An *hp* Finite Element Method for singularly perturbed transmission problems in smooth domains

Serge Nicaise
Université de Valenciennes et du Hainaut Cambrésis
LAMAV, FR CNRS 2956,
Institut des Sciences et Techniques de Valenciennes
F-59313 - Valenciennes Cedex 9 France
Serge.Nicaise@univ-valenciennes.fr

Christos Xenophontos
Department of Mathematics and Statistics
University of Cyprus
P.O. BOX 20537
Nicosia 1678 Cyprus
xenophontos@ucy.ac.cy

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Abstract

We consider a two-dimensional singularly perturbed transmission problem with two different diffusion coefficients, in a domain with smooth (analytic) boundary. The solution will contain boundary layers only in the part of the domain where the diffusion coefficient is high and interface layers along the interface. Utilizing existing and newly derived regularity results for the exact solution, we design a robust *hp* finite element method for its approximation. Under the assumption of analytic input data, we show that the method converges at an exponential rate, provided the mesh and polynomial degree distribution are chosen appropriately. Numerical results illustrating our theoretical findings are also included.

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1 Introduction

The approximation of singularly perturbed problems has retained the attention of many authors in recent years. Let us mention [5, 9, 10, 11, 13, 14] and the references quoted there. However, in all references quoted no analysis is carried out for differential operators with piecewise constant or piecewise smooth coefficients. On the other hand, in many real life applications, the differential operators have such piecewise coefficients that may have a very large discrepancy. In that case, the solution of the problem will contain boundary layers near the exterior boundary (as usual) but will also contain interface layers along the interface where the coefficients have a large jump. We refer to [4] for the description of this phenomenon in one and two dimensions and to [12] for several numerical methods for the robust approximation of such problems in one-dimension.

The goal of the present paper is to extend certain results from [12] to two-dimensions. In particular, we consider a singularly perturbed transmission problem in a domain with analytic boundary. Under the assumption of the data also being analytic, we provide an asymptotic expansion for the solution (in the style of [6]) that provides the necessary information for the design of a robust finite element method that converges at an exponential rate as the degree p of the approximating polynomials is increased. The expansion of the solution includes an outer (smooth) part, an inner (boundary layer) part, an interface layer and a (smooth) remainder. The regularity of each compoment is studied and known results from [8] allow us to treat the outer and inner parts, as well as the remainder (defined on one part of the domain). The results obtained for the regularity of the interface layer (and the remainder defined on the other part of the domain) are new and in line with those reported in [12] for the one-dimensional analog of our model problem. Our work closely follows what was done in [7] but also includes the additional analysis for the interface layer.

The paper is organized as follows: In Section 2 we present the singularly perturbed problem and describe the typical phenomena. Section 3 is devoted to the expansion of the solution of our model problem into the parts mentioned above (i.e. outer, inner, interface and remainder). The regularity of each component is also described in that section. Section 4 gives the main approximation result and in Section 5 we show the results of numerical computations illustrating our theoretical findings. We end with some conclusions in Section 6.

Throughout the paper the spaces $H^s(\Omega)$, with $s \geq 0$, are the standard Sobolev spaces on the domain $\Omega \subset \mathbb{R}^2$, with norm $\| \cdot \|_{s,\Omega}$ and semi-norm $| \cdot |_{s,\Omega}$. The space $H_0^1(\Omega)$ is defined, as usual, by $H_0^1(\Omega) := \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$. $L^p(\Omega)$, $p > 1$, are the usual Lebesgue spaces with norm $\| \cdot \|_{0,p,\Omega}$ (we drop the index p for $p = 2$). Finally, the notation $A \lesssim B$ means the existence of a positive constant C, which is independent of the quantities A and B under consideration and of the parameter ε, such that $A \leq CB$.

2 The model problem

Let Ω_+ and Ω_- be smooth domains in $\mathbb{R}^2$, with respective boundaries $\partial\Omega_+$ and $\partial\Omega_-$, such that $\partial\Omega_+ \cap \partial\Omega_- = \Sigma$; an example is shown in Figure 1 below. We assume that $\partial\Omega$ is an analytic curve, i.e. $\partial\Omega_\pm$ and Σ are analytic curves. Moreover, we assume that $\partial\Omega_+\backslash\Sigma$, as well as Σ are connected. We will write $\Omega = \Omega_+ \cup \Omega_-$, and for any function u defined on Ω we will denote by u_+ (resp. u_-) the restriction of u to Ω_+ (resp. Ω_-) and we will write $u \equiv (u_+, u_-)$.

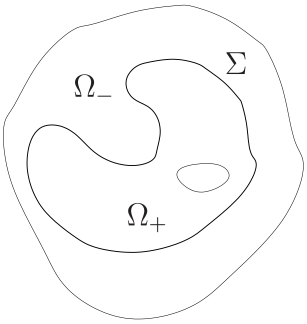

Figure 1: Example of the domains Ω_+ and Ω_-.

We consider the following singularly perturbed transmission problem: Find $u^\varepsilon = \left(u_+^\varepsilon, u_-^\varepsilon\right)$ such that

$$
\begin{aligned}
-\varepsilon^2\Delta u_+^\varepsilon + u_+^\varepsilon &= f_+ \text{ in } \Omega_+, &(1)\\
-\Delta u_-^\varepsilon + u_-^\varepsilon &= f_- \text{ in } \Omega_-, &(2)\\
u_+^\varepsilon &= 0 \text{ on } \partial\Omega_+\backslash\Sigma, &(3)\\
u_-^\varepsilon &= 0 \text{ on } \partial\Omega_-\backslash\Sigma, &(4)\\
u_+^\varepsilon - u_-^\varepsilon &= 0 \text{ on } \Sigma, &(5)\\
\varepsilon^2\frac{\partial u_+^\varepsilon}{\partial\nu} - \frac{\partial u_-^\varepsilon}{\partial\nu} &= h \text{ on } \Sigma, &(6)
\end{aligned}
$$

where Δ denotes the Laplacian operator, $\varepsilon \in (0,1]$ is a given parameter, $f_\pm, h$ are given smooth functions and ν denotes the outward normal vector along Σ oriented outside Ω_+. The formal limit problem of (1)–(6), as $\varepsilon \to 0$, is

$$
\begin{aligned}
u_+^0 &= f_+ \text{ in } \Omega_+,\\
-\Delta u_-^0 + u_-^0 &= f_- \text{ in } \Omega_-,\\
u_+^0 &= 0 \text{ on } \partial\Omega_+\backslash\Sigma,\\
u_-^0 &= 0 \text{ on } \partial\Omega_-\backslash\Sigma,\\
u_+^0 - u_-^0 &= 0 \text{ on } \Sigma,\\
-\frac{\partial u_-^0}{\partial\nu} &= h \text{ on } \Sigma.
\end{aligned}
$$

Since, in general, f_+ does not satisfy the boundary conditions $f_+ = u_+^0$ on $\partial\Omega_+\backslash\Sigma$ and $f_+ = u_-^0$ on Σ, we expect that the solution u^ε will contain boundary layers along $\partial\Omega_+\backslash\Sigma$ and an interface layer along Σ.

We assume that the data of our problem is analytic and satisfies

$$\|\nabla^p f_\pm\|_{\infty,\Omega_\pm} \le C_{f_\pm}\gamma_{f_\pm}^p p! \ \forall\ p = 0, 1, 2, ..., \tag{7}$$

$$\|\nabla_\Sigma^p h\|_{\infty,\Sigma} \le C_h\gamma_h^p p! \ \forall\ p = 0, 1, 2, ..., \tag{8}$$

for some positive constants $C_{f_\pm}, \gamma_{f_\pm}, C_h, \gamma_h$, where ∇_Σ denotes the tangential derivative along Σ. The following theorem gives bounds on the derivatives of the solution to (1)–(6) that are explicit in terms of the order of differentiation as well as the singular perturbation parameter ε.

Theorem 1 *Let $u^\varepsilon = \left(u_+^\varepsilon, u_-^\varepsilon\right)$ be the solution to (1)–(6) with the data satysfying (7), (8). Then there are constants $C, K > 0$ depending only of the data such that*

$$\varepsilon\left\|D^\alpha u_+^\varepsilon\right\|_{0,\Omega_+} + \left\|D^\alpha u_-^\varepsilon\right\|_{0,\Omega_-} \le C\varepsilon K^{|\alpha|}\max\left\{|\alpha|, \varepsilon^{-1}\right\}^{|\alpha|} \ \forall\ \alpha = 1, 2, ... \tag{9}$$

Proof. This follows from the local estimates

$$\begin{aligned} \varepsilon|u_+^\varepsilon|_{2,B_{x_0}\cap\Omega_+} + |u_-^\varepsilon|_{2,B_{x_0}\cap\Omega_-} &\le C(\varepsilon^{-1}\|f_+\|_{0,B'_{x_0}\cap\Omega_+} \\ +\|f_-\|_{0,B'_{x_0}\cap\Omega_-} + \|h\|_{B'_{x_0}\cap\Sigma} &+ \varepsilon\|u_+^\varepsilon\|_{1,B'_{x_0}\cap\Omega_+} + \|u_-^\varepsilon\|_{1,B'_{x_0}\cap\Omega_-}), \end{aligned}$$

for all sufficiently small balls $\bar{B}_{x_0} \subset B'_{x_0}$ centred at $x_0 \in \Sigma$ (proved by a local change of variables and some reflexions to reduce the transmission problem into a Dirichlet problem and a Neumann one in half-balls) and the use of Morrey-Nirenberg techniques (see Theorem 2.1 in [7] or Theorems 5.2.2 and 5.3.8 in [3]). ■

It should be noted that (9) gives sufficient information for the approximation to u^ε in the so-called asymptotic case, i.e. when the degree p of the approximating polynomials satisfies $p > O(\varepsilon^{-1})$. For the pre-asymptotic case, i.e. when $p \le O(\varepsilon^{-1})$, we will need the regularity results provided in the next section.

3 Expansion of the solution

The solution of (1)–(6) may be decomposed as

$$u^\varepsilon = w^\varepsilon + \chi_{BL}u_{BL}^\varepsilon + \chi_{IL}u_{IL}^\varepsilon + r^\varepsilon, \tag{10}$$

where w^ε denotes the outer (smooth) part, u^ε_{BL} denotes the boundary layer along $\partial\Omega_+\backslash\Sigma$, u^ε_{IL} denotes the interface layer along Σ and r^ε denotes the remainder. The functions χ_{BL},χ_{IL} denote smooth cut-off functions (see equations (17), (18) ahead) in order to account for the fact that the aforementioned components do not have support in the entire domain Ω.

In order to define the inner (boundary layer) expansion we introduce *boundary fitted coordinates* as follows: Let $(X(\theta), Y(\theta))\,, \theta \in [0, L]$ be an analytic $L-$periodic parametrization of $\partial\Omega_+\backslash\Sigma$ (by arc length), such that the normal vector $(-Y'(\theta), X'(\theta))$ always points into the domain Ω_+. Let $\kappa_+(\theta)$ denote the curvature of $\partial\Omega_+\backslash\Sigma$ and denote by $\mathbb{T}_L$ the one-dimensional torus of length L. By the analyticity of $\partial\Omega$ we have that the functions X, Y and κ are analytic. We also let $\rho_0 > 0$ be a fixed constant satisfying

$$(11) \qquad 0 < \rho_0 < \frac{1}{\|\kappa_+\|_{L^\infty([0,L))}}.$$

Then the mapping $\psi : [0, \rho_0] \times \mathbb{T}_L \to \overline{\Omega}_+$ given by

$$(12) \qquad \psi : (\rho, \theta) \to (X(\theta) - \rho Y'(\theta), Y(\theta) + \rho X'(\theta))$$

is real analytic on $[0, \rho_0] \times \mathbb{T}_L$. The function ψ maps the rectangle $(0, \rho_0) \times [0, L)$ onto a half-tubular neighborhood Ω^0_+ of $\partial\Omega_+\backslash\Sigma$, which may be described as

$$(13) \qquad \Omega^0_+ = \{z - \rho\mathbf{n}_z : z \in \partial\Omega_+\backslash\Sigma, 0 < \rho < \rho_0\}\,,$$

with $z = z(\theta) = (X(\theta), Y(\theta))$ and $\mathbf{n}_z$ the outward unit normal at $z \in \partial\Omega_+\backslash\Sigma$.

