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Two optimality results about sample path properties of Operator Scaling Gaussian Random Fields

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We study the sample paths properties of Operator scaling Gaussian random fields. Such fields are anisotropic generalizations of self-similar fields. Some characteristic properties of the anisotropy are revealed by the regularity of the sample paths. The sharpest way of measuring smoothness is related to these anisotropies and thus to the geometry of these fields.

1 Introduction and motivations

Random fields are now used for modeling in a wide range of scientific areas including physics, engineering, hydrology, biology, economics and finance (see [29] and its bibliography). An important requirement is that the data thus modelled present strong anisotropies which therefore have to be present in the model. Many anisotropic random fields have therefore been proposed as natural models in various areas such as image processing, hydrology, geostatistics and spatial statistics (see, for example, Davies and Hall ([14]), Bonami and Estrade ([4]), Benson and al.([3])). Let us also quote the example of Levy random fields, deeply studied by Durand and Jaffard (see [16]), which is the only known model of anisotropic multifractal random field. In many cases, Gaussian models have turned to be relevant when investigating anisotropic problems. For example the stochastic model of surface waves is usually assumed to be Gaussian and is surprisingly accurate (see [21]). More generally anisotropic Gaussian random fields are involved in many others concrete situations and then arise naturally in stochastic partial differential equations (see, e.g., Dalang [13], Mueller and Tribe [22], Ôksendal and Zhang [25], Nualart [24]).

In many situations, the data present invariant features across the scales (see for example [1]). These two requirements (anisotropy and self–similarity) may seem contradictory, since the classical notion of self-similarity defined for a random field $\{X(x)\}_{x\in\mathbb{R}^d}$ on $\mathbb{R}^d$ by

$$\{X(ax)\}_{x\in\mathbb{R}^d} \overset{\mathcal{L}}{=} \{a^{H_0}X(x)\}_{x\in\mathbb{R}^d}, \tag{1}$$

for some $H_0\in\mathbb{R}$ (called the Hurst index) is by construction isotropic and has then to be changed in order to fit anisotropic situations. To this end, several extensions of self-similarity property in an anisotropic setting have been proposed. In [19], Hudson and Mason defined operator self-similar processes $\{X(t)\}_{t\in\mathbb{R}}$ with values in $\mathbb{R}^d$.

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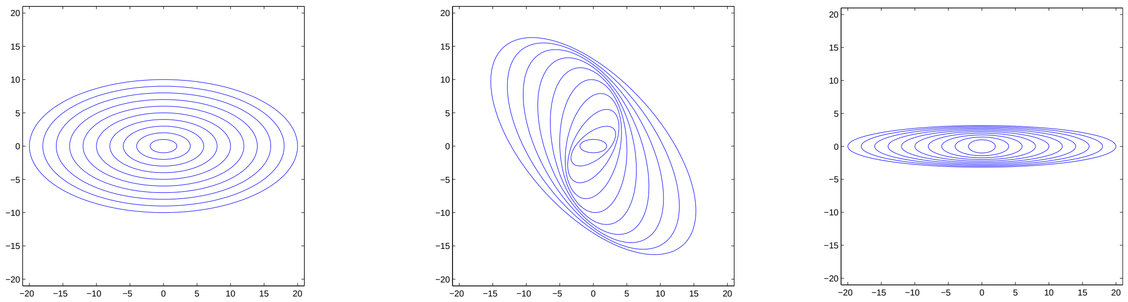
In [20], Kamont introduced Fractional Brownian Sheets which satisfies different scaling properties according the coordinate axes. More recently, in [8] Biermé, Meerschaert and Scheffler introduced the notion of Operator Scaling Random Fields (OSRF). These fields satisfy the following anisotropic scaling relation :

$$\{X(a^{E_0}x)\}_{x\in\mathbb{R}^d} \overset{\mathcal{L}}{=} \{a^{H_0}X(x)\}_{x\in\mathbb{R}^d}. \tag{2}$$

for some matrix E_0 (called an exponent or an anisotropy of the field) whose eigenvalues have a positive real part and some $H_0 > 0$ (called an Hurst index of the field). The usual notion of self-similarity is extended replacing **usual scaling**, (corresponding to the case $E_0 = Id$) by **a linear scaling** involving the matrix E_0 (see figure 1 below). It allows to define new classes of random fields with new geometry and structure.



$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \lambda \in \{1, \cdots, 10\}$ $E = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \lambda \in \{1, \cdots, 10\}$ $E = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}, \lambda \in \{1, \cdots, 10\}$

Fig. 1

Action of a linear scaling $x \mapsto \lambda^E x$ on the smallest ellipsis.

This new class of random fields have been introduced in order to model various phenomena such as fracture surfaces (see [27]) or sedimentary aquifers (see [3]). Furthermore in [8], the authors construct a large class of Operator Scaling Stable Random Fields with stationary increments presenting both a moving average and an harmonizable representation of these fields.

In order to use such models in practice, the first problem is to recover the parameters H_0 and E_0 from the inspection of one sample paths. Even if we consider the model mentioned above in the Gaussian case, the problem of identification of an exponent of self-similarity E_0 (which in some case is not unique) and of an Hurst index H_0 is an open problem.

The first step in the resolution of this question involves an identification of some specific features of exponents and indices which can be recovered on sample paths. This paper is a first step : we will prove that from the regularity point of view these exponents and Hurst indices satisfy what we call optimality properties. More precisely, we prove that (see Theorems 4.3 and 4.2), the Hurst index H_0 maximizes the local critical exponent of the field in specific functional spaces related with the anisotropy matrix E_0 among all possible critical exponents in general anisotropic functional spaces.

Therefore, the results of the present paper open the way to the following strategy to recover the Hurst index. One first have to consider a discretized version of the set of all possible anisotropies. In each case an estimator of the critical exponent related with these anisotropies has to be given. Therefore, one has to locate the maximum of all these estimators–which can be based on anisotropic quadratic variations–and to identify the corresponding values of the anisotropy. The problem can thus be reformulated in terms of finding extreme values of some multivariate Gaussian series related to the set of discrete anisotropies (see [17, 28] for some references about extremes and [32] for some reference about extremes of multivariate series). The study of these estimators from a statistical point of view will be the purpose of a forthcoming paper.

Our two optimality results come from sample paths properties of the model under study in an anisotropic setting. This approach is natural : In [20], Kamont studied the regularity of the sample paths of the well-known anisotropic Fractional Brownian Sheet in anisotropic Hölder spaces related to Fractional Brownian Sheet. Moreover, some results of regularity in specific anisotropic Hölder spaces related to matrix E_0 have already be established for operator scaling self-similar random fields (which may be not Gaussian) in [7] or in the more general setting of strongly non deterministic anisotropic Gaussian fields in [36]. We then extend already existing results by measuring smoothness in general anisotropic spaces not necessarily related to the exponent matrix E_0 of the field.

This paper is organized as follows. In Section 2, we briefly recall some facts about Operator Scaling Random Gaussian Fields (OSRGF) and describe the construction of [8] of the model. In Section 3, we present the different concepts used for measuring smoothness in an anisotropic setting and especially anisotropic Besov spaces. Section 4 is devoted to the statement of our optimality and regularity results. Finally, Section 5 contains proofs of the results stated in Section 4.

For any matrix M let us define

$$\rho_{\min}(M) = \min_{\lambda \in Sp(M)} (|Re(\lambda)|),\ \rho_{\max}(M) = \max_{\lambda \in Sp(M)} (|Re(\lambda)|).$$

where $Sp(M)$ denotes the spectrum of matrix M.
For any real $a > 0$, a^M denotes the matrix

$$a^M = \exp(M \log(a)) = \sum_{k \geq 0} \frac{M^k \log^k(a)}{k!}.$$

In the following pages, we denote $\mathcal{E}^+$ the collection of matrices of $M_d(\mathbb{R})$ whose eigenvalues have positive real part.

2 Presentation of the studied model

The existence of operator scaling stable random fields, that is random fields satisfying relationship (2), is proved in [8]. The following Theorem (Theorem 4.1 and Corollary 4.2 of [8]) completes this result by yielding a practical way to construct a Operator Scaling Stable Random Field (OSRF) with stationary increments for any $E_0 \in \mathcal{E}^+$ and $H_0 \in (0, \rho_{\min}(E_0))$. We state it only in the Gaussian case having in mind the problem of the estimation of the Hurst index H_0 and the anisotropy E_0.

Theorem 2.1 *Let E_0 be in $\mathcal{E}^+$ and ρ a continuous function with positive values such that for all $x \neq 0$, $\rho(x) \neq 0$. Assume that ρ is E_0^t-homogeneous that is :*

$$\forall a > 0,\ \forall \xi \in \mathbb{R}^d,\ \rho(a^{E_0^t}\xi) = a\rho(\xi).$$

Then the Gaussian field

$$X_\rho(x) = \int_{\mathbb{R}^d} (e^{i<x,\xi>} - 1)\rho(\xi)^{-H_0 - \frac{Tr(E_0)}{2}} d\widehat{W}(\xi), \tag{3}$$

exists and is stochastically continuous if and only if $H_0 \in (0, \rho_{\min}(E_0))$. Moreover this field has the following properties :

1. *Stationary increments :*

$$\forall h \in \mathbb{R}^d,\ \{X_\rho(x+h) - X_\rho(h)\}_{x \in \mathbb{R}^d} \overset{(fd)}{=} \{X_\rho(x)\}_{x \in \mathbb{R}^d}$$

2. *The operator scaling relation (2) is satisfied.*

Remark 2.2 The assumption of homogeneity on the function ρ is necessary to recover linear self-similarity properties of the Gaussian field $\{X_\rho(x)\}_{x\in\mathbb{R}^d}$. The assumption of continuity on ρ allows to ensure that the constructed field is stochastically continuous.

Remark 2.3 In general, the couple (H_0, E_0) of an OSRF is not unique. Indeed, if H_0 and E_0 are respectively an Hurst index and an exponent of the OSRF $\{X(x)\}_{x\in\mathbb{R}^d}$, then for any $\lambda > 0$ so do λH_0 and λE_0.
Uniqueness of the Hurst index H_0 can be recovered by choosing the following normalization for $E_0 : Tr(E_0) = d$. However, even under this assumption, E_0 is not necessarily unique. Nevertheless remark that the real diagonalizable real part of the matrix E_0 is unique (see Section 5.2 for a definition). We refer to Remark 2.10 of [8] for more details on the structure of the set of exponents of an OSRF.

