Anisotropic asymptotic behaviour for second order differential operators in non-divergential form

R. Tahraoui [1] [2]

Abstract :

For any matrix $A(\bullet)$ such that $\alpha I \leqslant A(x) \leqslant \beta I$, $\frac{N}{2} > \frac{\beta}{\alpha}$, we prove that the solution $u(\bullet)$ of

$$\begin{cases} -Tr\big(A(x)D^2u\big) = f \ , \quad f \geqslant 0 \quad \text{with compact support} \\ u(x) > 0 \ , \quad \lim_{||x|| \to +\infty} u(x) = 0 \end{cases}$$

behaves like the function $x \to \frac{1}{||x||^{2\left(\wedge(x)-1\right)}}$ where $\wedge(x) = \frac{Tr\left(A(x)\right)}{\frac{2}{||x||^2}\left(A(x).x,x\right)}$ satisfies a differential condition.

MSC : 35J25; 35K20; 35B50

[1] CEREMADE, Université de Paris 9, Dauphine, UMR7534, Place du Maréchal de Lattre de Tassigny, 75775, Paris CEDEX 16, France
[2] I.U.F.M., Rouen 2,rue du Tronquet, 76131 Mont-Saint-Aignan, France

I - Introduction. In this work we study the asymptotic behaviour with respect to x of the solution $u(\bullet)$ of the following equation

$$1 \quad \begin{cases} -Tr\big(A(x)D^2u\big) = f(x), \quad x \in \mathbb{R}^N, \quad N \geqslant 3 \\ \lim_{||x|| \to +\infty} u(x) = 0, \quad u(x) > 0, \quad \forall x \in \mathbb{R}^N \end{cases}$$

where $f(\bullet)$ is a bounded positive function with a compact support. The matrix $A(x) = \big(a_{ij}(x)\big)_{ij}$ is bounded symmetric and uniformly elliptic that is to say : for any $x \in \mathbb{R}^N$

$$\alpha||y||^2 \leqslant \big(A(x)y, y\big) \leqslant \beta||y||^2 \quad \forall y \in \mathbb{R}^N$$

where α and β are real constant such that $0 < \alpha < \beta$, D^2u stands for Hessian matrix of $u(\bullet)$, $Tr(B)$ stands for the trace of the matrix B and $x \to ||x||$ stands for the euclidian norme of $\mathbb{R}^N$. The result depends to the behaviour of the function [4]

$$\wedge(x) := \wedge_A(x) := \frac{Tr\big(A(x)\big)}{\frac{2}{||x||^2}\big(A(x).x, x\big)}$$

as $||x||$ goes to infinity. Without hypotheses concerning the matrix $A(\bullet)$ we prove the following result : for any $\varepsilon > 0$ there exist three real positive constants $c_1 := c_1(\varepsilon)$, $c_2 := c_2(\varepsilon)$ and $R := R(\varepsilon)$ such that for any $x \in \mathbb{R}^N$, $||x|| \geqslant R$, we have

$$\frac{c_1}{||x||^{2(\overline{\theta}-1-\varepsilon)}} \leqslant u(x) \leqslant \frac{c_2}{||x||^{2(\underline{\theta}-1-\varepsilon)}}$$

with

$$\overline{\theta} = \inf_{B \in Q(\alpha,\beta)} \lim_{R \to +\infty} \sup_{||x|| \geqslant R} \wedge(A,B)(x), \qquad \underline{\theta} = \sup_{B \in Q(\alpha,\beta)} \lim_{R \to +\infty} \inf_{||x|| \geqslant R} \wedge(A,B)(x).$$

where

$$\wedge(A,B)(x) = \frac{Tr\big(\sqrt{B^{-1}}.A(\sqrt{B}.x).\sqrt{B^{-1}}\big)}{\frac{2}{||x||^2}\big(\sqrt{B^{-1}}.A(\sqrt{B}.x).\sqrt{B^{-1}}x, x\big)}.$$

Let us remark that $\underline{\theta} > 1$. This function $\wedge(\bullet)$ plays a central role in our work. It is called the function of the spectral dispersion of the matrix $A(\bullet)$, in short the spectral dispersion of A. This name is justified by the following inequalities :

$$1 < \frac{N}{2}.\frac{\alpha}{\beta} \leqslant \frac{N}{2}\frac{\lambda_1(A)}{\lambda_N(A)} \leqslant \wedge(x) \leqslant \frac{N}{2}\frac{\lambda_N(A)}{\lambda_1(A)} \leqslant \frac{N}{2}.\frac{\beta}{\alpha}.$$

These inequalities give an idea of the distribution of the spectrum of the matrix $A(\bullet)$:

$$\text{oscillation } (\wedge) \leqslant \frac{N}{2}\left[\sup_x \frac{\lambda_N\big(A(x)\big)}{\lambda_1\big(A(x)\big)} - \inf_x \frac{\lambda_1\big(A(x)\big)}{\lambda_N\big(A(x)\big)}\right] \leqslant \frac{\beta^2-\alpha^2}{\alpha.\beta}.\frac{N}{2}$$

where the ordered real numbers $0 < \lambda_1(A) \leqslant \cdots \leqslant \lambda_N(A)$ stand for the eigenvalues of $A(\bullet)$. In some sens the previous result is optimal. Indeed in the case A is the identity matrix (or equivalently a constant matrix) we have $\underline{\theta} = \overline{\theta} = \frac{N}{2} > 1$ and we obtain the classical optimal result :

$$u(x) \sim \frac{c}{||x||^{N-2}}$$

as $||x||$ goes to infinity. Let us specify that there is no evident link between the asymptotic behaviour of $\wedge(\bullet)$ and the asymptotic behaviour of $A(\bullet)$, as $||x||$ goes to infinity. Indeed, for instance, there exist matrices $A(\bullet)$ such that, for any x, $A(x)$ belongs to $Q(\alpha,\beta)=\{M$ symmetric matrix / $\alpha I\leqslant M\leqslant \beta I\}$ and has oscillating coefficients and for which the associated function $\wedge(\bullet)$ goes to some constant as $||x||$ goes to infinity. And there exist matrices $A(\bullet)$ having a function $\wedge(\bullet)$ of spectral dispersion which is oscillating as $||x||$ goes to infinity i.e. which satisfies :

$$\lim_{R\to+\infty}\Big[\sup_{||x||\geqslant R}\wedge\big(A(x)\big)-\inf_{||x||\geqslant R}\wedge\big(A(x)\big)\Big]>0,$$

with the notation $\wedge\big(A(x)\big):=\wedge(x)$. These two examples are explained by the fact that there are many matrices belonging to $Q(\alpha,\beta)$ and having the same function $\wedge(\bullet)$ satisfying :

$$\frac{N}{2}.\frac{\alpha}{\beta}\leqslant\wedge(x)\leqslant\frac{N}{2}.\frac{\beta}{\alpha}\qquad\forall x.$$