The interface layer will also be defined in a neighborhood of the interface Σ. Quite analogously, let $(X_\Sigma(\theta), Y_\Sigma(\theta))\,, \theta \in [0, L_\Sigma]$ be an analytic $L_\Sigma-$periodic parametrization of Σ (as above), let $\kappa_\Sigma(\theta)$ denote the curvature of Σ and denote by $\mathbb{T}_{L_\Sigma}$ the one-dimensional torus of length L_Σ. With $\rho_\Sigma > 0$ a fixed constant satisfying

$$(14) \qquad 0 < \rho_\Sigma < \frac{1}{\|\kappa_\Sigma\|_{L^\infty([0,L_\Sigma))}},$$

we define, analogously to (13),

$$(15) \qquad \Omega^0_\Sigma = \{z - \rho\mathbf{n}_\Sigma : z \in \Sigma, 0 < \rho < \rho_\Sigma\}\,,$$

with $z = z(\theta) = (X_\Sigma(\theta), Y_\Sigma(\theta))$ and $\mathbf{n}_\Sigma$ the outward unit normal at $z \in \Sigma$.

The smooth cut-off functions χ_{BL}, χ_{IL} appearing in (10) are defined as follows: Let ρ_1, ρ_2 be given satisfying

$$(16) \qquad 0 < \rho_1 < \rho_0\;,\; 0 < \rho_2 < \rho_\Sigma$$

and let χ_{BL}, χ_{IL} be defined on $\overline{\Omega}_+$ via

$$\chi_{BL}(x) = \begin{cases} 1 & \text{for } 0 \leq \mathrm{dist}(x, \partial\Omega_+ \backslash \Sigma) \leq \rho_1 \\ 0 & \text{for } \mathrm{dist}(x, \partial\Omega_+ \backslash \Sigma) \geq (\rho_1 + \rho_0)/2 \end{cases} , \tag{17}$$

$$\chi_{IL}(x) = \begin{cases} 1 & \text{for } 0 \leq \mathrm{dist}(x, \Sigma) \leq \rho_2 \\ 0 & \text{for } \mathrm{dist}(x, \Sigma) \geq (\rho_2 + \rho_\Sigma)/2 \end{cases} . \tag{18}$$

The above will be utilized in sections 3.2 and 3.3 ahead.

3.1 Construction and regularity of the outer part

We begin by constructing the outer part w^ε in (10). To this end, we expand the solution $u^\varepsilon = \left(u_+^\varepsilon, u_-^\varepsilon\right)$ as a formal series in powers of ε,

$$u_\pm^\varepsilon = u_0^\pm + \varepsilon u_1^\pm + \varepsilon^2 u_2^\pm + ... \tag{19}$$

and insert it in the differential equations (1)–(6), equating like powers of ε. This allows us to get expressions for the functions $u_j^\pm, j = 0, 1, 2, ...$ In particular, for u_j^+ we obtain

$$u_0^+ = f_+, \ u_{2j}^+ = \Delta^{(2j+2)} f_+, \ u_{2j-1}^+ = 0, \ j = 1, 2, ... \tag{20}$$

where $\Delta^{(i)}$ denotes the iterated Laplacian. For u_0^- we obtain

$$-\Delta u_0^- + u_0^- = f_- \text{ in } \Omega_-, \tag{21}$$

$$u_0^- = 0 \text{ on } \partial\Omega_- \backslash \Sigma, \tag{22}$$

$$\frac{\partial u_0^-}{\partial \nu} = -h \text{ on } \Sigma. \tag{23}$$

For $j \geq 1$ we find $u_{2j-1}^- = 0$ and

$$-\Delta u_{2j}^- + u_{2j}^- = 0 \text{ in } \Omega_-, \tag{24}$$

$$u_{2j}^- = 0 \text{ on } \partial\Omega_- \backslash \Sigma, \tag{25}$$

$$\frac{\partial u_{2j}^-}{\partial \nu} = \frac{\partial u_{2j-2}^+}{\partial \nu} \text{ on } \Sigma. \tag{26}$$

Note that u_{2j}^- is not explicitly known but is solution of a Dirichlet-Neumann problem in Ω_-. Due to the analyticity assumption, u_{2j}^- is analytic as well (see equation (34) ahead).

Using the above, we can define the *outer expansion* as

$$w^\pm \equiv w_M^\pm = \sum_{j=0}^{M} \varepsilon^{2j} u_{2j}^\pm, \tag{27}$$

where M is the order of the expansion (i.e. the number of terms that we will include) and will ultimately be taken to be proportional to $1/\varepsilon$ (cf. [5], [7]). It is not difficult to see that

$$\left(-\varepsilon^2 \Delta w_M^+ + w_M^+\right) - f_+ = \varepsilon^{2M+2} \Delta^{(M+1)} f_+ \tag{28}$$

and

$$\left(-\Delta w_M^- + w_M^-\right) - f_- = 0. \tag{29}$$

Moreover, we have the following theorem.

Theorem 2 *Let* $w_M^\pm$ *be defined by (27). Then there exist positive constants* K_1 *and* C *depending only on the data of the problem, such that if* εM *is sufficiently small then*

$$\left\| D^\alpha w_M^+ \right\|_{\infty,\Omega_+} \lesssim K_1^{|\alpha|} |\alpha|! \; \forall \; \alpha \in \mathbb{N}_0^2, \tag{30}$$

$$\left\| w_M^- \right\|_{k,\Omega_-} \lesssim C^{k+1} k! \; . \tag{31}$$

Proof. From Theorem 2.2 of [8] we have that

$$\left\| D^\alpha w_M^+ \right\|_{\infty,\Omega_+} \lesssim K_1^{|\alpha|} |\alpha|! \left(1 + (2M\varepsilon K_2)^{2M}\right) \; \forall \; \alpha \in \mathbb{N}_0^2,$$

so if $2M\varepsilon K_2 < 1$ we get (30) . In order to establish (31), we first consider u_0^-, which satisfies the Dirichlet-Neumann problem (21)–(23). Since the data of this problem are analytic, we have that u_0^- is also analytic [3], and moreover

$$\left| u_0^- \right|_{k,\Omega_-} \leq C^{k+1} k! \; \forall \; k \in \mathbb{N}_0.$$

Next, we consider u_{2j}^-, $j = 0, 1, ...$, defined by (24)–(26), with again the data being analytic. Casting (24)–(26) into a variational formulation, allows us to write

$$\left\| u_{2j}^- \right\|_{1,\Omega_-}^2 = \int\limits_{\Sigma} \frac{\partial u_{2j-2}^+}{\partial \nu} u_{2j}^- \lesssim \left\| \frac{\partial u_{2j-2}^+}{\partial \nu} \right\|_{0,\Sigma} \left\| u_{2j}^- \right\|_{1,\Omega_-} ,$$

which, using (20), gives

$$\left\| u_{2j}^- \right\|_{1,\Omega_-} \lesssim \left\| \frac{\partial u_{2j-2}^+}{\partial \nu} \right\|_{0,\Sigma} \lesssim \left\| \frac{\partial \left(\Delta^{2j} f_+\right)}{\partial \nu} \right\|_{0,\Sigma} \lesssim \left\| \nabla^{4j+1} f_+ \right\|_{\infty,\Omega_+} . \tag{32}$$

From [3] we have that there exists $C \in \mathbb{R}^+$ such that

$$\frac{1}{k!} \left| u_{2j}^- \right|_{k,\Omega_-} \leq C^{k+1} \left\{ \sum_{\ell=1}^{k} \frac{1}{\ell!} \left\| \frac{\partial u_{2j}^-}{\partial \nu} \right\|_{\ell+\frac{1}{2},\Sigma} + \left\| u_{2j}^- \right\|_{1,\Omega_-} \right\} , \tag{33}$$

and we note that (see eq. (26)),

$$\begin{aligned}\left\|\frac{\partial u_{2j}^-}{\partial\nu}\right\|^2_{\ell+\frac{1}{2},\Sigma} &= \left\|\frac{\partial u_{2j-2}^+}{\partial\nu}\right\|^2_{\ell+\frac{1}{2},\Sigma} = \left\|\frac{\partial\left(\Delta^{2j}f_+\right)}{\partial\nu}\right\|^2_{\ell+\frac{1}{2},\Sigma} \lesssim \left\|\Delta^{2j}f_+\right\|^2_{\ell+1,\Omega_+}\\ &\lesssim \sum_{|\alpha|\le\ell+1}\int\limits_{\Omega_+}\left|D^\alpha\Delta^{2j}f_+\right|^2dx \lesssim \sum_{|\alpha|\le\ell+1}\left\|D^\alpha\Delta^{2j}f_+\right\|^2_{\infty,\Omega_+}.\end{aligned}$$

Hence, (33) becomes (with the aid of (7) and (32))

$$\begin{aligned}\frac{1}{k!}\left|u_{2j}^-\right|_{k,\Omega_-} &\le C^{k+1}\left\{\sum_{\ell=1}^{k}\frac{1}{\ell!}\sum_{|\alpha|\le\ell+1}\left\|D^\alpha\Delta^{2j}f_+\right\|_{\infty,\Omega_+}+\left\|\Delta^{2j}f_+\right\|_{\infty,\Sigma}\right\}\\ &\lesssim C^{k+1}\left\{\sum_{\ell=1}^{k}\frac{1}{\ell!}\sum_{|\alpha|\le\ell+1}\gamma^{|\alpha|+2j}\left(|\alpha|+2j\right)!+\gamma^{2j}\left(2j\right)!\right\}\\ &\lesssim C^{k+1}\sum_{\ell=1}^{k}\ell^2\gamma^{\ell+2j}\left(\ell+2j\right)!\\ &\lesssim C^{k+1}k^2\left(2j\right)!\sum_{\ell=1}^{k}\gamma^{k+2j}\binom{\ell+2j}{\ell}\\ &\lesssim C_1^{k+1}\left(2j\right)!\left(1+\gamma\right)^{k+2j}\gamma^{2j}\\ &\lesssim C_1^{k+1}\left(2j\right)!\gamma_1^{2j}\end{aligned}$$

for a suitable $C_1,\gamma_1>0$. This shows that u_{2j}^- are analytic and $\forall\ j=0,1,...$

(34) $$\left|u_{2j}^-\right|_{k,\Omega_-}\lesssim C_1^{k+1}k!\left(2j\right)!\gamma_1^{2j}\ ,\ k\in\mathbb{N}.$$

Thus, from the definition of w_M^- we have

$$\begin{aligned}\left|w_M^-\right|_{k,\Omega_-} &\le \sum_{j=0}^{M}\varepsilon^{2j}\left|u_{2j}^-\right|_{k,\Omega_-}\lesssim C_1^{k+1}k!\sum_{j=0}^{M}\varepsilon^{2j}\left(2j\right)!\gamma_1^{2j}\\ &\lesssim C_1^{k+1}k!\sum_{j=0}^{M}\varepsilon^{2j}\left(2M\right)^{2j}\gamma_1^{2j}\lesssim C_1^{k+1}k!\sum_{j=0}^{M}\left(2\varepsilon M\gamma_1\right)^{2j}\\ &\lesssim C^{k+1}k!,\end{aligned}$$

provided $2\varepsilon M\gamma_1<1$ (so that the above sum can be estimated by a converging geometric series). Estimate (31) follows. ■

Remark 1 *The above theorem gives bounds on the smooth (outer) part of the solution to (1)–(6) under the assumption that* εM *is sufficiently small. In the complementary case, the asymptotic expansion loses its meaning.*

