THE SCHWARZIAN DERIVATIVE ON SYMMETRIC SPACES OF CAYLEY TYPE

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Abstract. Let M be a symmetric space of Cayley type and f a conformal diffeomorphism of M. We study a relationship between the conformal factor of f and a generalized Shwarzian derivatine of f.

1. Introduction

Let $M = G/H$ be a Riemannian symmetric space and let $\underline{\mathbf{g}}$ be a $G-$invariant metric on M. Let f be a conformal transformation of $(M, \underline{\mathbf{g}})$, *i.e.*

$$f^*\underline{\mathbf{g}} = \mathbf{c}_f\underline{\mathbf{g}}$$

where $\mathbf{c}_f$ is the conformal factor of f.

When $M = \mathcal{H}$ is the single sheeted hyperboloid (with the unique Lorentz metric), then it is well known that $\mathcal{H}$ is conformally equivalent to $S^1 \times S^1 \setminus \Delta_{S^1}$ where S^1 is the unit cirle and Δ_{S^1} the null space. The group of (orientation preserving) conformal diffeomorphisms of $\mathcal{H}$ is $\mathrm{Diff}(S^1)$, the group of diffeomorphisms of the circle S^1. In [KS] Kostant and Sternberg, pointed out an interesting relationship between the Shwarzian derivative of a transformation $f \in \mathrm{Diff}(S^1)$ and the corresponding conformal factor $\mathbf{c}_f$ (which is a singular function on the null space). More precisely, they prove that $\mathbf{c}_f$ tends to 1 on $\mathcal{H}$ as we approach infinity, and that the Hessain of the (extended) $\mathbf{c}_f$ is the Schwarzian derivative of f.

The single sheeted hyperboloid is the simplest example of a large class of para-hermitian symmetric spaces, the Cayley type spaces. Those spaces are characterized as an open orbit in $S \times S$ where S is the Shilov boundary of a bounded symmetric domain of tube type [K_1]. In this paper, we will use this characterization to extend the results of Kostant and Sternberg to symmetric spaces of Cayley type.

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2. Causal symmetric spaces, Symmetric spaces of Cayley type

Let M be a smooth $n-$manifold. A causal structure on M is a cone field $\mathcal{C} = (C_p)_{p\in M}$ where

$$C_p \subset T_pM$$

is a causal cone *ie.* non-zero, closed convex cone which is pointed $(C_p \cap -C_p = \{0\})$, generating $(C_p - C_p = T_pM)$ and such that C_p depends smoothly on p.

If $M = G/H$ is a homogeneous space, where G is a Lie group and $H \subset G$ a closed subgroup, then the causal structure is said to be $G-$invariant if for any $g \in G$

$$C_{g\cdot x} = Dg(x)(C_x), \ \text{ for } x \in M,$$

where $Dg(x)$ is the derivative of g at x.

Let $M = G/H$ be a symmetric space, *ie.* there exists an involution σ of G such that $(G^\sigma)^\circ \subset H \subset G^\sigma$ where $(G^\sigma)^\circ$ is the identity component of G^σ.

Let $\mathfrak{g} = \mathrm{Lie}(G)$ be the Lie algebra of G and put

$$\mathfrak{h} = \mathfrak{g}(+1,\sigma), \ \ \mathfrak{q} = \mathfrak{g}(-1,\sigma)$$

the eigenspaces of σ. Then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ and the tangent space $T_{x_0}M$ at $x_0 = 1H$ can be identified with $\mathfrak{q}$. In this identification, the derivative $Dg(x_0)$, $g \in H$ corresponds to $\mathrm{Ad}(g)$. Therefore an invariant causal structure on M is determined by a causal cone C in $\mathfrak{q}$ which is $\mathrm{Ad}(H)-$invariant.

Suppose that G is be semi-simple with a finite centre and that the pair $(\mathfrak{g}, \mathfrak{h})$ is irreducible (*ie.* there is no non-trivial ideal in $\mathfrak{g}$ which is invariant under σ). Then, there exists a Cartan involution θ commuting with the given involution σ.

Let K be the corresponding maximal compact subgroup of G. Let $\mathfrak{k} = \mathfrak{g}(+1,\theta)$, $\mathfrak{p} = \mathfrak{g}(-1,\theta)$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the corresponding Cartan decomposition of $\mathfrak{g}$.