Remark that Theorem 2.1 relies on the existence of E_0^t homogeneous functions. Constructions of such functions have been proposed in [8] via an integral formula (Theorem 2.11). An alternative construction which is more fitted for numerical simulations can be found in [12].

3 Anisotropic concepts of smoothness

Our main goal here is to study the sample paths properties of this class of Gaussian fields in well adapted anisotropic functional spaces. This approach is quite natural (see [7, 20]) since the studied model is anisotropic. To this end, suitable concepts of anisotropic smoothness are needed. The aim of this Section is to give some background about the appropriate anisotropic functional spaces : Anisotropic Besov spaces. These spaces generalize classical (isotropic) Besov spaces and have been studied in parallel with them (see [5, 9] for a complete account on the results presented in this Section. The definition of anisotropic Besov spaces is based on the concept of pseudo-norm. We first recall some well known facts about pseudo-norms which can be found with more details in [26].

3.1 Preliminary results about pseudo-norms

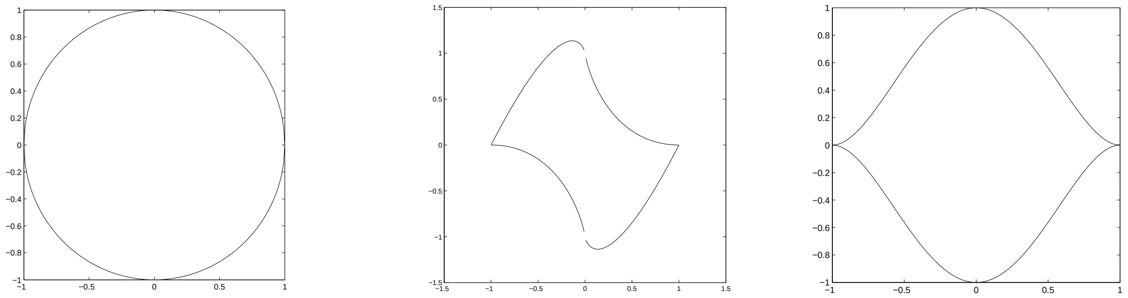
In order to introduce anisotropic functional spaces, an anisotropic topology on $\mathbb{R}^d$ is needed. We need to introduce a slight variant of the notion introduced by Lemarie in [26] since the one used in [26] is fitted to the case of discrete dilatations.

Definition 3.1 Let $E \in \mathcal{E}^+$. A function ρ defined on $\mathbb{R}^d$ is a $(\mathbb{R}^d, E)$ pseudo-norm if it satisfies the three following properties :

1. ρ is continuous on $\mathbb{R}^d$,
2. ρ is E-homogeneous, *i.e.* $\rho(a^E x) = a\rho(x) \quad \forall x \in \mathbb{R}^d, \forall a > 0,$
3. ρ is strictly positive on $\mathbb{R}^d \setminus \{0\}$.

For any $(\mathbb{R}^d, E)$ pseudo-norm, define the anisotropic sphere $S_0^E(\rho)$ as

$$S_0^E(\rho) = \{x \in \mathbb{R}^d;\, \rho(x) = 1\}. \tag{4}$$



$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $E = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $E = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$

Fig. 2

Examples of spheres $S_0^E(\rho)$.

Proposition 3.2 *For all* $x \in \mathbb{R}^d \setminus \{0\}$*, there exists an unique couple* $(r, \theta) \in \mathbb{R}_+^* \times S_0^{E_0}(\rho)$ *such that* $x = r^{E_0}\theta$*. Moreover* $S_0^{E_0}(\rho)$ *is a compact of* $\mathbb{R}^d$ *and the map*

$$(r, \theta) \to x = r^{E_0}\theta$$

is an homeomorphism from $\mathbb{R}_+^* \times S_0^{E_0}(\rho)$ *to* $\mathbb{R}^d \setminus \{0\}$.

The term ”pseudo-norm” is justified by the following Proposition :

Proposition 3.3 *Let* ρ *a* $(\mathbb{R}^d, E)$ *pseudo-norm. There exists a constant* $C > 0$ *such that*

$$\rho(x + y) \leq C(\rho(x) + \rho(y)), \quad \forall x,\, y \in \mathbb{R}^d.$$

The following key property allows to define an anisotropic topology on $\mathbb{R}^d$ based on pseudo-norms and then anisotropic functional spaces :

Proposition 3.4 *Let* ρ_1 *and* ρ_2 *be two* $(\mathbb{R}^d, E)$ *pseudo-norms. They are equivalent in the following sense : There exists a constant* $C > 0$ *such that*

$$\frac{1}{C}\rho_1(x) \leq \rho_2(x) \leq C\rho_1(x), \quad \forall x \in \mathbb{R}^d.$$

In particular, any topologies on $\mathbb{R}^d$ *related with two different* $(\mathbb{R}^d, E)$ *pseudo-norms are equivalent.*

3.2 Anisotropic Besov spaces

Let E be a matrix with positive real part of the eigenvalues. Let us fix a $(\mathbb{R}^d, E^*)$-pseudo-norm, denoted by $|\cdot|_{E^*}$. For $x_0 \in \mathbb{R}^d$ and $r > 0$, $B_{|\cdot|_{E^*}}(x_0, r)$ denotes the anisotropic ball of center x_0 and radius r

$$B_{|\cdot|_{E^*}}(x_0, r) = \{x \in \mathbb{R}^d, |x - x_0|_{E^*} \leq r\}.$$

Definition 3.5 Let $\psi_0^E \in \mathcal{S}(\mathbb{R}^d)$ be such that

$$\widehat{\psi_0^E}(\xi) = 1 \text{ if } |\xi|_{E^*} \leq 1, \widehat{\psi_0^E}(\xi) = 0 \text{ if } |\xi|_{E^*} \geq 2.$$

For $j \in \mathbb{N}$, let

$$\widehat{\psi_j^E}(\xi) = \widehat{\psi_0^E}(2^{-jE^*}\xi) - \widehat{\psi_0^E}(2^{-(j-1)E^*}\xi).$$

Then

$$\sum_{j=0}^{+\infty} \widehat{\psi_j^E} \equiv 1,$$

is an anisotropic partition of the unity with $supp(\widehat{\psi_j^E}) \subset B_{|\cdot|_{E^*}}(0, 2^{j+1}) \setminus B_{|\cdot|_{E^*}}(0, 2^{j-1})$.

The anisotropic Besov spaces $B^s_{p,q}(\mathbb{R}^d, E)$ are then defined as follows.

Definition 3.6 Let $0 < p, q \leq \infty$, $s \in \mathbb{R}$ and

$$\|f\|_{B^s_{p,q}(\mathbb{R}^d,E)} = \sum_{j=0}^{\infty} 2^{jsq} \|f * \psi_j^E\|^q_{L^p(\mathbb{R}^d)}.$$

Then

$$B^s_{p,q}(\mathbb{R}^d, E) = \{f \in \mathcal{S}'(\mathbb{R}^d), \|f\|_{B^s_{p,q}(\mathbb{R}^d,E)} < +\infty\}.$$

The matrix E is called the anisotropy of the Besov spaces $B^s_{p,q}(\mathbb{R}^d, E)$.
In a more general way, if $N \in \mathbb{R}$, we define

$$\|f\|_{B^s_{p,q,|\log|^N}(\mathbb{R}^n,E)} = \sum_{j=0}^{\infty} j^N 2^{jsq} \|f * \psi_j^E\|^q_{L^p(\mathbb{R}^n)},$$

and

$$B^s_{p,q,|\log|^N}(\mathbb{R}^d, E) = \{f \in \mathcal{S}'(\mathbb{R}^d), \|f\|_{B^s_{p,q,|\log|^N}(\mathbb{R}^d,E)} < +\infty\}.$$

Remark 3.7 Let $E \in \mathcal{E}^+$ and $|\cdot|_{E^*}$ a $(\mathbb{R}^d, E^*)$-pseudo-norm. For any $\lambda > 0$, $|\cdot|^{\frac{1}{\lambda}}_{E^*}$ is a $(\mathbb{R}^d, \lambda E^*)$-pseudo-norm. Hence for any $s > 0$, $B^{\frac{s}{\lambda}}_{p,q}(\mathbb{R}^d, \lambda E) = B^s_{p,q}(\mathbb{R}^d, E)$.

So, without loss of generality, we assume in the sequel that $Tr(E) = d$.

As it is the case for isotropic spaces, anisotropic Hölder spaces $\mathcal{C}^s(\mathbb{R}^d, E)$ can be defined as particular Besov spaces.

Definition 3.8 Let s be in $\mathbb{R}$ and $N \in \mathbb{R}$. The anisotropic Hölder spaces $\mathcal{C}^s(\mathbb{R}^d, E)$ and $\mathcal{C}^s_{|log|^N}(\mathbb{R}^d, E)$ are defined by

$$\mathcal{C}^s(\mathbb{R}^d, E) = B^s_{\infty,\infty}(\mathbb{R}^d, E) \quad \text{and} \quad \mathcal{C}^s_{|\log|^N}(\mathbb{R}^d, E) = B^s_{\infty,\infty,|\log|^N}(\mathbb{R}^d, E).$$

Proposition 3.9 *Let* $0 < s < \rho_{\min}(E)$ *and* $N \in \mathbb{R}$. *Then*

$$\|f\|_{L^\infty(\mathbb{R}^d)} + \sup_{|h|_E \leq 1} \sup_{x \in \mathbb{R}^d} \left(\frac{|f(x+h) - f(x)|}{|h|^s_E |\log(|h|_E)|^N}\right),$$

and the norm $\|f\|_{B^s_{\infty,\infty,|\log|^N}}$ *defined above are equivalent norms in* $\mathcal{C}^s_{|\log|^N}(\mathbb{R}^d, E)$.

Remark 3.10 Anisotropic Hölder spaces admit a characterization by finite differences under the general assumption $s > 0$. Here, we only need to deal with the case $0 < s < \rho_{\min}(E)$ and have thus stated Proposition 3.9 in this special setting.