3.2 Construction and regularity of the boundary layers along $\partial\Omega_+\backslash\Sigma$

Boundary layers are introduced in order to account for the fact that the function w_M^+ does not satisfy the boundary condition on $\partial\Omega_+\backslash\Sigma$ (cf. (28)). These are precisely the ones constructed and analyzed in [8], so we will only outline the procedure and quote the relevant results from [8]. The boundary layer correction u_{BL}^ε of w_M^+ is defined as the solution of

$$\begin{aligned} L_\varepsilon u_{BL}^\varepsilon &= 0 \text{ in } \Omega_+, && (35)\\ u_{BL}^\varepsilon &= -w_M^+ \text{ on } \partial\Omega_+\backslash\Sigma, && (36) \end{aligned}$$

where L_ε is defined as

$$L_\varepsilon u := -\varepsilon^2\Delta u + u. \tag{37}$$

With $\kappa_+(\theta)$ the curvature of $\partial\Omega_+\backslash\Sigma$ we set

$$\sigma_+(\rho,\theta) = \frac{1}{1-\kappa_+(\theta)\rho},$$

and we have (see, e.g. [1])

$$\Delta u(\rho,\theta) = \partial_\rho^2 u - \kappa_+(\theta)\sigma_+(\rho,\theta)\partial_\rho u + \sigma_+^2(\rho,\theta)\partial_\theta^2 u + \rho\kappa_+'(\theta)\sigma_+^3(\rho,\theta)\partial_\theta u.$$

Introducing the stretched variable $\widehat{\rho} = \rho/\varepsilon$, the operator L_ε becomes

$$L_\varepsilon = -\partial_{\widehat{\rho}}^2 + \mathrm{Id} + \varepsilon\kappa_+(\theta)\sigma_+(\varepsilon\widehat{\rho},\theta)\partial_{\widehat{\rho}} - \varepsilon^2\sigma_+^2(\varepsilon\widehat{\rho},\theta) - \varepsilon^3\widehat{\rho}\kappa_+'(\theta)\sigma_+^3(\varepsilon\widehat{\rho},\theta)\partial_\theta. \tag{38}$$

Expanding the above in power series of ε, we can formally write

$$L_\varepsilon = \sum_{i=0}^{\infty}\varepsilon^i L_i, \tag{39}$$

where the operators L_i have the form (see equations (2.12)–(2.14) in [8])

$$L_0 = -\partial_{\widehat{\rho}}^2 + \mathrm{Id},\ L_i = -\widehat{\rho}^{\,i-1}a_1^{i-1}\partial_{\widehat{\rho}} - \widehat{\rho}^{\,i-2}a_2^{i-2}\partial_\theta^2 - \widehat{\rho}^{\,i-2}a_3^{i-3}\partial_\theta,\ i \ge 1, \tag{40}$$

and the coefficients a_j^i are given by

$$a_1^i = -\left[\kappa_+(\theta)\right]^{i+1},\ a_2^i = (i+1)\left[\kappa_+(\theta)\right]^i,\ a_3^i = \frac{(i+1)(i+2)}{2}\left[\kappa_+(\theta)\right]^i\kappa_+'(\theta),\ i \in \mathbb{N}_0, \tag{41}$$

$$a_1^i = a_2^i = a_3^i = 0 \text{ for } i < 0. \tag{42}$$

We next make the formal ansatz

$$u_{BL}^{\varepsilon} = \sum_{i=0}^{\infty} \varepsilon^i \widehat{U}_i(\widehat{\rho}, \theta),$$

and insert it into (35). This yields

$$\sum_{i=0}^{\infty} \varepsilon^i \sum_{j=0}^{i} L_j \widehat{U}_{i-j} = 0, \tag{43}$$

allowing us to find the following problem for the functions $\widehat{U}_i(\widehat{\rho}, \theta), i = 0, 1, 2, ...$:

$$-\partial_{\widehat{\rho}}^2 \widehat{U}_i + \widehat{U}_i = \widehat{F}_i =: \widehat{F}_i^1 + \widehat{F}_i^2 + \widehat{F}_i^3, \tag{44}$$

$$\widehat{F}_i^1 = \sum_{k=0}^{i-1} \widehat{\rho}^k a_1^k \partial_{\widehat{\rho}} \widehat{U}_{i-1-k}\ ,\ \widehat{F}_i^2 = \sum_{k=0}^{i-2} \widehat{\rho}^k a_2^k \partial_{\theta}^2 \widehat{U}_{i-2-k}\ ,\ \widehat{F}_i^3 = \sum_{k=0}^{i-3} \widehat{\rho}^{k+1} a_3^k \partial_{\theta} \widehat{U}_{i-3-k}, \tag{45}$$

where empty sums are assumed to be zero. (See, also, equations (2.15)–(2.16) in [8]). The above are supplemented with boundary conditions

$$\begin{aligned} \widehat{U}_i &\ \to\ 0 \text{ as } \rho \to \infty, \\ \left[\widehat{U}_i\right]_{\partial\Omega\setminus\Sigma} &= \begin{cases} -[f]_{\partial\Omega\setminus\Sigma} & \text{if } \quad i = 0, \\ -\left[\Delta^{(i/2)} f\right]_{\partial\Omega\setminus\Sigma} & \text{if } \ i \in \mathbb{N} \text{ is even}, \\ 0 & \text{if } \ i \in \mathbb{N} \text{ is odd}. \end{cases} \end{aligned}$$

The boundary layer (inner) expansion in (10) is then defined as

$$u_{BL}^{\varepsilon} \equiv u_{BL}^{M}(\rho, \theta) = \sum_{j=0}^{2M+1} \varepsilon^j \widehat{U}_j(\widehat{\rho}, \theta) = \sum_{j=0}^{2M+1} \varepsilon^j \widehat{U}_j(\rho/\varepsilon, \theta), \tag{46}$$

and by construction, it satisfies the boundary condition

$$\left[u_{BL}^{M}\right]_{\partial\Omega\setminus\Sigma} = -\sum_{i=0}^{2M+1} \varepsilon^{2i} \left[\Delta^{(i)} f\right]_{\partial\Omega\setminus\Sigma}.$$

By Theorem 2.2 of [8] we have that for every $\alpha \in [0, 1)$ and all $p, m \in \mathbb{N}_0$,

$$\left|\partial_{\rho}^p \partial_{\theta}^m u_{BL}^{M}(\rho, \theta)\right| \lesssim \left(1 + \left(\frac{\varepsilon(2M+1)K_2}{1-\alpha}\right)^{2M+1}\right) m! K_1^{m+p} \varepsilon^{-p} e^{-\alpha\rho/\varepsilon}, \tag{47}$$

for $\theta \in \mathbb{T}_L, \rho \in [0, \rho_0]$, with $K_1, K_2 > 0$ independent of ε, p and m. Moreover, by Lemma 2.12 of [8], there exist constants $K, \Theta > 0$ independent of ε such that

$$\left|L_{\varepsilon} u_{BL}^{M}(\rho, \theta)\right| \lesssim K^{2M+2} \left(\varepsilon(2M+2) + |\rho|\right)^{2M+2} e^{-\rho/\varepsilon}\ \forall\ (\rho, \theta) \in B_{\rho_0}(0) \times S(\Theta), \tag{48}$$

where $B_{\delta}(z)$ denotes the (open) disc in the complex plane of radius δ centered at z, and

$$S(\Theta) = \{\theta \in \mathbb{C} : \operatorname{Im}(\theta) < \Theta\}. \tag{49}$$

3.3 Construction and regularity of the interface layer on Σ

For a function $\omega = (\omega_+, \omega_-)$ we denote the jump $[[\omega]]_\Sigma$ on Σ as

$$[[\omega]]_\Sigma := (\omega_+)|_\Sigma - (\omega_-)|_\Sigma\,. \tag{50}$$

We define the function $v_I(\rho,\theta) := \left(v_I^-, v_I^+\right)$ as the solution of the following problem:

$$\left\{\begin{array}{c} -\varepsilon^2 \Delta v_I^+ + v_I^+ = 0 \text{ in } \Omega_+ \\ -\Delta v_I^- + v_I^- = 0 \text{ in } \Omega_- \\ [[v_I]]_\Sigma = -\sum_{j=0}^{\infty} \varepsilon^{2j} [[u_{2j}]]_\Sigma \\ \left(\varepsilon^2 \frac{\partial v_I^+}{\partial \rho} - \frac{\partial v_I^-}{\partial \rho}\right)\Big|_\Sigma = -\sum_{j=0}^{\infty} \varepsilon^{2j} \left(\varepsilon^2 \frac{\partial u_{2I}^+}{\partial \rho} - \frac{\partial u_{2j}^-}{\partial \rho}\right)\Big|_\Sigma \end{array}\right. . \tag{51}$$

With $\widehat{\rho} = \rho/\varepsilon$ as before, we write

$$\widehat{v_I^+}(\rho,\theta) = v_I^+\left(\widehat{\rho},\theta\right),$$

and problem (51) becomes

$$\left\{\begin{array}{c} \left(-\partial^2_{\widehat{\rho}} + \mathrm{Id}\right)\widehat{v_I^+} + \varepsilon\kappa_+(\theta)\sigma(\varepsilon\widehat{\rho},\theta)\partial_{\widehat{\rho}}\widehat{v_I^+} - \varepsilon^2\sigma^2(\varepsilon\widehat{\rho},\theta)\widehat{v_I^+} - \\ -\varepsilon^3\widehat{\rho}\kappa'_+(\theta)\sigma^3(\varepsilon\widehat{\rho},\theta)\partial_\theta\widehat{v_I^+} = 0 \text{ in } \Omega_+ \\ -\Delta v_I^- + v_I^- = 0 \text{ in } \Omega_- \\ \left(\widehat{v_I^+} - v_I^-\right)|_\Sigma = -\sum_{j=0}^{\infty} \varepsilon^{2j}\,[[u_{2j}]]_\Sigma \\ \left(\varepsilon \frac{\partial \widehat{v_I^+}}{\partial \widehat{\rho}} - \frac{\partial v_I^-}{\partial \rho}\right)\Big|_\Sigma = -\sum_{j=0}^{\infty} \varepsilon^{2j} \left(\varepsilon^2 \frac{\partial u_{2j}^+}{\partial \rho} - \frac{\partial u_{2j}^-}{\partial \rho}\right)\Big|_\Sigma \end{array}\right. . \tag{52}$$

Now, we write

$$\widehat{v_I^+} = \sum_{j=0}^{\infty} \varepsilon^j \widehat{V}_j^+\,,\; v_I^- = \sum_{j=0}^{\infty} \varepsilon^j V_j^-, \tag{53}$$

and insert it in (52) equating like powers of ε, to get (utilizing again the expansion (39))

$$\left\{\begin{array}{c} -\partial^2_{\widehat{\rho}}\widehat{V}_j^+ + \widehat{V}_j^+ = \widehat{F}_j^1 + \widehat{F}_j^2 + \widehat{F}_j^3 \text{ in } \mathbb{R}_+ \;\forall\; j \geq 0 \\ -\Delta V_j^- + V_j^- = 0 \text{ in } \Omega_- \;\forall\; j \geq 0 \\ \left(\widehat{V}_{2j}^+ - V_{2j}^-\right) = -\left(u_{2j}^+ - u_{2j}^-\right) \text{ on } \Sigma \;\forall\; j \geq 0 \\ \left(\widehat{V}_{2j+1}^+ - V_{2j+1}^-\right) = 0 \text{ on } \Sigma \;\forall\; j \geq 0 \\ -\frac{\partial V_0^-}{\partial \rho} = \frac{\partial u_0^-}{\partial \rho} \text{ on } \Sigma \\ \left(\frac{\partial}{\partial \rho} V_{2j}^- - \frac{\partial}{\partial \widehat{\rho}} \widehat{V}_{2j-1}^+\right) = -\left(\frac{\partial}{\partial \rho} u_{2j}^- - \frac{\partial}{\partial \rho} u_{2j-2}^+\right) \text{ on } \Sigma \;\forall\; j \geq 1 \\ \left(\frac{\partial}{\partial \rho} V_{2j+1}^- - \frac{\partial}{\partial \widehat{\rho}} \widehat{V}_{2j}^+\right) = 0 \text{ on } \Sigma \;\forall\; j \geq 0 \end{array}\right. , \tag{54}$$

with $\widehat{F}_j^1, \widehat{F}_j^2, \widehat{F}_j^3$ given by (45) but with $\widehat{U}$ replaced by $\widehat{V}^+$. So for $j = 0$, we have

$$\begin{cases} -\Delta V_0^- + V_0^- = 0 \text{ in } \Omega_- \\ -\frac{\partial V_0^-}{\partial \rho} = \frac{\partial u_0^-}{\partial \rho} \text{ on } \Sigma \\ V_0^- = 0 \text{ on } \partial\Omega_- \backslash \Sigma \end{cases}, \tag{55}$$

$$\begin{cases} -\partial_{\widehat{\rho}}^2 \widehat{V}_0^+ + \widehat{V}_0^+ = 0 \text{ in } \mathbb{R}_+ \\ \widehat{V}_0^+ = V_0^- - \left(u_0^+ - u_0^-\right) \text{ on } \Sigma \end{cases}, \tag{56}$$

$$\begin{cases} -\Delta V_1^- + V_1^- = 0 \text{ in } \Omega_- \\ \frac{\partial}{\partial \rho} V_1^- = \frac{\partial}{\partial \widehat{\rho}} \widehat{V}_0^+ \text{ on } \Sigma \\ V_1^- = 0 \text{ on } \partial\Omega_- \backslash \Sigma \end{cases}, \tag{57}$$

$$\begin{cases} -\partial_{\widehat{\rho}}^2 \widehat{V}_1^+ + \widehat{V}_1^+ = \widehat{V}_0^+ \text{ in } \mathbb{R}_+ \\ \widehat{V}_1^+ = V_1^- \text{ on } \Sigma \end{cases}. \tag{58}$$