In section II we will give theses results with some details. To give a classification of this problem, we introduce the function which follows :

$$B\in Q(\alpha,\beta)\longrightarrow J_\infty(B)=\lim_{R\to+\infty}J_R(B)$$

where

$$J_R(B)=\sup_{||x||\geqslant R}\lambda_N\big(B^{-1}.A(x)\big)-\inf_{||x||\geqslant R}\lambda_1\big(B^{-1}.A(x)\big)\ .$$

There exists B_∞ belonging to $Q(\alpha,\beta)$ such that $0\leqslant J_\infty(B_\infty)\leqslant J_\infty(B)\ \forall B\in Q(\alpha,\beta)$. If $J_\infty(B_\infty)=0$, the matrix $A(\bullet)$ goes to a constant matrix as $||x||$ goes to infinity and after a change of variables by a suitable transformation in (1) we find that $\wedge(x)$ goes to the constant $\frac{N}{2}$ as $||x||$ goes to infinty. And roughly speeking the asymptotic behaviour of $u(\bullet)$ is given by [4] :

$$u(x)\sim\frac{c}{||x||^{N-2}}$$

This result is also given by [2] in the exterior problem with $f=0$ i.e. $u(x)=\bigcirc\Big(\frac{1}{||x||^{N-2}}\Big)$.

If $J_\infty(B_\infty)\neq 0$, we have two cases to study.

i) after a change of variable in (1) by a suitable transformation, if necessary, the behaviour of $\wedge(\bullet)$ is radially oscillating as $||x||$ goes to infinity. Let us suppose that this behaviour is given by, for instance, $\gamma(||x||^2)$. Then, under some hypothesis, the work in [4] give the result :

$$u(x)\sim c.v(||x||^2)\qquad\forall x,\quad ||x||\geqslant R,$$

for $R>0$ large enough and where

$$v(||x||^2)=\int_{||x||^2}^{+\infty}\exp\Big(-\int_{R^2}^{s}\frac{\gamma(\sigma)}{\sigma}\,\mathrm{d}\sigma\Big)\,\mathrm{d}s$$

- (cf remark 4.1 section III) -.

ii) the function $\wedge(\bullet)$ is oscillating anisotropically in a neighborhood of the infinity. This case is the aim of our study. Under a suitable hypothesis called $(\mathcal{H})$ and satisfied by $\wedge(\bullet)$ in a neighborhood of the infinity, we prove the following result :

$$u(x)\sim\frac{c(x)}{||x||^{2\left(\wedge(x)-1\right)}},\quad 0<c_0\leqslant c(x)\leqslant c_1,\quad\forall x,\quad ||x||\geqslant R\ ,$$

where $R>0$ is large enough.

Let us specify that our hypothesis $(\mathcal{H})$ is satisfied by the constant functions. A particulary interesting case is the following : Let us suppose that $\wedge(\bullet)$ verifies

$$(2) \quad \begin{cases} \wedge(\lambda x) = \wedge(x) & \forall x, \quad ||x|| \geqslant R, \quad R \text{ large enough} \\ \forall \lambda \geqslant 1 \ . \end{cases}$$

The relation (2) is satisfied, for instance, in the following case : There exists $R > 0$ large enough such that for any x_0, $||x_0|| \geqslant R$ we have $A(\lambda x_0) = A(x_0)$ $\forall \lambda \geqslant 1$.
Under some conditions, the result of this example can be explained as following : $\forall x$, $||x|| \geqslant R$ we have

$$\forall \lambda \geqslant 1 \qquad u(\lambda x) = \frac{c(\lambda x)}{|\lambda|^{2\left(\wedge(x)-1\right)}}$$

where the funtion $c(\bullet)$ is such that $0 < c_0 \leqslant c(x) \leqslant c_1$. Our assumption $(\mathcal{H})$ is a differential inequality satisfied by the function $\wedge(\bullet)$. Roughly speaking $\wedge(\bullet)$ satisfies

$$c(x).Tr\big(A(x)D^2 \wedge\big) + b(x).\big(A(x)D\wedge, D \wedge\big) + a(x).\big(A(x)D\wedge, x\big) \in \left[- \frac{\tilde{c}(c,b,a)}{1 + ||x||^{2(\wedge(x)+\delta(x))}} \, , \, \frac{\tilde{c}(c,b,a)}{1 + ||x||^{2(\wedge(x)+\delta(x))}} \right]$$

for any x, $||x|| \geqslant R > 0$, where $c(\bullet)$, $b(\bullet)$ and $a(\bullet)$ stands for some given functions and $\delta(x) \geqslant \delta_0 > 0$ such that $\delta_0 > \overline{\theta} - \underline{\theta}$.
Finally, note that after the publication of our work in [4], we discovered that when the matrix $A(\bullet)$ is constant at infinity - (i.e. there exists a constant matrix A_0 such that $\lim_{||x|| \to +\infty} A(x) = A_0$) - ,the behaviour $u(x) = \bigcirc\left(\frac{1}{||x||^{N-2}}\right)$ was established in [2] for the exterior domain with $f = 0$ and $\wedge(\bullet)$ was used. But this result [2] does not apply in the case $\wedge(x) := \wedge(A)(x) = \frac{N}{2}$, $\forall x$, and A is an oscillating matrix - (cf proposition 3 section II) -, where $u(\bullet)$ behaves like $\frac{1}{||x||^{N-2}}$ as $||x||$ goes to infinity. Finally we note that the most part of the paper [2] is devoted to the question of the well_posedness of problems like (1).
The paper is organised as follow. A wide part of this paper is devoted to justify our assumption $(\mathcal{H})$. That is the goal of the preparatory part in section II. In the third section, we give a classification of asymptotic behaviour of the matrix $x \longrightarrow A(x)$ when $||x||$ tends to infinity and we shall need later. And we give a new presentation of previous results [4]. Section IV will be devoted to establish the asymptotic behavior of the solution $u(\bullet)$ of (1) when the function $\wedge(\bullet)$ oscillates anisotropically as $||x||$ goes to infinity. The main result is obtained under the key hypothesis $(\mathcal{H})$. The guiding idea of our method is to consider that $A(\bullet)$ and $\wedge(\bullet)$ are two independant datas and to construct two functions $u_i(\bullet)$ - $(i = 1, 2)$- respectively solution of

$$(\mathcal{P}_i) \quad \begin{cases} -Tr(A_i D^2 u_i) = f \quad \text{in} \quad \mathbb{R}^N \\ u_i(x) > 0, \quad \lim_{||x|| \to +\infty} u_i(x) = 0, \quad i = 1, 2 \ . \end{cases}$$

and having the same asymptotic behaviour as the solution $u(\bullet)$ of (1). The matrices $A_i(\bullet)$ - $(i = 1, 2)$ - are two adequates matrices satisfying : $A_1(x) = A_2(x) = A(x)$ for any x, $||x|| \geqslant R$ and for $R > 0$ large enough. Next we construct a super-solution $H_1(\bullet)$ of $(\mathcal{P}_1)$ and a sub-solution $H_2(\bullet)$ of $(\mathcal{P}_2)$. Next we establish that $H_1(\bullet)$ and $H_2(\bullet)$ have the same asymptotic behavior. And the final result follows from the comparison principle [1], [3].

