Gaussian fields satisfying simultaneous operator scaling relations

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Summary. In this paper we define a special class of group self-similar Gaussian fields. We present an harmonizable representation of m parameter group self-similar Gaussian field by utilizing Haar measure of this group. These fields have also stationary rectangular increments according to special directions linked to co reduction of matrices of the considered m parameter group.

1 Introduction

Random fields are a useful tool for modelling spatial phenomenon like environmental fields, including for example, hydrology, geology, oceanography and medical images. Many times the chosen model has to include some statistical dependence structure that might be present across the scales. Thus, an usual assumption is self-similarity (see [Lamp62]), defined for a random field $\{X(x)\}_{x\in\mathbb{R}^d}$ on $\mathbb{R}^d$ by

$$\{X(ax)\}_{x\in\mathbb{R}^d} \overset{(f.d.)}{=} \{a^H X(x)\}_{x\in\mathbb{R}^d}$$

for some $H \in \mathbb{R}$ (called the Hurst index). As usual, $\overset{(f.d.)}{=}$ denotes equality of all finite-dimensional marginal distributions. The most famous example of self-similar processes is Fractional Brownian Motion (FBM) $\{B_H(x)\}_{x\in\mathbb{R}^d}$, introduced in 1940 by Kolmogorov (see [Kolm40]) and first studied in the famous paper of Mandelbrot and Van Ness (see [MVN68]).

Moreover in many cases, random fields have an anisotropic nature in the sense that they have different geometric characteristics along different directions (see, for example, Davies and Hall ([DH99]), Bonami and Estrade ([BE03]) and Benson, et al.([Ben06])). The classical notion of self-similarity–by construction isotropic–has then to be changed in order to fit anisotropic situations. For this reason, an increasing interest has been paid in defining a suitable concept for anisotropic self-similarity. Many authors have developed techniques to handle anisotropy in the scaling : In [HM82] Hudson and Mason introduced operator-self-similar processes $\{X(t)\}_{t\in\mathbb{R}}$ with values in $\mathbb{R}^d$. Moreover in [SL85, SL87] Schertzer and Lovejoy introduced a general concept of scaling with respect to a one-parameter group (which may be a matricial one) for random fields. These authors have in mind the particular case of linear case and its application to the study to atmospheric stratification.

In [Kam96], A.Kamont introduced a first example of anisotropic self-similar Gaussian field : Fractional Brownian Sheet (FBS). For any $(H_1,\cdots,H_d)$ in $(0,1)^d$, FBS with Hurst indices $(H_1,\cdots,H_d)$–denoted $\{B_{H_1,\cdots,H_d}(x)\}_{x\in\mathbb{R}^d}$– can be defined through its harmonizable representation (see [ALP02]) :

$$B_{H_1,\cdots,H_d}(x)=\int_{\mathbb{R}^d}\frac{(e^{i<x_1,\xi_1>}-1)\cdots(e^{i<x_d,\xi_d>}-1)}{|\xi_1|^{H_1+\frac{1}{2}}\cdots|\xi_d|^{H_d+\frac{1}{2}}}d\widehat{W}_{\xi_1,\cdots,\xi_d}, \quad (1)$$

where $dW_{x_1,\cdots,x_d}$ is a Brownian measure on $\mathbb{R}^d$ and $d\widehat{W}_{\xi_1,\cdots,\xi_d}$ its Fourier transform. This definition implies that FBS satisfies simultaneous scaling properties : For any $(a_1,\cdots,a_d)\in(\mathbb{R}_+^*)^d$

$$\begin{cases}\{B_{H_1,\cdots,H_d}(a_1x_1,\cdots,x_d)\}_{x\in\mathbb{R}^d}\overset{(f.d.)}{=}\{a_1^{H_1}B_{H_1,\cdots,H_d}(x)\}_{x\in\mathbb{R}^d}\\ \qquad\vdots\\ \{B_{H_1,\cdots,H_d}(x_1,\cdots,a_dx_d)\}_{x\in\mathbb{R}^d}\overset{(f.d.)}{=}\{a_d^{H_d}B_{H_1,\cdots,H_d}(x)\}_{x\in\mathbb{R}^d}\end{cases} \quad (2)$$

Moreover, it follows from definition (1) that FBS admits stationary rectangular increments according to the coordinate axes. For example in the bidimensional case ($d=2$) if we denote

$$\begin{aligned}\varDelta_{h_1,h_2}B_{H_1,H_2}(x_1,x_2)=&\,B_{H_1,H_2}(x_1+h_1,x_2+h_2)-B_{H_1,H_2}(x_1+h_1,x_2)\\&-B_{H_1,H_2}(x_1,x_2+h_2)+B_{H_1,H_2}(x_1,x_2)\end{aligned}$$

then (Proposition 2 of [ALP02]) for any $(x_1,x_2)\in\mathbb{R}^2$:

$$\{\varDelta_{h_1,h_2}B_{H_1,H_2}(x_1,x_2)\}_{(h_1,h_2)\in\mathbb{R}^2}\overset{(f.d.)}{=}\{\varDelta_{h_1,h_2}B_{H_1,H_2}(0,0)\}_{(h_1,h_2)\in\mathbb{R}^2}.$$

Conversely, any Gaussian field $\{X(x)\}_{x\in\mathbb{R}^d}$ of the form

$$X(x)=\int_{\mathbb{R}^d}(e^{i<x_1,\xi_1>}-1)\cdots(e^{i<x_d,\xi_d>}-1)\phi(\xi)d\widehat{W}_{\xi_1,\cdots,\xi_d} \quad (3)$$

with $\int\min(1,|\xi_1|^2)\cdots\min(1,|\xi_d|^2)|\phi(\xi)|^2d\xi<+\infty$, admits stationary rectangular increments according to the coordinate axes (see Section 3). Furthermore if, as in the case of FBS,

$$\forall(a_1,\cdots,a_d)\in(\mathbb{R}_+^*)^d,\ |\phi(a_1\cdots a_d\xi)|^2=a_1^{-2H_1-1}\cdots a_d^{-2H_d-1}|\phi(\xi)|^2, \quad (4)$$

then the Gaussian field $\{X(x)\}_{x\in\mathbb{R}^d}$ defined by (3) satisfies properties (2).

