_{t+s}(\omega) - V^{(n)}_t(\omega)\right)\ d\,\mathrm{P}(\omega)\\
&\quad = \lim_{n\to\infty} \int_\Omega \mathbb{1}_A\, g(V^{(n)})\, \mathrm{E}^{\mathcal{F}_t} \left(V^{(n)}_{t+s} - V^{(n)}_t\right)\ d\,\mathrm{P}\\
&\quad = 0.
\end{aligned}$$

Furthermore, if we set $g(v) = \|v(0)\| \wedge 1$, we get

$$\mathrm{E}\left(g(V)\right) = \lim_n \mathrm{E}\left(g(V^{(n)})\right) = 0$$

thus $V_0 = 0$ μ-a.e. $\square$

Lemma 3.17 *Let $\widehat{V}_t = \int_0^t Z_s\, dW_s$. The martingale $L := V - \widehat{V}$ is orthogonal to any $(\mathcal{F}_t)$-martingale.*

Proof Let M be an $\mathbb{R}^d$-valued $(\mathcal{F}_t)$-martingale: $M_t = \int_0^t H_s\, dW_s$ for some $(\mathcal{F}_t)$-adapted $\mathbb{L}$-valued process H with $\mathrm{E}\int_0^T \|H_s\|^2\ ds < \infty$. Let us denote the coordinates processes as in the following examples: $V = (V^{[i]})_{1\le i\le d}$, $Z^{(n)}_t = (Z^{(n),[i,k]}_t)_{1\le i\le d, 1\le k\le m}$, $Z_t = (Z^{[i,k]}_t)_{1\le i\le d, 1\le k\le m}$, $W_t = (W^{[k]})_{1\le k\le m}$.

Let $i, j \in \{1, \ldots, d\}$. Let us denote by $\langle P, Q \rangle$ the quadratic cross variation of two continuous processes P and Q. For each n, let

$$\begin{aligned} N_t^{(n),[i,j]} &= M_t^{[i]} \left(V_t^{(n),[j]} - \widehat{V}_t^{[j]} \right) - \left\langle M^{[i]}, V^{(n),[j]} - \widehat{V}^{[j]} \right\rangle_t \\ &= \sum_{k=1}^{m} \int_0^t H_r^{[i,k]} \, dW_r^{[k]} \int_0^t \left(Z_r^{(n),[j,k]} - Z_r^{[j,k]} \right) dW_r^{[k]} \\ &\qquad\qquad - \int_0^t H_r^{[i,k]} \left(Z_r^{(n),[j,k]} - Z_r^{[j,k]} \right) dr. \end{aligned}$$

Let $A \in \mathcal{F}_t$, and let $g : \mathrm{C}_{\mathbb{R}^d}[0,T] \to \mathbb{R}$ be a bounded $\mathcal{C}_t$-measurable continuous function. Let $t \in [0,T]$, and let $s \in [0, T-t]$. Using Lemma 3.13, the fact that $N^{(n),[i,j]}$ is a martingale, and the weak convergence of $Z^{(n)}$ to Z in $\mathrm{L}^2_{\mathbb{L}}(\Omega \times [0,T])$, we get

$$\begin{aligned} &\int M_{t+s}^{[i]}(\omega) \left(V_{t+s}^{[j]}(\omega, v) - \widehat{V}_{t+s}^{[j]}(\omega) \right) \mathbb{1}_A(\omega) g(v) \, d\mu(\omega, v) \\ &= \int M_{t+s}^{[i]}(\omega) \left(v^{[j]}(t+s) - \widehat{V}_{t+s}^{[j]}(\omega) \right) \mathbb{1}_A(\omega) g(v) \, d\mu(\omega, v) \\ &= \lim_n \mathrm{E} \left(M_{t+s}^{[i]} \left(V_{t+s}^{(n),[j]} - \widehat{V}_{t+s}^{[j]} \right) \mathbb{1}_A \, g(V^{(n)}) \right) \\ &= \lim_n \mathrm{E} \left(\left(N_{t+s}^{(n),[i,j]} + \sum_{1 \le k \le m} \int_0^{t+s} H_r^{[i,k]} \left(Z_r^{(n),[j,k]} - Z_r^{[j,k]} \right) dr \right) \mathbb{1}_A \, g(V^{(n)}) \right) \\ &= \lim_n \mathrm{E} \left(N_t^{(n),[i,j]} \, \mathbb{1}_A \, g(V^{(n)}) \right) \\ &= \int M_t^{[i]}(\omega) \left(V_t^{[j]}(\omega, v) - \widehat{V}_t^{[j]}(\omega) \right) \mathbb{1}_A(\omega) g(v) \, d\mu(\omega, v). \end{aligned}$$

By Lemma 3.15, this result remains true if g is only assumed to be a bounded $\mathcal{C}_t$-measurable function. Thus $M^{[i]} \left(V^{[j]} - \widehat{V}^{[j]} \right)$ is a martingale.

