

A Polynomial Solution to the Model Matching Problem of Nonlinear Time-delay Systems

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Abstract—In this paper the model matching problem is considered for single input single output nonlinear systems with delays. A full characterization for its solvability is provided within a simple class of compensators. The approach is developed by means of the transfer functions of nonlinear time-delay systems. The state elimination problem for the systems given by their state-space representation is discussed as well.

I. INTRODUCTION

An algebraic formalism of differential forms, originally developed for nonlinear systems without delays [6], [2], was recently extended to the case of time-delay systems [22], [24], [25], [26], [31] and was shown to be effective in solving control problems like accessibility and observability, disturbance decoupling, feedback linearization and others. On the other side, in the case of systems without delays, there exists, in comparison to the machinery of one-forms, an alternative approach in which the system properties are described by skew polynomials from non-commutative polynomial rings. Such polynomials act as differential [33], [34] or shift [20] operators on the differentials of the system inputs and outputs. The polynomial approach to nonlinear systems shows a great similarity to methods well known from the linear theory, see for instance [30]. In particular, one can introduce even a notion of a transfer function of a nonlinear system as was recently shown in [10], [13], [14], [15], [32]. Such a concept is equivalent to that of [7] for linear time-varying systems which allows us to associate to a nonlinear system the tangent (or variational) linear system, see for instance [8], over Kähler differentials [19] except that now the time-varying coefficients of the polynomials are not necessarily independent [21]. The transfer function formalism of nonlinear systems generalizes well known results valid for linear time invariant systems and was, for instance, already employed in [29] to investigate some structural properties of nonlinear systems, in [18] to study the nonlinear model matching problem, in [16] to study the observer design and in [17] to study the realization problem of nonlinear systems. The polynomial approach to nonlinear systems and the notion of a transfer function was recently carried over also to the case of time-delay systems [11], [12] where some basic

properties were discussed. In this paper, the model matching problem of nonlinear time-delay systems is solved within such a transfer function formalism. As it was done in [12], a more detailed discussion is provided below, concerning also structural properties as the notions of a relative degree and a relative shift. In addition, we also depict a possible solution to the state elimination problem for the nonlinear time-delay case.

II. TRANSFER FUNCTIONS OF NONLINEAR TIME-DELAY SYSTEMS

We will use the algebraic formalism of [22], [24], [25], [26], [31] which extends the concept of differential one-forms to the case of time-delay systems and of [11], [12] which introduces transfer functions of nonlinear time-delay systems.

In this paper we restrict our attention to the SISO nonlinear time-delay systems defined by an input-output equation of the form

$$y^{(n)}(t) = \varphi(\{y^{(k)}(t-i), u^{(l)}(t-j)\}) \quad (1)$$

where $0 \leq k \leq n-1$; $0 \leq l \leq n$; $i, j \geq 0$ and $u \in \mathbf{R}$ and $y \in \mathbf{R}$ denote input and output to the system and φ is assumed to be an element of the field of meromorphic functions \mathcal{K} .

Note that it is not restrictive to assume $i, j \in \mathbf{N}$ as all commensurable delays can be considered as multiples of an elementary delay h [25].

Remark 1: In the case of systems without delays even if one starts with a state-space representation it is always possible to eliminate the state variables to get an input-output equation, see for instance [6]. However, it is not completely clear how one can carry over the idea of the state elimination procedure to the time-delay case. A special case is discussed in Appendix I. In the general case, possible drawbacks may be met in the state elimination process for system (10). For the sake of simplicity, here we assume that the system (10) admits an input-output equation of the form (1).

The Ore algebra $\mathcal{K}[\delta, s]$ of polynomials in s and δ over \mathcal{K} with the usual addition, and the (non-commutative) multiplications given by the commutation rules

$$\begin{aligned} sa(t) &= a(t)s + \dot{a}(t) \\ \delta a(t) &= a(t-1)\delta \\ s\delta &= \delta s \end{aligned} \quad (2)$$

where $a(t) \in \mathcal{K}$, represents the ring of linear differential time-delay operators that act over the vector space of one-

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forms $\mathcal{E} = \text{span}_{\mathcal{K}}\{d\xi(t); \xi(t) \in \mathcal{K}\}$ in the following way

$$\left(\sum_{i,j} a_{ij} \delta^j s^i \right) v(t) = \sum_{i,j} a_{ij} v^{(i)}(t-j)$$

for any $v(t) \in \mathcal{E}$.