Let us comment Proposition 3.9. Let $0 < s < \rho_{\min}(E)$ and $N \in \mathbb{R}$. A bounded function f belongs to the Hölder space $\mathcal{C}^s_{|\log|^N}(\mathbb{R}^d, E)$ if and only if : For any $r \in (0, 1)$, $\Theta \in S^0_E(|\cdot|_E)$ and $x \in \mathbb{R}^d$

$$|f(x + r^E\Theta) - f(x)| \leq C_0 r^s |\log(r)|^N$$

for some $C_0 > 0$.
Hence, a function f belongs to the Hölder space $\mathcal{C}^s_{|\log|^N}(\mathbb{R}^d, E)$ if and only if its restriction f_Θ along any parametric curve of the form

$$r > 0 \mapsto r^E \Theta,$$

with $\Theta \in S^0_E(|\cdot|_E)$ is in the usual Hölder space $\mathcal{C}^s_{|log|^N}(\mathbb{R})$ and $\|f_\Theta\|_{\mathcal{C}^s_{|log|^N}(\mathbb{R})}$ does not depend on Θ. Roughly speaking, the anisotropic "directional" regularity in any anisotropic "direction" has to be larger than s. In other words, we replace straight lines of isotropic setting by curves with parametric equation $r > 0 \mapsto r^E\Theta$ adapted to anisotropic setting.

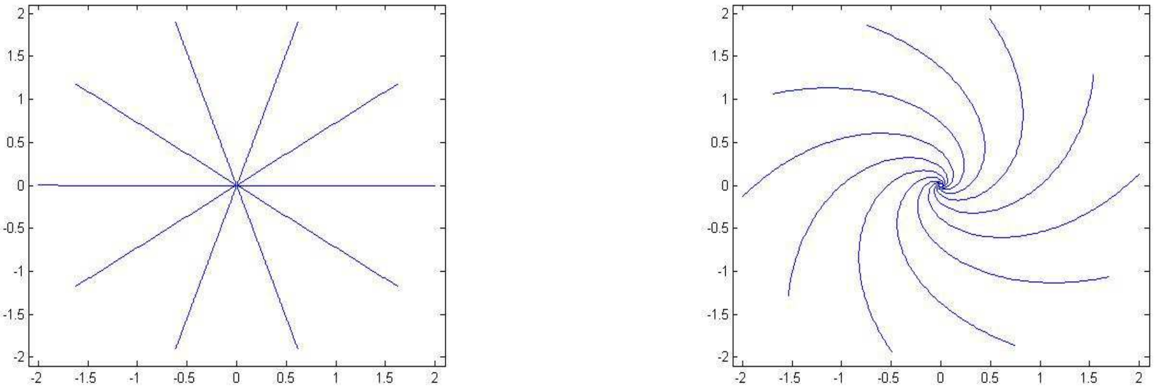


Fig. 3

"Isotropic lines" and "anisotropic lines" in the case $E = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

4 Statement of our results

First in Section 4.1, we state our optimality results and characterize in some sense an anisotropy E_0 and an Hurst index of the field H_0. These results come from an accurate study of sample paths properties of the OSRGF $\{X_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ in anisotropic Besov spaces (see in Section 4.2). But before any statement let us give some definitions and notations.
In this section ρ_{E_0} denotes a $(\mathbb{R}^d, E_0)$ pseudo-norm, $\{X_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ is the OSRGF with exponent E_0 and Hurst index H_0 defined by (3).

We assume - without loss of generality - that any anisotropy of the field E_0 and any anisotropy E of the analyzing spaces $B^s_{p,q}(\mathbb{R}^d, E)$ satisfy $Tr(E_0) = Tr(E) = d$. Let us denote by $\mathcal{E}^+_d$ the set of matrices of $M_d(\mathbb{R})$ satisfying $Tr(E) = d$ whose eigenvalues have positive real parts.

Our results are based on local sample path properties of the Gaussian field $\{X_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$. We first need some definitions.

Definition 4.1 Let $E \in \mathcal{E}^+_d$ be a fixed anisotropy, $(p, q, s) \in (1, +\infty]^2 \times (0, +\infty)$ and $f \in L^p_{loc}(\mathbb{R}^d)$.
The function f belongs to $B^\alpha_{p,q,loc}(\mathbb{R}^d, E)$ if for any $\varphi \in \mathcal{D}(\mathbb{R}^d)$, the function φf belongs to $B^\alpha_{p,q,|\log|^N}(\mathbb{R}^d, E)$.
The spaces $B^\alpha_{p,q,|\log|^N,loc}(\mathbb{R}^d, E)$ can be defined in an analogous way for any $(p, q, s, N) \in (0, +\infty]^2 \times (0, +\infty) \times \mathbb{R}$.
The anisotropic local critical exponent in anisotropic Besov spaces $B^s_{p,q}(\mathbb{R}^d, E)$ of the OSRGF $\{X_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ is then defined by

$$\alpha_{X_{\rho_{E_0},H_0},loc}(E, p, q) = \sup\{s,\ X_{\rho_{E_0},H_0}(\cdot) \in B^s_{p,q,loc}(\mathbb{R}^d, E)\}\ .$$

In the special case $p = q = \infty$, this exponent is also called the anisotropic local critical exponent in anisotropic Hölder spaces of the OSRGF $\{X_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ and is denoted by $\alpha_{X_{\rho_{E_0},H_0},loc}(E)$.

4.1 Two optimality results

We get a first general result :

Theorem 4.2 *Let $(p,q) \in (1,+\infty]^2$ and E_0 a matrix whose eigenvalues have positive real parts. Then almost surely*

$$\alpha_{X_{\rho_{E_0},H_0},loc}(E_0,p,q) = \sup\{\alpha_{X_{\rho_{E_0},H_0},loc}(E,p,q), E \in \mathcal{E}_d^+, E \textit{ commuting with } E_0\}.$$

that is the value $E = E_0$ maximizes the anisotropic local critical exponent of the OSRGF $\{X_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ among all possible anisotropic local critical exponent in anisotropic Besov spaces with an anisotropy E commuting with E_0.

Remark The assumption 'E and E_0 are commuting' implies that the two matrices D and D_0 of Theorem 4.2 admit the same spectral decomposition. Hence, in fact we proved that any anisotropy matrix E_0 maximize the critical exponent among matrices having the same spectral decomposition. Thus, in the general case dimension we implicitly assumed that the spectral decomposition of anisotropy matrix is well-known.
In dimension two, we have a stronger optimality result about anisotropy matrix E_0 and Hurst index H_0. Note, that this case is interesting when dealing with anisotropic images.

Theorem 4.3 *Let $(p,q) \in (1,+\infty]^2$ and E_0 a matrix whose eigenvalues have positive real parts. Then almost surely*

$$\alpha_{X_{\rho_{E_0},H_0},loc}(E_0,p,q) = \sup\{\alpha_{X_{\rho_{E_0},H_0},loc}(E,p,q), E \in \mathcal{E}_d^+\}.$$

In fact, Theorem 4.3 contains two main results :

- The critical exponent of the field $\{X_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ in anisotropic Besov space $B^s_{p,q}(\mathbb{R}^d,E_0)$, and more generally in anisotropic Besov space $B^s_{p,q}(\mathbb{R}^d,E_0)$ equals the associated Hurst index H_0.

- Any anisotropy E_0 of the field $\{X_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ maximizes this critical exponent among all possible anisotropy analysis matrix. In fact, the 'best way' of measuring smoothness of the field $\{X_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ is to measure smoothness along the anisotropic 'directions' $r>0 \mapsto r^{E_0}\Theta$ related to the genuine geometry of the field.

4.2 Sample paths properties of the OSRGF $\{X_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ in anisotropic Besov spaces

In order to prove Theorem 4.2 and Theorem 4.3, we investigate the local regularity of the sample path of the field $\{X_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ in general anisotropic Besov spaces. But before any statement, we first need some background about the concept of real diagonalizable part of a square matrix. This notion is based on real additive Jordan decomposition of a square matrix (see for e.g. to Lemma 7.1 chap 9 of [18] where a multiplicative version of Proposition 4.4 is given) :

Proposition 4.4 *Any matrix M of $M_d(\mathbb{R})$ can be decomposed into a sum of three commuting real matrices*

$$M = D + S + N\ ,$$

where D is a diagonalizable matrix in $M_d(\mathbb{R})$, S is a diagonalizable matrix in $M_d(\mathbb{C})$ with zero or imaginary complex eigenvalues, and N is a nilpotent matrix. Matrix D is called the real diagonalizable part of M, S its imaginary semi-simple part, and N its nilpotent part.

Now we are given two **commuting** matrices E_0, E of $\mathcal{E}_d^+$. Let D_0 (resp D) be the real diagonalizable part of matrix E_0 (resp E). Since matrices E_0 and E are commuting, so do matrices D_0 and D. Furthermore, matrices D_0 and D are diagonalizable in $M_d(\mathbb{R})$ then they are simultaneously diagonalizable. Up to a change of basis, we may assume that D_0 and D are two diagonal matrices. More precisely, suppose that

$$D_0 = \begin{pmatrix} \lambda_1^0 Id_{d_1} & & 0 \\ & \ddots & \\ 0 & & \lambda_m^0 Id_{d_m} \end{pmatrix},\ D = \begin{pmatrix} \lambda_1 Id_{d_1} & & 0 \\ & \ddots & \\ 0 & & \lambda_m Id_{d_m} \end{pmatrix}, \qquad (5)$$

with

$$\frac{\lambda_m}{\lambda_m^0} \leq \cdots \leq \frac{\lambda_1}{\lambda_1^0}. \tag{6}$$

Since $Tr(E_0) = Tr(E) = d$, one has $\lambda_m/\lambda_m^0 \leq 1$.

The regularity results about sample path of the field $\{X_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ are summed up in the following theorem.

Theorem 4.5 *Let $1 < p \leq +\infty$, $1 < q \leq +\infty$. Almost surely the anisotropic local critical exponent $\alpha_{X_{\rho_{E_0},H_0},loc}(E,p,q)$ in anisotropic Besov spaces $B^s_{p,q}(\mathbb{R}^d,E)$ of the OSRGF $\{X_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ satisfies*

$$\alpha_{X_{\rho_{E_0},H_0},loc}(E,p,q) = H_0\frac{\lambda_m}{\lambda_m^0} \leq H_0.$$

In particular, in the special case $E = E_0$, $\alpha_{X_{\rho_{E_0},H_0},loc}(E,p,q)H_0$.

In other words Theorem 4.5 asserts that when one measures local regularity of the sample paths along anisotropic directions different from those associated to an anisotropy of the field E_0, one loses smoothness. The further the anisotropic direction of measure from the genuine anisotropic direction associated to the field are, the smaller the anisotropic local critical exponent is. This anisotropic local critical exponent can take any value in the range $(0, H_0]$.

The special case $p = q = +\infty$ yields us the following result about anisotropic Hölderian regularity of the sample paths.