II - Preliminary results preparing to justify the main assumption $(\mathcal{H})$**.**
Let us give two real positive numbers α and β, $0 < \alpha < \beta$ such that $\frac{N}{2} > \frac{\beta}{\alpha}$. Let us set

$$Q(\alpha,\beta) = \left\{ M \in S_+^{N\times N} \ / \ \alpha||x||^2 \leqslant (Mx,x) \leqslant \beta||x||^2, \quad \forall x \in \mathbb{R}^N \right\}$$

where $S_+^{N\times N}$ stands for the set of symmetric definite positive matrices. For any x and y belonging to $\mathbb{R}^N$, (x,y) stands for the inner product of $\mathbb{R}^N$ and $||x||$ the associated norm of x. Let us give a function $A : x \longrightarrow A(x) \in Q(\alpha,\beta)$. In order to simplify the text we note $A(x) \in Q(\alpha,\beta)$ $\forall x \in \mathbb{R}^N$ or $A(\bullet) \in Q(\alpha,\beta)$. We have the following result.

Theorem 1. Let E be a subset of $\mathbb{R}^N$. Let $\wedge(\bullet)$ be a function satisfying

$$\frac{N.\alpha}{2\beta} \leqslant \wedge(x) \leqslant \frac{N.\beta}{2\alpha} \ , \qquad \forall x \in E \subseteq \mathbb{R}^N \setminus \{0\}.$$

Then for any $x \in E$ there exists an uncountable sub-set $\mathcal{M}(x)$ of $Q(\alpha,\beta)$ such that for any $A(x) \in \mathcal{M}(x)$,

$$\frac{Tr\big(A(x)\big)}{\frac{2}{||x||^2}\cdot\big(A(x).x,x\big)} = \wedge(x) \qquad \forall x \in E.$$

This means that $\wedge(\bullet)$ is the spectral dispersion of $A(\bullet)$.

Proof. To proove this result we need two steps.

First step. For any $x \in E$ let us give a partition $\big\{I_1(x), I_2(x), I_3(x)\big\}$ of the set $[1, 2, \cdots, N]$. Let us set

$$\gamma_j(x) := \sum_{i\in I_j(x)} x_i^2 \ , \quad n_j = \text{card}\, I_j(x) \ , \quad \forall j = 1,2,3 \ .$$

First, we look for a diagonal matrix $A(\bullet)$ belonging to $Q(\alpha,\beta)$ such that for any $x \in E$

$$a_{ii}(x) = d_j \ , \qquad \forall i \in I_j(x) \ , \qquad \forall j \in [1,2,3] \ .$$

The unknown real numbers d_j satisfies $\alpha \leqslant d_j \leqslant \beta$. Let us give an arbitrary function $\theta(\bullet)$ such that $\alpha N \leqslant \theta(x) \leqslant \beta N$, $\forall x \in E$. The real numbers d_j satisfy the following equations :

$$(3) \qquad TrA := \theta = \sum_{i=1}^{3} n_i.d_i \ , \qquad \sum_{i=1}^{3} \lambda_i.d_i = \frac{\theta}{2\wedge}$$

where

$$\wedge(x) = \frac{Tr\big(A(x)\big)}{2\dfrac{(A(x).x,x)}{||x||^2}} \qquad \text{and} \qquad \lambda_i(x) = \frac{\gamma_i(x)}{||x||^2} \ , \quad \sum_{i=1}^{3} \lambda_i = 1$$

To solve the equations (3) we consider the following problem for any fixed $x \in E$:

$$(\mathcal{P}) \ : \ \inf \left[J(d) = \sum_{i=1}^{3} d_i^2 \ / \ d = (d_1,d_2,d_3) \ , \ \alpha \leqslant d_i \leqslant \beta \ , \ \sum_{i=1}^{3} n_i d_i = \theta \ , \ \sum_i \lambda_i d_i = \frac{\theta}{2\wedge} \right]$$

This problem has a unique solution. So we found a diagonal matrix $A(\bullet)$ that the spectral dispersion is $\wedge(\bullet)$.

Second step. To find an unreduced form of $A(\bullet)$, we introduce an arbitrary orthogonal transformation whose the matrix is $B(\bullet)$. Let us set $\tilde{\theta}(x) := \theta\big(B^{-1}(x).x\big)$ and $\tilde{\wedge}(x) = \wedge\big(B^{-1}(x).x\big)$, where $\theta(\bullet)$ is the arbitrary function introduced in the first step. We denote by $D(x)$ the diagonal matrix founded in the previous step and corresponding to the two functions $\tilde{\theta}(x)$ and $\tilde{\wedge}(x)$. After some elementary computations the matrix $A(x) = B^{-1}(x).D\big(B(x).x\big).B(x)$ satisfies :

$$\wedge(x) = \wedge_A(x) := \frac{Tr\big(A(x)\big)}{2\dfrac{(A(x).x, x)}{||x||^2}} \qquad \forall x \in E$$

and thus $A(x)$ solve our problem. In addition $x \longrightarrow A(x)$ is regular in $\mathbb{R}^N \setminus \{0\}$. $\square$

For any $A(\bullet) \in Q(\alpha, \beta)$ we denote by $\wedge(A)(\bullet)$ the associated spectral dispersion. There is no evident link between the asymptotic behavior of $A(\bullet)$ and the asymptotic behavior of $\wedge(A)(\bullet) = \wedge(\bullet)$. The following results show this fact.

Corollary 2. For any $k \in \big[N\frac{\alpha}{\beta}, N\frac{\beta}{\alpha}\big]$ there exists a non constant matrix $A(\bullet)$ belonging to $Q(\alpha, \beta)$ such that $\wedge(A)(x) = k$, $\forall x \neq 0$. In addition $A(\bullet)$ does not have a limit when $||x||$ goes to infinity. In fact we can prove a more precise result.

Proposition 3. There exists a matrix $A(\bullet)$ having oscillating coefficients and such that $\wedge(A)(x) =$ constant for any x.