The commutation rules (2) actually represent the rule for differentiating and, respectively, time-delaying.

Proposition 1 (Ore condition): For all non-zero $a, b \in \mathcal{K}[\delta, s]$, there exist non-zero $a_1, b_1 \in \mathcal{K}[\delta, s]$ such that $a_1 b = b_1 a$.

Thus, the ring $\mathcal{K}[\delta, s]$ can be embedded to the non-commutative quotient field $\mathcal{K}\langle\delta, s\rangle$ by defining quotients [27], [28] as

$$\frac{a}{b} = b^{-1} \cdot a$$

The addition and multiplication in $\mathcal{K}\langle\delta, s\rangle$ are defined as

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} = \frac{\beta_2 a_1 + \beta_1 a_2}{\beta_2 b_1}$$

where $\beta_2 b_1 = \beta_1 b_2$ by Ore condition and

$$\frac{a_1}{b_1} \cdot \frac{a_2}{b_2} = \frac{\alpha_1 a_2}{\beta_2 b_1} \quad (3)$$

where $\beta_2 a_1 = \alpha_1 b_2$ again by Ore condition.

Due to the non-commutative multiplications (2) they, of course, differ from the usual rules. In particular, in case of the multiplication (3) we, in general, cannot simply multiply numerators and denominators, nor cancel them in a usual manner. We neither can commute them as the multiplication in $\mathcal{K}\langle\delta, s\rangle$ is non-commutative as well.

Once the fraction of two skew polynomials is defined we can introduce the transfer function of the nonlinear time-delay system (1) as an element $F(\delta, s) \in \mathcal{K}\langle\delta, s\rangle$ such that $dy(t) = F(\delta, s)du(t)$.

After differentiating (1) we get

$$\begin{aligned} dy^{(n)}(t) - \sum_{\substack{k=0 \\ i \geq 0}}^{n-1} \frac{\partial \phi}{\partial y^{(k)}(t-i)} dy^{(k)}(t-i) &= \\ &= \sum_{\substack{k=0 \\ j \geq 0}}^{n-1} \frac{\partial \phi}{\partial u^{(k)}(t-j)} du^{(k)}(t-j) \end{aligned}$$

or alternatively

$$a(\delta, s)dy(t) = b(\delta, s)du(t) \quad (4)$$

where $a(\delta, s) = s^n - \sum_{k=0, i \geq 0}^{n-1} \frac{\partial \phi}{\partial y^{(k)}(t-i)} \delta^i s^k$ and $b(\delta, s) = \sum_{k=0, j \geq 0}^{n-1} \frac{\partial \phi}{\partial u^{(k)}(t-j)} \delta^j s^k$ are in $\mathcal{K}[\delta, s]$. Then

$$\begin{aligned} F(\delta, s) &= \frac{b(\delta, s)}{a(\delta, s)} \quad (5) \\ &= \frac{b_m(\delta)s^m + \dots + b_1(\delta)s + b_0(\delta)}{s^n + a_{n-1}(\delta)s^{n-1} + \dots + a_1(\delta)s + a_0(\delta)} \end{aligned}$$

where $m = \max\{i \geq 0; \partial \phi / \partial u^{(i)} \neq 0\}$.

Example 1: Consider the system

$$\dot{y}(t) = \dot{y}(t-1)u(t-1)$$

After differentiating

$$\begin{aligned} d\dot{y}(t) - u(t-1)d\dot{y}(t-1) &= \dot{y}(t-1)du(t-1) \\ (s^2 - u(t-1)\delta s)dy(t) &= \dot{y}(t-1)\delta du(t) \end{aligned}$$

and the transfer function is

$$F(\delta, s) = \frac{\dot{y}(t-1)\delta}{s^2 - u(t-1)\delta s} \quad (6)$$

The reader is referred to [10], [11], [12], [13], [14], [15], [18] for more detailed description of the transfer function formalism in nonlinear control theory and to [3], [4], [5], [11], [12] for more detailed description of Ore rings and Ore algebras.