Corollary 4.6 *Almost surely the anisotropic local critical exponent of the sample paths of $\{X_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ in anisotropic Hölder spaces equals $H_0\frac{\lambda_m}{\lambda_m^0}$ and is always lower than H_0. In particular, if $E = E_0$ this critical exponent equals the Hurst index H_0.*

Remark 4.7 This estimate on anisotropic local critical exponent was already known in the case $E = E_0$ (see [7]).

Theorem 4.5 allows us to obtain regularity results which extend those proved in the case $p = q = \infty$ in the usual isotropic setting. Since matrices E_0 and Id are commuting, we can apply the above result to the case $E = Id$. Note that in this case $\lambda_m^0 = \rho_{\max}(E_0)$. We obtain the following Proposition :

Proposition 4.8 *Almost surely the local critical exponent of the sample paths of $\{X_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ in classical Besov spaces equals $H_0\frac{1}{\rho_{\max}(E_0)}$.*

In particular, for $p = q = \infty$, almost surely the local critical exponent of the sample paths of $\{X_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ in classical Hölder spaces equals $H_0\frac{1}{\rho_{\max}(E_0)}$.

Remark 4.9 In the special case $p = q = \infty$, we recover results about classic Hölderian regularity already established in Theorem 5.4 of [8]. Recall that Theorem 5.4 of [8] is based on directional regularity results about the Gaussian field $\{X_{\rho_{E_0},H_0}\}$ and comes from an accurate estimate of the variogram $v_{X_{\rho_{E_0},H_0}}(h) = \mathbb{E}(|X_{\rho_{E_0},H_0}(h)|^2)$ along special directions linked to the spectral decomposition of matrix E_0. Here our approach is based on wavelet technics.

5 Complements and proofs

5.1 Role of the real diagonalizable part of the anisotropy E of the analysing spaces $B^s_{p,q}(\mathbb{R}^d,E)$

We will first prove that measuring smoothness in the general Besov spaces $B^s_{p,q}(\mathbb{R}^d,E)$ may actually be deduced from the special case where matrix E is diagonalizable. To this end, we show the following embedding property:

Proposition 5.1 *Assume that* $E_1 \in \mathcal{E}_d^+$ *and* $E_2 \in \mathcal{E}_d^+$ *have the same real diagonalizable part* D*. For any* $\alpha > 0$ *and any* $(p,q) \in (1,+\infty]^2$ *one has,*

$$B^{\alpha}_{p,q,|\log|^{\frac{d}{\rho_{\min}(D)}}}(\mathbb{R}^d, E_1) \hookrightarrow B^{\alpha}_{p,q}(\mathbb{R}^d, E_2) \hookrightarrow B^{\alpha}_{p,q,|\log|^{-\frac{d}{\rho_{\min}(D)}}}(\mathbb{R}^d, E_1).$$

As a direct consequence, we obtain Corollary 5.2. Note that this result does not depend on the studied Gaussian field but of the involved functional spaces. Hence, it does not give any information about the anisotropic properties of the field.

Corollary 5.2 *The anisotropic local critical exponent*

$$\alpha_{X,loc}(E,p,q) = \sup\{s > 0,\ X(\cdot) \in B^s_{p,q,loc}(\mathbb{R}^d, E)\},$$

of any Gaussian field $\{X(x)\}_{x\in\mathbb{R}^d}$ *in anisotropic Besov spaces* $B^s_{p,q}(\mathbb{R}^d, E)$ *depends only on the real diagonalizable part of* E*.*

Proof of Proposition 5.1 relies on the following lemma :

Lemma 5.3 *Assume that* E_1 *and* E_2 *are two matrices of* $\mathcal{E}^+$ *having the same real diagonalizable part* D*. Then there exists two positive constants* c_1 *and* c_2 *such that, for all* $x \in \mathbb{R}^d$*,*

$$c_1 |x|_{E_2^*}(1 + |\log(|x|_{E_2^*})|)^{-\frac{d}{\rho_{\min}(D)}} \le |x|_{E_1^*} \le c_2 |x|_{E_2^*}(1 + |\log(|x|_{E_2^*})|)^{\frac{d}{\rho_{\min}(D)}}.$$

Proof 5.4 Using polar coordinates associated to E_1^*, one has, for $x \in \mathbb{R}^d$,

$$x = r^{E_1^*}\Theta, (r,\Theta) \in \mathbb{R}_+^* \times S_0(E_1^*).$$

Denote $F_1 = E_1 - D$, $F_2 = E_2 - D$ and remark that those two matrices have only pure imaginary eigenvalues. By Lemma 2.1 of [8], it comes that for any $\varepsilon > 0$

$$\begin{aligned} |x|_{E_2^*} &= |r^{E_2^*} r^{-D} r^{-F_2^*} r^{D} r^{F_1^*}\Theta|_{E_2^*} \\ &\le r |r^{-F_2^*} r^{F_1^*}\Theta|_{E_2^*} \\ &\le Cr \max(|r^{-F_2^*} r^{F_1^*}\Theta|^{\frac{1}{\rho_{\min}(D)-\varepsilon}}, |r^{-F_2^*} r^{F_1^*}\Theta|^{\frac{1}{\rho_{\max}(D)+\varepsilon}}) \\ &\le Cr \max(\|r^{-F_2} r^{F_1}\|^{\frac{1}{\rho_{\min}(D)-\varepsilon}}, \|r^{-F_2} r^{F_1}\|^{\frac{1}{\rho_{\max}(D)+\varepsilon}}) \\ &\le Cr(1 + |\log(r)|)^{\frac{d-1}{\rho_{\min}(D)-\varepsilon}} \\ &\le Cr(1 + |\log(r)|)^{\frac{d}{\rho_{\min}(D)}}. \end{aligned}$$

Using two anisotropic Littlewood-Paley analysis associated respectively to matrices E_1, E_2 and D and the lemma above we deduce the following embedding stated in Proposition 5.1 :

$$B^{\alpha}_{p,q,-|\log|^{\frac{d}{\rho_{\min}(D)}}}(\mathbb{R}^d, E_1) \hookrightarrow B^{\alpha}_{p,q}(\mathbb{R}^d, E_2) \hookrightarrow B^{\alpha}_{p,q,|\log|^{\frac{d}{\rho_{\min}(D)}}}(\mathbb{R}^d, E_1).$$

for any $\alpha > 0$, $1 < p, q \le +\infty$.
Indeed, for any $i \in \{1,2\}$, let $(\psi_j^{E_i})_{j\in\mathbb{Z}}$ an anisotropic Littlewood-Paley analysis of Besov spaces $B^{\alpha}_{p,q}(\mathbb{R}^d, E_i)$. By definition

$$supp(\widehat{\psi_1^{E_i}}) \subset \{\xi, 1 \le |\xi|_{E_i^*} \le 4\}$$

for $j \in \{1,2\}$. Then there exists $j_0 \in \mathbb{Z}$ such that for any $j \in \mathbb{Z}$, one has

$$\begin{aligned} \mathrm{supp}(\widehat{\psi_j^{E_2}}) &\subset \{\xi, 2^{j-1} \le |\xi|_{E_2^*} \le 2^{j+1}\} \\ &\subset \bigcup_{l=j-j_0-\frac{d}{\rho_{\min}(D)}\log_2(j)}^{j+j_0+\frac{d}{\rho_{\min}(D)}\log_2(j)} \{\xi, 2^{l-1} \le |\xi|_{E_1^*} \le 2^{l+1}\} \end{aligned}$$

Hence

$$\widehat{\psi_j^{E_2}}\widehat{f}(\xi) = \widehat{\psi}_j^{E_2} \sum_{l=j-j_0-\frac{d}{\rho_{\min}(D)}\log_2(j)}^{j+j_0+\frac{d}{\rho_{\min}(D)}\log_2(j)} \widehat{\psi}_l^{E_1}\widehat{f}(\xi),$$

which implies

$$\begin{aligned} \|f * \psi_j^{E_2}\|_{L^p} &\leq \sum_{l=j-j_0-\frac{d}{\rho_{\min}(D)}\log_2(j)}^{j+j_0+\frac{d}{\rho_{\min}(D)}\log_2(j)} \|\psi_j^{E_2} * (\psi_l^{E_1} * f)\|_{L^p} \\ &\leq \|\psi\|_{L^1} \sum_{l=j-j_0-\frac{d}{\rho_{\min}(D)}\log_2(j)}^{j+j_0+\frac{d}{\rho_{\min}(D)}\log_2(j)} \|\psi_l^{E_1} * f\|_{L^p} \end{aligned}$$

Then we can give the following upper bound of $\sum_{j=1}^{J} \frac{2^{jsq}}{j^{\frac{d}{\rho_{\min}(D)}}}\|f * \psi_j^{E_2}\|_{L^p}$:

$$\begin{aligned} \sum_{j=1}^{J} \frac{2^{jsq}}{j^{\frac{d}{\rho_{\min}(D)}}}\|f * \psi_j^{E_2}\|_{L^p} &\leq \sum_{j=1}^{J} \frac{2^{jsq}}{j^{\frac{d}{\rho_{\min}(D)}}} \sum_{l=j-j_0-\frac{n}{\rho_{\min}(D)}\log_2(j)}^{j+j_0+\frac{d}{\rho_{\min}(D)}\log_2(j)} \|(f * \psi_l^{E_1})\|_{L^p} \\ &\leq \sum_{l=1}^{J+j_0+\frac{d}{\rho_{\min}(D)}\log_2(J)} \|f * \psi_l^{E_1}\|_{L^p} \sum_{j=l-j_0-\frac{n}{\rho_{\min}(D)}\log_2(l)}^{l+j_0+\frac{d}{\rho_{\min}(D)}\log_2(l)} \frac{2^{jsq}}{j} \\ &\leq \sum_{l=1}^{J+j_0+\log_2(J)} \|f * \psi_l^{E_1}\|_{L^p} 2^{lsq}\log_2(l) < +\infty. \end{aligned}$$

Let us now assume that $J \to \infty$ which yields the inclusion

$$B^{\alpha}_{p,q}(\mathbb{R}^d, E_2) \hookrightarrow B^{\alpha}_{p,q,|\log|^{\frac{d}{\rho_{\min}(D)}}}(\mathbb{R}^d, E_1).$$

Permuting E_1 and E_2 yields the other inclusion.

5.2 Local regularity in anisotropic Besov spaces of the studied field

In the previous section, we proved that we can restrict our study to diagonal Besov spaces. This point is crucial for the proof of the regularity results enounced in Section 4. Indeed it allows us to use tools that are only defined in the diagonal case, as anisotropic multi-resolution analysis and anisotropic wavelet bases.