Another model of anisotropic self-similar random field is the class of Operator Scaling Random Fields (OSRF) introduced by H.Biermé, M.Meerschaert and H.P.Scheffler in [BMS07]. These fields satisfy the following scaling relation :

$$\forall a>0,\ \{X(a^E x)\}_{x\in\mathbb{R}^d} \overset{(f.d.)}{=} \{a^H X(x)\}_{x\in\mathbb{R}^d}. \tag{5}$$

for some matrix E (called an anisotropy of the field) whose eigenvalues have a positive real part and some $H>0$ (called an Hurst index of the field). Remind that for any real $a>0$, a^E denotes the matrix $a^E=\exp(E\log(a))=\sum_{k\geq 0}\frac{E^k\log^k(a)}{k!}$. Moreover H.Biermé, M.Meerschaert and H.P.Scheffler defined a special class of OSRF with stationary increments : For any matrix E with positive real parts of the eigenvalues, any $H\in(0,\rho_{\min}(E))$–where $\rho_{\min}(E)=\min_{\lambda\in Sp(E)}(Re(\lambda))$–Gaussian fields with stationary increments satisfying (5) can be defined in the following way

$$X(x)=\int_{\mathbb{R}^d}(e^{i<x,\xi>}-1)\rho(\xi)^{-(H+\frac{Tr(E)}{2})}d\widehat{W}_\xi,$$

where ρ is a $(\mathbb{R}^d,E^t)$ pseudo-norm (see [PGL94]) that is a continuous function defined on $\mathbb{R}^d\setminus\{0\}$ with positives values satisfying :

$$\forall\xi\in\mathbb{R}^d\setminus\{0\},\ \rho(a^{E^t}\xi)=a\rho(\xi).$$

Then, the main difficulty to overcome is to define a suitable $(\mathbb{R}^d,E^t)$ pseudo-norm. In [BMS07], for any matrix E whose eigenvalues have positive real parts, is proved that

$$\rho(\xi)=(\int_{S_0}\int_0^\infty(1-\cos(<\xi,r^E\theta>))\frac{dr}{r^2}d\mu(\theta))$$

is a $(\mathbb{R}^d,E^t)$ pseudo-norm (S_0 denotes the unit sphere of $\mathbb{R}^d$ for a well-chosen norm, and μ a finite measure on S_0). We will generalize this result using another approach based on Haar measure of an m parameter group.

Here, our purpose is to introduce another class of anisotropic Gaussian fields satisfying given simultaneous operator scaling relations : For any $(a_1,\cdots,a_m)\in(\mathbb{R}_+^*)^m$

$$\begin{cases}\{X(a_1^{E_1}x)\}_{x\in\mathbb{R}^d} \overset{(f.d.)}{=} \{a_1^{H_1}X(x)\}_{x\in\mathbb{R}^d},\\ \qquad\vdots\\ \{X(a_m^{E_m}x)\}_{x\in\mathbb{R}^d} \overset{(f.d.)}{=} \{a_m^{H_m}X(x)\}_{x\in\mathbb{R}^d},.\end{cases} \tag{6}$$

where $E_1,\cdots,E_m$ are m given pairwise commuting matrices. We now illustrate through an example the potential usefulness of this model.

Example 1. Let us consider the two commuting matrices

$$E_1 = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; E_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For any $\theta \in \mathbb{R}$, denote $R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ and remark that for any positive numbers a_1, a_2 :

$$a_1^{E_1} = \begin{pmatrix} a_1 R_{\log(a_1)} & 0 \\ 0 & 1 \end{pmatrix}, \; a_2^{E_2} = \begin{pmatrix} R_{\log(a_2)} & 0 \\ 0 & a_2 \end{pmatrix}.$$

As a consequence of Theorem 1 (see Section 4), for all $(H_1, H_2) \in (0,1)^2$ we can define an anisotropic field $\{X(x)\}_{x\in\mathbb{R}^3} = \{X(x_1, x_2, t)\}_{(x_1,x_2,t)\in\mathbb{R}^3}$ such that :

$$\begin{cases} \forall a_1 \in \mathbb{R}_+^*, \{X(a_1^{E_1} x)\}_{x\in\mathbb{R}^3} \overset{(f.d.)}{=} \{a_1^{H_1} X(x)\}_{x\in\mathbb{R}^3}, \\ \forall a_2 \in \mathbb{R}_+^*, \{X(a_2^{E_2} x)\}_{x\in\mathbb{R}^3} \overset{(f.d.)}{=} \{a_2^{H_2} X(x)\}_{x\in\mathbb{R}^3}. \end{cases} \tag{7}$$

Let us make some comments about these two scaling properties of field $\{X(x)\}_{x\in\mathbb{R}^3}$. Assume that x_1, x_2 denote two space variables whereas t denotes times. Then, the first scaling property means that at fixed time $\{X(x)\}_{x\in\mathbb{R}^3}$ is a (maybe anisotropic) operator scaling Gaussian field. The second scaling property means that anisotropy of the field evolves throughout time. In fact, we defined a fixed time anisotropic Gaussian field whose anisotropy rotates with time.