III. STRUCTURAL PROPERTIES

Transfer functions of control systems are well known from linear theory. Their use is the most natural for a several control problems, for they give the insight into the system structure. Recently, it was shown that such an algebraic formalism can be generalized also to nonlinear systems without delays [10], [14] and even to the time-delay case [12]. In this section we, in comparison to [12], discuss additional structural properties of transfer functions of nonlinear time-delay systems.

In dealing with polynomials from $\mathcal{K}[\delta, s]$ we define their (highest) degrees and lowest degrees in s and in δ as

$$\begin{aligned} \deg_s a(\delta, s) &= \max \left\{ k \geq 0; \frac{\partial a(\delta, s)}{\partial s^k} \neq 0 \right\} \\ \deg_\delta a(\delta, s) &= \max \left\{ k \geq 0; \frac{\partial a(\delta, s)}{\partial \delta^k} \neq 0 \right\} \\ \text{low deg}_s a(\delta, s) &= \min \left\{ k \geq 0; \frac{\partial a(\delta, s)}{\partial s^k} \neq 0 \right\} \\ \text{low deg}_\delta a(\delta, s) &= \min \left\{ k \geq 0; \frac{\partial a(\delta, s)}{\partial \delta^k} \neq 0 \right\} \end{aligned}$$

where $0 \neq a(\delta, s) \in \mathcal{K}[\delta, s]$. If $a(\delta, s) = 0$ then define $\deg_s a(\delta, s) = \deg_\delta a(\delta, s) = -\infty$ and $\text{low deg}_s a(\delta, s) = \text{low deg}_\delta a(\delta, s) = \infty$.

A. Structural indices

Certain structural indices of a control system play a key role in solving a number of control problems. One of them is represented by the notion of a relative degree, originally introduced for the case of systems without delays. It can be easily extended also to the time-delay case [26]:

Definition 1: The relative degree of the system (10) is defined as

$$\rho = \min \left\{ k \geq 0; \frac{\partial y^{(k)}(t)}{\partial u(t-j)} \neq 0 \text{ for some } j \geq 0 \right\}$$

It is straightforward to show that the relative degree is given as the difference between orders of polynomials in s in denominator and numerator of the transfer function (5)

$$\rho = \text{rel deg}_s F(\delta, s) = \deg_s a(\delta, s) - \deg_s b(\delta, s)$$

as in the linear case.

Another important structural index in the time-delay case is given by the notion of a relative shift [24], [26]:

Definition 2: Assume that the relative degree ρ of the system (10) is finite. Then the relative shift of this system is defined as

$$\mu = \min \left\{ k \geq 0; \frac{\partial y^{(\rho)}(t)}{\partial u(t-k)} \neq 0 \right\}$$

It can be shown that in terms of the transfer function (5) it means

$$\mu = \deg_{\delta} b_m(\delta)$$

where $b_m(\delta)$ is the leading coefficient of $b(\delta, s)$ in s .

Example 2: Consider the system from Example 1 with the transfer function (6). Then

$$\begin{aligned} \rho &= \text{rel deg}_s F(\delta, s) = 2 \\ \mu &= \deg_{\delta} \dot{y}(t-1)\delta = 1 \end{aligned}$$

Remark 2: Note that in comparison to the relative degree, the relative shift is not the difference between orders of polynomials in δ in denominator and numerator of the transfer function (5), $\mu \neq \text{rel deg}_{\delta} F(\delta, s) = \deg_{\delta} a(\delta, s) - \deg_{\delta} b(\delta, s)$, which does not play such an important role as the notion of the relative shift.

B. Accessibility

An important notion, concerning additional structural properties of a given system, is the notion of accessibility. It is related mainly to the possibility of controlling the system and can be carried over also to the time-delay case [22], [23]:

Definition 3: The system (1) is said to be accessible if there does not exist any non-constant autonomous function.