The aim of the following subsection is to recall the constructions of these wavelet bases.

5.2.1 Orthonormal Wavelet bases of (diagonal) anisotropic spaces

In this section, we assume that the anisotropy D of the space is diagonal (with positive eigenvalues). We assume that $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_d \end{pmatrix}$ and - as it is the case for general anisotropic Besov spaces - that $Tr(D) = d$. Let us first recall the definition given by Triebel in [35] of an anisotropic multi-resolution analysis.

Let $\{V_j, j \geq 0\}$ be a one-dimensional multi-resolution analysis of $L^2(\mathbb{R})$ and let us denote by ψ^F (resp. ψ^M) the corresponding scaling function (resp. wavelet function).

Notation 5.5 We denote by $\{F,M\}^{d^*}$ the set

$$\{F,M\}^{d^*}=\{F,M\}^d\setminus\{(F,\cdots,F)\}.$$

For $j\in\mathbb{N}$, we define the set $I^j(D)$ of $\{F,M\}^d\times\mathbb{N}^d$ in the following way.

- If $j=0$, $I^0(D)=\{((F,\cdots,F),(0,\cdots,0))\}$.

- If $j\geq 1$, $I^j(D)$ is the set of all the elements (G,γ) with $G\in\{F,M\}^{d^*}$ and $\gamma\in\mathbb{N}^d$ such that for any $r\in\{1,\cdots,d\}$:

  If $G_r=F, \gamma_r=[(j-1)\lambda_r]$,
  If $G_r=M, [(j-1)\lambda_r]\leq\gamma_r<[j\lambda_r]$.

  Finally, for $j\in\mathbb{N}$ and $(G,\gamma)\in I^j(D)$, we will denote by $D_{j,G,\gamma}$ the matrix defined by

$$D_{j,G,\gamma}=\begin{pmatrix}\gamma_1 & & 0\\ & \ddots & \\ 0 & & \gamma_d\end{pmatrix}$$

Finally, let us define the family of wavelets as follows. For $j\in\mathbb{N}$, $(G,\gamma)\in I^j(D)$ and $k\in\mathbb{Z}^d$, we set

$$\Psi^k_{j,G,\gamma}(x)=(\psi^{(G)})(2^{D_{j,G,\gamma}}x-k)\ ,$$

with

$$\psi^{(G)}=\psi_{G_1}\otimes\cdots\otimes\psi_{G_d}\ .$$

The anisotropic wavelet bases yield a wavelet characterisation of anisotropic Besov spaces ([34] and [35], Theorem 5.23).

Theorem 5.6 1. *The family* $\left\{2^{\frac{Tr(D_{j,G,\gamma})}{2}}\Psi^k_{j,G,\gamma},\ j\in\mathbb{N},\ (G,\gamma)\in I^j(D),\ k\in\mathbb{Z}^d\right\}$ *is an orthonormal basis of* $L^2(\mathbb{R}^d)$.

2. *Let* $(\Psi^{j,G,\gamma}_k)_{j\in\mathbb{N},(G,\gamma)\in I^j(D),k\in\mathbb{Z}^d}$ *be the family constructed from* ψ_F *and* ψ_M *Daubechies wavelets with, for some* $u\in\mathbb{N}$,

$$\psi_F\in\mathcal{C}^u(\mathbb{R}),\psi_M\in\mathcal{C}^u(\mathbb{R}).$$

Let $0<p,q\leq\infty$ *and* $s\in\mathbb{R}$. *There exists an integer* $u(s,p,D)$ *such that if* $u>u(s,p,D)$, *for any tempered distribution* f *the two following assertions are equivalent*

(a) $f\in B^s_{p,q}(\mathbb{R}^d,D)$,

(b) $f=\sum c^k_{j,G,\gamma}\Psi^k_{j,G,\gamma}$ *with*

$$\sum_{j,G,\gamma}2^{j(s-\frac{d}{p})q}\left(\sum_k|c^k_{j,G,\gamma}|^p\right)^{\frac{q}{p}}<+\infty,$$

the convergence being in $\mathcal{S}'(\mathbb{R}^d)$.

The above expansion is then unique and

$$c^k_{j,G,\gamma}=<f,2^{Tr(D_{j,G,\gamma})}\Psi^k_{j,G,\gamma}>\ .$$

Remark 5.7 An analogous result is stated ([35], Theorem 5.24) replacing Daubechies wavelets by Meyer wavelets. In that case, $u=+\infty$.

We now prove regularity results about the sample path of $\{X_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ based on wavelet characterization of Besov spaces.

5.2.2 Local regularity of the field $\{X_{E_0,H_0}(x)\}_{x\in\mathbb{R}^d}$ in anisotropic Besov spaces $B^s_{p,q}(\mathbb{R}^d, D_0)$

Assume that we are given a Gaussian field of the form (3) $\{X_{E_0,H_0}(x)\}_{x\in\mathbb{R}^d}$ where E_0 is a matrix whose eigenvalues have positive real part and $H_0 \in (0, \rho_{\min}(E_0))$.
Define ε on $(0, +\infty]$ as follows : $\varepsilon(p) = 1/2$ if $p = +\infty$, 0 otherwise. The aim of this section is to prove :

Proposition 5.8 *Let $1 < p \leq +\infty$, $1 < q \leq +\infty$. Then one has*

1. *For any $\beta > 1/q+d/\rho_{\min}(E_0)$, almost surely, the sample path of $\{X_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ belongs to $B^{H_0}_{p,q,|\log|^{\beta+\varepsilon(p)+1},loc}(\mathbb{R}^d, D_0)$,*
2. *For $\beta = 1/q + d/\rho_{\min}(E_0)$, almost surely, the sample path of $\{X_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ does not belong to $B^{H_0}_{p,q,|\log|^{-\beta-\varepsilon(p)-1},loc}(\mathbb{R}^d, D_0)$.*

Adapting to our setting a result of [26], we first remark that there exists $\mathcal{C}^\infty(\mathbb{R}^d\setminus\{0\})$ $(\mathbb{R}^d, E_0)$ pseudo-norms

Lemma 5.9 *Let E_0 be a $d\times d$ matrix with positive real parts of the eigenvalues. Let φ be a $\mathcal{C}^\infty$ function compactly supported in $\mathbb{R}^d\setminus\{0\}$.The function ρ defined, for $x\in\mathbb{R}^d$, by $\rho(x) = \int_{\mathbb{R}^d}\varphi(a^{-E_0}x)da$ is a $(\mathbb{R}^d, E_0)$ pseudo-norm belonging to $\mathcal{C}^\infty(\mathbb{R}^d\setminus\{0\})$.*

In [11], we proved that the sample path properties of the Gaussian field $\{X_{\rho_{E_0},H_0}\}_{x\in\mathbb{R}^d}$ do not depend on the chosen $(\mathbb{R}^d, E_0)$ pseudo-norm. Thus, we assume from now that the $(\mathbb{R}^d, E_0)$ pseudo-norm $|\cdot|_{E_0}$ used in the construction of the field $\{X_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ belongs to $\mathcal{C}^\infty(\mathbb{R}^d\setminus\{0\})$.

Our results come from the series expansion of $X_{\rho_{E_0},H_0}$ in a Meyer anisotropic wavelet basis (see Section 5.2.1 just above).
Denote for any $j\in\mathbb{N}$, $(G,\gamma)\in I_j$, $c^k_{j,G,\gamma} =< X_{\rho_{E_0},H_0}, 2^{Tr(D_{j,G,\gamma})}\Psi^k_{j,G,\gamma} >$ as above. Thereafter set

$$X^{(1)}_{\rho_{E_0},H_0}(x) = \sum_{j,G,\gamma}\sum_{|k|<j2^{jd}} c^k_{j,G,\gamma}(\omega)\Psi^k_{j,G,\gamma}(x),$$

and

$$X^{(2)}_{\rho_{E_0},H_0}(x) = \sum_{j,G,\gamma}\sum_{|k|>j2^{jd}} c^k_{j,G,\gamma}(\omega)\Psi^k_{j,G,\gamma}(x)\ .$$

We will investigate separately the sample path properties in anisotropic Besov spaces of the Gaussian fields $X^{(1)}_{\rho_{E_0},H_0}$ and $X^{(1)}_{\rho_{E_0},H_0}$. We first prove that

Proposition 5.10 *Let $(p,q)\in(1,+\infty]^2$.*

1. *Almost surely, for any $\beta > 1/q + d/\rho_{\min}(E_0)$, the sample path of the field $\{X^{(1)}_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ belongs to $B^{H_0}_{p,q,|\log|^{\beta+\varepsilon(p)+1}}(\mathbb{R}^d, D_0)$.*
2. *Almost surely, for $\beta = 1/q + d/\rho_{\min}(E_0)$ the sample path of the field $\{X^{(1)}_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ does not belong to $B^{H_0}_{p,q,|\log|^{-\beta-\varepsilon(p)-1}}(\mathbb{R}^d, D_0)$.*

Proof 5.11 The proof uses several technics introduced in [10]. Set

$$g^k_{j,G,\gamma} = \frac{c^k_{j,G,\gamma}}{\mathbb{E}(|c^k_{j,G,\gamma}|^2)^{1/2}} \tag{7}$$

for any $j\in\mathbb{N}$, $(G,\gamma)\in I_j$, $k\in\Gamma_j(D_0) = \{k\in\mathbb{Z}^2,\ |k|_{D_0}\leq j2^j\}$.
Let us distinguish the two cases $p\neq\infty$ and $p=\infty$.