Proof. First, we are going to construct a such matrix in $\mathbb{R}^3$, and next we give its extension in $\mathbb{R}^N$. For this let us set $y := (t, x) = (t, x_1, x_2) \in \mathbb{R}^3$. Let us give two functions $y \longrightarrow a(y)$ such that $0 < 1 \leqslant a(y) \leqslant 2$, $\forall\, y$, for instance, and $x \longrightarrow \delta(x) = \sin x_1. \sin x_2$. Let us set

$$\gamma(x) := \begin{cases} \dfrac{\delta(x)}{2} \dfrac{x_1.x_2}{||x||^2} & \forall\ x = (x_1, x_2) \in \mathbb{R}^2, \quad x \neq 0; \\ 0 \quad \text{if} \quad x = 0 \end{cases}$$

The searched matrix is

$$A_0(y) = \begin{bmatrix} a(y) + \gamma(x) & \lambda(x) & 0 \\ \lambda(x) & a(y) - \gamma(x) & 0 \\ 0 & 0 & a(y) \end{bmatrix}$$

where $\lambda(x) = -\dfrac{\delta(x)}{2}.\dfrac{x_1^2 - x_2^2}{||x||^2}$, $\forall\, x \in \mathbb{R}^2$, $x \neq 0$ and $\lambda(0) = 0$. Indeed after some computations we obtain that $\wedge(A_0)(y) := \wedge(y) = \dfrac{3}{2}$, $\forall\, y \in \mathbb{R}^3$. In the case $N > 3$ we chose $a(y) = a_0$, $\forall\, y \in \mathbb{R}^3$ and, for instance, the searched matrix is

$$A = \begin{bmatrix} A_0 & 0 & \cdots & 0 & 0 \\ 0 & A_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_0 & 0 \\ 0 & 0 & \cdots & 0 & I \end{bmatrix}$$

where I is some identity matrix (n, n).

Proposition 4 . There exists a definite positive matrix A such that $x \longrightarrow \wedge(A)(x)$ is an oscillating function as $||x||$ goes to infinity.

Proof . We can chose the following example

$$A(y) = A(x,t) = \begin{bmatrix} k+\lambda^2(x) & \lambda\mu & 0 \\ \lambda\mu & k+\mu^2(x) & 0 \\ 0 & 0 & k \end{bmatrix}$$

where $y = (x,t) = (x_1, x_2, t) \in \mathbb{R}^3$, $k =$ constant > 0, $\lambda(x) = \dfrac{x_2}{||x||}\big[\cos(||x||) + 2\big]$,

$\mu(x) = \dfrac{x_1}{||x||}\big[\cos(||x||) + 2\big]$.

We have $\wedge(A)(y) = \wedge(y) = \dfrac{3k + (\cos(||x||) + 2)^2}{2k}$. This case of $\wedge(\bullet)$ is interesting for the section III-B. □

Approximation lemma. Let us give two regular matrices $A_1(\bullet)$ and $A_2(\bullet)$ belonging to $Q(\alpha,\beta)$. Then for any R_1 and R_2, $0 < R_1 < R_2$ there exists a regular matrix $A(\bullet)$ belonging to $Q(\alpha,\beta)$ such that :

$$A(x) = \begin{cases} A_1(x) & \forall x, \quad ||x|| \leqslant R_1, \\ A_2(x) & \forall x, \quad ||x|| \geqslant R_2. \end{cases}$$

In addition if $d \geqslant \wedge_{A_1}(x) \geqslant c \; \forall x$, $||x|| \leqslant R$, $d \geqslant \wedge_{A_2}(x) \geqslant c \; \forall x$, $||x|| \geqslant R$, we have $d \geqslant \wedge_A(x) \geqslant c \; \forall x$, with $R_1 < R < R_2$.

Proof. Let us give a regular increasing function $\Psi : \mathbb{R}^+ \longrightarrow [0,1]$ such that $\Psi(t) = 0$, $\forall t \leqslant R_1$; $\Psi(t) = 1$, $\forall t \geqslant R_2$. Let us set $\varphi(x) = \Psi(||x||)$. Then we are going to prove that the matrix $A(x) = \big(1-\varphi(x)\big)A_1(x) + \varphi(x)A_2(x)$ is the matrix that is our solution. Indead $A(\bullet)$ is regular and satisfies $\alpha||x||^2 \leqslant \big(A(x).x, x\big) \leqslant \beta||x||^2$ and thus $A(\bullet) \in Q(\alpha,\beta)$. The last result follows from the following remark : for any real numbers $(a_1, b_1, a_2, b_2, c) \in (\mathbb{R}^{+,*})^5$ such that $\frac{a_1}{b_1} \geqslant c$, $\frac{a_2}{b_2} \geqslant c$ we have $\frac{ta_1+sa_2}{tb_1+sb_2} \geqslant c$, for any $(t,s) \in (\mathbb{R}^+)^2$ such that $tb_1 + sb_2 > 0$. The second inequality can be established in the same way. □

Proposition 5. Let M be a symmetric positive definite matrix. Let us consider the half-straight line of matrices $\mathcal{D}(O,M) = \big\{B/B = t.M \;,\; t > 0\big\}$. Then we have $\mathcal{D}(O,M) \cap Q(\alpha,\beta) \neq \emptyset$ if and only if $\dfrac{\beta}{\alpha} \geqslant \dfrac{\lambda_N(M)}{\lambda_1(M)}$. In addition we have

$$\mathcal{D}(O,M) \cap Q(\alpha,\beta) = \Big\{B = t.M \;/\; t \in \Big[\frac{\alpha}{\lambda_1(M)}, \frac{\beta}{\lambda_N(M)}\Big]\Big\}.$$

If $\dfrac{\beta}{\alpha} = \dfrac{\lambda_N(M)}{\lambda_1(M)}$ we have

$$\mathcal{D}(O,M) \cap Q(\alpha,\beta) = \Big\{B = \frac{\alpha}{\lambda_1(M)}.M\Big\} = \Big\{B = \frac{\beta}{\lambda_N(M)}.M\Big\}.$$

And in this case $\mathcal{D}(O,M)$ is called an extremal ray passing through the zero matrix O.