One possibility how to study the accessibility is to introduce the notion of an autonomous element. In the case of system without delays this results in the equivalence of the two following conditions:

- There does not exist any non-constant autonomous function for system.
- System does not have any autonomous elements.

For linear time-delay systems both definitions are equivalent to the property that the system is torsion free over the ring $\mathbf{R}[\delta, d/dt]$, [9]. However, in the case of nonlinear time-delay systems their equivalence remains an open problem [23].

The accessibility filtration, employing the notion of an autonomous element, was introduced for nonlinear systems without delays in [6]. Recently, it was extended to the time-delay case as well [22], [23]. However, it should be noticed that the submodules \mathcal{H}_k are, in general, infinite dimensional for time-delay systems, which represents a major drawback of their practical computation. Therefore, an alternative way to compute this filtration was studied [23].

However, another alternative accessibility condition can be stated in terms of polynomials $a(\delta, s)$ and $b(\delta, s)$ in (4).

Proposition 1: The system (1) is accessible if polynomials $a(\delta, s)$ and $b(\delta, s)$ have no nontrivial common left factors.

Proof: It was shown in [11] that $a(\delta, s)$ and $b(\delta, s)$ have no common left factors if and only if system does not have any autonomous elements, following the same line as in [33].

□

C. Observability

Following the lines in [29] also the observability condition can be stated in terms of polynomials.

Proposition 2: The system (10) is observable if and only if

$$\deg_s a(\delta, s) = n$$

Sketch of the proof: If the system is not observable one obtains, by eliminating the state variables in (10), see Remark 1 and Appendix I, an input-output equation of the form

$$\begin{aligned} y^{(r)}(t) &= \varphi(\{y^{(k)}(t-i), u^{(l)}(t-j); \\ &0 \leq k \leq r-1; 0 \leq l \leq r; i, j \geq 0\}) \end{aligned}$$

where $r < n$ and then $\deg_s a(\delta, s) = r$. □

IV. MODEL MATCHING PROBLEM

The transfer function approach to the model matching problem is the most natural, in comparison with the state space approaches. For the nonlinear system without delays it was recently considered in [18]. As in linear case, it is stated as the equality of the transfer functions of the model and of the compensated system. Here, we extend some basic ideas of [18] to the time-delay case.

Consider a nonlinear system F and a model G described by the transfer functions

$$\begin{aligned} F(\delta, s) &= \frac{b_F(\delta, s)}{a_F(\delta, s)} \\ G(\delta, s) &= \frac{b_G(\delta, s)}{a_G(\delta, s)} \end{aligned}$$

respectively. Find a (proper) feedforward compensator R , described by the transfer function

$$R(\delta, s)$$

such that the transfer function of the compensated system coincides with that of the model G

$$G(\delta, s) = F(\delta, s) \cdot R(\delta, s)$$

Proposition 3: Given $F(\delta, s) \neq 0$ and $G(\delta, s)$, there exists a feedforward compensator $R(\delta, s)$ which solves the model matching problem if $a_R(\delta, s)du - b_R(\delta, s)dv$ is integrable, where $\frac{b_R(\delta, s)}{a_R(\delta, s)} = F^{-1}(\delta, s) \cdot G(\delta, s)$.

Proof: By the transfer function algebra [12] we get

$$G(\delta, s) = F(\delta, s) \cdot R(\delta, s)$$

Hence, the compensator

$$R(\delta, s) = F^{-1}(\delta, s) \cdot G(\delta, s) = \frac{b_R(\delta, s)}{a_R(\delta, s)} \quad (7)$$

Clearly, the existence of such a compensator is determined by the integrability of the compensator's equation $a_R(\delta, s)du = b_R(\delta, s)dv$. □

Example 3: Given the system F

$$\dot{y}(t) = \dot{u}(t-1) + u^2(t-2)$$

with the transfer function

$$F(\delta, s) = \frac{\delta s + 2u(t-2)\delta^2}{s}$$

Consider the following three models

$$\begin{aligned} G(\delta, s) &= \frac{\delta}{s} \\ G'(\delta, s) &= \frac{\delta}{s+1} \\ G''(\delta, s) &= \frac{\delta}{s+2y(t)} \end{aligned}$$

