

# Added mass in density-stratified fluids

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## Résumé :

*En fluide stratifié en densité, la masse ajoutée est modifiée par les forces de flottabilité. Sont considérés le mouvement le plus simple – les petites oscillations d'un fluide illimité uniformément stratifié dans l'approximation de Boussinesq – et les corps les plus simples – un cylindre circulaire horizontal et une sphère. Deux masses ajoutées distinctes sont mises en évidence, l'une liée à la pression et l'énergie, l'autre à l'impulsion et au moment dipolaire. Leur application au rayonnement d'ondes internes de gravité est discutée pour des oscillations forcées et libres.*

## Abstract :

*In a density-stratified fluid, the modification of added mass by buoyancy is investigated for the simplest possible flow – the small-amplitude motion of an unbounded Boussinesq uniformly stratified fluid – and the simplest possible bodies – a horizontal circular cylinder and a sphere. Two distinct added masses arise, one involved in pressure and energy and the other in momentum and dipole strength. Their application to the radiation of internal gravity waves by forced and free oscillations is discussed.*

**Mots clefs :** masse ajoutée, ondes internes, flottabilité, stratification

## 1 Introduction

Added mass characterizes the interaction of a body with a fluid flowing around it, or a topography with a fluid flowing over it. One such interaction is the generation of internal gravity waves, called internal tides, by the oscillation of the surface tide over bottom topography in the density-stratified ocean. Internal tides are currently under scrutiny worldwide, as they are thought to dissipate up to 30% of the total surface tidal energy (at a rate of about 1 TW). In this context, the power radiated to the internal tidal field is called the conversion rate and has been the focus of many studies, reviewed in [1], all for specific topographies or under restrictive approximations. We investigate the influence of density stratification on added mass for arbitrary bodies or topographies, and discuss the relation of added mass to energy radiation (including conversion rate) and float oscillation. Application to circular cylinders and spheres is performed explicitly.

## 2 Added mass in homogeneous flow

The concept of added mass pertains to the irrotational flow of a homogeneous fluid past a rigid body [2]. It follows from the linearity of the flow, such that the translation of the body at the velocity  $\mathbf{U}$  creates a velocity potential  $\phi = \phi_i U_i$ . By defining the added mass tensor  $m_{ij}$  according to

$$m_{ij} = \rho \oint_S n_i \phi_j d^2S, \quad (1)$$

with  $\rho$  the density of the fluid,  $S$  the surface of the body and  $\mathbf{n}$  the inward normal to  $S$ , the pressure force on the body may be expressed as

$$F_i = \oint_S p n_i d^2S = -\rho \frac{d}{dt} \oint_S n_i \phi d^2S = -m_{ij} \frac{dU_j}{dt}, \quad (2)$$

the impulse and (kinetic) energy of the fluid as, respectively,

$$I_i = \rho \oint_S n_i \phi d^2S = m_{ij} U_j, \quad E = \frac{1}{2} \rho U_i \oint_S n_i \phi d^2S = \frac{1}{2} m_{ij} U_i U_j, \quad (3)$$

and the dipole strength of the body as

$$\mathcal{D}_i = \oint_S \left( n_i \phi - x_i \frac{\partial \phi}{\partial n} \right) d^2 S = \frac{1}{\rho} (m_f \delta_{ij} + m_{ij}) U_j, \quad (4)$$

with  $\delta_{ij}$  the Kronecker delta symbol,  $V$  the volume of the body and  $m_f = \rho V$  the mass of the displaced fluid. Accordingly, added mass characterizes the flow fully, being involved in the dynamics of the body through  $\mathbf{F}$ , in the global dynamics of the fluid as a whole through  $\mathbf{I}$  and  $E$ , and in the local dynamics of the distant fluid through  $\mathcal{D}$ .