Our objective is now to define such Gaussian fields. In the two following Sections, we present our approach. We first consider the special case of Gaussian fields satisfying simultaneous uncoupled relations.

2 A first example of field satisfying simultaneous operator scaling properties

We first consider a particular case. Let $(d_1, \cdots, d_m) \in \mathbb{N}^m$ such that $d_1 + \cdots + d_m = d$, $(E_1, \cdots, E_m)$ m given matrices in $(M_{d_1}(\mathbb{R}), \cdots, M_{d_m}(\mathbb{R}))$ whose eigenvalues have positive real parts and $(H_1, \cdots, H_m)$ in $\prod_{\ell=1}^{m}(0, \rho_{min}(E_\ell))$.

Combining the model of FBS and this of OSRF, one can easily define a Gaussian field $\{X(x_1, \cdots, x_m)\}_{(x_1,\cdots,x_m)\in\mathbb{R}^{d_1}\times\cdots\times\mathbb{R}^{d_m}}$ satisfying the following simultaneous operator scaling relations : For any $(a_1, \cdots, a_m) \in (\mathbb{R}_+^*)^m$

$$\begin{cases} \{X(a_1^{E_1} x_1, \cdots, x_m)\}_{x\in\mathbb{R}^d} \overset{(f.d.)}{=} \{a_1^{H_1} X(x_1, \cdots, x_m)\}_{x\in\mathbb{R}^d} \\ \qquad \vdots \\ \{X(x_1, \cdots, a_m^{E_m} x_m)\}_{x\in\mathbb{R}^d} \overset{(f.d.)}{=} \{a_m^{H_m} X(x_1, \cdots, x_m)\}_{x\in\mathbb{R}^d} \end{cases} \tag{8}$$

Indeed, the Gaussian field $\{X(x)\}_{x\in\mathbb{R}^d}$ can be defined through its harmonizable representation

$$X(x) = Re(\int_{\mathbb{R}^{d_1}\times\cdots\times\mathbb{R}^{d_m}} \prod_{\ell=1}^{m}((e^{i<x_\ell,\xi_\ell>}-1)(e^{-i<y_\ell,\xi_\ell>}-1)|\phi_\ell(\xi_\ell)|^2)d\widehat{W}_\xi) \quad (9)$$

where for any ℓ, ϕ_ℓ denotes a $(\mathbb{R}^{d_\ell}\setminus\{0\}, E_\ell^t)$ pseudo-norm. By construction, the Gaussian field $\{X(x)\}_{x\in\mathbb{R}^d}$ satisfies simultaneously the m operator scaling relationships (8).
As FBS, this field does not admit stationary increments but satisfies a weaker stationarity property : It admits rectangular stationary increments according to some special directions.
Before giving a precise statement about stationarity properties of the field $\{X(x)\}_{x\in\mathbb{R}^d}$, we first recall the notion of Gaussian field with rectangular stationary increments. Let us begin by defining the notion of rectangular increments of a function :

Definition 1. *Let $M_1,\cdots,M_m$ m subspaces of $\mathbb{R}^d$ in direct sum, f a function defined on $\mathbb{R}^d$. For any $x\in\mathbb{R}^d$ and any $(h_1,\cdots,h_m)\in M_1\times\cdots\times M_m$ define*

$$\Delta_{h_1,\cdots,h_m}f(x) = \sum_{\ell=0}^{m}\sum_{1\le i_1<\cdots<i_\ell\le m}(-1)^{m-\ell}f(x+h_{i_1}+\cdots+h_{i_\ell}).$$

with for $\ell=0$, $f(x+h_{i_1}+\cdots+h_{i_\ell}) = f(x)$.

Example 2. In the case $m=1$, $M_1=\mathbb{R}^d$ $\Delta_h f(x) = f(x+h)-f(x)$.
In the case $m=d=2$, $M_1=\mathbb{R}\times\{0\}$, $M_2=\{0\}\times\mathbb{R}$, if $(h,k)\in M_1\times M_2$

$$\Delta_{h,k}f(x_1,x_2) = f(x_1+h,x_2+k)-f(x_1+h,x_2)-f(x_1,x_2+k)+f(x_1,x_2).$$

Definition 2. *Let $M_1,\cdots,M_m$ m subspaces of $\mathbb{R}^d$ in direct sum. A Gaussian field $\{X(x)\}_{x\in\mathbb{R}^d}$ admits stationary rectangular increments according to the directions $M_1,\cdots,M_m$ if for any $x\in\mathbb{R}^d$*

$$\{\Delta_{h_1,\cdots,h_m}X(x)\}_{(h_1,\cdots,h_m)\in M_1\times\cdots\times M_m} \overset{(\mathcal{L})}{=} \{\Delta_{h_1,\cdots,h_m}X(0)\}_{(h_1,\cdots,h_m)\in M_1\times\cdots\times M_m}.$$

Example 3. In the case $m=1$ we recover the classical notion of a random field with stationary increments.
Bidimensional FBS admits stationary rectangular increments according to the directions $M_1=\mathbb{R}\times\{0\}$ and $M_2=\{0\}\times\mathbb{R}$.