By (7) and (3) we get the following transfer functions of the compensators

$$\begin{aligned} R(\delta, s) &= \frac{s}{\delta s + 2u(t-2)\delta^2} \cdot \frac{\delta}{s} \\ &= \frac{\delta}{\delta s + 2u(t-2)\delta^2} \\ R'(\delta, s) &= \frac{s}{\delta s + 2u(t-2)\delta^2} \cdot \frac{\delta}{s+1} \\ &= \frac{\delta s}{(s+1)(\delta s + 2u(t-2)\delta^2)} \\ R''(\delta, s) &= \frac{s}{\delta s + 2u(t-2)\delta^2} \cdot \frac{\delta}{s+2y(t)} \\ &= \frac{\delta s - \frac{\dot{y}(t)}{y(t)}\delta}{(s+2y(t) - \frac{\dot{y}(t)}{y(t)})\delta s + 2u(t-2)\delta^2} \end{aligned}$$

While $R(\delta, s)$ and $R'(\delta, s)$ result in the integrable compensators

$$\begin{aligned} R &: \dot{u}(t) = -u^2(t-1) + v(t) \\ R' &: \ddot{u}(t) = -2u(t-1)\dot{u}(t-1) - \dot{u}(t) - \\ &\quad -u^2(t-1) + \dot{v}(t) \end{aligned}$$

$R''(\delta, s)$ does not.

Remark that in multiplying transfer functions one always has to follow the rule (3) which, in general, yields a different result from the standard multiplication, as can be seen for instance in the case of $R''(\delta, s)$.

Remark 3: Note that in contrast to what happens in the linear case, a class of nonlinear systems for which the solution in terms of a feedforward compensator exists is, due to the integrability condition, quite restricted.

A. Properness of the compensator

Obviously, we are interested in finding a solution in a class of proper compensators. However, in comparison to what happens in the case of systems without delays [18] here, it is not sufficient to restrict just the relative degrees of the model and of the system by the standard inequality

$$\text{rel deg}_s G(\delta, s) \geq \text{rel deg}_s F(\delta, s)$$

even in the linear case.

Example 4: Consider the system

$$F(\delta, s) = \frac{\delta^2}{s-\delta}$$

and the model

$$G(\delta, s) = \frac{\delta}{s^2}$$

However, even if $\text{rel deg}_s G(\delta, s) = 2 \geq \text{rel deg}_s F(\delta, s) = 1$ the compensator

$$R(\delta, s) = F^{-1}(\delta, s) \cdot G(\delta, s) = \frac{s-\delta}{\delta^2} \cdot \frac{\delta}{s^2} = \frac{s-\delta}{\delta s^2}$$

is nonproper

$$\begin{aligned} \ddot{u}(t-1) &= \dot{v}(t) - v(t-1) \\ \ddot{u}(t) &= \dot{v}(t+1) - v(t) \end{aligned}$$

Here, the necessary and sufficient condition, under which the compensator becomes proper, is more involved. Clearly, to satisfy the properness of the compensator R the highest derivative of its output has to depend only on the past values of its output, input and their derivatives

$$\begin{aligned} du^{(k)}(t-l) \in \text{span}_{\mathcal{K}}\{du^{(i_1)}(t-j), dv^{(i_2)}(t-j); \\ 0 \leq i_1 \leq k-1; 0 \leq i_2 \leq k, j \geq l\} \end{aligned} \quad (8)$$

for some $l \geq 0$.

Remark 4: Note that in the linear time-delay systems the properness of the compensator (7) is satisfied by the requirement that $a_{n_R}(\delta)$, which is the leading coefficient of $a_R(\delta, s)$ in s , contains a nonzero constant term [30]. That is, $\text{low deg}_\delta a_{n_R}(\delta) = 0$. However, in that case the solution is, in general, given by a compensator from the class of systems of a neutral type. Such solutions are out of the scope of this paper, as we restricted our attention to systems, models and compensators respectively of the form (1). From this point of view, the condition (8) means that the leading coefficient contains *only* a nonzero constant term; that is, $\text{deg}_\delta a_{n_R}(\delta) = 0$.