If $p \neq \infty$, the definition of the sequence $(g^k_{j,G,\gamma})$ and Lemma A.1 imply that surely there exists some $C_1, C_2 > 0$ such that for any j, G, γ

$$\left(\sum_{k\in\Gamma_j} |c^k_{j,G,\gamma}|^p\right)^{1/p} \geq C_1 2^{j/p} 2^{-2jH_0/p} j^{-d/\rho_{\min}(E_0)-1} \left(\frac{1}{n_j}\sum_{k\in\Gamma_j} |g^k_{j,G,\gamma}|^p\right)^{1/p} , \tag{8}$$

and

$$\left(\sum_{k\in\Gamma_j} |c^k_{j,G,\gamma}|^p\right)^{1/p} \leq C_2 2^{j/p} 2^{-2jH_0/p} j^{d/\rho_{\min}(E_0)+1} \left(\frac{1}{n_j}\sum_{k\in\Gamma_j} |g^k_{j,G,\gamma}|^p\right)^{1/p} . \tag{9}$$

Lemma A.5 and inequalities (8),(8) then yield the required results for the case $p < \infty$.
If $p = \infty$, the definition of the sequence $(g^k_{j,G,\gamma})$ and Lemma A.1 imply that surely there exists some $C_1, C_2 > 0$ such that for any j, G, γ

$$\sup_{k\in\Gamma_j} |c^k_{j,G,\gamma}| \geq C_1 2^{-2jH_0} j^{1/2-d/\rho_{\min}(E_0)-1} \left(\frac{1}{\sqrt{\log(n_j)}} \sup_{k\in\Gamma_j} |g^k_{j,G,\gamma}|\right) , \tag{10}$$

and

$$\sup_{k\in\Gamma_j} |c^k_{j,G,\gamma}| \leq C_2 2^{-2jH_0} j^{1/2+d/\rho_{\min}(E_0)+1} \left(\frac{1}{\sqrt{\log(n_j)}} \sup_{k\in\Gamma_j} |g^k_{j,G,\gamma}|\right) . \tag{11}$$

Lemma A.7 and inequalities (10), (11) then yield the required results for the case $p = \infty$.

Proposition 5.8 can then be directly deduced from the following proposition :

Proposition 5.12 *Almost surely, the sample path of the field* $\{X^{(2)}_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ *are* $B^{H'}_{p,q,loc}(\mathbb{R}^d, E_0)$ *for any*

$$0 < H_0 < H' < \rho_{\min}(D_0) = \rho_{\min}(E_0) .$$

Then, the sample path smoothness of $\{X_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ *in anisotropic Besov spaces of anisotropy* D_0 *are those of the field* $\{X^{(1)}_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$.

Proof 5.13 Using the transference results of [35] (see Theorem 5.28) and the usual embedding of isotropic Besov spaces defined on bounded domains one remarks that

$$\mathcal{C}^{s+\varepsilon}_{loc}(\mathbb{R}^d, E_0) \subset B^s_{p,q,loc}(\mathbb{R}^d, E_0) ,$$

for any $(p, q) \in (1, +\infty]^2$ and any $(s, \varepsilon) \in (0, +\infty)^2$. It then suffices to prove the result for $p = q = \infty$.
Let now consider $0 < \overline{H} < H' < 1$, $\varepsilon > 0$ and $\varphi \in \mathcal{D}(\mathbb{R}^d)$. We may assume that $supp(\varphi) \subset B_{E_0}(0, 1) = \{x, |x|_{E_0} \leq 1\}$ and $0 \leq \varphi \leq 1$. We denote by Y the random field $\varphi X^{(2)}_{\rho_{E_0},H_0}$. We want to give an upper bound of $|Y(x + h) - Y(x)|$ for any given x, h in $B_{E_0}(0, 1)$.
Let us first remark that

$$\begin{aligned} Y(x+h) - Y(x) &= \sum_{j,G,\gamma}\sum_{|k|_{E_0}>j2^j} c^k_{j,G,\gamma}(\omega)(\varphi(x+h) - \varphi(x))\Psi^k_{j,G,\gamma}(x) \\ &\quad + \sum_{j,G,\gamma}\sum_{|k|>j2^{jd}} c^k_{j,G,\gamma}(\omega)\varphi(x+h)(\Psi^k_{j,G,\gamma}(x+h) - \Psi^k_{j,G,\gamma}(x)) . \end{aligned}$$

Let $\varepsilon = 1 - H'/\rho_{\min}(E_0)$. Since $\varphi \in B^{1-\varepsilon}_{\infty,\infty,loc}(\mathbb{R}^d)$ and x, h belong to the compact set $B_{E_0}(0, 1)$, Lemma A.7 and the fast decay of the wavelets imply that almost surely

$$\begin{aligned} &\left|\sum_{j,G,\gamma}\sum_{|k|_{E_0}>j2^j} c^k_{j,G,\gamma}(\omega)(\varphi(x+h) - \varphi(x))\Psi^k_{j,G,\gamma}(x)\right| \\ \leq\ & |h|^{1-\varepsilon}\|\varphi\|_{B^{1-\varepsilon}_{\infty,\infty}(B_{E_0}(0,1))} \sum_{j,G,\gamma} j^{1/2+d/\rho_{\min}(E_0)} 2^{-jH_0} \sum_{|k|_{E_0}>j2^j} \frac{1}{(1+|k-2^{D_{j,G,\gamma}}x|)^M} \end{aligned}$$

for some $M > 0$.
Remark now that

$$|k|_{E_0} \geq j2^j \geq 2^j \geq |x|_{E_0}$$

Then there exists some $\alpha > 0$ such that for j sufficiently large and any x in $B_{E_0}(0,1)$

$$|k - 2^{D_{j,G,\gamma}} x| \geq |k|^\alpha / 2$$

Then

$$\begin{array}{ll} & \left| \sum_{j,G,\gamma} \sum_{|k|_{E_0} > j2^j} c^k_{j,G,\gamma}(\omega)(\varphi(x+h) - \varphi(x)) \Psi^k_{j,G,\gamma}(x) \right| \\ \leq & |h|^{1-\varepsilon} \|\varphi\|_{B^{1-\varepsilon}_{\infty,\infty}(B_{E_0}(0,1))} \sum_{j,G,\gamma} j^{1/2+d/2\rho_{\min}(E_0)} 2^{-jH_0} \sum_{|k|_{E_0} > j2^j} \frac{1}{(1+|k|^\alpha)^M} \leq C|h|^{1-\varepsilon} \leq C|h|^{H'}_{E_0} \end{array}$$

Further, by the same approach we prove that almost surely

$$\begin{array}{ll} & \left| \sum_{j,G,\gamma} \sum_{|k| > j2^{jd}} c^k_{j,G,\gamma}(\omega) \varphi(x+h) (\Psi^k_{j,G,\gamma}(x+h) - \Psi^k_{j,G,\gamma}(x)) \right| \\ \leq & \|\varphi\|_{L^\infty(B_{E_0}(0,1))} \sum_{j,G,\gamma} j^{1/2+d/(2\rho_{\min}(E_0))} 2^{-jH_0} |2^{D_{j,G,\gamma}} h| \sup_{2^{-D_{j,G,\gamma}} y \in [x,x+h]} \sum_{|k|_{E_0} > j2^j} \frac{1}{(1+|k-2^{D_{j,G,\gamma}} y|)^M} . \end{array}$$

The end of the proof is exactly the same as above remarking that

$$|2^{D_{j,G,\gamma}} h| \leq j^\delta 2^j |h|^{\rho_{\min}(E_0)}_{E_0}$$

for some $\delta > 0$.

5.3 Proof of regularity results in anisotropic Besov spaces with an anisotropy unrelated to the one of the field

The following Proposition extends the results of Proposition 5.8

Proposition 5.14 *Let* $1 < p \leq +\infty$, $1 < q \leq +\infty$ *and* $\beta > 1/q + d/\rho_{\min}(D) + 2d/\rho_{\min}(E_0) + \varepsilon(p)$.

1. *Almost surely the sample path of* $\{X_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ *belongs to* $B^{H_0 \frac{\lambda_m}{\lambda^0_m}}_{p,q,|\log|^\beta,loc}(\mathbb{R}^d, E)$,
2. *Almost surely the sample path of* $\{X_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ *does not belong to* $B^{H_0 \frac{\lambda_m}{\lambda^0_m}}_{p,q,|\log|^{-\beta},loc}(\mathbb{R}^d, E)$.

The proof is made in several steps. First we need to compare Besov spaces with different anisotropies.

5.3.1 A comparison result between Besov spaces with different anisotropies

Recall that D_0 and D are assumed to be two diagonal matrices of the form :

$$D_0 = \begin{pmatrix} \lambda^0_1 Id_{d_1} & & 0 \\ & \ddots & \\ 0 & & \lambda^0_m Id_{d_m} \end{pmatrix},\ D = \begin{pmatrix} \lambda_1 Id_{d_1} & & 0 \\ & \ddots & \\ 0 & & \lambda_m Id_{d_m} \end{pmatrix}, \qquad (12)$$

with

$$\frac{\lambda_m}{\lambda^0_m} \leq \cdots \leq \frac{\lambda_1}{\lambda^0_1}. \qquad (13)$$

In this section the assumption $Tr(D_0) = Tr(D) = d$ is not required. We first need a comparison result between between Besov spaces with different anisotropies in the diagonal case :

Proposition 5.15 *The notations and assumptions are as above. For any* $\alpha > 0$, $\beta \in \mathbb{R}$ *and* $p, q \in (0, +\infty]$, *one has the following embedding*

$$B^{\alpha}_{p,q,|\log|^{\beta}}(\mathbb{R}^d, D_0) \hookrightarrow B^{\alpha\frac{\lambda_m}{\lambda_m^0}}_{p,q,|\log|^{\beta}}(\mathbb{R}^d, D).$$

The proof is straightforward and based on finite differences characterization of Besov spaces given in Theorem 5.8 (ii) of [35].

5.3.2 Proof of Proposition 5.14

1. It is a straigthforward consequence of Propositions 5.1 and 5.15.

2. Recall that the Gaussian field $\{X_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ is defined by its harmonizable representation :

$$X_{\rho_{E_0},H_0}(x) = \int \frac{(e^{i<x,\xi>}-1)}{\rho_{E_0}(\xi)^{H_0+\frac{d}{2}}} dW_\xi$$

where the pseudo-norm ρ_{E_0} can be chosen of the form

$$\rho_{E_0} = \rho_1 + \cdots + \rho_m,$$

where for any $1 \leq \ell \leq m$, ρ_ℓ is a $(\mathbb{R}^{d_\ell}, E_\ell)$ pseudo-norm.
Set $\{X_m(x_m)\}_{x_m\in\mathbb{R}^{d_m}} = \{X(0, \cdots, 0, x_m)\}_{x_m\in\mathbb{R}^{d_m}}$. Remark that this field has the same finite dimensional margin than the field Y_m defined as follows

$$Y_m(x_m) = \int_{\mathbb{R}^{d_m}} \frac{(e^{i<x_m,\xi_m>}-1)}{\widetilde{\rho_m}(\xi_m)^{H+\frac{d_m}{2}}} d\widehat{W}_{\xi_m}$$

with

$$\widetilde{\rho_m}^{-(H+\frac{d_m}{2})}(\xi_m) = \int_{\mathbb{R}^{d_1}\times\cdots\times\mathbb{R}^{d_{m-1}}} \frac{d\xi_1\cdots d\xi_{m-1}}{\rho(\xi)^{H+\frac{d}{2}}}.$$