### 3 Added mass in stratified flow

The small-amplitude Boussinesq motion of a uniformly stratified fluid of buoyancy frequency  $N$  can similarly be described in terms of a scalar function  $\chi$  [3, 4, 5], satisfying the internal wave equation

$$\left( \frac{\partial^2}{\partial t^2} \nabla^2 + N^2 \nabla_h^2 \right) \chi = 0, \quad (5)$$

with the  $z$ -axis directed vertically upwards and the subscript  $h$  denoting a horizontal projection. The velocity  $\mathbf{u}$  and the disturbances  $p$  in pressure and  $\rho$  in density are related to the wave function through

$$\mathbf{u} = \left( \frac{\partial^2}{\partial t^2} \nabla + N^2 \nabla_h \right) \chi, \quad p = -\rho_0 \left( \frac{\partial^2}{\partial t^2} + N^2 \right) \frac{\partial \chi}{\partial t}, \quad \rho = \rho_0 \frac{N^2}{g} \frac{\partial^2 \chi}{\partial t \partial z}, \quad (6)$$

with  $g$  the acceleration due to gravity, while the pressure  $p_0$  and density  $\rho_0$  at rest satisfy  $dp_0/dz = -\rho_0 g$  and  $d\rho_0/dz = -\rho_0 N^2/g$ . The hydrodynamic force on a moving body follows immediately as

$$F_i = \oint_S p n_i d^2 S = -\rho_0 \frac{d}{dt} \oint_S n_i \left( \frac{\partial^2}{\partial t^2} + N^2 \right) \chi d^2 S. \quad (7)$$

The momentum and total (kinetic plus potential) energy of the fluid have densities  $\rho_0 u_i$  and  $\frac{1}{2} \rho_0 u_i^2 + \frac{1}{2} \rho_0 N^2 \zeta^2$ , respectively, with  $\zeta = \partial^2 \chi / \partial t \partial z$  the vertical displacement. By integrating the associated fluxes over the surface of the body, the momentum and power outputs of this body are obtained as, respectively,

$$I_i = \rho_0 \oint_S \left( n_i \frac{\partial^2}{\partial t^2} + n_{hi} N^2 \right) \chi d^2 S, \quad P = \rho_0 U_i \frac{d}{dt} \oint_S n_i \left( \frac{\partial^2}{\partial t^2} + N^2 \right) \chi d^2 S. \quad (8)$$

Introduction of the Green's function of the wave equation [6] combined with application of standard techniques [7, 8] yields a Kirchhoff–Helmholtz integral for internal waves, correcting [9] and generalizing [10]. Expansion of this integral at large distances yields the dipole strength

$$\mathcal{D}_i = \oint_S \left[ \left( n_i \frac{\partial^2}{\partial t^2} + n_{hi} N^2 \right) \chi - x_i \left( \frac{\partial^2}{\partial t^2} \frac{\partial}{\partial n} + N^2 \frac{\partial}{\partial n_h} \right) \chi \right] d^2 S. \quad (9)$$

Now, the linearity of the wave equation implies a linear relation between the velocity  $\mathbf{U}$  of a rigid body and the wave function that it creates, in the form of a temporal convolution  $\chi = \chi_i * U_i$ . Two distinct definitions of added mass arise: one,

$$m_{ij}^{(1)} = \rho_0 \oint_S n_i \left( \frac{\partial^2}{\partial t^2} + N^2 \right) \chi_j d^2 S, \quad (10)$$

involved in pressure and power through

$$F_i = -m_{ij}^{(1)} * \frac{dU_j}{dt}, \quad P = U_i \left[ m_{ij}^{(1)} * \frac{dU_j}{dt} \right], \quad (11)$$

and the other,

$$m_{ij}^{(2)} = \rho_0 \oint_S \left( n_i \frac{\partial^2}{\partial t^2} + n_{hi} N^2 \right) \chi_j d^2 S, \quad (12)$$

involved in momentum and dipole strength through

$$I_i = m_{ij}^{(2)} * U_j, \quad \mathcal{D}_i = \frac{1}{\rho_0} \left[ m_f \delta_{ij} \delta(t) + m_{ij}^{(2)} \right] * U_j, \quad (13)$$

with  $\delta(t)$  the Dirac delta function and  $m_f = \rho_0 V$  the mass of the displaced fluid. The relation between the two is more easily analysed in the monochromatic case, when the excitation  $\mathbf{U}$  and the responses  $\mathbf{u}$ ,  $p$  and  $\rho$  depend on time through the factor  $e^{-i\omega t}$  which is suppressed in the following. Added masses are replaced by their temporal Fourier transforms and the above relations simplify to

$$F_i = i\omega m_{ij}^{(1)} U_j, \quad \langle P \rangle = \frac{\omega}{2} \text{Im} \left[ m_{ij}^{(1)} \overline{U_i} U_j \right], \quad I_i = m_{ij}^{(2)} U_j, \quad \mathcal{D}_i = \frac{1}{\rho_0} \left[ m_f \delta_{ij} + m_{ij}^{(2)} \right] U_j, \quad (14)$$

with  $\bar{\phantom{x}}$  denoting a complex conjugate and  $\langle \phantom{x} \rangle$  a time average, so that

$$m_{ij}^{(1)} = \left( 1 - \frac{N^2}{\omega^2} \delta_{i3} \right) m_{ij}^{(2)}. \quad (15)$$

As did [11, 12, 13], in the following we will consider the first definition only and omit the superscript <sup>(1)</sup>.