Here, in the example above, the Gaussian field defined by (9) admits stationary rectangular increments according to

$$M_1=\mathbb{R}^{d_1}\times\cdots\times\{0_{\mathbb{R}^{d_m}}\},\cdots,M_m=\{0_{\mathbb{R}^{d_1}}\}\times\cdots\times\mathbb{R}^{d_m}.$$

We now use the special case introduced in this section in order to present a general approach to define Gaussian fields satisfying the m given scaling properties (6).

3 A general approach

In the general case, we will formulate the problem as in Section 2 and define a Gaussian field $\{X(x)\}_{x\in\mathbb{R}^d}$ which admits stationary rectangular increments according to some special directions $(M_1,\cdots,M_{m'})$ in direct sum with $m'\geq m$. These special directions follow from the simultaneous reduction of matrices $E_1,\cdots,E_m$ (see Section 6.2 above) and then are invariant through these matrices, that is : For any j in $\{1,\cdots,m\}$, for any ℓ in $\{1,\cdots,m'\}$ $E_jM_\ell\subset M_\ell$. Subspaces $(M_1,\cdots,M_{m'})$ will be called renormalization directions of Gaussian field $\{X(x)\}_{x\in\mathbb{R}^d}$ and have to be defined.

After definition of these renormalization directions, the Gaussian field $\{X(x)\}_{x\in\mathbb{R}^d}$ will be defined as follows. We are given a function ϕ with positive values satisfying the m following simultaneous properties : For any $a_1,\cdots,a_m$, for almost any ξ in $\mathbb{R}^d$

$$\begin{cases}\phi(a_1^{-E_1}\xi) &= a_1^{H_1+\frac{Tr(E_1)}{2}}\phi(\xi)\\ &\vdots\\ \phi(a_m^{-E_m}\xi) &= a_m^{H_m+\frac{Tr(E_m)}{2}}\phi(\xi)\end{cases} \tag{10}$$

Furthermore, in order that integral (12) exists, we assume that

$$\forall x\in\mathbb{R}^d,\int_{\mathbb{R}^d}|\phi(\xi)|^2\prod_\ell(\min(1,|<\xi,x_\ell>|^2))d\xi<+\infty. \tag{11}$$

Function $|\phi(\xi)|^2$ is then called a spectral density of Gaussian field $\{X(x)\}_{x\in\mathbb{R}^d}$. Thereafter, for any $x=(x_1,\cdots,x_{m'})\in M_1\times\cdots\times M_{m'}$ we set

$$X(x)=Re\left(\int_{\mathbb{R}^d}\prod_{\ell=1}^{m'}(e^{i<x_\ell,\xi>}-1)\phi(\xi)d\widehat{W}_\xi\right). \tag{12}$$

The following proposition proves that the Gaussian field $\{X(x)\}_{x\in\mathbb{R}^d}$ satisfies the required properties

Proposition 1. *Let $(E_1,\cdots,E_m)$ m pairwise commuting matrices. Assume that there exist $(M_1,\cdots,M_{m'})$ m' subspaces of $\mathbb{R}^d$ in direct sum, invariant through the action of matrices $(E_1,\cdots,E_m)$ and ϕ a function with positive values satisfying properties (10) and condition (11). Then $\{X(x)\}_{x\in\mathbb{R}^d}$ defined by (12) is with stationary rectangular increments according to directions $M_1,\cdots,M_{m'}$ and satisfies the m simultaneous operator scaling relations (6).*

Proof Indeed, since condition (11) is satisfied, $\{X(x)\}_{x\in\mathbb{R}^d}$ defined by (12) exists for any $x\in\mathbb{R}^d$. Denote

$$X_1=Re\left(\int_{\mathbb{R}^d}h_1(\xi)d\widehat{W}_\xi\right),\ X_2=Re\left(\int_{\mathbb{R}^d}h_2(\xi)d\widehat{W}_\xi\right),$$

and remark that using corollary 6.3.2 of [ST94], in the special case of Gaussian random variables

$$\mathbb{E}(X_1X_2) = Re\left(\int_{\mathbb{R}^d} h_1(\xi)\overline{h_2(\xi)}d\xi\right).$$

It is then sufficient to prove that $\{Y(x)\}_{x\in\mathbb{R}^d}$ defined as

$$Y(x) = \int_{\mathbb{R}^d} \prod_{\ell=1}^{m'} (e^{i<x_\ell,\xi>} - 1)\phi(\xi)d\widehat{W}_\xi,$$

satisfies the required properties.
Then remark that the homogeneity properties (10) satisfied by ϕ imply that $\{Y(x)\}_{x\in\mathbb{R}^d}$ satisfies (6). Moreover, $\{Y(x)\}_{x\in\mathbb{R}^d}$ admits stationary rectangular increments according to directions $M_1, \cdots, M_{m'}$. Indeed for any $x \in \mathbb{R}^d$, $(h_1, \cdots, h_{m'}) \in M_1 \times \cdots \times M_{m'}$, $(k_1, \cdots, k_{m'}) \in M_1 \times \cdots \times M_{m'}$

$$\mathbb{E}(\Delta_{h_1,\cdots,h_{m'}}Y(x)\overline{\Delta_{k_1,\cdots,k_{m'}}Y(x)}) = \int \prod_{\ell=1}^{m'} (e^{i<h_\ell,\xi>}-1)(e^{-i<k_\ell,\xi>}-1)|\phi(\xi)|^2 d\xi.$$

This expression does not depend on x, then the result follows.
Let us illustrate this proposition by giving an explicit construction of a Gaussian field satisfying the two simultaneous operator scaling properties introduced in example 1 :