Remark that Y_m is in fact an OSRGF of the form (3) with exponent $E_0 = \lambda_m^0 Id_{\mathbb{R}^{d_m}}$ and Hurst index H_0. Then apply the non local regularity results of Section 5.2 to the OSRGF Y_m. Then, deduce that

$$a.s.\ Y_m(\cdot) \notin B^{H_0}_{p,q,loc,\frac{1}{|\log|^d}}(\mathbb{R}^d, \lambda_m^0 Id_{\mathbb{R}^{d_m}})$$

and thus using Proposition 5.15 with $D_0 = \lambda_m^0 Id_{\mathbb{R}^{d_m}}$ and $D = \lambda_m Id_{\mathbb{R}^{d_m}}$

$$a.s.\ Y_m(\cdot) \notin B^{H_0\frac{\lambda_m}{\lambda_m^0}}_{p,q,loc,\frac{1}{|\log|^d}}(\mathbb{R}^d, \lambda_m Id_{\mathbb{R}^{d_m}}).$$

The conclusion comes from Proposition 5.1 and from the following Fubini Lemma which can be derived from the characterization by difference of Besov spaces $B^s_{p,p,|\log|^N}(\mathbb{R}^d, D)$ (see [35]) :

Lemma 5.16 *Let* $s > 0$, $N \in \mathbb{R}$ *and* $0 < p \leq +\infty$. *If* f *is a continuous function belonging to* $B^s_{p,p}(\mathbb{R}^d, D)$ *then* f_ℓ *defined on* $\mathbb{R}^{d_\ell}$ *by*

$$f_\ell(x_\ell) = f(x_1^0, \cdots, x_\ell)$$

belongs to $B^s_{p,p}(\mathbb{R}^{d_\ell}, D_\ell)$.

5.4 Proof of the two optimality results

Theorem 4.2 results directly from Proposition 5.14. We just prove Theorem 4.3. Remark that since we are dealing with critical exponent of the field $\{X_{E_0,H_0}(x)\}_{x\in\mathbb{R}^d}$ in anisotropic Besov spaces, we use the results of [11] and assume that E_0 equals its real diagonalizable part $D_0 = \begin{pmatrix} \lambda^0_{\min} & 0 \\ 0 & \lambda^0_{\max} \end{pmatrix}$.
Since $Tr(D_0) = Tr(D)$ only two cases may appear: $\rho_{\max}(D) \le \rho_{\max}(D_0)$ and $\rho_{\min}(D) \le \rho_{\min}(D_0)$.
First case : $\rho_{\max}(D) \le \rho_{\max}(D_0)$**.** Assume that for $\alpha > H_0$, the sample paths of $\{X_{D_0,H_0}(x)\}_{x\in\mathbb{R}^d}$ belong to $B^{\alpha}_{p,q}(\mathbb{R}^d, D)$. Then, by Proposition 5.15 almost surely the sample paths of $\{X_{D_0,H_0}(x)\}_{x\in\mathbb{R}^d}$ belong to $B^{\overline{\frac{\alpha}{\rho_{\max}(D)}}}_{p,q}(\mathbb{R}^d)$. By a similar approach to Proposition 5.14 above, consider $\{Y(h)\}_{h\in\mathbb{R}} = \{X_{D_0,H_0}(he^0_{\max})\}_{h\in\mathbb{R}}$ with $e^0_{\max}$ an eigenvector of D_0 related to the higher eigenvalue $\rho_{\max}(D_0)$. Then by Fubini Lemma 5.16, the critical exponent is $H_0/\rho_{\max}(D_0)$. Thus $\alpha/\rho_{\max}(D) > H_0/\rho_{\max}(D_0)$ that yields a contradiction.
Second case $\rho_{\min}(D) \le \rho_{\min}(D_0)$ **:**
One can apply a similar approach for $\{Y(h)\}_{h\in\mathbb{R}} = \{X_{D_0,H_0}(he_{\min})\}_{h\in\mathbb{R}}$ with $e_{\min}$ an eigenvector of D related to the lower eigenvalue of D.
Remark that the critical exponent of Y in any isotropic Besov spaces is lower than $H_0/\rho_{\min}(D_0)$. Indeed,

$$Y(h) = \int_{\mathbb{R}^d} \frac{e^{ihe_{\min}.\xi} - 1}{|\xi|_D^{H_0+1}} d\widehat{W}(\xi).$$

Hence Y has the same finite dimensional margin than

$$\widetilde{Y}(h) = \int_{\mathbb{R}} (e^{iht} - 1)\phi(t) d\widehat{W}(t)$$

where $\phi(t) = \left(\int_{V^{\perp}_{\min}} \frac{1}{|\zeta|_{D_0}^{2H_0+2} + |te_{\min}|_{D_0}^{2H_0+2}} d\zeta\right)^{\frac{1}{2}}$ and $V^{\perp}_{\min} = Vect < e_{\min} >^{\perp}$. Remark then that for $|t|$ sufficiently large, $\phi(t) \ge \frac{C(e_{\min})}{|t|^{\frac{H_0}{\rho_{\min}(D_0)} + \frac{1}{2}}}$. By the regularity comparison results of [11] about Gaussian fields, we deduce the required result. By the Fubini Lemma 5.16, if we assume that $\alpha > H_0$ it yields the required contradiction.

A Technical lemmas

Our results about smoothness of the sample path are based on the following lemma

Lemma A.1 *The wavelet coefficients of the random field* $\{X_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ *are weakly dependent in the following sense*

1. *There exists some* $C_0 > 0$ *such for any* $j \ge 1$, $(G, \gamma) \in I_j$ *and* $(k, k') \in (\mathbb{Z}^d)^2$

$$|\mathbb{E}(c^k_{j,G,\gamma} c^{k'}_{j,G,\gamma})| \le C_0 \frac{j^{2d/\rho_{\min}} 2^{-2jH_0}}{1 + |k - k'|}. \tag{14}$$

2. *There exists some* $C_1, C_2 > 0$ *such that for any* $j \ge 1$, $(G, \gamma) \in I_j$ *and any* $k \in (\mathbb{Z}^d)$

$$C_1 j^{-d/\rho_{\min}(E_0)} 2^{-2jH_0} \le \mathbb{E}(|c^k_{j,G,\gamma}|^2) \le C_2 j^{d/\rho_{\min}(E_0)} 2^{-2jH_0} . \tag{15}$$

Proof A.2 We use the Meyer wavelet basis whose Fourier transform is compactly supported with $0 \notin \operatorname{supp}(\widehat{\psi}^{(G)})$. One has for any $j \ge 1$, $(G, \gamma) \in I_j$ and for all $k \in \mathbb{Z}^d$

$$c^k_{j,G,\gamma} = \int_{\mathbb{R}^d} e^{i2^{-D^*_{j,G,\gamma}} k\xi} \overline{\widehat{\psi^{(G)}}}(2^{-D^*_{j,G,\gamma}}\xi) \rho_{E_0}(\xi)^{-H_0-d/2} d\widehat{W}(\xi).$$

This formula implies that (set $\zeta = 2^{-D^*_{j,G,\gamma}}\xi$)

$$\mathbb{E}(|c^k_{j,G,\gamma}|^2) = 2^{jTr(D_{j,G,\gamma})} \int_{\mathbb{R}^d} |\widehat{\psi}^{(G)}(\zeta)|^2 \rho_{E_0^*}(2^{D^*_{j,G,\gamma}}\zeta)^{-2H_0-d} d\zeta \ .$$

Since $2^{(j-2)d} \leq Tr(D_{j,G,\gamma}) \leq 2^{jd}$, using Lemma 5.3 and the inequalities $C_1 2^j \leq |2^{D^*_{j,G,\gamma}}\zeta|_{D_0} \leq C_2 2^j$ imply that

$$\mathbb{E}(|c^k_{j,G,\gamma}|^2) \geq C_1 2^{-2j(H_0+d)} 2^{jd} \int_{\mathbb{R}^d} |\widehat{\psi}^{(G)}(\zeta)|^2 |\zeta|^{-2H_0-d}_{D_0^*} (1+\log(|\zeta|_{D_0^*})+j)^{-d/\rho_{\min}(E_0)} d\zeta \ ,$$

and

$$\mathbb{E}(|c^k_{j,G,\gamma}|^2) \leq C_2 2^{-2j(H_0+d)} 2^{jd} \int_{\mathbb{R}^d} |\widehat{\psi}^{(G)}(\zeta)|^2 |\zeta|^{-2H_0-d}_{D_0^*} (1+\log(|\zeta|_{D_0^*})+j)^{d/\rho_{\min}(E_0)} d\zeta \ .$$

We then proved inequalities (15).
To prove inequality (14) remark that for any $\ell \in \{1, \cdots, d\}$

$$(k_\ell - k'_\ell)\mathbb{E}(c^k_{j,G,\gamma} c^{k'}_{j,G,\gamma}) = \int_{\mathbb{R}^d} (k_\ell - k'_\ell) e^{i 2^{-D^*_{j,G,\gamma}}(k-k')\xi} |\widehat{\psi}^{(G)}(2^{-D^*_{j,G,\gamma}}\xi)|^2 \rho_{E_0}(\xi)^{-2H_0-d} d\xi.$$

Set $\zeta = 2^{-D^*_{j,G,\gamma}}\xi$ and integrate by parts with respect to ζ_ℓ. Hence

$$(k_\ell - k'_\ell)\mathbb{E}(c^k_{j,G,\gamma} c^{k'}_{j,G,\gamma}) = 2^{jTr(D_{j,G,\gamma})} \int_{\mathbb{R}^d} (k_\ell - k'_\ell) e^{i(k-k')\zeta} |\widehat{\psi}^{(G)}(\zeta)|^2 \rho_{E_0}(2^{D^*_{j,G,\gamma}}\zeta)^{-2H_0-d} d\zeta.$$