## 4 Oscillating bodies

We consider two particular oscillating bodies, a horizontal circular cylinder and a sphere, of radius  $a$ , oscillating with surface velocity distribution  $\mathbf{U}(\mathbf{x})$ . We represent them by source terms  $q = \sigma \delta_S$  in the wave equation [14], namely by surface distributions of singularities of density  $\sigma(\mathbf{x})$ . The condition of fixed normal velocity  $U_n$  on  $S$  becomes an integral equation for  $\sigma$ , namely, for all  $\mathbf{x} \in S$ ,

$$U_n(\mathbf{x}) = \frac{1}{4\pi(\omega^2 - N^2)^{1/2}} \left( \omega^2 \frac{\partial}{\partial n} - N^2 \frac{\partial}{\partial n_h} \right) \oint_S \frac{\sigma(\mathbf{x}')}{[\omega^2(\mathbf{x} - \mathbf{x}')^2 - N^2(z - z')^2]^{1/2}} d^2 S'. \quad (16)$$

It is solved by the combination of stretched orthogonal curvilinear coordinates [15, 16] and eigenfunction expansion [17]. Specifically, for the sphere, of radial velocity  $U_r(\theta, \varphi)$  with  $\theta$  the colatitude and  $\varphi$  the azimuth, we consider frequencies  $\omega > N$  and introduce stretched oblate spheroidal coordinates  $(\xi, \eta, \varphi)$  by

$$r_h = \frac{N}{\omega} a \cosh \xi \sin \eta, \quad z = \frac{N}{(\omega^2 - N^2)^{1/2}} a \sinh \xi \cos \eta, \quad (17)$$

with  $r_h = |\mathbf{x}_h|$ . The sphere is turned into the surface  $\xi = \xi_0 = \text{arccosh}(\omega/N)$  while the kernel of the integral equation is turned into Coulomb's potential, expanded in spherical harmonics  $Y_{lm}(\eta, \varphi)$  as

$$\frac{1}{[\omega^2(\mathbf{x} - \mathbf{x}')^2 - N^2(z - z')^2]^{1/2}} = i \frac{4\pi}{Na} \sum_{l=0}^{\infty} \sum_{m=-l}^l (-1)^m \frac{(l-m)!}{(l+m)!} \times P_{lm}(i \sinh \xi_{<}) Q_{lm}(i \sinh \xi_{>}) Y_{lm}(\eta, \varphi) \overline{Y_{lm}(\eta', \varphi')}, \quad (18)$$

with  $P_{lm}$  and  $Q_{lm}$  associated Legendre functions and  $\xi_{<}$  ( $\xi_{>}$ ) the smaller (larger) of  $\xi$  and  $\xi'$ . The solution of the problem is then immediate, in the form

$$\sigma(\theta, \varphi) = \frac{N^2}{\omega^2} \sum_{l=0}^{\infty} \sum_{m=-l}^l (-1)^m \frac{(l+m)!}{(l-m)!} \frac{U_{lm}}{P_{lm}(i \sinh \xi_0) Q'_{lm}(i \sinh \xi_0)} Y_{lm}(\theta, \varphi), \quad (19)$$

with  $U_{lm} = \iint U_r(\theta, \varphi) \overline{Y_{lm}(\theta, \varphi)} \sin \theta d\theta d\varphi$ . Causality allows its analytic continuation onto the upper half of the complex  $\omega$ -plane, which amounts to replacing  $\omega$  by  $\omega + i0$  namely to adding to the real  $\omega$  a positive imaginary part which is later allowed to tend to zero.