In general, for $j \geq 0$ odd we have

$$\begin{cases} -\Delta V_{2j+1}^- + V_{2j+1}^- = 0 \text{ in } \Omega_- \\ \frac{\partial}{\partial \rho} V_{2j+1}^- = \frac{\partial}{\partial \widehat{\rho}} \widehat{V}_{2j}^+ \text{ on } \Sigma \\ V_{2j+1}^- = 0 \text{ on } \partial\Omega_- \backslash \Sigma \end{cases}, \tag{59}$$

$$\begin{cases} -\partial_{\widehat{\rho}}^2 \widehat{V}_{2j+1}^+ + \widehat{V}_{2j+1}^+ = \widehat{F}_{2j+1}^1 + \widehat{F}_{2j+1}^2 + \widehat{F}_{2j+1}^3 \text{ in } \mathbb{R}_+ \\ \widehat{V}_{2j+1}^+ = V_{2j+1}^- \text{ on } \Sigma \end{cases} \tag{60}$$

and for $j \geq 0$ even we have

$$\begin{cases} -\Delta V_{2j}^- + V_{2j}^- = 0 \text{ in } \Omega_- \\ \left(\frac{\partial}{\partial \rho} V_{2j}^- - \frac{\partial}{\partial \widehat{\rho}} \widehat{V}_{2j-1}^+\right) = -\left(\frac{\partial}{\partial \rho} u_{2j}^- - \frac{\partial}{\partial \rho} u_{2j-2}^+\right) \text{ on } \Sigma \\ V_{2j}^- = 0 \text{ on } \partial\Omega_- \backslash \Sigma \end{cases}, \tag{61}$$

$$\begin{cases} -\partial_{\widehat{\rho}}^2 \widehat{V}_{2j}^+ + \widehat{V}_{2j}^+ = \widehat{F}_{2j}^1 + \widehat{F}_{2j}^2 + \widehat{F}_{2j}^3 \text{ in } \mathbb{R}_+, \\ \widehat{V}_{2j}^+ = V_{2j}^- - \left(u_{2j}^+ - u_{2j}^-\right) \text{ on } \Sigma \end{cases}. \tag{62}$$

The regularity of the functions $V_j^-, \widehat{V}_j^+$ is given by Theorem 4 below. For its proof, we will need the following lemma.

Lemma 3 *Let* $U_j\left(\widehat{\rho},\theta\right), j=0,1,2,...,$ *be the solutions to*

$$\left.\begin{array}{c} -\partial_{\widehat{\rho}}^2 U_j + U_j = F_j\left(\widehat{\rho},\theta\right) \ \ in\ \mathbb{R}_+ \\ U_j = G_j\left(\theta\right) \ \ on\ \Sigma \end{array}\right\}, \tag{63}$$

where $F_j\left(\widehat{\rho},\theta\right) = F_j^1\left(\widehat{\rho},\theta\right) + F_j^2\left(\widehat{\rho},\theta\right) + F_j^2\left(\widehat{\rho},\theta\right)$ *is given by (44)–(45) and* G_j *satisfy*

$$\left|G_j\left(\theta\right)\right| \leq C_G \gamma_G^j j^j, \tag{64}$$

for some positive constants C_G, γ_G *depending only on the data. Then, there exist positive constants* Θ, C_U, γ_U*, depending only on the data, such that*

$$\left|U_j\left(\widehat{\rho},\theta\right)\right| \leq C_U \gamma_U^j \left(1+j+\widehat{\rho}\right)^j e^{-\widehat{\rho}}\ \forall\ \left(\widehat{\rho},\theta\right) \in \mathbb{R}_+ \times S\left(\Theta\right), \tag{65}$$

where $S(\Theta)$ *is given by (49). Moreover, for any* $\alpha \in [0,1)$ *there exists* $K \in \mathbb{R}_+$ *depending only on the data, such that*

$$|U_j(\widehat{\rho},\theta)| \lesssim K^j j^j \left(1-\alpha\right)^{-j} e^{-\alpha\widehat{\rho}}, \tag{66}$$

and

$$\left|\partial_\rho^p \partial_\theta^q U_j\left(\rho/\varepsilon,\theta\right)\right| \lesssim \varepsilon^{-p} e^{(1-\alpha)p}\left(p+1\right)^{1/2} q!\left(2/\Theta\right)^q \gamma_U^j j^j \left(1-\alpha\right)^{-j} e^{-\rho/\varepsilon}\ \forall\ p,q \in \mathbb{N}_0. \tag{67}$$

Proof. This is essentially a combination of Lemmas 2.9 and 2.11 in [8]. Estimate (65) follows directly from Lemma 2.11 in [8], while (66) follows from (65) and Lemma 2.8 in [8]. Finally, (67) follows from Cauchy's integral formula, in exactly the same way as in the proof of (2.24) in [8]. ■

Theorem 4 *Let* V_j^- *satisfy (59), (61) and* $\widehat{V}_j^+$ *satisfy (60), (62) . Then there exist constants* $C,\gamma,K,\Theta > 0$ *depending only on the data, such that*

$$\left|V_j^-\right|_{k,\Omega_-} \lesssim k! C^{k+1} j^j \gamma^j, \tag{68}$$

while for $\theta \in \mathbb{T}_{L_\Sigma}, \rho \in [0,\rho_\Sigma], \alpha \in [0,1)$,

$$\left|\widehat{V}_j^+(\rho/\varepsilon,\theta)\right| \lesssim K^j j^j \left(1-\alpha\right)^{-j} e^{-\alpha\rho/\varepsilon}, \tag{69}$$

and

$$\left|\partial_\rho^p \partial_\theta^q \widehat{V}_j^+(\rho/\varepsilon,\theta)\right| \lesssim \varepsilon^{-p} e^{(1-\alpha)p}\left(p+1\right)^{1/2} q!\left(2/\Theta\right)^q K^j j^j \left(1-\alpha\right)^{-j} e^{-\alpha\rho/\varepsilon}, \tag{70}$$

for $p,q \in \mathbb{N}_0$ *and* $\theta \in S(\Theta)$ *given by (49).*

Proof. The proof is by induction on j. First we note that estimates (69), (70) follow from Lemma 3, provided we show that (64) is satisfied, i.e. on Σ the functions $\widehat{V}_j^+$ are bounded by $Cj^j\gamma^j$ for suitable constants $C, \gamma > 0$. This will be verified during our induction argument; in fact it will be the only thing we will show for $\widehat{V}_j^+$, with the understanding that an application of Lemma 3 gives the desired result.

For $j = 0$ we see from the variational formulation of (55) that $\left\|V_0^-\right\|_{1,\Omega_-} \lesssim \left\|u_0^-\right\|_{1,\Omega_-}$, hence by (34) and [3]

$$\begin{aligned}
\frac{1}{k!}\left|V_0^-\right|_{k,\Omega_-} &\leq C^{k+1}\left\{\sum_{\ell=0}^{k-2}\frac{1}{\ell!}\left\|\frac{\partial u_0^-}{\partial y}\right\|_{\ell+\frac{1}{2},\Sigma} + \left\|V_0^-\right\|_{1,\Omega_-}\right\} \\
&\lesssim C^{k+1}\left\{\sum_{\ell=1}^{k}\frac{1}{\ell!}\left\|u_0^-\right\|_{\ell+1,\Sigma} + \left\|u_0^-\right\|_{1,\Omega_-}\right\} \\
&\lesssim C^{k+1}\left\{\sum_{\ell=1}^{k}\frac{1}{\ell!}C_1^{\ell+1}(\ell+1)!\right\} \\
&\lesssim C^{k+1}.
\end{aligned}$$

Next, for $\widehat{V}_0^+$ we see from (56) that $\widehat{V}_0^+(\rho,\theta) = G_0(\theta)\,e^{-\rho/\varepsilon}$ for some function $G_0(\theta)$ that depends on V_0^-, u_0^+, u_0^-. By the above, (20) and (34), we have that in the case $j = 0$, the boundary data for $\widehat{V}_j^+$ is bounded by $Cj^j\gamma^j$ for suitable constants $C, \gamma > 0$, hence by Lemma 3 the bounds (69), (70) hold for $\widehat{V}_0^+$.

Now, from the variational formulation of (57) and the fact that $V_0^+(\hat{\rho},\theta) = G_0(\theta)e^{-\hat{\rho}}$, we have $\frac{\partial}{\partial\hat{\rho}}V_0^+(\hat{\rho},\theta) = -G_0(\theta)e^{-\hat{\rho}}$ and then $\frac{\partial}{\partial\hat{\rho}}V_0^+(0,\theta) = -G_0(\theta)$, hence

$$\int_{\Omega_-}\left(\nabla V_1^- \cdot \nabla V + V_1^- V\right)dx = -\int_{\Sigma} G_0 V\,dx \;\forall\; V \in H_*^1(\Omega_-),$$

where

$$H_*^1(\Omega_-) = \left\{u \in H^1(\Omega_-) : u|_{\partial\Omega_-\setminus\Sigma} = 0\right\}. \tag{71}$$

Thus,

$$\begin{aligned}
\left\|V_1^-\right\|_{1,\Omega_-} &\lesssim \|G_0\|_{0,\Sigma} = \left\|\widehat{V}_0^+\right\|_{0,\Sigma} = \left\|V_0^- - (u_0^+ - u_0^-)\right\|_{0,\Sigma} \\
&\leq \left\|V_0^-\right\|_{0,\Sigma} + \left\|u_0^+\right\|_{0,\Sigma} + \left\|u_0^-\right\|_{0,\Sigma} \leq C_1 \in \mathbb{R}^+.
\end{aligned}$$

From [3] and the above result, we get

$$\begin{aligned}
\frac{1}{k!}\left|V_1^-\right|_{k,\Omega_-} &\le C^{k+1}\left\{\sum_{\ell=0}^{k-2}\frac{1}{\ell!}\left\|\frac{\partial \widehat{V}_0^+}{\partial\rho}\right\|_{\ell+\frac{1}{2},\Sigma}+\left\|V_1^-\right\|_{1,\Omega_-}\right\}\\
&\lesssim C^{k+1}\left\{\sum_{\ell=0}^{k-2}\frac{1}{\ell!}\left\|G_0\right\|_{\ell+\frac{1}{2},\Sigma}+C_1\right\}\\
&\lesssim C^{k+1}\left\{\sum_{\ell=0}^{k-2}\frac{1}{\ell!}\left(\left\|V_0^-\right\|_{\ell+\frac{1}{2},\Sigma}+\left\|u_0^+\right\|_{\ell+\frac{1}{2},\Sigma}+\left\|u_0^-\right\|_{\ell+\frac{1}{2},\Sigma}\right)+C_1\right\}\\
&\lesssim C^{k+1}\left\{\sum_{\ell=1}^{k}\frac{1}{\ell!}(\ell+1)!C^{\ell+1}+C_1\right\},
\end{aligned}$$

which leads to

$$\left|V_1^-\right|_{k,\Omega_-}\lesssim C^{k+1}k!. \tag{72}$$

In an analogous way as $\widehat{V}_0^+(\widehat{\rho},\theta)=G_0\left(\theta\right)e^{-\widehat{\rho}}$, we find that $\widehat{V}_1^+(\widehat{\rho},\theta)=-\frac{\widehat{\rho}}{2}a_1^0(\theta)e^{-\widehat{\rho}}+G_1\left(\theta\right)e^{-\widehat{\rho}}$, for some function $G_1\left(\theta\right)$ that depends on V_1^-, hence in view of (72) we see that the boundary data for $\widehat{V}_1^+$ satisfy the appropriate bound. As a result, (69)–(70) hold for $\widehat{V}_1^+$ as well.