Recall that the pseudo–norm may be assumed to be $\mathcal{C}^\infty(\mathbb{R}^d \setminus \{0\})$. Since $\widehat{\psi}^{(G)}$ is compactly supported

$$\begin{aligned} (k_\ell - k'_\ell)\mathbb{E}(c^k_{j,G,\gamma} c^{k'}_{j,G,\gamma}) &= -2^{jTr(D_{j,G,\gamma})} \int_{\mathbb{R}^d} e^{i(k-k')\zeta} \frac{\partial}{\partial \zeta_\ell} \left(|\widehat{\psi}^{(G)}(\zeta)|^2 \rho_{E_0}(2^{D^*_{j,G,\gamma}}\zeta)^{-2H_0-d} \right) d\zeta \\ &= -2^{jTr(D_{j,G,\gamma})} \int_{\mathbb{R}^d} e^{i(k-k')\zeta} \left(\frac{\partial}{\partial \zeta_\ell} |\widehat{\psi}^{(G)}(\zeta)|^2 \right) \rho_{E_0}(2^{D^*_{j,G,\gamma}}\zeta)^{-2H_0-d} d\zeta \\ &\quad -2^{jTr(D_{j,G,\gamma})} \int_{\mathbb{R}^d} \frac{e^{i(k-k')\zeta} |\widehat{\psi}^{(G)}(\zeta)|^2}{\rho_{E_0}(2^{D^*_{j,G,\gamma}}\zeta)^{2H_0+d+1}} \left(2^{\gamma_\ell} \frac{\partial}{\partial \zeta_\ell}(\rho_{E_0})(2^{D^*_{j,G,\gamma}}\zeta) \right) d\zeta \ . \end{aligned}$$

An approach similar to the proof of inequalities (15) yields

$$2^{jTr(D_{j,G,\gamma})} \left| \int_{\mathbb{R}^d} e^{i(k-k')\zeta} \left(\frac{\partial}{\partial \zeta_\ell} |\widehat{\psi}^{(G)}(\zeta)|^2 \right) \rho_{E_0}(2^{D^*_{j,G,\gamma}}\zeta)^{-2H_0-d} d\zeta \right| \leq C j^{d/\rho_{\min}(E_0)} 2^{-2jH_0} \ . \quad (16)$$

Further, differentiate the homogeneity relationship satisfied by $\rho_{E_0^*}$ and deduce that for any $a > 0$ and $z \in \mathbb{R}^d$

$$a^{E_0^*}(\overrightarrow{\mathrm{grad}}(\rho_{E_0^*}))(a^{E_0^*}z) = a(\overrightarrow{\mathrm{grad}}(\rho_{E_0^*}))(z) \ . \quad (17)$$

For any $y \in \mathbb{R}^d \setminus \{0\}$, let $r = |y|_{E_0^*}$. Then set $j = \log_2(r)$ and remark that $|\Theta|_{E_0^*} = |2^{-jE_0^*}y|_{E_0^*} \in [1/2, 2]$ and hence that Θ belongs to the compact set $\mathcal{C}(1/2, 2, E_0^*) = \{\theta, \ |\theta|_{E_0^*} \in [1/2, 2]\}$. Relationship (17) applied with $a = 2^j$ and $z = \Theta$ then implies

$$2^{D^*_{j,G,\gamma}} \overrightarrow{\mathrm{grad}}(\rho_{E_0^*}))(2^{jE_0^*}\Theta) = 2^{-jE_0^*+D^*_{j,G,\gamma}} 2^j (\overrightarrow{\mathrm{grad}(\rho_{E_0^*})}(\Theta))$$

Take the norm of each member of the equality and deduce that for any $y \in \mathbb{R}^d \setminus \{0\}$ satisfying $j = \log_2 |y|_{E_0^*}$

$$|2^{D^*_{j,G,\gamma}} \overrightarrow{\mathrm{grad}}(\rho_{E_0^*}))(y)| \leq C 2^j |2^{-jE_0^*+D^*_{j,G,\gamma}}|$$

where $C = \sup_{\Theta \in \mathcal{C}(1/2,2,E_0^*)} |\overrightarrow{\mathrm{grad}}(\rho_{E_0^*})(\Theta)|$.
Lemma 2.1 of [7] and the definition of j imply that

$$|2^{D^*_{j,G,\gamma}} \overrightarrow{\mathrm{grad}}(\rho_{E_0^*}))(y)| \leq C 2^j |j|^{d/\rho_{\min}} \leq C |y|^*_{E_0} |\log(|y|^*_{E_0})|^{d/\rho_{\min}(E_0)}$$

Set now $y = 2^{D^*_{j,G,\gamma}}\zeta$. One has

$$|2^{D^*_{j,G,\gamma}}(\overrightarrow{\mathrm{grad}}(\rho_{E_0^*}))(2^{D^*_{j,G,\gamma}}\zeta)| \leq C|2^{D^*_{j,G,\gamma}}\zeta|^*_{E_0}|\log(|2^{D^*_{j,G,\gamma}}\zeta|^*_{E_0})|^{d/\rho_{\min}(E_0)} \leq C2^j(j+|\log(|\zeta|^*_{E_0}|)|)^{2d/\rho_{\min}(E_0)}|\zeta|^*_{E_0}$$

Since for any $\ell \in \{1, \cdots, d\}$

$$2^{\gamma_\ell}\left|\left(\frac{\partial}{\partial\zeta_\ell}(\rho_{E_0^*})\right)(2^{D^*_{j,G,\gamma}}\zeta)\right| \leq \left|2^{D^*_{j,G,\gamma}}(\overrightarrow{\mathrm{grad}}(\rho_{E_0^*}))(2^{D^*_{j,G,\gamma}}\zeta)\right|$$

it yields the following inequality

$$\left|2^{jTr(D_{j,G,\gamma})}\int_{\mathbb{R}^d}\frac{e^{i(k-k')\zeta}|\widehat{\psi}^{(G)}(\zeta)|^2}{\rho_{E_0}(2^{D^*_{j,G,\gamma}}\zeta)^{2H_0+d+1}}\left(2^{\gamma_\ell}\frac{\partial}{\partial\zeta_\ell}(\rho_{E_0})(2^{D^*_{j,G,\gamma}}\zeta)\right)d\zeta\right| \leq 2^{-2jH_0}|j|^{2d/\rho_{\min}(E_0)} \ . \tag{18}$$

Combining inequalities (16) and (18) then yield inequality (14).

Remark now that

Lemma A.3 *Let* D_0 *an admissible diagonal anisotropy satisfying* $Tr(D_0) = d$ *and* $\Gamma_j(D_0) = \{k \in \mathbb{Z}^2,\ |k|_{D_0} \leq j2^j\}$. *There exists some* $C_1, C_2 > 0$ *such that*

$$C_1 j^d 2^{jd} \leq \mathrm{card}(\Gamma_j(D_0)) \leq j^d 2^{jd} \ .$$

Proof A.4 Indeed, since the norms $|\cdot|_{\ell_1}$ and $|\cdot|_{\ell^\infty}$ on $\mathbb{R}^d$ are equivalent, there exists some $C_1, C_2 > 0$ such that

$$C_1 \max_\ell |k_\ell|^{1/\lambda_\ell} \leq |k|_{D_0} \leq C_2 \max_\ell |k_\ell|^{1/\lambda_\ell} \ .$$

The conclusion follows since it is quite clear since that

$$\mathrm{card}\{k, \max_\ell |k_\ell|^{1/\lambda_\ell} \leq j2^j\} = \mathrm{card}\{k, \max_\ell |k_\ell| \leq j^{\lambda_\ell}2^{j\lambda_\ell}\} = \prod_\ell \left(j^{\lambda_\ell}2^{j\lambda_\ell}\right) = j^d 2^d$$

using the fact that $\lambda_1 + \cdots + \lambda_\ell = d$.

The proof of Proposition 5.10 is then based on the two following results which are a slight modification of Theorem II.1 and II.7 of [10]. We recall the proofs for completeness.
We denote

$$c_p = \mathbb{E}(|g_{j,G,\gamma}|^p) \ .$$

We can thus state a central limit theorem for the sequence $(g^k_{j,G,\gamma})_{j\in\mathbb{N},(G,\gamma)\in I_j,k\in\Gamma_j}$ which is a slight modified version of Lemma II.4 of [10]

Lemma A.5 *Let* $p \in (1, +\infty)$ *and* $(g^k_{j,G,\gamma})$ *the Gaussian sequence defined by (7). Then almost surely when* $j \to \infty$

$$2^{-n_j}\left(\sum_{k\in\Gamma_j}|g_{j,G,\gamma}|^p\right) \to c_p \ .$$

Proof A.6 By Lemma A.1 the sequence $(g^k_{j,G,\gamma})$ is weakly correlated in the sense of [10]–that is satisfies the assumption (H) of [10]. We follow the main line of [10] and first give an upper bound of

$$\mathbb{E}\left|\sum_{k\in\Gamma_j}(|g_{j,G,\gamma}|^p - c_p)\right|^2 \ .$$

Using the same approach that [10] (see Lemma II.3) we get that

$$\mathbb{E}\left|\sum_{k\in\Gamma_j}\left(|g_{j,G,\gamma}|^p-c_p\right)\right|^2\le C_j c_{2p}\sum_{(k,k')\in\Gamma_j^2}\frac{1}{(1+|k-k'|)^2}\ ,$$

with $C_j=j^{2d/\rho_{\min}(E_0)}$ by weak correlation of the wavelet coefficients. Set $\ell=k-k'$. Hence

$$\sum_{(k,k')\in\Gamma_j^2}\frac{1}{(1+|k-k'|)^2}\le\sum_{k\in\Gamma_j}\sum_{\ell\in 2.\Gamma_j}\frac{1}{(1+|\ell|)^2}\le C_j j2^j\sum_{\ell\in 2.\Gamma_j}\frac{1}{(1+|\ell|)^2}$$

Remark now that

$$\sum_{\ell\in 2.\Gamma_j}\frac{1}{(1+|\ell|)^2}\le\sum_{\ell\in 2.\Gamma_j}\frac{1}{(1+|\ell|_{D_0})^{2/\rho_{\max}}}\le j^{d-\delta}2^{j(d-\delta)}$$

with $\delta=2/\rho_{\max}(E_0)>0$ by comparison with an integral and Proposition 2.3 of [8]. Thereafter the end of the proof is exactly the same than in Theorem II.1 of [10].

In an analogous way, one can give a result on the asymptotic behavior of

$$\frac{1}{\sqrt{|\log(n_j)|}}\left(\max_{k\in\Gamma_j}|g_{j,G,\gamma}|\right)\ .$$

Lemma A.7 *Almost surely*

$$0<\liminf_{j\to\infty}\frac{1}{\sqrt{|\log(n_j)|}}\left(\max_{k\in\Gamma_j}|g_{j,G,\gamma}|\right)\le\limsup_{j\to\infty}\frac{1}{\sqrt{|\log(n_j)|}}\left(\max_{k\in\Gamma_j}|g_{j,G,\gamma}|\right)<\infty$$

Proof A.8 The proof is exactly the same than these of Lemmas II.8 and II.10 of [10].

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