For rigid oscillations at the uniform velocity  $\mathbf{U}$ , only the dipolar terms  $l = 1$  remain. We obtain

$$q(\mathbf{x}) = \left[ \frac{2}{1 + B(\omega/N)} \mathbf{U}_h \cdot \frac{\mathbf{x}_h}{a} + \frac{1}{1 - B(\omega/N)} U_z \frac{z}{a} \right] \delta(r - a), \quad (20)$$

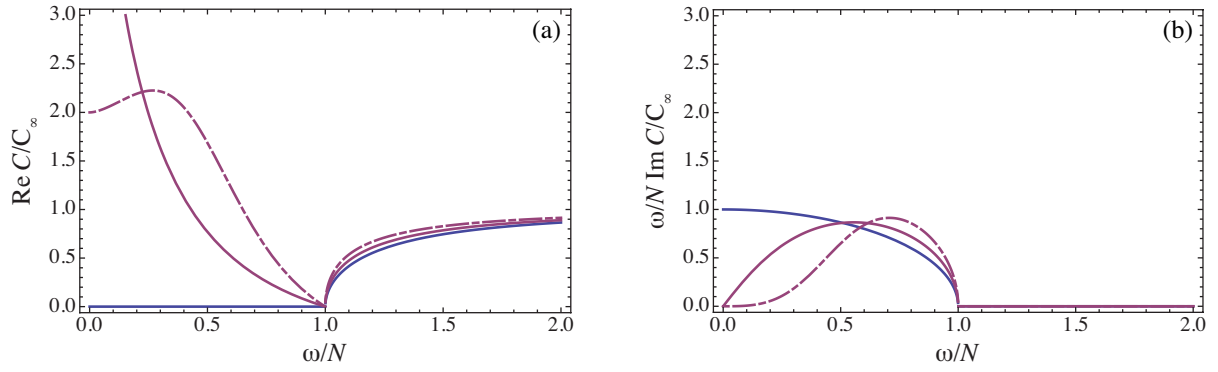


FIG. 1 – Coefficients of (a) added mass and (b) damping for a circular cylinder (—) and a sphere oscillating either vertically (—) or horizontally (---).

where  $r = |\mathbf{x}|$  and

$$B\left(\frac{\omega}{N}\right) = \frac{\omega^2}{N^2} \left[ 1 - \left( \frac{\omega^2}{N^2} - 1 \right)^{1/2} \arcsin\left(\frac{N}{\omega}\right) \right], \quad (21)$$

with analytic continuation, at the frequencies  $|\omega| < N$  of propagative internal waves,

$$B\left(\frac{\omega}{N}\right) = \frac{\omega^2}{N^2} \left\{ 1 - \left( 1 - \frac{\omega^2}{N^2} \right)^{1/2} \left[ \operatorname{arccosh}\left(\frac{N}{|\omega|}\right) + i\frac{\pi}{2} \operatorname{sign} \omega \right] \right\}. \quad (22)$$

The circular cylinder is treated in the same way, with elliptical coordinates replacing spheroidal coordinates, and trigonometric functions replacing spherical harmonics. We obtain, for rigid oscillations,

$$q(\mathbf{x}) = \left\{ \left[ 1 + \left( 1 - \frac{N^2}{\omega^2} \right)^{1/2} \right] U_x \frac{x}{a} + \left[ 1 + \left( 1 - \frac{N^2}{\omega^2} \right)^{-1/2} \right] U_z \frac{z}{a} \right\} \delta(r - a), \quad (23)$$

with continuation  $(1 - N^2/\omega^2)^{1/2} = i(N^2/\omega^2 - 1)^{1/2} \operatorname{sign} \omega$  at  $|\omega| < N$ . In the limit  $\omega/N \rightarrow \infty$  the effect of the stratification vanishes and the classical results  $q(\mathbf{x}) = (3/2)\mathbf{U} \cdot (\mathbf{x}/a)\delta(r - a)$  for the sphere and  $2\mathbf{U} \cdot (\mathbf{x}/a)\delta(r - a)$  for the cylinder in a homogeneous fluid [14] are recovered.