So, we assume that (68)–(70) hold for j and we will establish them for $j+1$.

The case of odd j: If j is odd, then $j+1$ is even and we would like to establish bounds for V_{2s}^- and $\widehat{V}_{2s}^+$ (with $2s=j+1$), which satisfy (61), (62) respectively. First, for V_{2s}^- we see from the variational formulation of (61) that

$$\left\|V_{2s}^-\right\|_{1,\Omega_-}\lesssim\left\|\frac{\partial}{\partial\widehat{\rho}}\widehat{V}_{2s-1}^+\right\|_{0,\Sigma}+\left\|\frac{\partial}{\partial\rho}u_{2s}^-\right\|_{0,\Sigma}+\left\|\frac{\partial}{\partial\rho}u_{2s-2}^+\right\|_{0,\Sigma},$$

hence by (20), (34), a trace theorem and the induction hypothesis, we have

$$\left\|V_{2s}^-\right\|_{1,\Omega_-}\lesssim K^{2s-1}(2s-1)^{2s-1}+C_{u^-}\gamma_{u^-}^{2s}(2s)!+C_{f_+}\gamma_{f_+}^{2s}(2s)!\lesssim C(2s)!\gamma^{2s},$$

for suitable constants $C,\gamma>0$ independent of s. Therefore, from [3] we obtain for $k\ge 2$

$$\begin{aligned}
\frac{1}{k!}\left|V_{2s}^-\right|_{k,\Omega_-} &\le C^{k+1}\left\{\sum_{\ell=0}^{k-2}\frac{1}{\ell!}\left(\left\|\frac{\partial}{\partial\widehat{\rho}}\widehat{V}_{2s-1}^+\right\|_{\ell+\frac{1}{2},\Sigma}+\left\|\frac{\partial}{\partial\rho}u_{2s}^-\right\|_{\ell+\frac{1}{2},\Sigma}+\left\|\frac{\partial}{\partial\rho}u_{2s-2}^+\right\|_{\ell+\frac{1}{2},\Sigma}\right)+\left\|V_{2s}^-\right\|_{1,\Omega_-}\right.\\
&\lesssim C^{k+1}\left\{\sum_{\ell=0}^{k-2}\frac{1}{\ell!}\left(\left\|\widehat{V}_{2s-1}^+\right\|_{\ell+\frac{1}{2},\Sigma}+\left\|u_{2s}^-\right\|_{\ell+2,\Omega_-}+\left\|u_{2s-2}^+\right\|_{\ell+2,\Omega_+}\right)+(2s)!\gamma^{2s}\right\}.
\end{aligned}\tag{73}$$

Now, $u^+_{2s-2} = \Delta^{(2s)} f_+$ (see eq. (20)), hence using (7) we have

$$\begin{aligned} \left\|u^+_{2s-2}\right\|_{\ell+1,\Omega_+} &= \left\|\Delta^{(2s)} f^+\right\|_{\ell+1,\Omega_+} \lesssim \sum_{\ell=1}^{k} \frac{1}{\ell!} \sum_{|\alpha|\le \ell+1} \left\|D^\alpha \Delta^{(2s)} f^+\right\|_{\infty,\Omega_+} \\ &\lesssim \sum_{\ell=1}^{k} \frac{1}{\ell!}\sum_{|\alpha|\le\ell+1} \gamma_f^{|\alpha|+2s}\left(|\alpha|+2s\right)! \lesssim \sum_{\ell=1}^{k} \ell^2 \gamma_f^{\ell+2s}\left(\ell+2s\right)! \\ &\lesssim k^2\left(2s\right)!\sum_{\ell=1}^{k} \gamma_f^{k+2s}\binom{\ell+2s}{\ell} \lesssim (2s)!\left(1+\gamma_f\right)^{k+2s}\gamma_f^{2s} \\ &\lesssim (2s)!\widetilde{\gamma}^{2s}, \end{aligned}$$

for suitable $\widetilde{\gamma} > 0$ independent of ε. Also, by (62) we get

$$\begin{aligned} \left\|\widehat{V}^+_{2s-1}\right\|_{\ell+\frac{1}{2},\Sigma} &= \left\|V^-_{2s-1} - \left(u^+_{2s-1} - u^-_{2s-1}\right)\right\|_{\ell+\frac{1}{2},\Sigma} \\ &\le \left\|V^-_{2s-1}\right\|_{\ell+1,\Omega_-} + \left\|u^+_{2s-1}\right\|_{\ell+1,\Omega_+} + \left\|u^-_{2s-1}\right\|_{\ell+1,\Omega_-}. \end{aligned}$$

Equation (20) gives $u^+_{2s-1} = 0$, and by (20), (34) and the induction hypothesis, we obtain

$$\left\|\widehat{V}^+_{2s-1}\right\|_{\ell+\frac{1}{2},\Sigma} \lesssim (\ell+1)!C^{\ell+1}(2s-1)^{2s-1}\gamma^{2s-1}.$$

Thus, (73) becomes

$$\begin{aligned} \frac{1}{k!}\left|V^-_{2s}\right|_{k,\Omega_-} &\lesssim C^{k+1}\sum_{\ell=1}^{k}\frac{1}{\ell!}\left((\ell+1)!C^{\ell+1}(2s-1)^{2s-1}\gamma^{2s-1} + C_{u^-}^{\ell+1}\left(\ell+1\right)!(2s)!\gamma_{u^-}^{2s} + (2s)!\widetilde{\gamma}_f^{2s}\right) + \\ &\quad + C^{k+1}(2s)!\gamma^{2s} \\ &\lesssim C_1^{k+1}(2s)^{2s}\gamma_1^{2s}, \end{aligned}$$

for suitable constants $C_1, \gamma_1 > 0$ independent of ε. This establishes (68); to establish (69)–(70) we will simply check that the boundary data in (62) satisfies the appropriate bound (so that we may apply Lemma 3). Since

$$\widehat{V}^+_{2s} = V^-_{2s} - \left(u^+_{2s} - u^-_{2s}\right) \text{ on } \Sigma,$$

we see that for $\theta \in [0, L_\Sigma]$, (cf. (62))

$$\begin{aligned} \left|\widehat{V}^+_{2s}\left(0,\theta\right)\right| &\le \left|V^-_{2s}\left(0,\theta\right)\right| + \left|u^+_{2s}\left(0,\theta\right)\right| + \left|u^-_{2s}\left(0,\theta\right)\right| \\ &\lesssim \widehat{C}(2s)^{2s}\gamma^{2s}, \end{aligned}$$

which is the bound that allows us to apply Lemma 3 and conclude that for $\widehat{V}_{2j}^{+}(\widehat{\rho},\theta)$, the estimates (69), (70) hold as desired.

The case of even j: If j is even, then $j+1$ is odd and we would like to establish bounds for V_{2s+1}^{-} and $\widehat{V}_{2s+1}^{+}$ (with $2s+1=j+1$), which satisfy (59), (60) respectively. First, for V_{2s+1}^{-} we see from the variational formulation of (59) that

$$\left\|V_{2s+1}^{-}\right\|_{1,\Omega_-} \lesssim \left\|\widehat{V}_{2s}^{+}\right\|_{0,\Sigma} \lesssim C_+(2s)^{2s}\gamma^{2s} \le C_+(2s+1)^{2s+1}\gamma^{2s+1},$$

and, in a similar fashion as above, we obtain

$$\begin{aligned}
\frac{1}{k!}\left|V_{2s+1}^{-}\right|_{k,\Omega_-} &\le C^{k+1}\left\{\sum_{\ell=0}^{k-2}\frac{1}{\ell!}\left\|\widehat{V}_{2s}^{+}\right\|_{\ell+1/2,\Sigma} + \left\|V_{2s+1}^{-}\right\|_{1,\Omega_-}\right\}\\
&\lesssim C^{k+1}\left\{\sum_{\ell=0}^{k-2}\frac{(\ell+1)!}{\ell!}\widehat{C}^{\ell+1}(2s)^{2s}\gamma^{2s} + C_+(2s+1)^{2s+1}\gamma^{2s+1}\right\}\\
&\lesssim C_2^{k+1}(2s+1)^{2s+1}\gamma_2^{2s+1},
\end{aligned}$$

for suitable constants $C_2,\gamma_2>0$ independent of ε. Finally, from the above result we see that the boundary data of (60) satisfies the appropriate bound, hence by Lemma 3, $\widehat{V}_{2s+1}^{+}$ satisfies (69) and (70) as desired. ■

In view of the previous theorem, we define the (truncated) interface layer expansion(s) as

$$u_{IL}^{\varepsilon} := \left(\widehat{v}_{I,M}^{+}, v_{I,M}^{-}\right) \tag{74}$$

where

$$\widehat{v}_{I,M}^{+} = \sum_{j=0}^{2M+1}\varepsilon^j\widehat{V}_j^{+}\ ,\ v_{I,M}^{-} = \sum_{j=0}^{2M+1}\varepsilon^j V_j^{-}. \tag{75}$$

The following corollary follows from Theorem 4.

Corollary 5 *There exist constants $C,\gamma,\Theta,K>0$ depending only on the data, such that under the assumption $\varepsilon(2M+1)\max\{\gamma,K\}<1$, the functions $\widehat{v}_{I,M}^{+}, v_{I,M}^{-}$ defined by (75) satisfy*

$$\left|v_{I,M}^{-}\right|_{k,\Omega_-} \lesssim C^{k+1}k!,$$

$$\left|\partial_\rho^p\partial_\theta^q\widehat{v}_{I,M}^{+}(\rho,\theta)\right| \lesssim \varepsilon^{-p}e^p(p+1)^{1/2}q!\left(\frac{2}{\Theta}\right)^q,$$

for $p,q\in\mathbb{N}_0,\rho\in[0,\rho_\Sigma]$ and $\theta\in S(\Theta)$ given by (49).