Proof. It is sufficient to remark that tM belongs to $Q(\alpha,\beta)$ if and only if $\frac{\alpha}{t} \leqslant \lambda_1(M) \leqslant \cdots \leqslant \lambda_N(M) \leqslant \frac{\beta}{t}$. □

Let us consider the set $\mathcal{P}$ of supporting hyperplanes of $Q(\alpha,\beta)$ passing through the zero matrix O. We set :

$$\begin{aligned} Q_c(\alpha,\beta) &:= \bigcup_{P\in\mathcal{P}} \big[P \cap Q(\alpha,\beta)\big] \\ &= \Big\{T \in Q(\alpha,\beta)\ /\ \alpha = \lambda_1(T) \leqslant \lambda_i \leqslant \lambda_N(T) = \beta \quad i = 2,\cdots,N-1\Big\} \end{aligned}$$

Theorem 6. Let V be a (N,N)-symmetric matrix. Then there exists a matrix A belonging to $Q(\alpha,\beta)$ such that $Tr(A.V) = 0$ if and only if

$$(4) \qquad \left\{ \begin{array}{l} \big[\beta\lambda_N(V) - \alpha\lambda_1(V)\big]\big[\beta\lambda_1(V) - \alpha\lambda_N(V)\big] \leqslant 0 \\ (\beta-\alpha)|||V|||^2 \geqslant \big[\beta\lambda_N(V) - \alpha\lambda_1(V)\big]TrV \quad \text{if} \quad TrV \geqslant 0 \\ (\beta-\alpha)|||V|||^2 \geqslant \big[\beta\lambda_1(V) - \alpha\lambda_N(V)\big]TrV \quad \text{if} \quad TrV \leqslant 0 \end{array}\right.$$

Proof. Let us consider the matrices plan Vect$\{I,V\}$ generated by the two matrices I and V. We have Vect$\{I,V\} \cap Q(\alpha,\beta) \cap Q_c(\alpha,\beta) = \{W_1,W_2\} \subseteq Q_c(\alpha,\beta)$ i.e. $\lambda_1(W_i) = \alpha$, $\lambda_N(W_i) = \beta \ i = 1,2$ and $\alpha \leqslant \lambda_j(W_i) \leqslant \beta$, $\forall\ i = 1,2$; $\forall\ j = 2,\cdots,N-1$. We have $aI + bV = W_1$ with a and b belonging to $\mathbb{R}$. This entails that V and W have the sames eigenspaces. We equip then $\mathbb{R}^N$ with the basis of the eigenvectors of V. We obtain

$$a = \frac{\alpha\lambda_N(V) - \beta\lambda_1(V)}{\lambda_N(V) - \lambda_1(V)}\ , \qquad b = \frac{\beta-\alpha}{\lambda_N(V) - \lambda_1(V)} > 0\ ,$$

$\alpha \leqslant \lambda_i(W_1) = a + b\lambda_i(V) \leqslant \beta, \quad i = 2,\cdots,N-1$. And thus $Tr(W_1.V) = aTrV + b|||V|||^2$. In the same way we have $cW_1 + dV = W_2$. The matrices W_1, V and W_2 have the same eigenspaces. And as previously we obtain :

$$d = \frac{\beta^2 - \alpha^2}{\beta\lambda_1(V) - \alpha\lambda_N(V)}\ , \quad c = \frac{\beta\lambda_N(V) - \alpha\lambda_1(V)}{\alpha\lambda_N(V) - \beta\lambda_1(V)}\ ,$$

$$\alpha \leqslant \lambda_i(W_2) = c\lambda_i(W_1) + d\lambda_i(V) \leqslant \beta\ , \quad i = 2,\cdots,N-1.$$

Since Vect$V \bigcap \Big\{Q(\alpha,\beta) \bigcup \big[-Q(\alpha,\beta)\big]\Big\} = \emptyset$, we have $c \geqslant 0$. And thus $T(V.W_2) = acTrV + (d+bc)|||V|||^2$ i.e. after some calculations

$$Tr(V.W_2) = \frac{\beta\lambda_N(V) - \alpha\lambda_1(V)}{\lambda_N(V) - \lambda_1(V)}.TrV - \frac{(\beta-\alpha).|||V|||^2}{\lambda_N(V) - \lambda_1(V)}\ .$$

It is easy to see that a necessary and sufficient condition that there exists $A \in Q(\alpha,\beta)$ such that $Tr(A.V) = 0$ is that we have

$$(5) \qquad Tr(W_1.V).Tr(W_2.V) \leqslant 0, \quad \text{and} \quad c \geqslant 0.$$

And (5) is equivalent to (4). □

Remarks 1.

1) Let us set $X := \big\{M \in S^{N\times N}\ /\ Tr(M.V) = 0\big\}$. Then any M belonging to $X \cap Q(\alpha,\beta)$ satisfies $Tr(V.M) = 0$ that is to say there exist many M belonging to $Q(\alpha,\beta)$ such that $Tr(M.V) = 0$ if (4) is satisfied, where $S^{N\times N}$ stands for the set of symmetric matrices.

2) Let V and R be two (N,N)-symmetric matrices. We can prove that there exists $A \in Q(\alpha,\beta)$ such that $Tr(AV) = Tr(AR) = 0$ if we have :

$$(6) \quad \begin{cases} |||V|||^2.|||R|||^2 - (R:V)^2 \neq 0 \qquad \text{and} \\ \quad 1 - \dfrac{1}{N}\dfrac{|||(TrV).R - (TrR).V|||^2}{|||V|||^2.|||R|||^2 - (R:V)^2} \geqslant \\ \dfrac{1}{N} - \dfrac{\left[(\alpha+\beta)^2 + (N-2)(\alpha^2+\beta^2)\right]^2}{(\alpha+\beta)^2(\alpha^2+\beta^2) + (N-2)(\alpha^2+\beta^2)^2} \end{cases}$$

The proof follows from a straightforward calculation and classical idea. □

Application. Let $B(O,\varepsilon)$ be the open ball of $\mathbb{R}^N$, of radius ε, centred at O. Let us give a C^2 bounded function $\varphi : \mathbb{R}^N \setminus B(O,\varepsilon) \longrightarrow \mathbb{R}^{+,*}$ such that $\varphi(x) \geqslant \varphi_0 > 0\ \forall x$, where φ_0 is a real positive number. Let us define the following matrix

$$x \longrightarrow M(\varphi)(x) := D^2\varphi - \gamma(x)\nabla\varphi(x) \otimes \nabla\varphi(x) + kH\big(x, \nabla\varphi(x)\big)$$

where

$$\Big(H\big(x,\nabla\varphi(x)\big)\Big)_{ij} = \frac{\varphi(x)}{2}\Big(\frac{x_i}{||x||^2}.\frac{\partial\varphi}{\partial x_j} + \frac{x_j}{||x||^2}.\frac{\partial\varphi}{\partial x_i}\Big)_{ij},$$

k is a real number and $\gamma(\bullet)$ is a given function defined on $\mathbb{R}^N \setminus B(O,\varepsilon)$. In section IV we use the following corollary to justify our essential hypothesis $(\mathcal{H})$.