The associated added mass coefficients  $C_{ij} = m_{ij}/m_f$ , deduced from  $\mathcal{D}_i = \int x_i q(\mathbf{x}) d^3x = V[\delta_{ij} + C_{ij}/(1 - \delta_{i3}N^2/\omega^2)]U_j$ , are complex. Their real part represents added inertia and their imaginary part, only present for  $|\omega| < N$ , represents wave damping. For the cylinder and the sphere only the diagonal coefficients are nonzero. They are given for the cylinder by

$$C_x = C_z = \left( 1 - \frac{N^2}{\omega^2} \right)^{1/2}, \quad (24)$$

and for the sphere by

$$C_h = \frac{1 - B(\omega/N)}{1 + B(\omega/N)}, \quad C_z = \left( 1 - \frac{N^2}{\omega^2} \right) \frac{B(\omega/N)}{1 - B(\omega/N)}. \quad (25)$$

They coincide with those obtained by direct solution of the equations of motion in [11, 13] for the sphere and [18, 19] for the cylinder. Their variations, represented in figure 1, have been confirmed by the experiments in [20, 21] for horizontal oscillations. In the limit  $\omega/N \rightarrow \infty$  the values  $C_\infty = 1$  for the cylinder and  $1/2$  for the sphere in a homogeneous fluid are recovered.

## 5 Forced oscillations

A first application is the power output  $\langle P \rangle$  of a body oscillating with displacement amplitude  $A$  at the angle  $\alpha$  to the vertical (becoming a conversion rate for  $\alpha = \pi/2$ ). The calculation of  $\langle P \rangle$  usually involves complicated integration [5, 22, 23], made unnecessary by added mass. Assuming symmetry around the vertical and about the horizontal, so that  $C_{11} = C_{22} = C_h$  and  $C_{33} = C_z$  with all other coefficients zero, we have simply

$$\langle P \rangle = \frac{1}{2} m_f \omega^3 A^2 \operatorname{Im} \left[ C_h \sin^2 \alpha + C_z \cos^2 \alpha \right]. \quad (26)$$

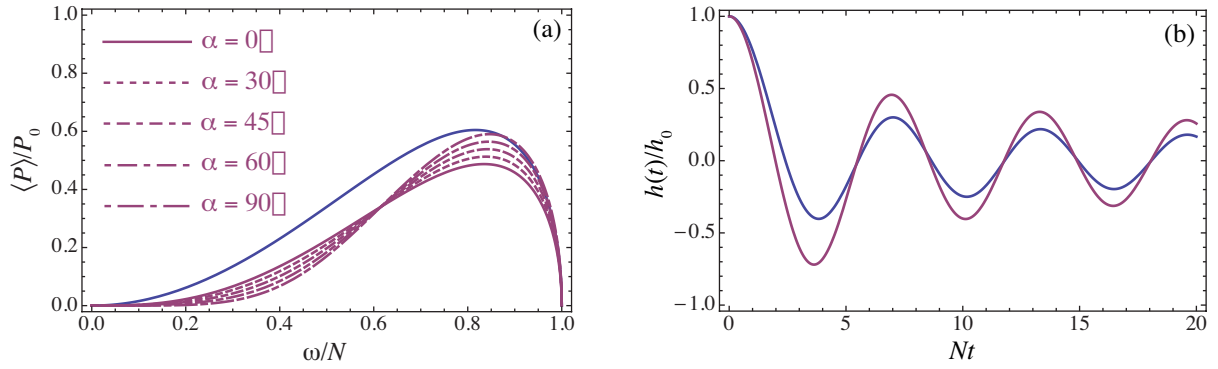


FIG. 2 – For a circular cylinder (—) and a sphere (—), (a) power output of forced oscillations, normalized by  $P_0 = \rho_0 N^3 a^2 A^2$  for the cylinder and  $\rho_0 N^3 a^3 A^2$  for the sphere, and (b) free buoyant oscillations.

For  $\omega > N$ , the waves are evanescent and no power is radiated. For  $0 < \omega < N$ , the waves are propagative and the power output is for the cylinder, per unit length along its span,

$$\langle P \rangle = \frac{\pi}{2} \rho_0 a^2 A^2 \omega^2 (N^2 - \omega^2)^{1/2}, \quad (27)$$

consistent with direct solution of the equations of motion in [18], and for the sphere

$$\langle P \rangle = \frac{\pi^2}{3} \rho_0 a^3 A^2 \omega^3 \left(1 - \frac{\omega^2}{N^2}\right)^{1/2} \left[ 2 \frac{\omega^2}{N^2} \frac{\sin^2 \alpha}{|1 + B(\omega/N)|^2} + \left(1 - \frac{\omega^2}{N^2}\right) \frac{\cos^2 \alpha}{|1 - B(\omega/N)|^2} \right]. \quad (28)$$

Its variations with  $\omega$  are represented in figure 2a, consistent with experimental measurements for horizontal oscillations [20, 21]. It is a maximum at  $\omega/N = \sqrt{2/3} \approx 0.82$  for the cylinder and  $\omega/N$  varying weakly, between 0.84 and 0.85, with  $\alpha$  for the sphere.