□

Proof of the main result In order to check that (Y, Z) is a solution to (2), we prove in the next lemma that we can replace $\widetilde{Z}^{(n)}$ by $Z^{(n)}$ in the limit of $\int_t^T f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)}) \, ds$. This is where we use the continuity hypothesis (H_2-ii).

Lemma 3.18 **For each** $t \in [0,T]$, *the sequence*

$$\int_t^T f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds - \int_t^T f(s, X_s, Y_s^{(n)}, Z_s^{(n)})\, ds$$

converges to 0 in probability.

Proof By Lemma 3.4, there exists $\mathfrak{M} > 0$ such that $\left\| Z_t^{(n)} \right\|_{\mathrm{L}^2_{\mathbb{R}^d}} \leq \mathfrak{M}$ for all $t \in [0,T]$ and $n \geq 1$. Thus, by Lemma 3.8, we have

$$\mathrm{E} \left\| Z_{t+1/n}^{(n)} - \widetilde{Z}_t^{(n)} \right\|^2 \leq \alpha_2 \frac{1}{\sqrt{n}} \mathfrak{M},$$

which implies by the continuity assumption (H_2-ii) that

$$f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)}) - f(s, X_s, Y_s^{(n)}, Z_{s+1/n}^{(n)})$$

converges in probability to 0 for all $s \in [0,T]$. But, from Proposition 3.5 and the growth condition (H_1), the sequence

$$\left(f(., X, Y^{(n)}, \widetilde{Z}^{(n)}) - f(., X, Y^{(n)}, Z_{.+1/n}^{(n)}) \right)_{n \geq 1}$$

of random variables defined on $\Omega \times [0,T]$ is uniformly integrable. So we can replace $\widetilde{Z}_s$ by $Z_{s+1/n}$:

$$\lim_{n\to\infty} \mathrm{E} \int_0^T \left\| f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)}) - f(s, X_s, Y_s^{(n)}, Z_{s+1/n}^{(n)}) \right\| ds = 0. \tag{28}$$

The conclusion follows from hypothesis (H_2-ii). □

Now we use the linearity of f with respect to Z:

Lemma 3.19 *There exists a subsequence of* $(\int_.^T f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds)$ *which converges in law to* $\int_.^T f(s, X_s, Y_s, Z_s^{(n)})\, ds$.

Proof By Lemma 3.18, we only need to prove that $(\int_.^T f(s, X_s, Y_s^{(n)}, Z_s^{(n)})\, ds)$ converges in law to $\int_.^T f(s, X_s, Y_s, Z_s)\, ds$.

By hypothesis (H_2-i), f has the form

$$f(s,x,y,z) = \alpha(s,x,y)z + \beta(s,x,y),$$

where α and β are bounded and continuous in (x, y), and α takes its values in the space of linear mappings from $\mathbb{L}$ to $\mathbb{R}^d$.

Extracting if necessary a further sequence, we may assume that $(X, Y^{(n)})$ jointly converges in law to (X, Y). Then $(\int_.^T \beta(s, X_s, Y_s^{(n)})\, ds)$ converges in law to $\int_.^T \beta(s, X_s, Y_s)\, ds$. If

$$(29) \quad (\int_.^T \alpha(s, X_s, Y_s^{(n)}) Z_s^{(n)}\, ds) \text{ converges in law to } \int_.^T \alpha(s, X_s, Y_s) Z_s\, ds$$

then, at least for an extracted sequence,

$$\left(\int_.^T \alpha(s, X_s, Y_s^{(n)}) Z_s^{(n)}\, ds, \int_.^T \beta(s, X_s, Y_s^{(n)})\, ds\right)$$

converges in law to

$$\left(\int_.^T \alpha(s, X_s, Y_s) Z_s\, ds, \int_.^T \beta(s, X_s, Y_s)\, ds\right)$$

and the conclusion of Lemma 3.19 follows by continuity of the addition. So, we only need to prove (29).

Let us show that the mapping

$$\Phi : \begin{cases} \mathrm{L}^0_{\mathrm{C}_{\mathbb{M}}[0,T]}(\Omega) \times \mathrm{L}^2_{\mathbb{R}^d}(\Omega \times [0,T]) \times \mathrm{L}^2_{\mathbb{L}}(\Omega \times [0,T]) & \to \quad \mathbb{R}^d \\ (F, K, H) & \mapsto \quad \mathrm{E}\left(\int_0^T \alpha(s, F_s, K_s) H_s\, ds\right) \end{cases}$$

is continuous in (F, K), uniformly with respect to H in bounded subsets of $\mathrm{L}^2_{\mathbb{L}}(\Omega \times [0,T])$.