Example 4. The notations are those of example 1. Our objective is to define a Gaussian field satisfying the two simultaneous scaling properties (7). Having Proposition 1 in mind, we consider

$$X(x_1,x_2,t) = \int_{\mathbb{R}^3} (e^{i(x_1\xi^1_{space}+x_2\xi^2_{space})}-1)(e^{it\xi_{time}}-1)\phi(\xi^1_{space},\xi^2_{space},\xi_{time})d\widehat{W}_\xi.$$

It will be required that function ϕ satisfies (10) and (11). Let us set

$$\begin{aligned}&\phi(\xi^1_{space},\xi^2_{space},\xi_{time})\\ &= \frac{\left|\xi^1_{space}\cos(\log(|\xi_{space}|\cdot|\xi_{time}|)) - \xi^2_{space}\cos(\log(|\xi_{space}|\cdot|\xi_{time}|))\right|}{|\xi_{space}|^{H_1+1}|\xi_{time}|^{H_2+\frac{1}{2}}}\end{aligned}$$

One can easily check that ϕ fullfills the assumptions (10) and (11) of Proposition 1 and thus the Gaussian field $\{X(x)\}_{x\in\mathbb{R}^3}$ satisfies the two simultaneous scaling properties (7).

We now state the existence results proved in this paper.

4 Existence results

In order to state our existence results some hypotheses are needed about matrices $(E_1, \cdots, E_m)$. We first recall what is real diagonalizable part of a matrix.

4.1 Real diagonalizable part of a matrix

As usual (see [HM82] ou [MS01]), our main tool will be complete additive Jordan decomposition of a matrix. We refer to lemma 7.1, chap 9 of [Helg78] :

Proposition 2. *Any matrix M of $M_d(\mathbb{R})$ can be decomposed into a sum of commuting real matrices $M = D+S+N$ where D is a diagonalizable matrix in $M_d(\mathbb{R})$, S is a diagonalizable matrix in $M_d(\mathbb{C})$ with zero or imaginary complex eigenvalues, and N is a nilpotent matrix.*
Matrix D is called the real diagonalizable part of M.

Below, we give examples of real diagonalizable part of a matrix E.

1. If $E = \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix}$ or $E = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$ then E admits λ as unique eigenvalue and as real diagonalizable part $D = \lambda Id$ where Id denotes the identity matrix.
2. If $E = \begin{pmatrix} \Delta & & 0 \\ & \ddots & \\ 0 & & \Delta \end{pmatrix}$ or $E = \begin{pmatrix} \Delta & I_2 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & I_2 \\ 0 & & & \Delta \end{pmatrix}$ with $\Delta = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$, then E admits exactly two conjugate complex eigenvalues $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$. It implies that $D = \alpha Id$.

Recall that in general, a square real matrix is similar to a block diagonal matrix where each block is a square matrix of the form above. Thus, from the previous examples we can deduce the complete additive Jordan decomposition of any square real matrix. We now state our assumptions on matrices $E_1, \cdots, E_m$.

4.2 Assumptions on matrices $E_1, \cdots, E_m$

Denote $D_1, \cdots, D_m$ the real diagonalizable parts of matrices $E_1, \cdots, E_m$.

Hypotheses 4.1 *We assume that*

1. *Matrices $E_1, \cdots, E_m$ are pairwise commuting matrices.*
2. *Matrices $(D_1, \cdots, D_m)$ are linearly independent matrices in $M_d(\mathbb{R})$.*

Under these assumptions we now state our main result :

Theorem 1. *Assume that Hypotheses (4.1) are satisfied. Then for some* $C_0 \in (0,1)$ *depending only on matrices* $E_1, \cdots, E_m$*, for any* $\ell \in \{1, \cdots, m\}$ *and any* $0 < H_\ell < C_0 \rho_{\min}(E_\ell)$ *there exists* $m' \geq m$*,* m' *subspaces* $M_1, \cdots, M_{m'}$ *of* $\mathbb{R}^d$ *in direct sum and a Gaussian field* $\{X(x)\}_{x \in \mathbb{R}^d}$ *with rectangular stationary increments according to the directions* $M_1, \cdots, M_{m'}$ *satisfying the* m *simultaneous operator scaling properties :*

$$\forall (a_1, \cdots, a_m) \in (\mathbb{R}_+^*)^m, \begin{cases} \{X(a_1^{E_1} x)\}_{x \in \mathbb{R}^d} \overset{(f.d.)}{=} \{a_1^{H_1} X(x)\}_{x \in \mathbb{R}^d} \\ \vdots \\ \{X(a_m^{E_m} x)\}_{x \in \mathbb{R}^d} \overset{(f.d.)}{=} \{a_m^{H_m} X(x)\}_{x \in \mathbb{R}^d} \end{cases} \tag{13}$$

Before any proof of this result, we need first to reformulate our problem in terms of group self-similarity. It will then allow us to use the concept of Haar measure in order to define a spectral density of Gaussian field $\{X(x)\}_{x \in \mathbb{R}^d}$.

5 Reformulation of the problem in terms of group self-similarity

In [KR03], S.Kolodynski and J.Rosinski defined the notion of group self-similar random field. We adapt this definition to our setting

Definition 3. *Let* $\mathcal{A}$ *a subgroup of* $Gl_d(\mathbb{R})$ *and* χ *a continuous mapping from* $\mathcal{A}$ *into* $\mathbb{R}_+^*$*.*
The random field $\{X(x)\}_{x \in \mathbb{R}^d}$ *is* $\mathcal{A}$*-self-similar with coefficient* χ *if*

$$\forall A \in \mathcal{A},\ \{X(Ax)\}_{x \in \mathbb{R}^d} \overset{(f.d.)}{=} \{\chi(A) X(x)\}_{x \in \mathbb{R}^d}. \tag{14}$$

Remark 1. Remark that (see [KR03]) the self-similarity coefficient of a Gaussian field is necessary an homomorphism from $\mathcal{A}$ into $\mathbb{R}_+^*$.