Consider now an assembly of incoherent elementary excitations of approximately constant size, oscillating with approximately fixed excursion in random directions. In a uniformly stratified fluid, this assembly will generate waves with a power spectrum peaked at  $\omega/N \approx 0.8$ . If a region of disorganized turbulent motion can indeed be considered as an assembly of this type, then the above might explain the peaks observed at  $\omega/N = 0.8$  and 0.7 for two-dimensional regions of mixed fluid collapsing after release at [24] and above [25] their neutral buoyancy level, respectively.

## 6 Free oscillations

A second application is the free buoyant oscillations of a body displaced slightly from its neutral buoyancy level then released abruptly. Under an arbitrary external force  $\mathbf{F}_e$ , the position  $\mathbf{X} = (X, Y, Z) = (X_1, X_2, X_3)$  of the centroid of the body relative to this level satisfies the equation

$$m \frac{d^2 X_i}{dt^2} = -m N^2 \delta_{i3} Z(t) - m_{ij}(t) * \frac{d^2 X_j}{dt^2} + F_{e_i}(t), \quad (29)$$

with  $\rho_0$  the density at rest at the neutral buoyancy level and  $m = \rho_0 V$  the mass of the body. The first term on the right-hand side represents the hydrostatic force, namely the combination of weight  $-mg\mathbf{e}_z$  and Archimedes' force  $\oint_S p_0 \mathbf{n} d^2 S = m_f g \mathbf{e}_z \sim mg\mathbf{e}_z - mN^2 Z \mathbf{e}_z$ , and the second term the hydrodynamic force  $\oint_S p \mathbf{n} d^2 S$ . By temporal Fourier transformation we obtain, in terms of the added mass coefficients,

$$\{\omega^2 [\delta_{ij} + C_{ij}(\omega)] - N^2 \delta_{i3} \delta_{ij}\} X_j(\omega) = -\frac{F_{e_i}(\omega)}{m}. \quad (30)$$

For release at  $t = 0$  after initial displacement  $h_0$ , the external force is  $mN^2 h_0 H(-t)\mathbf{e}_z$  with  $H(t)$  the Heaviside step function. Assuming the body to be symmetric around the vertical so that only the coefficient  $C_{33} = C_z$  comes into play, and writing the position of the body as  $Z(t) = h_0 H(-t) + h(t)$  with  $h(t)$  causal, we have

$$\frac{h(\omega)}{h_0} = \frac{i}{\omega} \frac{1 + C_z(\omega)}{1 + C_z(\omega) - N^2/\omega^2}, \quad (31)$$

yielding for the cylinder and the sphere, respectively,

$$\frac{h(\omega)}{h_0} = \frac{i}{(\omega^2 - N^2)^{1/2}}, \quad \frac{h(\omega)}{h_0} = \frac{i}{(\omega^2 - N^2)^{1/2}} \frac{\omega}{N} \arcsin\left(\frac{N}{\omega}\right), \quad (32)$$

and by Fourier inversion

$$\frac{h(t)}{h_0} = H(t)J_0(Nt), \quad \frac{h(t)}{h_0} = \frac{\pi}{2} H(t)E_1(Nt), \quad (33)$$

with  $J_0$  a Bessel function and  $E_1$  a Weber function. The oscillations, represented in figure 2b, are consistent with direct solution of the equations of motion and experiments for small initial displacement in [26]. Experiments for larger initial displacements [27, 28] have pointed out the importance of viscous damping, either laminar or turbulent, and the topic remains an area of active research, both theoretically [29] and experimentally [30, 31, 32].

## 7 Conclusion

The concept of added mass has been generalized to the small Boussinesq oscillations of a density-stratified fluid. Two distinct added masses arise, one associated with pressure and energy and the other with impulse and dipole strength. The added mass tensor loses its symmetry and becomes frequency-dependent. The power output of an oscillating body in a fluid at rest, or the conversion rate of an oscillatory flow over a fixed obstacle, are expressed in terms of added mass coefficients alone, as are the buoyant oscillations of a float.

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