Let $(F^{(1)}, K^{(1)}), (F^{(2)}, K^{(2)}) \in \mathrm{L}^0_{\mathrm{C}_{\mathbb{M}}[0,T]}(\Omega) \times \mathrm{L}^2_{\mathbb{R}^d}(\Omega \times [0,T])$ and let H in the unit ball of $\mathrm{L}^2_{\mathbb{L}}(\Omega \times [0,T])$. We have

$$\begin{aligned} &\left\| \Phi\left(F^{(1)}, K^{(1)}, H\right) - \Phi\left(F^{(2)}, K^{(2)}, H\right) \right\| \\ &\qquad \le \left(\mathrm{E} \int_0^T \left\| \alpha(s, F_s^{(1)}, K_s^{(1)}) - \alpha(s, F_s^{(2)}, K_s^{(2)}) \right\|^2 ds\right)^{1/2} \|H\|_{\mathrm{L}^2}, \end{aligned}$$

and the continuity property of Φ follows from Hypothesis (H_2-i).

Now, by (22), we can extract a further sequence such that $(X, Y^{(n)}, Z^{(n)})$ converges in law to (X, Y, Z) in $\mathrm{C}_{\mathbb{M}}[0,T] \times \mathrm{C}_{\mathbb{R}^d}[0,T] \times \mathbb{H}_\sigma$. By Jajubowski's version of Skorokhod's representation theorem for random variables in nonnecessarily metrizable spaces [16], extracting if necessary a further sequence, we can find another probability space $(\Omega^*, \mathcal{F}^*, \mathrm{P}^*)$ and a sequence $(X^{(n*)}, Y^{(n*)}, Z^{(n*)})$ of $\mathrm{C}_{\mathbb{M}}[0,T] \times \mathrm{C}_{\mathbb{R}^d}[0,T] \times \mathbb{H}_\sigma$-valued random variables defined on Ω^* which converges a.e. to a limit (X^*, Y^*, Z^*), and such that $(X^{(n*)}, Y^{(n*)}, Z^{(n*)})$ has the same law as $(X, Y^{(n)}, Z^{(n)})$ for each n. Then, by Proposition 3.5, $(Z^{(n*)})$ is bounded in $\mathrm{L}^2_{\mathbb{L}}(\Omega^* \times [0,T])$. By the continuity property of Φ, for each $m \geq 1$, and with obvious notations, the sequence

$$\left(\mathrm{E}^*\big(\int_0^T \alpha(s, X_s^{(n*)}, Y_s^{(n*)}), Z_s^{(m*)}\, ds\big)\right)_n = \left(\mathrm{E}\big(\int_0^T \alpha(s, X_s, Y_s^{(n)}), Z_s^{(m)}\, ds\big)\right)_n$$

converges to

$$\mathrm{E}^*\big(\int_0^T \alpha(s, X_s^*, Y_s^*), Z_s^{(m*)}\, ds\big) = \mathrm{E}\big(\int_0^T \alpha(s, X_s, Y_s), Z_s^{(m)}\, ds\big)$$

uniformly with respect to m, when $n \to \infty$. But, for each n, as α is bounded, we have

$$\lim_m \mathrm{E}\big(\int_0^T \alpha(s, X_s, Y_s^{(n)}), Z_s^{(m)}\, ds\big) = \mathrm{E}\left(\int_0^T \alpha(s, X_s, Y_s^{(n)}), Z_s\, ds\right).$$

We deduce that

$$\lim_{n,m} \mathrm{E}\big(\int_0^T \alpha(s, X_s, Y_s^{(n)}), Z_s^{(m)}\, ds\big) = \mathrm{E}\big(\int_0^T \alpha(s, X_s, Y_s), Z_s\, ds\big),$$