Our problem can now be reformulated in terms of group self-similarity. Let us define the following m parameter group :

$$\mathcal{A} = \{a_1^{E_1} \cdots a_m^{E_m}, (a_1, \cdots, a_m) \in (\mathbb{R}_+^*)^m\}. \tag{15}$$

Proposition 3. *The two problems are equivalent :*

1. *Find sufficient conditions on* $H_1, \cdots, H_m$ *for the existence of a Gaussian field* $\{X(x)\}_{x \in \mathbb{R}^d}$ *satisfying the simultaneous scaling properties (6).*

2. *Find sufficient conditions on homomorphism* χ *for the existence of a Gaussian field* $\{X(x)\}_{x \in \mathbb{R}^d}$ $\mathcal{A}$*-self-similar with coefficient* χ *where* $\mathcal{A}$ *is the* m *parameter group defined by (15).*

The proof of Proposition 3 is based on the complete description of the continuous homomorphisms from $\mathcal{A}$ into $\mathbb{R}_+^*$. More precisely, one can easily check that (a more detailed proof may be found in [Clau08]):

Proposition 4. *If Hypotheses (4.1) are satisfied, the mapping* $(a_1, \cdots, a_m) \mapsto a_1^{E_1} \cdots a_m^{E_m}$ *is a bicontinous isomorphism from* $(\mathbb{R}_+^*)^m$ *into* $\mathcal{A}$.

Since the homomorphisms from $(\mathbb{R}_+^*)^m$ into $\mathbb{R}_+^*$ are well-known, if Hypotheses (4.1) are satisfied, Proposition 4 directly implies that (a more detailed proof may be found in [Clau08])

Proposition 5. *Let* $\mathcal{A}$ *a subgroup of* $Gl_d(\mathbb{R})$ *of the form (15). Assume that Hypotheses (4.1) are satisfied. Then for any continuous homomorphism* χ *from* $\mathcal{A}$ *into* $\mathbb{R}_+^*$*, there exists a unique* $(H_1, \cdots, H_m) \in (\mathbb{R}_+^*)^m$ *such that*

$$\forall (a_1, \cdots, a_m) \in (\mathbb{R}_+^*)^m,\ \chi(a_1^{E_1} \cdots a_m^{E_m}) = a_1^{H_1} \cdots a_m^{H_m}.$$

Then any $\mathcal{A}$-self-similar Gaussian field with coefficient χ, where χ is a continuous mapping from $\mathcal{A}$ into $\mathbb{R}_+^*$, necessary satisfies

$$\forall (a_1, \cdots, a_m) \in (\mathbb{R}_+^*)^m,\ \{X(a_1^{E_1} \cdots a_m^{E_m} x)\}_{x \in \mathbb{R}^d} \overset{(\mathcal{L})}{=} \{a_1^{H_1} \cdots a_m^{H_m} X(x)\}_{x \in \mathbb{R}^d}$$

for some $(H_1, \cdots, H_m) \in (\mathbb{R}_+^*)^m$. We then proved Proposition 3.

Example 5. We now give examples of $\mathcal{A}$-group self-similar Gaussian fields with coefficient χ.

- FBM is group self-similar with $\mathcal{A} = \{aId\}_{a \in \mathbb{R}_+^*}$ and $\chi(aId) = a^H$.
- FBS is group self-similar with $\mathcal{A}$ being the group of diagonal matrices with positive coefficients and $\chi\left(\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}\right) = \lambda_1^{H_1} \cdots \lambda_n^{H_n}$.
- Any OSRF is group self-similar with $\mathcal{A} = \{a^E, a > 0\}$, $\chi(a^E) = a^H$.
- The Gaussian field defined in Section 2 above is also group self-similar with $\mathcal{A} = \left\{\begin{pmatrix} a_1^{E_1} & & 0 \\ & \ddots & \\ 0 & & a_m^{E_m} \end{pmatrix}, \forall i\, a_i > 0\right\}$, $\chi\left(\begin{pmatrix} a_1^{E_1} & & 0 \\ & \ddots & \\ 0 & & a_m^{E_m} \end{pmatrix}\right) = \prod_\ell a_\ell^{H_\ell}$.

6 Definition of the desired Gaussian field

As announced in Section 3, we now define the desired Gaussian field in two steps. We first define a suitable spectral density (see Section 6.1). Thereafter using simultaneous reduction of matrices $E_1, \cdots, E_m$ we define renormalization directions $M_1, \cdots, M_{m'}$ (see Section 6.2). Finally in Section 6.3, we will use Proposition 1 to prove that the Gaussian field we just defined satisfies the required properties.

6.1 Definition of a suitable spectral density

We reformulate the properties required on spectral density using group $\mathcal{A}$ defined by (15)

Proposition 6. *Let ϕ a function defined on $\mathbb{R}^d$ with positive values. The two properties are equivalent :*

1. *Function $|\phi(\xi)|^2$ satisfies the m simultaneous relations (10).*
2. *Function $|\phi(\xi)|^2$ is $\mathcal{A}$ homogeneous with coefficient $\chi^2(\cdot)|\det(\cdot)|$ that is :*

$$\forall A \in \mathcal{A},\ a.e.\xi \in \mathbb{R}^n,\ |\phi((A^{-1})^t\xi)|^2 = |\chi(A)|^2|\det(A)||\phi(\xi)|^2, \tag{16}$$

with $\chi(a_1^{E_1}\cdots a_m^{E_m}) = a_1^{H_1}\cdots a_m^{H_m}$.