Proof. By (75) and Theorem 4 we have

$$\left|v^-_{I,M}\right|_{k,\Omega_-} \le \sum_{j=0}^{2M+1} \varepsilon^j \left|V_j^-\right|_{k,\Omega_-} \lesssim \sum_{j=0}^{2M+1} \varepsilon^j k! C^{k+1} j^j \gamma^j \lesssim k! C^{k+1} \sum_{j=0}^{2M+1} (\varepsilon(2M+1)\gamma)^j \lesssim k! C^{k+1}$$

and

$$\begin{aligned}
\left|\partial_\rho^p \partial_\theta^q \widehat{v}^+_{I,M}(\rho,\theta)\right| &\le \sum_{j=0}^{2M+1} \varepsilon^j \left|\partial_\rho^p \partial_\theta^q \widehat{V}_j^+(\rho,\theta)\right| \lesssim \sum_{j=0}^{2M+1} \varepsilon^j \varepsilon^{-p} e^p (p+1)^{1/2} q! \left(\frac{2}{\Theta}\right)^q K^j j^j \\
&\lesssim \varepsilon^{-p} e^p (p+1)^{1/2} q! \left(\frac{2}{\Theta}\right)^q \sum_{j=0}^{2M+1} (\varepsilon K(2M+1))^j \\
&\lesssim \varepsilon^{-p} e^p (p+1)^{1/2} q! \left(\frac{2}{\Theta}\right)^q .
\end{aligned}$$

■

Finally in this section, we wish to see what the contribution of the interface layers is, to the remainder of the expansion. For the interface layers in Ω_- we easily see that

$$-\Delta\left(v^-_{I,M}\right) + \left(v^-_{I,M}\right) = 0. \tag{76}$$

Now, by construction of the functions $\widehat{V}^+_{2j}$ we have (with the aid of (37) and (43))

$$\begin{aligned}
L_\varepsilon\left(\widehat{v}^+_{I,M}\right) &= \sum_{i=2M+2}^{\infty} \varepsilon^i \sum_{j=0}^{2M+1} L_{i-j} \widehat{V}_j^+ \\
&= -\sum_{j=0}^{2M+1} \sum_{i=2M+2}^{\infty} \varepsilon^i \widehat{\rho}^{\,i-1-j} a_1^{j-1-j} \partial_{\widehat{\rho}} \widehat{V}_j^+ - \sum_{j=0}^{2M+1} \sum_{i=2M+3}^{\infty} \varepsilon^i \widehat{\rho}^{\,i-2-j} a_2^{j-2-j} \partial_\theta^2 \widehat{V}_j^+ - \\
&\quad -\sum_{j=0}^{2M+1} \sum_{i=2M+4}^{\infty} \varepsilon^i \widehat{\rho}^{\,i-2-j} a_3^{j-3-j} \partial_\theta \widehat{V}_j^+ .
\end{aligned}$$

By Lemma 2.12 of [8], we have the bound

$$\left|L_\varepsilon \widehat{v}^+_{I,M}(\rho,\theta)\right| \lesssim K^{2M+2} \left(\varepsilon(2M+2) + |\rho|\right)^{2M+2} e^{-\rho/\varepsilon} \ \forall\, (\rho,\theta) \in B_{\rho_0}(0) \times S(\Theta), \tag{77}$$

for some $K, \Theta > 0$ independent of ε. (As before, $B_\delta(z)$ denotes the open disc in the complex plane of radius δ centered at z, and $S(\Theta)$ is given by (49).)

Remark 2 *Corollary 5 shows that the interface layer functions in Ω_+ behave just like the boundary layers, while the interface layers in Ω_- are smooth. This will be taken into consideration in the design of the approximation scheme in Section 4 ahead.*

3.4 Remainder estimates

We now consider the remainder $r^\varepsilon \equiv \left(r_+^\varepsilon, r_-^\varepsilon\right)$ in the decomposition (10), which is given by

$$r_+^\varepsilon = u_+^\varepsilon - w_+^\varepsilon - \chi_{BL} u_{BL}^\varepsilon - \chi_{IL} \widehat{v}_{I,M}^+, \tag{78}$$

$$r_-^\varepsilon = u_-^\varepsilon - w_-^\varepsilon - \chi_{IL} v_{I,M}^-, \tag{79}$$

and by construction, satisfies the equivalent (but homogeneous) boundary conditions as u^ε on $\partial\Omega$. To see this note that on $\partial\Omega_-\backslash\Sigma$ we have

$$\left(r_-^\varepsilon\right)\big|_{\partial\Omega_-\backslash\Sigma} = \left(u_-^\varepsilon - w_-^\varepsilon - \chi_{IL} v_{I,M}^-\right)\big|_{\partial\Omega_-\backslash\Sigma} = 0,$$

by (4), (22) and (25). On $\partial\Omega_+\backslash\Sigma$ we have

$$\left(r_+^\varepsilon\right)\big|_{\partial\Omega_+\backslash\Sigma} = \left(u_+^\varepsilon - w_+^\varepsilon - \chi_{BL} u_{BL}^\varepsilon - \chi_{IL} \widehat{v}_{I,M}^+\right)\big|_{\partial\Omega_+\backslash\Sigma} = 0,$$

by (3), (36) and (18). Finally on Σ we have

$$\left(r_+^\varepsilon - r_-^\varepsilon\right)\big|_\Sigma = \left(u_+^\varepsilon - u_-^\varepsilon - w_-^\varepsilon + w_+^\varepsilon + \chi_{IL} v_{I,M}^- - \chi_{IL} \widehat{v}_{I,M}^+\right)\big|_\Sigma = 0$$

by (5), (54) and

$$\left(\varepsilon^2 \frac{\partial r_+^\varepsilon}{\partial \nu} - \frac{\partial r_-^\varepsilon}{\partial \nu}\right)\bigg|_\Sigma = \left(\varepsilon^2 \frac{\partial u_+^\varepsilon}{\partial \nu} - \frac{\partial u_-^\varepsilon}{\partial \nu} - \frac{\partial w_-^\varepsilon}{\partial \nu} + \varepsilon^2 \frac{\partial w_+^\varepsilon}{\partial \nu} + \frac{\partial v_{I,M}^-}{\partial \nu} - \varepsilon^2 \frac{\partial \widehat{v}_{I,M}^+}{\partial \nu}\right)\bigg|_\Sigma = 0,$$

by (6), (23), (26) and (54). Moreover, we have the following.

Theorem 6 *Let $r^\varepsilon = \left(r_+^\varepsilon, r_-^\varepsilon\right)$ be given by (78)–(79) and let $L_\varepsilon r_+^\varepsilon = -\varepsilon^2 \Delta r_+^\varepsilon + r_+^\varepsilon$ and $L_1 r_-^\varepsilon = -\Delta r_-^\varepsilon + r_-^\varepsilon$. Then there exist constants $K_1, K_2 > 0$ independent of ε, such that*

$$\left\| L_\varepsilon r_+^\varepsilon \right\|_{0,\Omega_+} \lesssim \left(\varepsilon(2M+2)K_1\right)^{2M+2} \tag{80}$$

and

$$\left\| L_1 r_-^\varepsilon \right\|_{0,\Omega_-} \lesssim \left(\varepsilon(2M+2)K_2\right)^{2M+2}. \tag{81}$$

Proof. We first consider (80) and we have

$$\begin{aligned} L_\varepsilon r_+^\varepsilon &= L_\varepsilon \left(u_+^\varepsilon - w_+^\varepsilon - \chi_{BL} u_{BL}^\varepsilon - \chi_{IL} \widehat{v}_{I,M}^+\right) \\ &= L_\varepsilon \left(u_+^\varepsilon - w_+^\varepsilon\right) - L_\varepsilon \left(\chi_{BL} u_{BL}^\varepsilon\right) - L_\varepsilon \left(\chi_{IL} \widehat{v}_{I,M}^+\right). \end{aligned} \tag{82}$$

From (28) we notice that

$$L_\varepsilon\left(u_+^\varepsilon - w_+^\varepsilon\right) = \varepsilon^{2M+2}\Delta^{(M+1)} f_+, \tag{83}$$

and also

$$L_\varepsilon\left(\chi_{BL} u_{BL}^\varepsilon\right) = \varepsilon^2\left(\Delta\chi_{BL}\right) u_{BL}^\varepsilon - 2\varepsilon^2 \nabla\chi_{BL}\cdot\nabla u_{BL}^\varepsilon + \chi_{BL} L_\varepsilon u_{BL}^\varepsilon,$$

where the function χ_{BL} equals 1 for $0 < \rho < \rho_0$ and 0 for $\rho > (\rho_1+\rho_0)/2$. Hence by (47),

$$\left\|\varepsilon^2\left(\Delta\chi_{BL}\right) u_{BL}^\varepsilon\right\|_{0,\Omega_+} \lesssim \varepsilon^2\left(1 + \left(\varepsilon(2M+1)K\right)^{2M+1}\right) e^{-\alpha\rho/\varepsilon}$$

and

$$\left\|\varepsilon^2\nabla\chi_{BL}\cdot\nabla u_{BL}^\varepsilon\right\|_{0,\Omega_+} \lesssim \varepsilon\left(1 + \left(\varepsilon(2M+1)K\right)^{2M+1}\right) e^{-\alpha\rho/\varepsilon},$$

for some appropriate constant $K > 0$. Therefore, by (48) and the previous two inequalities, we obtain

$$\left\|L_\varepsilon\left(\chi_{BL} u_{BL}^\varepsilon\right)\right\|_{0,\Omega_+} \lesssim \left(\varepsilon(2M+2)K\right)^{2M+2}. \tag{84}$$

In a completely analogous way, we may obtain bounds for $L_\varepsilon\left(\chi_{IL}\widehat{v}_{I,M}^+\right)$, viz.

$$\left\|L_\varepsilon\left(\chi_{IL}\widehat{v}_{I,M}^+\right)\right\|_{0,\Omega_+} \lesssim \left(\varepsilon(2M+2)\widehat{K}\right)^{2M+2}, \tag{85}$$

for some appropriate constant $\widehat{K} > 0$. Combining (82)–(85) we have

$$\begin{aligned}
\left\|L_\varepsilon r_+^\varepsilon\right\|_{0,\Omega_+} &\lesssim \left\|\varepsilon^{2M+2}\Delta^{(M+1)} f_+\right\|_{0,\Omega_+} + \left(\varepsilon(2M+2)K\right)^{2M+2} + \left(\varepsilon(2M+2)\widehat{K}\right)^{2M+2} \\
&\lesssim \varepsilon^{2M+2}\gamma_{f_+}^{2M+2}(2M+2)! + \left(\varepsilon(2M+2)K\right)^{2M+2} + \left(\varepsilon(2M+2)\widehat{K}\right)^{2M+2} \\
&\lesssim \left(\varepsilon(2M+2)K_1\right)^{2M+2},
\end{aligned}$$

for a suitable $K_1 > 0$ independent of ε. This establishes (80).

Turning our attention to (81), we have

$$\begin{aligned}
L_1 r_-^\varepsilon &= L_1\left(u_-^\varepsilon - w_-^\varepsilon - \chi_{IL} v_{I,M}^-\right) \\
&= L_1\left(u_-^\varepsilon - w_-^\varepsilon\right) - L_1\left(\chi_{IL} v_{I,M}^-\right).
\end{aligned} \tag{86}$$

We have from (86) (with the aid of (29))

$$L_1 r_-^\varepsilon = -L_1\left(\chi_{IL} v_{I,M}^-\right),$$

hence by (76),

$$\begin{aligned}
\left\|L_1 r_-^\varepsilon\right\|_{0,\Omega_-} &= \left\|L_1\left(\chi_{IL} v_{I,M}^-\right)\right\|_{0\Omega_-} \\
&= \left\|\left(\Delta\chi_{IL}\right) v_{I,M}^- - 2\nabla\chi_{IL}\cdot\nabla v_{I,M}^- + \chi_{IL} L_1 v_{I,M}^-\right\|_{0,\Omega_-} \\
&\le \left\|\left(\Delta\chi_{IL}\right) v_{I,M}^-\right\|_{0,\Omega_-} + \left\|2\nabla\chi_{IL}\cdot\nabla v_{I,M}^-\right\|_{0,\Omega_-}.
\end{aligned}$$

Since the function χ_{IL} equals 1 for $0 < \rho < \rho_\Sigma$ and 0 for $\rho > (\rho_2 + \rho_0)/2$ (cf. (18)), we further get (using (68))

$$\begin{aligned}\left\|L_1 r_-^\varepsilon\right\|_{0,\Omega_-} &\lesssim \sum_{j=0}^{2M+1} \varepsilon^j \left(\left\|V_j^-\right\|_{0,\Omega_-} + \left\|\nabla V_j^-\right\|_{0,\Omega_-}\right) \lesssim \sum_{j=0}^{2M+1} \varepsilon^j j^j \gamma^j \\ &\lesssim \sum_{j=0}^{2M+1} \left(\varepsilon\gamma(2M+1)\right)^j \lesssim \left(\varepsilon K_2(2M+2)\right)^{2M+2}\end{aligned}$$

for a suitable $K_2 > 0$ independent of ε. Thus (81) is established and this completes the proof. ■

Remark 3 *Theorem 6 shows that for εM sufficiently small, the remainder in (10) is exponentially small, hence it need not be approximated. This information will be utilized in the next section when we will construct the approximation to u^ε.*