which implies (29). □

Remark 3.20 An alternative proof of (29) is possible using Young measures tools on the locally convex space $\mathrm{C}_{\mathbb{R}^d}[0,T] \times \mathbb{H}_\sigma$. Indeed, $(Y^{(n)})$ converges in $\mathcal{Y}(\Omega, \mathcal{F}, \mathrm{P}; \mathrm{C}_{\mathbb{R}^d}[0,T])$ to a measure ν, whereas, by (22), $(Z^{(n)})$ converges in probability to Z in $\mathbb{H}_\sigma$. By the fiber product lemma (see e.g. [7] or [11, Theorem 3.3.1 and Corollary 3.3.5]), $(Y^{(n)}, Z^{(n)})$ converges in $\mathcal{Y}(\Omega, \mathcal{F}, \mathrm{P}; \mathrm{C}_{\mathbb{R}^d}[0,T] \times \mathbb{H}_\sigma)$ to the measure λ defined by $\lambda_\omega = \nu_\omega \otimes \delta_{Z(\omega)}$ for each $\omega \in \Omega$. Furthermore, by Proposition 3.5, the sequence $(Y^{(n)}, Z^{(n)})$ is uniformly integrable. Thus we obtain (29) by application of the same reasoning as in Lemma 3.13 to the integrand

$$\phi : \begin{cases} \Omega \times \mathrm{C}_{\mathbb{M}}[0,T] \times \mathrm{C}_{\mathbb{R}^d}[0,T] \times \mathbb{H}_\sigma & \to \mathbb{R}^d \\ (\omega, x, y, z) & \mapsto \alpha(., x(.), y(.))z. \end{cases}$$

Proof of Theorem 3.1 By Lemma 3.14, W is a Brownian motion on $(\underline{\Omega}, \underline{\mathcal{F}}, (\underline{\mathcal{F}}_t)_t, \mu)$. Let $L_t = M_t - V_t$, $0 \le t \le T$. By Lemma 3.16, L is a (continuous) martingale, with $L_0 = 0$, and, by Lemma 3.17, L is orthogonal to every $(\mathcal{F}_t)$-martingale. Thus there only remains to prove that (Y, Z, L) satisfies (1).

Thanks to Lemmas 3.11 and 3.12, we know that the sequence

$$\left(X, Y^{(n)}, \int_\cdot^T f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds, \int_\cdot^T Z_s^{(n)}\, dW_s\right)_{n\ge1} \tag{30}$$

is tight in $\mathrm{C}_{\mathbb{M}}[0,T]\times\mathrm{C}_{\mathbb{R}^d}[0,T]\times\mathrm{C}_{\mathbb{R}^d}[0,T]\times\mathrm{C}_{\mathbb{R}^d}[0,T]$. Furthermore, $\left(\int_\cdot^T Z_s^{(n)}\, dW_s\right)_{n\ge1}$ converges in law to $V_T - V_\cdot$, and, by Lemma 3.19, $\left(\int_\cdot^T f(s, X_s, Y_s^{(n)}, \widetilde{Z}_s^{(n)})\, ds\right)_{n\ge1}$ converges in law to $\int_\cdot^T f(s, X_s, Y_s, Z_s)\, ds$. Extracting if necessary a further subsequence, we can thus assume that the sequence (30) jointly converges in law on $\mathrm{C}_{\mathbb{M}}[0,T]\times\mathrm{C}_{\mathbb{R}^d}[0,T]\times\mathrm{C}_{\mathbb{R}^d}[0,T]\times\mathrm{C}_{\mathbb{R}^d}[0,T]$ to

$$\left(X, Y, \int_\cdot^T f(s, X_s, Y_s^{(n)}, Z_s)\, ds, V_T - V_\cdot\right).$$

Then the process

$$U_\cdot^{(n)} = Y_\cdot^{(n)} - \xi - \int_\cdot^T f(s, X_s, Y_s^{(n)}, Z_s^{(n)})\, ds + \int_\cdot^T Z_s^{(n)}\, dW_s$$

converges in law to

$$\begin{aligned} U_\cdot :=& Y_\cdot - \xi - \int_\cdot^T f(s, X_s, Y_s, Z_s)\, ds + V_T - V_\cdot \\ =& Y_\cdot - \xi - \int_\cdot^T f(s, X_s, Y_s, Z_s)\, ds + \int_\cdot^T Z_s\, dW_s + L_T - L_\cdot \end{aligned}$$

But, by Lemma 3.9, $(\sup_{0\le t\le T} U_t^{(n)})$ converges to 0 in probability, thus $U = 0$ a.e., which proves Theorem 3.1. □

References

[1] David J. Aldous. Stopping times and tightness. *Ann. Probab.*, 6:335–340, 1978.

[2] Fabio Antonelli and Jin Ma. Weak solutions of forward-backward SDE's. *Stochastic Anal. Appl.*, 21(3):493–514, 2003.

[3] K. Bahlali, B. Mezerdi, M. N'zi, and Y. Ouknine. Weak solutions and a Yamada-Watanabe theorem for FBSDEs. *Random Oper. Stoch. Equ.*, 15(3):271–285, 2007.