Let $\mathrm{Cone}_H(\mathfrak{q})$ be the set of $\mathrm{Ad}(H)-$ invariant causal cones in $\mathfrak{q}$. Then following Ólafsson (see [H-Ó]) the irreducible symmetric space M is called

(CC): Compactly Causal space if there exists a $C \in \mathrm{Cone}_H(\mathfrak{q})$ such that $C^\circ \cap \mathfrak{k} \neq \emptyset$.
(NCC): Non-Compactly Causal space if there exists a $C \in \mathrm{Cone}_H(\mathfrak{q})$ such that $C^\circ \cap \mathfrak{p} \neq \emptyset$.
(CT): Cayley Type space if it is (CC) and (NCC).

The simplest example of Cayley type symmetric spaces is the space $M = SO_0(2,1)/SO_0(1,1)$ realized as the one-sheeted hyperboloid

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 - x_3^2 = 1\}.$$

3. Causal compactification of symmetric spaces of Cayley type

Let $D = G/K \subset V_{\mathbb{C}} = V + iV \simeq \mathbb{C}^n$ be a bounded symmetric domain in $\mathbb{C}-$vector space. The group G is the identity component of the group of holomorphic automorphisms of D, and $K = \{g \in G \mid g \cdot o = o\}$ the stabilizer of the base point $o = 1K \in D$, which is a maximal compact subgroup of G.

Suppose that D is of tube type,

$$D \simeq T_\Omega = V + i\Omega$$

where Ω is the symmetric cone of the Euclidean Jordan algebra V.

Let $z \mapsto \bar{z}$ the complex conjugation in $V_{\mathbb{C}}$ with respect to V and e the unit element of V. The set

$$S = \{z \in V_{\mathbb{C}} \mid \bar{z}z = e\}$$

is a connected submanifold of $V_{\mathbb{C}}$ which is a Riemannian symmetric space of compact type

$$S \simeq U/U_e,$$

where U is the identity component of the group of linear transformations $g \in GL(V_{\mathbb{C}})$ such that $gS = S$, and U_e is the stabilizer subgroup of $e \in S$.

There exists on $V_{\mathbb{C}}$ a $U-$invariant spectral norm $z \mapsto |z|$, and one can prove (see [F-K]) that D is the unit disc

$$D = \{z \in V_{\mathbb{C}} \mid |z| < 1\}$$

and S is its Shilov boundary.

Let

$$G(\Omega) = \{g \in \mathrm{GL}(V) \mid g\Omega = \Omega\}.$$

It is a reductive Lie group which acts transitively on Ω. Let $G_0 = G(\Omega)^\circ$ be the identity component of $G(\Omega)$.

Let G^c be the identity component of the group of holomorphic automorphisms of T_Ω. Then G_0 is a Lie subgroup of G^c. The subgroups G_0 and $N^+ = \{t_v : z \mapsto z + v,\ v \in V\}$, together with the inversion $j : z \mapsto -z^{-1}$, generate the group G^c.

For any $x \in V$ we define the quadratic representation $P(x)$ of Jordan algebra V given by $P(x) = 2L(x)^2 - L(x^2)$, where $L(x)$ is the multiplication by x.
The Lie algebra $\mathfrak{g}^c$ of G^c is the set of vector fields on V of the form

$$X(z) = u + Tz + P(z)v \simeq (u, T, v),$$

where T is linear and $u, v \in V$.

Consider on G^c the involutions

$$\begin{aligned} \sigma^c(g) &= \nu \circ g \circ \nu \\ \theta^c(g) &= (-\nu) \circ g \circ (-\nu) \end{aligned}$$

where $\nu : z \mapsto \bar{z}^{-1}$. We use the same letters for the corresponding involutions on the Lie algebra $\mathfrak{g}^c$.
If $X = (u, T, v) \in \mathfrak{g}^c$, then

$$\sigma^c(X) = (v, -T^*, u) \text{ and } \theta^c(X) = (-v, -T^*, -u),$$

Therefore,

$$\begin{aligned} \mathfrak{h}^c &:= \mathfrak{g}^c(\sigma^c, +1) = \{(u, T, u) \mid u \in V,\ T \in \mathfrak{k}_0\} \\ \mathfrak{q}^c &:= \mathfrak{g}^c(\sigma^c, -1) = \{(u, L(v), -u) \mid u, v \in V\},\ (L(v)x = vx) \\ \mathfrak{k}^c &:= \mathfrak{g}^c(\theta^c, +1) = \{(u, T, -u) \mid u \in V,\ T \in \mathfrak{k}_0\} \\ \mathfrak{p}^c &:= \mathfrak{g}^c(\theta^c, -1) = \{(u, L(v), u) \mid u \in V\}. \end{aligned}$$