Corollary 7. Suppose that the matrix $M(\varphi)$ verify the differential inequalities (4) of the theorem 6. Then there exists $x \longrightarrow A(x) \in Q(\alpha,\beta)$ such that

$$-Tr(AD^2\varphi) - \gamma(x)\big(A\nabla\varphi, \nabla\varphi\big) + \frac{k}{||x||^2}\big(A\nabla\varphi, x\big) = 0 \quad \forall x \in \mathbb{R}^N \setminus B(O,\varepsilon)$$

Proof. For any B_1 and B_2 belonging to $Q(\alpha,\beta)$, we have :

$$(B_1V, V) = Tr(B_1.C_1) \qquad \forall V = (v_1, \cdots, v_N) \in \mathbb{R}^N$$

with

$$C_1 = V \otimes V = (v_i.v_j); \quad \text{and} \quad Tr\big(B_2.H(x,\nabla\varphi)\big) = \frac{1}{||x||^2}\big(B_2\nabla\varphi, x\big)$$

From this we apply the theorem 6 to obtain the result. □

Remark 2. If we add the additional condition that $\varphi(\bullet)$ is the spectral dispersion of the matrix A that we are looking for, we can use (6) instead of corollary 7, with

$$V = M(\varphi) \quad \text{and} \quad R = I - 2\frac{\varphi(x)}{||x||^2}x \otimes x,$$

to conclude that there is A belonging to $Q(\alpha,\beta)$ such that $Tr(A.V) = Tr(A.R) = 0$ if V and R satisfy (6).

Remark 3. Given a matrix $x \longrightarrow A(x) \in Q(\alpha,\beta)$, we can ask if there exists a bounded and positive function $\varphi(\bullet)$ oscillating as $||x||$ goes to infinity - (i.e. such that $\lim_{R\to+\infty}\sup_{||x||\geqslant R}\varphi(x) - \lim_{R\to+\infty}\inf_{||x||\geqslant R}\varphi(x) > 0$) - and such that we have

$$(7) \quad -Tr(AD^2\varphi) - \gamma(x)\big(A\nabla\varphi, \nabla\varphi\big) + \frac{k.\varphi}{||x||^2}\big(A\nabla\varphi, x\big) = 0$$

for any x, $||x|| > R > 0$. Since any constant function is a solution of (7), it seems difficult to answer this question in general. However there are some encouraging examples which show that (7) has nontrivial solutions in general. A particular case of (7) is the following equation that we will encounter later -(section IV)- :

$$(8) \qquad -\triangle u + \mathrm{Log}\big(||x||^2\big).|\nabla u|^2 + u\frac{x.\nabla u}{||x||^2} = 0, \qquad ||x|| > R.$$

Proposition 8. The equation (8) has many nontrivial solutions, for $R > 0$ large enough.

Proof. The result is technical, but it is straightforward. So we will give the essential steps of the proof. We construct a family of solutions $u(\bullet)$ in the form $u(x) = \varphi\big(\mathrm{Log}(||x||^2)\big)$. We establish an o.d.e. satisfied by the function $t \longrightarrow \varphi(t)$, where t plays the role of $\mathrm{Log}(||x||^2) > t_0 > 1$. We solve this o.d.e. by setting $\varphi(t) = f\big(v(t)\big)$ where

$$(9) \quad f(\sigma) = -\frac{2}{c}\mathrm{Arctg}\Big(\frac{\sigma}{c}\Big) + d \qquad \forall\,\sigma \geqslant \sigma_0 > 0$$

with $c > 0$ and $d > \frac{\pi}{2} + \frac{N-1}{2}$. The function $v(\bullet)$ is solution of the following Cauchy-Lipschitz problem :

$$(10) \quad \begin{cases} v'(t) = G\big(t, v(t)\big), & \forall\, t \geqslant t_0 > 1 \\ v(t_0) = v_0, & v_0 > \sigma_0 > 0 \end{cases}$$

where

$$G(t,\sigma) = \int_{t_0}^{t} g(\theta)\,\mathrm{d}\theta - \frac{N-2}{2}\sigma + F(\sigma) \quad \forall\,\sigma \geqslant t_0$$

and

$$\Big|\frac{\partial G}{\partial \sigma}(t,\sigma)\Big| \leqslant \frac{N-2}{2} + \frac{\pi}{2} + d\ ;$$

$g(\bullet)$ is an arbitrary continuous function with compact support and $F(\bullet)$ is a primitive of $f(\bullet)$:

$$F(\sigma) = -\frac{2\sigma}{c}\mathrm{Arctg}\Big(\frac{\sigma}{c}\Big) + d.\sigma + \frac{1}{c}\mathrm{Log}\Big(1 + \Big(\frac{\sigma}{c}\Big)^2\Big)\ .$$

Thus we obtain :

$$(11) \quad u(x) = -\frac{2}{c}\mathrm{Arctg}\Big(v\big(\mathrm{Log}(||x||^2)\big)\Big) + d$$

is a solution of equation (8) in $\mathbb{R}^N \setminus B(0,R)$ where c,d and g are somewhat arbitrary datas. $\square$

III - Isotropic asymptotic behavior. This section gives a new presentation of the results already established in [4] and allows us to understand the classification of the behavior at infinity of matrices and associated solutions. As we have already mentioned, the asymptotic behavior of the solution $u(\bullet)$ of

$$(12) \quad \begin{cases} -Tr\big(A(x)D^2u\big) = f(x) \quad \text{in} \quad \mathbb{R}^N, \\ \lim_{||x||\to+\infty} u(x) = 0\ , \quad u(x) > 0 \quad \forall\, x\, , \\ \text{with} \quad f(x) \geqslant 0\, , \quad \text{support} f = \text{compact}, \end{cases}$$

depends on the behavior of the function $\wedge(A)(x) = \dfrac{Tr\big(A(x)\big)}{\frac{2}{||x||^2}(A(x).x,x)}$. From the results of section II there is no evident links between the behavior of $A(\bullet)$ and that of $\wedge(A)(\bullet)$. However there are some precise situations to be distinguished.

First case. The matrix $A(\bullet)$ is constant. By a change of variables $x = B^{-1}y$ with $B = \sqrt{A^{-1}}$, the equation (12) becomes

$$(13) \quad \begin{cases} -\triangle\,\omega(y) = g(y) = f(B^{-1}y) \\ u(x) = \omega(Bx)\,, \quad \lim_{||y||\to+\infty}\omega(y) = 0\,, \quad \omega(y) > 0\,. \end{cases}$$

For (13) the asymptotic behavior of $\omega(\bullet)$ is known - (cf [4] for instance) - : $\omega(y) \simeq \dfrac{c}{||y||^{N-2}}$.

And thus $u(x) \simeq \dfrac{c}{||x||^{N-2}}$, since $\wedge(I)(x) = \dfrac{N}{2} \quad \forall\, x$.