[4] E. J. Balder. On Prohorov's theorem for transition probabilities. *Sém. Anal. Convexe*, 19:9.1–9.11, 1989.

[5] E. J. Balder. New sequential compactness results for spaces of scalarly integrable functions. *J. Math. Anal. Appl.*, 151:1–16, 1990.

[6] E. J. Balder. Lectures on Young measure theory and its applications in economics. *Rend. Istit. Mat. Univ. Trieste*, 31, suppl.:1–69, 2000. Workshop di Teoria della Misura et Analisi Reale Grado, 1997 (Italia).

[7] Erik J. Balder. Generalized equilibrium results for games with incomplete information. *Math. Oper. Res.*, 13(2):265–276, 1988.

[8] R. Buckdahn and H.-J. Engelbert. A backward stochastic differential equation without strong solution. *Teor. Veroyatn. Primen.*, 50(2):390–396, 2005.

[9] R. Buckdahn and H.-J. Engelbert. On the continuity of weak solutions of backward stochastic differential equations. *Teor. Veroyatn. Primen.*, 52(1):190–199, 2007.

[10] R. Buckdahn, H.-J. Engelbert, and A. Răşcanu. On weak solutions of backward stochastic differential equations. *Teor. Veroyatn. Primen.*, 49(1):70–108, 2004.

[11] Charles Castaing, Paul Raynaud de Fitte, and Michel Valadier. *Young measures on Topological Spaces. With Applications in Control Theory and Probability Theory*, volume 571 of *Mathematics and its Applications*. Kluwer Academic Publishers, Dordrecht, 2004.

[12] F. Delarue and G. Guatteri. Weak existence and uniqueness for forward-backward SDEs. *Stochastic Process. Appl.*, 116(12):1712–1742, 2006.

[13] Claude Dellacherie and Paul André Meyer. *Probabilités et Potentiel. Chapitres I à IV.* Hermann, Paris, 1975.

[14] Jean Jacod. Théorèmes limite pour les processus. In P. L. Hennequin, editor, *École d'Été de Probabilités de Saint–Flour XIII-1983*, number 1117 in Lecture Notes in Math., pages 298–409, Berlin, 1985. Springer Verlag.

[15] Jean Jacod and Jean Mémin. Weak and strong solutions of stochastic differential equations: existence and stability. In *Stochastic integrals (Proc. Sympos., Univ. Durham, Durham, 1980)*, volume 851 of *Lecture Notes in Math.*, pages 169–212. Springer, Berlin, 1981.

[16] A. Jakubowski. The almost sure Skorokhod representation for subsequences in nonmetric spaces. *Theory Proba. Appl.*, 42(1):209–216, 1997.

[17] Adam Jakubowski, Mikhail I. Kamenskiĭ, and Paul Raynaud de Fitte. Existence of weak solutions to stochastic differential inclusions. *Stoch. Anal. Appl.*, 23(4):723–749, 2005.

[18] N. V. Krylov. *Introduction to the theory of diffusion processes*, volume 142 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1995. Translated from the Russian manuscript by Valim Khidekel and Gennady Pasechnik.

[19] Thomas G. Kurtz. The Yamada-Watanabe-Engelbert theorem for general stochastic equations and inequalities. *Electron. J. Probab.*, 12:951–965 (electronic), 2007.

[20] J. P. Lepeltier and J. San Martin. Backward stochastic differential equations with continuous coefficient. *Statist. Probab. Lett.*, 32(4):425–430, 1997.

[21] Jin Ma, Jianfeng Zhang, and Ziyu Zheng. Weak solutions for forward-backward SDEs—a martingale problem approach. *Ann. Probab.*, 36(6):2092–2125, 2008.

[22] É. Pardoux and S. G. Peng. Adapted solution of a backward stochastic differential equation. *Systems Control Lett.*, 14(1):55–61, 1990.

[23] J. Pellaumail. Weak solutions for semimartingales. *Canad. J. Math.*, 33(5):1165–1181, 1981.

[24] Alfred Rényi. On stable sequences of events. *Sankhyā Ser. A*, 25:293–302, 1963.

[25] Michael Sharpe. *General Theory of Markov Processes.* Number 133 in Pure and Applied Math. Academic Press, Boston, MA, 1988.

[26] Michel Valadier. Désintégration d'une mesure sur un produit. *C. R. Acad. Sci. Paris Sér. I*, 276:A33–A35, 1973.

[27] Michel Valadier. A course on Young measures. *Rendiconti dell'istituto di matematica dell'Università di Trieste*, 26, suppl.:349–394, 1994. Workshop di Teoria della Misura et Analisi Reale Grado, 1993 (Italia).