Consider the convex cones in $\mathfrak{q}^c$

$$\begin{aligned} C_1 &= \{(u, L(v), -u) \mid u + v \in -\bar{\Omega}, u - v \in \bar{\Omega}\}, \\ C_2 &= \{(u, L(v), -u) \mid u + v \in \bar{\Omega}, u - v \in \bar{\Omega}\}. \end{aligned}$$

Then C_1 and C_2 are $\mathrm{Ad}(H^c)-$invariant causal cones and

$$C_1 \cap \mathfrak{p}^c \neq \emptyset,\ C_2 \cap \mathfrak{k}^c \neq \emptyset.$$

Let $c : z \mapsto i(e + z)(e - z)^{-1}$ be the Cayley transform corredponding to the bounded symmetric domain D. Then we have

Theorem 3.1 ([K$_1$, K$_2$]). (1) *$H := c^{-1} \circ G_0 \circ c = H^c := G \cap G^c$.*

(2) *$M = G/H \simeq G^c/H^c$ is a symmetric space of Cayley type, and every Cayley type space is given in this way.*

Let

$$\Delta_S = \{(z,w) \in S \times S \mid \Delta(z-w) = 0\}$$

be the null space of $S \times S$, where Δ is the determinant function of V, extended to $V_{\mathbb{C}}$.

The group G acts diagonally on $S \times S$. Furthermore,

Theorem 3.2 ([K$_1$, K$_2$]). *G acts transitively on $S \times S \setminus \Delta_S$ and the stabilizer of the base point $(e,-e) \in S \times S \setminus \Delta_S$ is the subgroup H. Therefore, $M = G/H \simeq S \times S \setminus \Delta_S$ and $S \times S$ is the (causal) compactification of M.*

For example,

$$\begin{aligned} D &= SU(n,n)/S(U(n)\times U(n)) \\ &= \{z \in \mathrm{Mat}(n,\mathbb{C}) \mid I_n - z^*z \gg 0\} \\ S &= U(n) \qquad\qquad\qquad\qquad \text{and} \\ M &= SU(n,n)/GL(n,\mathbb{C}) \times \mathbb{R}^+ \\ &\simeq \{(z,w) \in U(n)\times U(n) \mid \mathrm{Det}(z-w) \neq 0\}, \end{aligned}$$

$$\begin{aligned} D &= Sp(n,\mathbb{R})/U(n) \\ &= \{z \in \mathrm{Sym}(n,\mathbb{C}) \mid I_n - z^*z \gg 0\} \\ S &= U(n)/O(n) \\ &= \{z \in U(n) \mid z^t = z\} \\ M &= Sp(n,\mathbb{R})/GL(n,\mathbb{R}) \times \mathbb{R}^+ \\ &\simeq \{(z,w) \in U(n)\times U(n) \mid z^t = z, w^t = w,\ \mathrm{Det}(z-w) \neq 0\}. \end{aligned}$$

4. The Schwarzian derivative on the one-sheeted hyperboloid

Recall the classical cross-ratio of four points in the complex plane

$$[z_1,z_2,z_3,z_4] = \frac{z_1-z_3}{z_2-z_3} : \frac{z_1-z_4}{z_2-z_4}$$

Then the Élie Cartan formula for the cross-ratio is given by

Theorem 4.1 (Élie Cartan). *Consider $f : S^1 \to S^1$ and four points $x_1,x_2,x_3,x_4 \in S^1$ tending to $x \in S^1$. Then*

$$\begin{aligned} \tfrac{[f(x_1),f(x_2),f(x_3),f(x_4)]}{[x_1,x_2,x_3,x_4]} - 1 &= \tfrac{1}{6}S(f)(x)(x_1-x_2)(x_3-x_4) \\ &\qquad + [\text{ higher order terms }] \end{aligned}$$

where $S(f)$ denote the Schwarzian derivative of f,

$$S(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2.$$

Consider now the the single sheeted hyperboloid

$$\begin{aligned}\mathcal{H} &= SL(2,\mathbb{R})/\mathbb{R}_+^* \\ &\simeq SU(1,1)/SO(1,1) \\ &\simeq S^1\times S^1\setminus \Delta_{S^1} \\ &= \{(e^{i\theta_1},e^{i\theta_2})\ :\ \theta_1\not\equiv\theta_2\}.\end{aligned}$$

$\mathcal{H}$ carries a (unique up to multiplicative constant) Lorentz metric, $\underline{\mathbf{g}}$, invariant under $SL(2,\mathbb{R})$,

$$\underline{\mathbf{g}} = \frac{d\theta_1 d\theta_2}{|e^{i\theta_1}-e^{i\theta_2}|^2}.$$