We now define a suitable spectral density using a Haar measure $\mu_{\mathcal{A}}$ of group $\mathcal{A}$. Let

$$|\phi(\xi)|^2 = \int_{\mathcal{A}} \chi(A)^2|\det(A)||\widehat{\psi}(-A^t\xi)|^2 d\mu_{\mathcal{A}}(A), \tag{17}$$

where $\psi \in H^{m+1}(\mathbb{R}^d)$. As usual

$$H^{m+1}(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d), |\xi|^{m+1}\widehat{f} \in L^2(\mathbb{R}^d)\}.$$

Then, the invariance property of any Haar measure of group $\mathcal{A}$ implies that :

Proposition 7. *Function $|\phi(\xi)|^2$ defined by (17), is an $\mathcal{A}^t$ homogeneous function with coefficient $\chi^2(\cdot)|\det(\cdot)|$.*

Remark 2. Proposition 7 does not prove that function $|\phi(\xi)|^2$ defined by (17) is finite almost everywhere. The finitness of function $|\phi(\xi)|^2$ comes from the existence of covariance function of the required Gaussian field.

Function $|\phi(\xi)|^2$ defined by (17) then satisfies the required properties of a spectral density. We now give some details about the definition of renormalization directions.

6.2 Definition of renormalization directions

In this Section, our purpose is to define m' special subspaces of $\mathbb{R}^d$ $M_1, \cdots, M_{m'}$ in direct sum, invariant through matrices $E_1, \cdots, E_m$ such that hypotheses of Proposition 1 are fullfilled. These subspaces will be called renormalization directions and are invariant through group $\mathcal{A}$. Proposition 1 then ensures the existence of a Gaussian field satisfying the required properties. Moreover (see Section 2) if

$$\mathcal{A} = \{diag(a_1^{E_1}, \ldots, a_m^{E_m}), (a_1, \cdots, a_m) \in (\mathbb{R}_+^*)^m\}, \tag{18}$$

with $(E_1, \cdots, E_m) \in M_{d_1}(\mathbb{R}) \times \cdots \times M_{d_m}(\mathbb{R})$, whose eigenvalues have positive real parts (see Section 2), we can choose as renormalization directions

$$M_1 = \mathbb{R}^{d_1} \times \cdots \times \{0\}, \cdots, M_m = \{0\} \times \cdots \times \mathbb{R}^{d_m}.$$

Now we want to extend the approach of Section 2 to the general case. In order to define renormalization directions, we simultaneously diagonalize matrices $D_1, \cdots, D_m$ using the following Proposition :

Proposition 8. *Let $E_1, \cdots, E_m$ be m pairwise commuting square matrices. Denote $D_1, \cdots, D_m$ their real diagonalizable parts. Then*

1. *Matrices $D_1, \cdots, D_m$ are pairwise commuting and then simultaneously diagonalizable.*
2. *Matrices $E_1, \cdots, E_m$ are all commuting with matrices $D_1, \cdots, D_m$.*

Definition of renormalization directions will follow from this simultaneous reduction of matrices $D_1, \cdots, D_m$. The following notation will be needed :

Notation 2 *Let $\mathcal{A}$ the group defined by (15). Then denote*

$$\mathcal{A}_D = \{a_1^{D_1} \cdots a_m^{D_m}, (a_1, \cdots, a_m) \in (\mathbb{R}_+^*)^m\}.$$

We can reduce simultaneously matrices of group $\mathcal{A}_D$:

Proposition 9. *Assume that Hypotheses 4.1 are fullfilled. There exists an invertible matrix P such that*

$$\mathcal{A}_D = \{P \times diag(a_1^{\Delta_1}, \ldots, a_m^{\Delta_m}, a_1^{D_1^{m+1}} .. a_m^{D_m^{m+1}}) \times P^{-1}, (a_1, \cdots, a_m) \in (\mathbb{R}_+^*)^m\}$$

where

1. *For any $k \in \{1, \cdots, m\}$, $\Delta_k = \begin{pmatrix} \Delta_k^+ & 0 \\ 0 & \Delta_k^- \end{pmatrix}$.*
2. *For any $k \in \{1, \cdots, m\}$, matrices Δ_k^+ (resp Δ_k^-, D_ℓ^{m+1}) are diagonal matrices with positive coefficients (resp negative, unspecified).*
3. *Matrices Δ_k^+ always exists for any $k \in \{1, \cdots, m\}$ whereas matrices Δ_k^-, D_ℓ^{m+1} can possibly not exist for some values of k or ℓ.*

Proof Proof detailed in in [Clau08].
Let us illustrate Proposition 9 through an example.