Clearly, the condition (8), in terms of the transfer function (7), means that in addition to $\text{deg}_s a_R(\delta, s) \geq \text{deg}_s b_R(\delta, s)$ also powers of δ in $a_R(\delta, s)$ and $b_R(\delta, s)$ have to be taken into account. Namely

$$\text{deg}_\delta a_{n_R}(\delta) \leq \min \{ \text{low deg}_\delta a_R(\delta, s), \text{low deg}_\delta b_R(\delta, s) \} \quad (9)$$

where $a_{n_R}(\delta)$ is the leading coefficient of $a_R(\delta, s)$ in s . When coming back to the transfer functions $F(\delta, s)$ and $G(\delta, s)$ one concludes that

Proposition 4: $R(\delta, s)$ is proper (causal) if and only if

- $\text{rel deg}_s G(\delta, s) \geq \text{rel deg}_s F(\delta, s)$
- $\mu_F \leq \min \{ \text{low deg}_\delta b_F(\delta, s), \text{low deg}_\delta b_G(\delta, s) \}$

where μ_F is the relative shift of the system F .

Proof: For $\text{rel deg}_s G(\delta, s) \geq \text{rel deg}_s F(\delta, s)$ it follows the same line as in [18]. For the second part note firstly that $\mu_F = \text{deg}_\delta b_{m_F}(\delta)$ where $b_{m_F}(\delta)$ is the leading coefficient of $b_F(\delta, s)$ in s . Now, from (7) it follows that

$$\begin{aligned} \text{deg}_\delta a_{n_R}(\delta) &= \text{deg}_\delta b_{m_F}(\delta) + \text{deg}_\delta a_{n_G}(\delta) \\ \text{low deg}_\delta a_R(\delta) &= \text{low deg}_\delta b_F(\delta) + \text{low deg}_\delta a_G(\delta) \\ \text{low deg}_\delta b_R(\delta) &= \text{low deg}_\delta a_F(\delta) + \text{low deg}_\delta b_G(\delta) \end{aligned}$$

where $a_{n_G}(\delta)$ is the leading coefficient of $a_G(\delta, s)$ in s . Since both the system and the model are considered to

be of the form (1) note, however, that $\deg_{\delta} a_{n_G}(\delta) = \text{low deg}_{\delta} a_G(\delta) = \text{low deg}_{\delta} a_F(\delta) = 0$ which means that $\mu_F \leq \min \{\text{low deg}_{\delta} b_F(\delta, s), \text{low deg}_{\delta} b_G(\delta, s)\}$ is equivalent to (9). \square

Note that in Example 4 we have $\mu_F = 2$, $\text{low deg}_{\delta} b_F(\delta, s) = 2$ and $\text{low deg}_{\delta} b_G(\delta, s) = 1$ and in Example 3 we have $\mu_F = 1$, $\text{low deg}_{\delta} b_F(\delta, s) = 1$ and for all G, G' and G'' $\text{low deg}_{\delta} b_G(\delta, s) = \text{low deg}_{\delta} b_{G'}(\delta, s) = \text{low deg}_{\delta} b_{G''}(\delta, s) = 1$.

V. CONCLUSIONS

In this paper, several problems are discussed within the transfer function formalism of nonlinear time-delay systems. Mainly, the structural properties of the transfer functions and the model matching problem. In the model matching problem it was shown that the existence of the compensator requires a restrictive integrability condition and that its properness requires to restrict not only the relative degrees, as in the case without delays [18], but the relative shift and a zero structure of both the system and the model as well. A preliminary result relating the state elimination problem for the nonlinear time-delay case was also given.

In conclusion, the paper depicts basic ideas in a quite wide range of problems relating the nonlinear time-delay systems and opens the others that are worth to be studied, as for instance finding a solution to the model matching problem of nonlinear time-delay systems allowing the compensator to be a system of a neutral type, or considering the more general class of feedback compensators, etc.