4 Approximation results

We begin this section with the variational formulation of (1)–(6), which reads: Find $u^\varepsilon = \left(u_+^\varepsilon, u_-^\varepsilon\right) \in H_0^1(\Omega)$ such that

$$B_\varepsilon\left(u^\varepsilon, v\right) = F(v) \;\forall\; v = \left(v_+^\varepsilon, v_-^\varepsilon\right) \in H_0^1(\Omega), \tag{87}$$

where

$$B_\varepsilon\left(u^\varepsilon, v\right) = \int_{\Omega_+} \left\{\varepsilon^2 \nabla\left(u_+^\varepsilon\right) \cdot \nabla\left(v_+^\varepsilon\right) + u_+^\varepsilon v_+^\varepsilon\right\} + \int_{\Omega_-} \left\{\nabla\left(u_-^\varepsilon\right) \cdot \nabla\left(v_-^\varepsilon\right) + u_-^\varepsilon v_-^\varepsilon\right\}, \tag{88}$$

$$F(v) = \int_{\Omega_+} f_+ v_+^\varepsilon + \int_{\Omega_-} f_- v_-^\varepsilon + \int_\Sigma hv. \tag{89}$$

It is straight forward to show that the bilinear form (88) is coercive and continuous on $H_0^1(\Omega)$, hence the variational problem (87) admits a unique solution thanks to the Lax-Milgram lemma. The discrete version of (87) reads: Find $u_N^\varepsilon = \left(u_+^N, u_-^N\right) \in V_N \subset H_0^1(\Omega)$ such that

$$B_\varepsilon\left(u_N^\varepsilon, v\right) = F(v) \;\forall\; v = \left(v_+^\varepsilon, v_-^\varepsilon\right) \in V_N \subset H_0^1(\Omega), \tag{90}$$

and by Céa's Lemma we have

$$\left\|u^\varepsilon - u_N^\varepsilon\right\|_\varepsilon \le \inf_{v \in V_N} \left\|u^\varepsilon - v\right\|_\varepsilon, \tag{91}$$

where the energy norm $\|\cdot\|_\varepsilon$ is defined as

$$\|u\|_\varepsilon^2 = B_\varepsilon(u,u)\,. \tag{92}$$

We now describe the subspace V_N. For simplicity, we will focus on quadrilateral elements, even though triangular elements are also possible (see [7] for this and other choices of a suitable mesh). Since the behavior of the solution u^ε depends on the value of ε (cf. Theorem 1), we distinguish between the cases $\kappa p\varepsilon \geq 1/2$ and $\kappa p\varepsilon < 1/2$ (with $\kappa \in \mathbb{R}$ a fixed constant) as follows: If $\kappa p\varepsilon \geq 1/2$ then the mesh does not need any special design, as in this case the polynomial degree p of the approximating functions is high enough to ensure good approximability. Hence, in this case the mesh Δ only needs to be regular in the sence of [2] (or satisfy conditions M1–M3 in [7]). In the case $\kappa p\varepsilon < 1/2$ the mesh will include elements of size $O(p\varepsilon)$ along $\partial\Omega_+$ in order for the boundary and interface layer effects to be captured – these are referred to as *needle elements* in [7]. We now describe one such possible construction: Let Ω_+^0 be given by (13), and divide $\partial\Omega_+\backslash\Sigma$ into subintervals $(\theta_j,\theta_{j+1})\,, j=1,...,m-1, \theta\in\partial\Omega_+$. Then draw the inward normal at θ_j of length ρ_0 (see eq. (11)) and connect each point $(\rho_j,\theta_j) = (\rho_0,\theta_j)$ using the curve $\rho=\rho_0$ (=constant). Further, divide each

$$\left(\Omega_+^0\right)_j := \left\{(\rho,\theta) : 0 \leq \rho \leq \rho_0, \theta_j \leq \theta \leq \theta_{j+1}\right\}, j=1,...,m \tag{93}$$

into $\left(\Omega_+^{0,1}\right)_j, \left(\Omega_+^{0,2}\right)_j$, where

$$\begin{aligned}\left(\Omega_+^{0,1}\right)_j &= \left\{(\rho,\theta) : \theta_j \leq \theta \leq \theta_{j+1}, 0 \leq \rho \leq \frac{1}{2}\rho_0\kappa p\varepsilon\right\},\\ \left(\Omega_+^{0,2}\right)_j &= \left(\Omega_+^0\right)_j \setminus \left(\Omega_+^{0,1}\right)_j .\end{aligned}$$

In the above definitions, $\kappa\in\mathbb{R}$ is a fixed constant, p is the degree of the approximating polynomials and we recall that we assume $\kappa p\varepsilon < 1/2$. This will define a mesh

$$\Delta_+^0 := \left\{\left(\Omega_+^{0,1}\right)_j, \left(\Omega_+^{0,2}\right)_j\right\}_{j=1}^m$$

over Ω_+^0. We may define a completely analogous mesh Δ_Σ^0 over Ω_Σ^0 (see eq. (15)), as

$$\Delta_\Sigma^0 := \left\{\left(\Omega_\Sigma^{0,1}\right)_k, \left(\Omega_\Sigma^{0,2}\right)_k\right\}_{k=1}^n,$$

with $\left(\Omega_\Sigma^{0,1}\right)_k, \left(\Omega_\Sigma^{0,2}\right)_k$ defined in an analogous way as $\left(\Omega_+^{0,1}\right)_j, \left(\Omega_+^{0,2}\right)_j$. Next, let $\left\{\left(\Omega_+^1\right)_i\right\}_{i=1}^\ell$ be some subdivision of Ω_+^1 that is compatible with Δ_+^0 and Δ_Σ^0, and define the mesh

$$\Delta_+ = \left\{\left(\Omega_\Sigma^{0,1}\right)_k, \left(\Omega_\Sigma^{0,2}\right)_k, \left(\Omega_+^{0,1}\right)_j, \left(\Omega_+^{0,2}\right)_j, \left(\Omega_+^1\right)_i, k=1,...,n, j=1,...,m, i=1,...,\ell\right\}, \tag{94}$$

over Ω_+. The mesh Δ_- over Ω_- is simply be chosen to be compatible with Δ_+ , and regular, in the sense of [2]. The mesh over the entire domain Ω is then taken to be

$$\Delta = \Delta_+ \cup \Delta_-, \tag{95}$$

and we assume that the number of elements in Δ is bounded independently of ε. The above mesh satisfies the definition of a *regular admissible boundary layer mesh* (Definition 3.2 in [7]), which implies the following: With $S := [0,1] \times [0,1]$ the usual reference square, we associate with each quadrilateral $\Omega_i^{\pm} \in \Delta$ a differentiable, bijective element mapping

$$M_i^{\pm} : S \to \overline{\Omega}_i^{\pm},$$

which, in this case, satisfies $\forall\ i$

$$\left\| D^{\alpha} M_i^{\pm} \right\|_{L^{\infty}(S)} \lesssim \gamma^{|\alpha|}\, |\alpha|!\ \forall\ \alpha \in \mathbb{N}_0^2.$$

The space V_N is then defined as

$$V_N = \left\{ u \in H^1(\Omega) : u|_{\Omega_i^{\pm}} = \phi_p \circ \left(M_i^{\pm} \right)^{-1} \text{ for } \phi_p \in Q_p(S) \right\} \cap H_0^1(\Omega), \tag{96}$$

where $Q_p(S)$ denotes the space of all polynomials of degree p in each variable defined on the reference square S. Note that

$$N = \dim V_N = O\left(p^2\right).$$

Now, for $p \geq 1$ we define on the space of continuous function $C([0,1])$, the operator π_p by interpolation in the $p+1$ Gauss-Lobatto points, and on S we introduce the interpolation operator Π_p as the tensor product of the two one-dimensional operators π_p^x and π_p^y. Then, by Lemma 3.8 in [7], we have that for any $u \in C^{\infty}(S)$ with $\|D^{\alpha} u\|_{0,S} \lesssim \gamma^{|\alpha|}\, |\alpha|!\ \forall\ \alpha \in \mathbb{N}_0^2$, there exists a constant $\sigma > 0$ depending only on γ, such that

$$\|u - \Pi_p u\|_{L^{\infty}(S)} + \|\nabla (u - \Pi_p u)\|_{L^{\infty}(S)} \lesssim e^{-\sigma p}. \tag{97}$$

Moreover, there holds (see, e.g., Lemma 3.7 in [7]),

$$\left\{ \begin{array}{c} \|\Pi_p u\|_{L^{\infty}(S)} \lesssim (1+\ln p)^2 \|u\|_{L^{\infty}(S)} \\ \|\partial_x \Pi_p u\|_{L^{\infty}(S)}, \|\partial_y \Pi_p u\|_{L^{\infty}(S)} \lesssim p^2 (1+\ln p)^2 \|u\|_{L^{\infty}(S)} \end{array} \right. . \tag{98}$$

We now prove our main approximation result.

Theorem 7 *Let $u^{\varepsilon} \in H_0^1(\Omega)$, $u_N^{\varepsilon} \in V_N$ be the solutions of (88) and (90), respectively, with V_N defined by (96) on the mesh Δ given by (95). Further, assume that $\partial\Omega$ is analytic and the functions $f_{\pm}$ are analytic on $\Omega_{\pm}$ while the function h is analytic on Σ. Then, for κ sufficiently small, we have*

$$\|u^{\varepsilon} - u_N^{\varepsilon}\|_{\varepsilon} \lesssim N^2 e^{-b\sqrt{N}},$$

for some constant $b > 0$ independent of ε and p.

Proof. We consider the cases $\kappa p\varepsilon > 1/2$ (asymptotic case) and $\kappa p\varepsilon \leq 1/2$ (pre-asymptotic case) separately.

Case 1: $\kappa p\varepsilon > 1/2$ (asymptotic case)

By Theorem 1 there exist constants $C, K > 0$ depending only on the data such that

$$\varepsilon \left\| D^\alpha u_+^\varepsilon \right\|_{0,\Omega_+} + \left\| D^\alpha u_-^\varepsilon \right\|_{0,\Omega_-} \leq C\varepsilon K^{|\alpha|} \max\left\{ |\alpha|, \varepsilon^{-1} \right\}^{|\alpha|} \ \forall\ \alpha = 1, 2, ...$$

Now, by Lemma 3.10 of [7] we have that for each element map $M_i^\pm$,

$$\left\| D^\alpha \left(u_\pm^\varepsilon \circ M_i^\pm \right) \right\|_{0,S} \lesssim \gamma_\pm^{|\alpha|} |\alpha|! e^{1/\varepsilon},$$

for some constants $\gamma_\pm > 0$ independent of ε and i. Hence, by (97) and Lemma 3.6 of [7] we have

$$\left\| \left(u_\pm^\varepsilon \circ M_i^\pm \right) - \Pi_p \left(u_\pm^\varepsilon \circ M_i^\pm \right) \right\|_{L^\infty(S)} + \left\| \nabla \left(\left(u_\pm^\varepsilon \circ M_i^\pm \right) - \Pi_p \left(u_\pm^\varepsilon \circ M_i^\pm \right) \right) \right\|_{L^\infty(S)} \lesssim e^{-\sigma p + 1/\varepsilon}.$$

Since $2\kappa p > 1/\varepsilon$, we have $-\sigma p + 1/\varepsilon \leq -\sigma p + 2\kappa p$ and, under the assumption that $\kappa < \sigma/2$,

$$\left\| \left(u_\pm^\varepsilon \circ M_i^\pm \right) - \Pi_p \left(u_\pm^\varepsilon \circ M_i^\pm \right) \right\|_{L^\infty(S)} + \left\| \nabla \left(\left(u_\pm^\varepsilon \circ M_i^\pm \right) - \Pi_p \left(u_\pm^\varepsilon \circ M_i^\pm \right) \right) \right\|_{L^\infty(S)} \lesssim e^{-bp},$$

for some constant $b > 0$. By (91) and (92) we get the desired result.