Second step. The matrix $A(\bullet)$ is not constant. In this case we see that it suffices to examine the behavior at infinity of $A(\bullet)$ and $\wedge(A)(\bullet)$ [4]. In the following our aim is to examine the oscillations of $A(\bullet)$ and $\wedge(A)(\bullet)$. The matrix $A(\bullet)$ belongs to $Q(\alpha,\beta)$. Let B be a constant matrix belonging to $Q(\alpha,\beta)$. For any x, $B^{-1}A(x)$ has n eigenvalues $\lambda_i(x) = \lambda_i(B^{-1}A)$ such that

$$\frac{\alpha}{\beta} \leqslant \lambda_1(x) \leqslant \cdots \leqslant \lambda_N(x) \leqslant \frac{\beta}{\alpha}$$

where for any $x \in \mathbb{R}^N$, any $i,\ 1 \leqslant i \leqslant N\ ,\ \lambda_i(x) = \lambda_i\big(B^{-1}A(x)\big)$ stands for the i^{th} eigenvalue of the following problem :

$$A(x).r = \lambda_i(x)B.r$$

where $r \in \mathbb{R}^N$ is an eigenvector associated to $\lambda_i(x)$. For any $R > 0$, we introduce the following function $J_R : Q(\alpha,\beta) \longrightarrow \mathbb{R}^+$ defined by

$$J_R(B) = \sup_{||x||\geqslant R} \lambda_N\big(B^{-1}A(x)\big) - \inf_{||x||\geqslant R} \lambda_1\big(B^{-1}A(x)\big).$$

Proposition 9. The set of functions $\big(J_R(\bullet)\big)_R$ is $\dfrac{2\beta^2}{\alpha^3}$-lipschitz.

Proof. It is sufficient to prove that the function $B \longrightarrow \sup_{||x||\geqslant R}\lambda_N\big(B^{-1}A(x)\big)$ and $B \longrightarrow \inf_{||x||\geqslant R}\lambda_1\big(B^{-1}A(x)\big)$ are uniformly lipschitz. For any $y \neq 0$ we have

$$\left|\frac{(B_1y,y)}{(A(x)y,y)} - \frac{(B_2y,y)}{(A(x)y,y)}\right| \leqslant \frac{|||B_1 - B_2|||}{\alpha}$$

or again

$$-\frac{|||B_1 - B_2|||}{\alpha} + \inf_{y\neq0}\frac{(B_1y,y)}{(A(x)y,y)} \leqslant \inf_{y\neq0}\frac{(B_2y,y)}{(A(x)y,y)} \leqslant \inf_{y\neq0}\frac{(B_1y,y)}{(A(x)y,y)} + \frac{|||B_1 - B_2|||}{\alpha}$$

i.e.

$$\forall\, x \in \mathbb{R}^N\,, \quad -\frac{|||B_1 - B_2|||}{\alpha} + \frac{1}{\lambda_N\big(B_1^{-1}A(x)\big)} \leqslant \frac{1}{\lambda_N\big(B_2^{-1}A(x)\big)} \leqslant \frac{1}{\lambda_N\big(B_1^{-1}A(x)\big)} + \frac{|||B_1 - B_2|||}{\alpha}\,.$$

This implies

$$\left| \sup_{||x||\geqslant R} \lambda_N\big(B_1^{-1}A(x)\big) - \sup_{||x||\geqslant R} \lambda_N\big(B_2^{-1}A(x)\big)\right| \leqslant \Big(\frac{\beta}{\alpha}\Big)^2 \frac{|||B_1 - B_2|||}{\alpha}$$

From the same way we show that

$$\left| \inf_{||x||\geqslant R} \lambda_1\big(B_1^{-1}A(x)\big) - \inf_{||x||\geqslant R} \lambda_1\big(B_2^{-1}A(x)\big)\right| \leqslant \Big(\frac{\beta}{\alpha}\Big)^2 \frac{|||B_1 - B_2|||}{\alpha}$$

and thus the result follows : for any $R > 0$

$$\Big|J_R(B_1) - J_R(B_2)\Big| \leqslant \frac{2\beta^2}{\alpha^3}|||B_1 - B_2|||$$

□

Proposition 10. $J_R(\bullet)$ converges uniformly to $J_\infty(\bullet)$ in $Q(\alpha,\beta)$ as R goes to infinity. In addition $J_\infty(\bullet)$ is $\dfrac{2\beta^2}{\alpha^3}$-lipschitz and there exists B_∞ belonging to $Q(\alpha,\beta)$ such that, for any $R > 0$, we have

$$0 \leqslant J_\infty(B_\infty) \leqslant J_\infty(B) \leqslant J_R(B)\ , \quad \forall\, B \in Q(\alpha,\beta).$$

Proof. It is easy. We use Ascoli-Arzelà's theorem, the compactness of $Q(\alpha,\beta)$ and the fact that $R \longrightarrow J_R(\bullet)$ is decreasing. Thus it follows that there exists $B_\infty \in Q(\alpha,\beta)$ such that

$$0 \leqslant J_\infty(B_\infty) = \inf_{B\in Q(\alpha,\beta)} J_\infty(B) \leqslant J_R(B)\ , \quad \forall\, B \in Q(\alpha,\beta).$$

□

Definition 1. The real number $J_\infty(B_\infty) = \inf_{B\in Q(\alpha,\beta)} J_\infty(B)$ is called the amplitude of the oscillations of the spectrum of $A(\bullet)$ as $||x||$ goes to infinity.

Study of a classification : We have two cases to examine.

A - First case : the amplitude $J_\infty(B_\infty)$ is null. We have :

$$0 = J_\infty(B_\infty) \leqslant J_R(B_\infty) \qquad \forall\, R > 0\ ,$$

$$(14) \qquad \begin{cases} \lim_{R\to+\infty} \sup_{||x||\geqslant R} \lambda_N\big(B_\infty^{-1}A(x)\big) = \gamma_\infty^1 \\ \lim_{R\to+\infty} \inf_{||x||\geqslant R} \lambda_1\big(B_\infty^{-1}A(x)\big) = \gamma_\infty^2\ , \end{cases}$$

since the sequences $R \longrightarrow \sup_{||x||\geqslant R} \lambda_N\big(B_\infty^{-1}A(x)\big)$ and $R \longrightarrow \inf_{||x||\geqslant R} \lambda_1\big(B_\infty^{-1}A(x)\big)$ are monotonic and bounded. Thus we have :

$$\lim_{R\to+\infty} J_R(B_\infty) = \gamma_\infty^1 - \gamma_\infty^2 = J_\infty(B_\infty) = 0\ ,$$

that is to say

$$(15) \qquad \gamma^\infty := \gamma_\infty^1 = \gamma_\infty^2$$

□

Proposition 11. Let us assume that $J_\infty(B_\infty) = 0$. Then we have

1) $\lim_{R\to+\infty} \sup_{||x||\geqslant R} Tr\big(B_\infty^{-1}A(x)\big) = N\gamma^\infty$,

2) $\lim_{R\to+\infty} \sup_{||x||\geqslant R} |||B_\infty^{-1}A(x) - \gamma^\infty.I||| = 0$,

$\lim_{R\to+\infty} \sup_{||x||\geqslant R} |||A(x) - \gamma^\infty B_\infty||| = 0$.