Let $f : S^1 \to S^1$ be a diffeomorphism viewed as a conformal transformation of $(\mathcal{H},\underline{\mathbf{g}})$, $f^*\underline{\mathbf{g}} = \mathbf{c}_f\underline{\mathbf{g}}$. Applying the Cartan formula, when $\theta_1,\theta_2\to\theta$, we get

$$\begin{aligned}\mathbf{c}_f(\theta_1,\theta_2)-1 &= \tfrac{f^*\underline{g}(\theta_1,\theta_2)}{\underline{g}(\theta_1,\theta_2)}-1 \\ &= \tfrac{1}{6}S(f)(e^{i\theta})(e^{i\theta_1}-e^{i\theta_2})^2+\dots\end{aligned}$$

Then we have

Theorem 4.2 (Kostant-Sternberg [KS])**.** *The conformal factor $\boldsymbol{c}_f\to 1$ as $(\theta_1,\theta_2)\to\Delta_{S^1}$. In the other word $\boldsymbol{c}_f$ tends to 1 on $\mathcal{H}$ as we approach the infinity.*
So let us extend $\boldsymbol{c}_f$ to be defined on $S^1\times S^1$ by setting it equal to 1 on Δ_{S^1}. Then $\boldsymbol{c}_f$ is twice differentiable on $S^1\times S^1$, and has Δ_{S^1} as critical manifold and the Hessian Hess$(\boldsymbol{c}_f)$ is equal to $S(f)$.

5. The Schwarzian derivative on symmetric spaces of Cayley type

The Kantor [Kan] cross-ratio for z_1,z_2,z_3,z_4 in $V_{\mathbb{C}}$, is the rational function

$$[z_1,z_2,z_3,z_4] = \frac{\Delta(z_1-z_3)}{\Delta(z_2-z_3)} : \frac{\Delta(z_1-z_4)}{\Delta(z_2-z_4)}$$

where Δ is the determinant function of V (extented to $V_{\mathbb{C}}$).

The cross-ratio is invariant under the group G^c (when it is well defined) : The invariance under translations is clear. The invariance under the group G_0 follows from the relation $\Delta(gz)=\chi(g)\Delta(z)$ where χ is a character of G_0. The invariance under the inversion follows from the Hua identity $\Delta(w^{-1}-z^{-1})=\Delta(z)^{-1}\Delta(z-w)\Delta(w)^{-1}$.

On the Cayley type symmetric space $M \simeq S \times S \setminus \Delta_S$ there exists a $G-$invariant measure

$$\underline{\mathbf{g}} = |\Delta(z-w)|^{-2\frac{n}{r}} d\sigma(z) d\sigma(w),$$

where n is the dimension of V and r its rank.

Let $\langle \cdot, \cdot \rangle$ be the inner product of the Euclidean Jordan algebra V extented to a Hermitian inner product of $V_{\mathbb{C}}$.

Let $f : V_{\mathbb{C}} \to V_{\mathbb{C}}$ be a map of class C^3. Let $z_j = z + ta_j u$ be four points tending to $z \in \bar{D}$, where $t \in \mathbb{R}$ and $a_i \in \mathbb{R}$ for $j = 1, 2, 3, 4$.

Theorem 5.1. *For any $\alpha \in \mathbb{R}$ we have*

$$\frac{[f(z_1), f(z_2), f(z_3), f(z_4)]^\alpha}{[z_1, z_2, z_3, z_4]^\alpha} - 1 = \alpha t^2 (a_1 - a_2)(a_3 - a_4) S(f)(z) + o(t^3)$$

where

$$S(f) = \frac{1}{6}\langle f''', f'^{-1}\rangle - \frac{1}{4}\langle P(f'') f'^{-1}, f'^{-1}\rangle$$

with $f' = Df(z)u$, $f'' = D^2 f(z)(u,u)$ and $f''' = D^3 f(z)(u,u,u)$

One can also prove

Theorem 5.2. *Let f be an orientation-preserving diffeomorphism of $(M, \underline{\boldsymbol{g}})$. Then*

(1) *$\boldsymbol{c}_f(z,w) \to 1$ as $z \to w$ and $\boldsymbol{c}_f$ extends smoothly to $S \times S$ and has, moreover, Δ_S as its critical set.*
(2) *The Schwarzian $S(f)$ completely determines $\boldsymbol{c}_f$.*

The complete proofs will appear in a forthcoming paper.

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