Case 2: $\kappa p\varepsilon \leq 1/2$ (pre-asymptotic case)

In this case we utilize the expansion and regularity results of Section 3 which state that the solution $u^\varepsilon = \left(u_+^\varepsilon, u_-^\varepsilon \right)$ can be written as

$$\begin{aligned} u_+^\varepsilon &= w_M^+ + \chi_{BL} u_{BL}^M + \chi_{IL} \widehat{v}_{I,M}^+ + r_+^\varepsilon, \\ u_-^\varepsilon &= w_M^- + \chi_{IL} v_{I,M}^- + r_-^\varepsilon, \end{aligned}$$

with each term defined and analyzed in subsections 3.1–3.4. We begin be selecting M in such a way that εM is sufficiently small for all the regularity results of subsections 3.1–3.4 to hold true. (The lack of concreteness on our part is due to the careful constant selection made in [7], i.e. such a choice for M is possible by [7]). The proof relies on the following observations:

1. The terms $w_M^\pm, v_{I,M}^-$ are analytic in their respective domains, hence (97) may be applied.
2. The construction of the mesh allows us to approximate u_{BL}^M and $\widehat{v}_{I,M}^+$ at an exponential rate.
3. The choice of M renders the term $\|r^\varepsilon\|_{\varepsilon,\Omega}$ negligible (exponentially small), hence the remainder need not be approximated.

Let us first consider item 1 above. For the term $v_{I,M}^{-}$ we have, by Corollary 5, that for each element map M_i^-

$$\left\| D^\alpha \left(v_{I,M}^{-} \circ M_i^- \right) \right\|_{0,S} \lesssim C^{|\alpha|} \, |\alpha|!,$$

hence by (97),

$$\left\| \left(v_{I,M}^{-} \circ M_i^- \right) - \Pi_p \left(v_{I,M}^{-} \circ M_i^- \right) \right\|_{L^\infty(S)} + \left\| \nabla \left(\left(v_{I,M}^{-} \circ M_i^- \right) - \Pi_p \left(v_{I,M}^{-} \circ M_i^- \right) \right) \right\|_{L^\infty(S)} \lesssim e^{-bp}.$$

The same works for the other two terms (the details are ommitted).

Next let us comment on item 2; since the steps are very similar for both u_{BL}^M and $\widehat{v}_{I,M}^{+}$, we will only consider the latter. Without loss of generality we assume that χ_{IL} is 1 in Ω_Σ^0 and 0 otherwise. Hence we only need to approximate $\widehat{v}_{I,M}^{+}$ within Ω_Σ^0. To this end, we note that the mesh in Ω_Σ^0 consists of two types of elements (cf. (94)):

$$\left(\Omega_\Sigma^{0,1} \right)_j = \left\{ (\rho, \theta) : \theta_j \le \theta \le \theta_{j+1}, 0 \le \rho \le \frac{1}{2} \rho_0 \kappa p \varepsilon \right\} \quad \text{and} \quad \left(\Omega_\Sigma^{0,2} \right)_j = \left(\Omega_+^0 \right)_j \setminus \left(\Omega_+^{0,1} \right)_j .$$

For $\left(\Omega_\Sigma^{0,1} \right)_j$, with associated mapping $M_j^{0,1}$, we have from Proposition 3.11 in [7]

$$\left\| D^\alpha \left(\widehat{v}_{I,M}^{+} \circ \psi^{-1} \circ M_j^{0,1} \right) \right\|_{L^\infty(S)} \lesssim e^{\kappa p} K^{|\alpha|} \, |\alpha|! \; \forall \alpha \in \mathbb{N}_0^2,$$

where ψ was defined by (12). Therefore, by (97)

$$\left\| \left(\widehat{v}_{I,M}^{+} \circ \psi^{-1} \circ M_j^{0,1} \right) - \Pi_p \left(\widehat{v}_{I,M}^{+} \circ \psi^{-1} \circ M_j^{0,1} \right) \right\|_{L^\infty(S)} +$$

$$+ \left\| \nabla \left(\left(\widehat{v}_{I,M}^{+} \circ \psi^{-1} \circ M_j^{0,1} \right) - \Pi_p \left(\widehat{v}_{I,M}^{+} \circ \psi^{-1} \circ M_j^{0,1} \right) \right) \right\|_{L^\infty(S)} \lesssim e^{\kappa p} e^{-bp},$$

from which the desired result follows *provided* $\kappa < b$. Now, let us consider the approximation of $\widehat{v}_{I,M}^{+}$ over the elements $\left(\Omega_\Sigma^{0,2} \right)_j$, with associated mapping $M_j^{0,1}$. From (47) we have

$$\left\| \chi_{IL} \widehat{v}_{I,M}^{+} \circ \psi^{-1} \circ M_j^{0,1} \right\|_{L^\infty(S)} \lesssim C_\alpha e^{-\alpha \kappa p} \; , \; \left\| \nabla \left(\chi_{IL} \widehat{v}_{I,M}^{+} \circ \psi^{-1} \circ M_j^{0,1} \right) \right\|_{L^\infty(S)} \lesssim C_\alpha \varepsilon^{-1} e^{-\alpha \kappa p}.$$

Therefore, from (98) we get

$$\left\| \chi_{IL} \widehat{v}_{I,M}^{+} \circ \psi^{-1} \circ M_j^{0,1} \right\|_{L^\infty(S)} \lesssim \left(1 + \ln p \right)^2 e^{-bp}$$

and

$$\left\| \nabla \left(\chi_{IL} \widehat{v}_{I,M}^{+} \circ \psi^{-1} \circ M_j^{0,1} \right) \right\|_{L^\infty(S)} \lesssim \varepsilon^{-1} p^2 \left(1 + \ln p \right)^2 e^{-bp},$$

from which the desired result follows once we use (92).

Turning to item 3, we have from the variational formulation (87)–(89), that the remainder $r^\varepsilon = \left(r_+^\varepsilon, r_-^\varepsilon \right)$ satisfies

$$B_\varepsilon(r^\varepsilon, v) = \int_{\Omega_+} L_\varepsilon r_+^\varepsilon v \, dx + \int_{\Omega_-} L_1 r_-^\varepsilon v \, dx, \quad \forall v \in H_0^1(\Omega).$$

Hence by Cauchy-Schwarz's inequality we get

$$B_\varepsilon(r^\varepsilon, r^\varepsilon) \le \|L_\varepsilon r_+^\varepsilon\|_{0,\Omega_+}^2 + \|L_1 r_-^\varepsilon\|_{0,\Omega_-}^2,$$

and by Theorem 6 we obtain

$$\|r^\varepsilon\|_\varepsilon = [B_\varepsilon(r^\varepsilon, r^\varepsilon)]^{1/2} \lesssim (\varepsilon(2M+2)K)^{2M+2} \lesssim e^{-bp},$$

for some suitable constant $b > 0$, depending only on the data. This completes the proof. ■

5 Numerical results

In this section we will illustrate our theoretical findings for the model problem (1)–(6), in the case when $f_+ = f_- = 1, h = 0$ and the domain Ω consists of the two subdomains Ω_+ and Ω_-, delimited by the three concentric circles with radii 1, 2 and 3. In other words, Ω_+ is the domain inside the two concentric circles of radii 1 and 2, while Ω_- is the domain inside the two concentric circles of radii 2 and 3, as shown in figure 2.

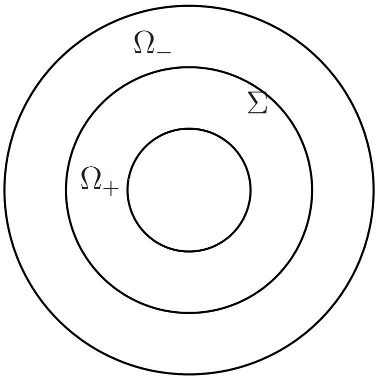

Figure 2: Domains Ω_+ and Ω_- used for the computations.

We expect to have a boundary layer along $\partial\Omega_+ \backslash \Sigma$ (the circle of radius 1) and an interface layer along Σ (the circle of radius 2). The mesh, shown in figure 3, accounts for the presence of the layers by including thin elements of size $p\varepsilon$ along $\partial\Omega_+ \backslash \Sigma$ and Σ – the value of the constant κ appearing in the definition of the mesh in the previous section was taken to be 1 (a value known to produce almost the same results as those obtained with the "optimal " value of κ, see, e.g., [14]). An exact solution is available for this problem, hence our computations are reliable.

The computations were performed with the commercial package StressCheck (E.S.R.D., St. Louis, MO) which is a p-version FEM software package allowing the polynomial degree to

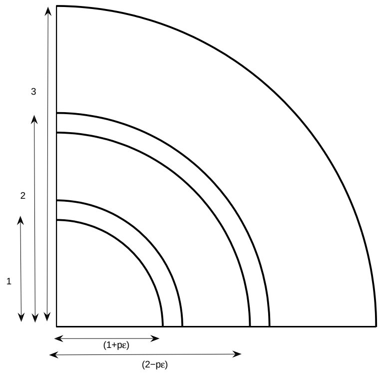

Figure 3: Design of the mesh (on a quarter of the domain).

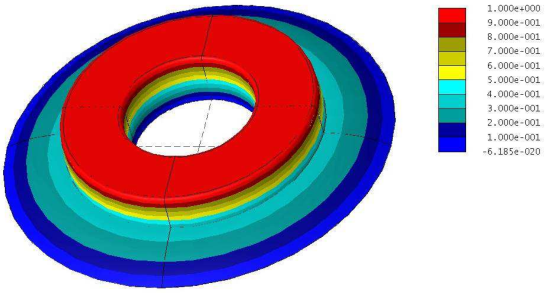

Figure 4: Approximate solution for $p = 8, \varepsilon = 0.01$.

vary from $p = 1$ to $p = 8$ (on a fixed mesh). Figure 4 below shows the approximate solution for $p = 8, \varepsilon = 0.01$, and figure 5 shows the convergence (in the energy norm) as p is increased – the exponential convergence is readily visible.

6 Conclusions

We have studied the finite element approximation of a singularly perturbed transmission problem posed on a (smooth) domain with analytic boundary. Upon obtaining appropriate regularity results, via asymptotic expansions, we were able to design and analyze an *hp* finite element method for the robust approximation of the solution to the singularly perturbed transmission problem. We showed that under the assumption of analytic data, our method converges at an exponential rate, independently of the singular perturbation parameter. This is in line with our one-dimensional results [12], as well as with two-dimensional results for non-

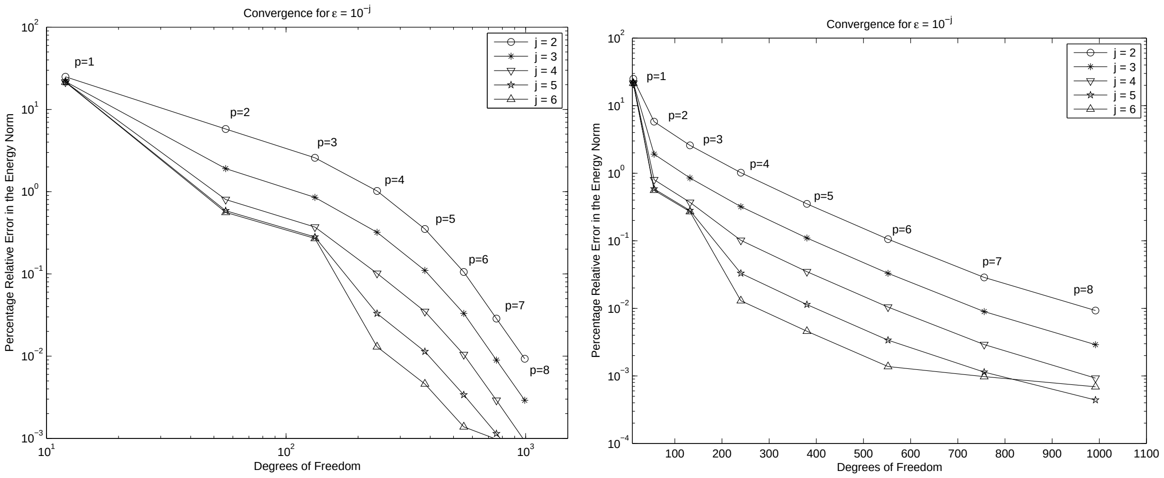

Figure 5: Convergence of the approximate solution: loglog plot(left); semilog plot (right).

transmission problems [7]. The approximation of singularly perturbed transmission problems on non-smooth domains is the focus of our current research efforts.

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