Example 6. Let us consider the following group

$$\mathcal{A} = \{a_1^{E_1} a_2^{E_2}, (a_1, a_2) \in (\mathbb{R}_+^*)^2\},$$

with $E_1 = D_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{pmatrix}, E_2 = D_2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
Here $\mathcal{A} = \mathcal{A}_D$, matrices D_1 et D_2 are diagonal and

$$Vect < D_1, D_2 >= Vect < \begin{pmatrix} \Delta & 0 \\ 0 & 5 \end{pmatrix}, \begin{pmatrix} 2\Delta & 0 \\ 0 & 1 \end{pmatrix} >, \text{ with } \Delta = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}.$$

To prove that matrices (D_1, D_2) are linearly independent matrices remark that $(\begin{pmatrix} 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix})$ are linearly independent vectors. Since

$$Vect < \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} >= Vect < \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} >,$$

it implies that

$$Vect < \begin{pmatrix} \Delta & 0 \\ 0 & 5 \end{pmatrix}, \begin{pmatrix} 2\Delta & 0 \\ 0 & 1 \end{pmatrix} >= Vect < \begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} > .$$

Hence we deduce that $\mathcal{A} = \{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_1^{-2} & 0 \\ 0 & 0 & a_2 \end{pmatrix}, (a_1, a_2) \in (\mathbb{R}_+^*)^2 \}$. Thus we recover the result of Proposition 9 with $\Delta_1^+ = \left(1\right), \Delta_1^- = \left(-2\right), \Delta_2^+ = \left(2\right)$.

Proposition 9 implies the following description of group $\mathcal{A}$:

Proposition 10. *We use notations of Proposition 9. Group $\mathcal{A}$ is of the form*

$$\mathcal{A} = \{ P a_1^{F_1} \cdots a_m^{F_m} P^{-1}, (a_1, \cdots, a_m) \in (\mathbb{R}_+^*)^m \},$$

where for any ℓ, matrix F_ℓ admits as real diagonalizable part the diagonal matrix $diag(0, \cdots, 0, \Delta_\ell, 0, \cdots, D_\ell^{m+1})$.

Proof Proof detailed in [Clau08].
Proposition 10 will allow us to define renormalization directions. Let us define for any $\ell \in \{1, \cdots, m\}$, $W_\ell^+ = \mathbb{R}^{d_1^+}$, $W_\ell^- = \mathbb{R}^{d_1^-}, W_{m+1} = \mathbb{R}^{d_{m+1}}$. We then choose as renormalization directions

$$\forall \ell \in \{1, \cdots, m\}, V_\ell^+ = P^{-1} W_\ell^+, V_\ell^- = P^{-1} W_\ell^-, V_{m+1} = P^{-1} W_{m+1}. \quad (19)$$

These subspaces are all invariant through matrices $E_1, \cdots, E_m$. The sets V_l^-, V_{m+1} can be possibly equal to $\{0\}$.

Notation 3 *Denote $M_1, \cdots, M_{m'}$ the m' non zero sets within the spaces $V_1^+, \cdots, V_m^+, V_1^-, \cdots, V_m^-, V_{m+1}$. Subspaces $M_1, \cdots, M_{m'}$ are called non trivial renormalization directions.*

Example 7. In example 6 above, the renormalization directions are

$$M_1^+ = \mathbb{R} \times \{0\} \times \{0\}, \; M_1^- = \{0\} \times \mathbb{R} \times \{0\}, \; M_2^+ = \{0\} \times \{0\} \times \mathbb{R}.$$

In the following Section, we prove that this construction method is effective.

6.3 Proof of Theorem 1

Consider function $|\phi(\xi)|^2$ defined by (17) and the renormalization directions $M_1, \cdots, M_{m'}$ defined in Section 6.2. We want to find sufficient conditions on χ in order that condition (11) of Proposition 1 is fullfilled. Let us first remark that :

Lemma 1. *Let $\mathcal{A}_P$ the following m parameter group*

$$\mathcal{A}_P = P^{-1}\mathcal{A}P = \{a_1^{F_1} \cdots a_m^{F_m}, (a_1, \cdots, a_m) \in (\mathbb{R}_+^*)^m\},$$

and χ_P defined as $\chi_P(A_P) = \chi(PA_PP^{-1})$ for any $A_P \in \mathcal{A}_P$. Condition (11) is satisfied iff for any $x \in \mathbb{R}^d$ the following integral

$$\int_{\mathbb{R}^d} \prod_{\ell} (\min(1, |< x_{W_l^+}, \zeta > |^2) \min(1, |< x_{W_l^-}, \zeta > |^2)) |\phi_{P^t}(\zeta)|^2 d\zeta, \quad (20)$$

is finite with $|\phi_{P^t}(\cdot)|^2 = |\phi(P^t \cdot)|^2 = \int \chi_P^2(A_P) |\det(A_P)| |\widehat{\psi_P}(-A_P^t \cdot)|^2 d\mu_{\mathcal{A}_P}(A_P)$ and $\psi_P(\cdot) = \psi(P\cdot)$.

Remark 3. In the proof of the existence of the desired Gaussian field, one can then replace $\mathcal{A}$ by $\mathcal{A}_P$, χ by χ_P and ψ by ψ_P.

Proof To prove this result, we perform the changes of variable $\xi = P^t\zeta$, $A_P = PAP^{-1}$.
This Lemma leads us to consider a special case :

Proposition 11. *Notations are those of proposition 10. Assume that for any $i \in \{1, \cdots, m\}$, $-\rho_{\min}(\Delta_i^-) < H_i' < \rho_{\min}(\Delta_i^+)$, with $\rho_{min}(\Delta_i^-) = 0$ if matrix Δ_i^- does not exist. Then condition 20 is satisfied for χ_P defined from $\mathcal{A}_P$ into $\mathbb{R}_+^*$ as $\chi(a_1^{F_1} \cdots a_m^{F_m}) = a_1^{H_1'} \cdots a_m^{H_m'}$.*

Proof Proof detailed in [Clau08].
Lemma 1 and Proposition 11 then implies Theorem 1 (see [Clau08] for more details).

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