APPENDIX I

STATE ELIMINATION FOR SISO NONLINEAR TIME-DELAY SYSTEMS

Consider the nonlinear time-delay system of the form

$$\begin{aligned} \dot{x}(t) &= f(\{x(t-i), u(t-j); i, j \geq 0\}) \\ y(t) &= g(\{x(t-i), u(t-j); i, j \geq 0\}) \end{aligned} \quad (10)$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}$ and $y \in \mathbf{R}$.

To find an input-output equation for the system (10) we will try to carry over the idea of the state elimination procedure known for the systems without delays, where an input-output representation is constructed by applying a suitable change of coordinates [6], also to the time-delay case. However, as the system (10) is the subject of two operators, in comparison to what happens in the case of systems without delays we may not be able to find such a change of coordinates.

Example 5: Consider the system

$$\begin{aligned} \dot{x}(t) &= u(t) \\ y(t) &= x(t)x(t-1) \end{aligned}$$

Following the lines of [6]

$$\dot{y}(t) = u(t)x(t-1) + x(t)u(t-1)$$

However, one cannot go any further, since from $y(t) = x(t)x(t-1)$ one cannot express $x(t)$.

Remark 5: It was shown in [1] that it is possible to derive N independent equations over \mathcal{K} in N variables $x_i(t-j)$,

$i = 1, \dots, n, j \geq 0$, for some N which may be greater than n . For instance, in the previous example one considers

$$\begin{aligned} y(t) &= x(t)x(t-1) \\ y(t-1) &= x(t-1)x(t-2) \\ \dot{y}(t-1) &= u(t-1)x(t-2) + x(t-1)u(t-2) \end{aligned}$$

However, in that case it yields, in general, an input-output equation representing a system of a neutral type, i.e. not being of the form (1). Thus, for our purpose a stronger result is needed.

In what follows we present the state elimination procedure extended to a class of systems with delays.

State elimination procedure:

Let s denote the minimum nonnegative integer such that

$$\text{rank}_{\mathcal{K}[\delta]} \begin{pmatrix} c_0(\delta) \\ \vdots \\ c_{s-1}(\delta) \end{pmatrix} = \text{rank}_{\mathcal{K}[\delta]} \begin{pmatrix} c_0(\delta) \\ \vdots \\ c_s(\delta) \end{pmatrix}$$

where $c_i(\delta) \in \mathcal{K}^{1 \times n}[\delta]$ and $d_i(\delta) \in \mathcal{K}[\delta]$ are such that

$$dy^{(i)}(t) = c_i(\delta)dx(t) + d_i(\delta)du(t)$$

for $i = 0, \dots, s$. If $c_0(\delta) = 0$ we define $s = 0$ and $y(t) = g(\cdot)$ is the input-output equation. Note that

$$\begin{pmatrix} c_0(\delta) \\ \vdots \\ c_{s-1}(\delta) \end{pmatrix}$$

is, in fact, an observability matrix and if $s < n$ the system (10) is not observable.

Clearly, to eliminate the state variables in $y^{(s)}(t) = g^{(s)}(\cdot)$ we must have that $c(\delta) \in \text{span}_{\mathcal{K}[\delta]} \{c_0(\delta), \dots, c_{s-1}(\delta)\}$. Then the input-output relation we are looking for can be found. That is

$$y^{(s)}(t) = \varphi(\{y^{(k)}(t-i), u^{(l)}(t-j)\}) \quad (11)$$

where $0 \leq k \leq s-1; 0 \leq l \leq s; i, j \geq 0$.

Example 6: Consider the system

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)x_2(t-1) \\ \dot{x}_2(t) &= x_2(t)u(t) \\ y(t) &= 1/x_1(t) \end{aligned}$$

We have $\dot{y}(t) = -x_2(t-1)/x_1(t)$, $\ddot{y}(t) = -x_2(t-1)u(t-1)/x_1(t) + x_2^2(t-1)/x_1(t)$ and

$$\begin{aligned} x_1(t) &= 1/y(t) \\ x_2(t) &= -\dot{y}(t+1)/y(t+1) \end{aligned}$$

Finally

$$\ddot{y}(t) = \dot{y}(t)u(t-1) - \dot{y}^2(t)/y(t)$$

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