Proof. Le us prove the first result. We have :

$$N.\inf_{||x||\geqslant R} \lambda_1\big(B_\infty^{-1}A(x)\big) \leqslant \inf_{||x||\geqslant R} Tr\big(B_\infty^{-1}A(x)\big) \\ \leqslant \sup_{||x||\geqslant R} Tr\big(B_\infty^{-1}A(x)\big) \leqslant N.\sup_{||x||\geqslant R} \lambda_N\big(B_\infty^{-1}A(x)\big)$$

By definition of $J_R(\bullet)$, we obtain :

$$0 \leqslant \sup_{||x||\geqslant R} Tr\big(B_\infty^{-1}A(x)\big) - \inf_{||x||\geqslant R} Tr\big(B_\infty^{-1}A(x)\big) \leqslant N.J_R(B_\infty) \ .$$

And from (14), (15) and the proposition 10, the first resultat is obtained. Let us prove the second result. By the very definition of $\lambda_i\big(B_\infty^{-1}A(x)\big)$, $i = 1,\ldots,N$, we have :

$$\forall\, y \neq 0 \ , \quad \inf_{||x||\geqslant R} \lambda_1\big(B_\infty^{-1}A(x)\big) \leqslant \frac{\big(A(x).y,y\big)}{\big(B_\infty y,y\big)} \leqslant \sup_{||x||\geqslant R} \lambda_N\big(B_\infty^{-1}A(x)\big) \ ,$$

and for any $i_0 \in [1,\ldots,N]$

$$(16) \quad \forall\, x \quad \inf_{||t||\geqslant R} \lambda_1\big(B_\infty^{-1}A(t)\big) \leqslant \lambda_{i_0}\big(B_\infty^{-1}A(x)\big) \leqslant \sup_{||t||\geqslant R} \lambda_N\big(B_\infty^{-1}A(t)\big) \ .$$

And thus, for any $y \neq 0$ and any x, we have :

$$\left| \frac{\big(A(x).y,y\big) - \lambda_{i_0}(x)\big(B_\infty y,y\big)}{\big(B_\infty y,y\big)} \right| \leqslant J_R(B_\infty) \ ,$$

$$\frac{1}{\beta} . \frac{\Big|\Big(\big(A(x) - \lambda_{i_0}(x)B_\infty\big)y,y\Big)\Big|}{||y||^2} \leqslant J_R(B_\infty) \ ,$$

that is to say, for any x, we obtain

$$\sup_{y\neq 0} \frac{\Big|\Big(\big(A(x) - \lambda_{i_0}(x)B_\infty\big)y,y\Big)\Big|}{||y||^2} \leqslant \beta J_R(B_\infty) \ .$$

And since the matrix $A(x) - \lambda_{i_0}(x)B_\infty$ is symmetric we obtain

$$\sup_{||x||\geqslant R} |||A(x) - \lambda_{i_0}(x)B_\infty||| \leqslant \beta.J_R(B_\infty) \ .$$

This entails

$$\sup_{||x||\geqslant R} |||A(x) - \gamma^\infty B_\infty||| \leqslant \beta.J_R(B_\infty) + \sup_{||x||\geqslant R} \big|\lambda_{i_0}(x) - \gamma^\infty\big|.|||B_\infty||| \ .$$

And from (16), (14) and (15) we obtain

$$(17) \qquad \lim_{R\to+\infty} \sup_{||x||\geqslant R} |||A(x) - \gamma^\infty B_\infty||| = 0 \ .$$

And the second result 2) follows from

$$|||B_\infty^{-1}A(x) - \gamma^\infty I||| \equiv |||B_\infty^{-1}\big(A(x) - \gamma^\infty B_\infty\big)||| \leqslant |||B_\infty^{-1}|||.|||A(x) - \gamma^\infty B_\infty||| \ .$$

Corollary 12. We have

$$\lim_R \sup_{||x||\geqslant R}\left[\frac{Tr\big(B_\infty^{-1}A(x)\big)}{\frac{2}{||x||^2}.\big(B_\infty^{-1}A(x).x,x\big)}\right] = \frac{N}{2} = \lim_R \sup_{||x||\geqslant R}\left[\frac{Tr\big(\hat{A}(x)\big)}{\frac{2}{||x||^2}.\big(\hat{A}(x).x,x\big)}\right] \text{ where } \left\{\begin{array}{l}\hat{A}(x)=BA(B^{-1}x)B\ ,\\ B=\sqrt{B_\infty^{-1}}\end{array}\right. .$$

Proof. For any x we have :

$$\big(B_\infty^{-1}A(x).x,x\big) = \Big(\big(B_\infty^{-1}A(x)-\gamma^\infty I\big).x,x\Big) + \big(\gamma^\infty I.x,x\big)\ .$$

This entails

$$\frac{\big(B_\infty^{-1}A(x).x,x\big)}{||x||^2} \geqslant \gamma^\infty - \sup_{||x||\geqslant R}|||B_\infty^{-1}A(x)-\gamma^\infty I|||$$

and

$$\frac{\big(B_\infty^{-1}A(x).x,x\big)}{||x||^2} \leqslant \gamma^\infty + \sup_{||x||\geqslant R}|||B_\infty^{-1}A(x)-\gamma^\infty I|||\ .$$

Then it is easy to obtain

$$\begin{array}{c}\frac{1}{2}\sup_{||x||\geqslant R} Tr\big(B_\infty^{-1}A(x)\big).\Big[\gamma^\infty + \sup_{||x||\geqslant R}|||B_\infty^{-1}A(x)-\gamma^\infty I|||\Big]^{-1}\\ \leqslant \sup_{||x||\geqslant R}\left[\frac{Tr\big(B_\infty^{-1}A(x)\big)}{\frac{2}{||x||^2}\big(B_\infty^{-1}A(x)x,x\big)}\right]\\ \leqslant \frac{1}{2}\sup_{||x||\geqslant R} Tr\big(B_\infty^{-1}A(x)\big).\Big[\gamma^\infty - \sup_{||x||\geqslant R}|||B_\infty^{-1}A(x)-\gamma^\infty I|||\Big]^{-1}\ .\end{array}$$

Thanks to proposition 11 and passing to the limit as R goes to infinity the first result follows. Now let us prove the second equality. We have

$$\begin{array}{rl}BAB - B_\infty^{-1}A & = \big(BAB-\gamma^\infty I\big) + \big(\gamma^\infty I - B_\infty^{-1}A\big)\\ & = B\big(A-\gamma^\infty B^{-2}\big)B + \big(\gamma^\infty I - B_\infty^{-1}A\big)\\ & = B\big(A-\gamma^\infty B_\infty\big)B + \big(\gamma^\infty I - B_\infty^{-1}A\big)\ ,\end{array}$$

and thus

$$|||BAB - B_\infty^{-1}A||| \leqslant |||B|||^2\ |||A-\gamma^\infty B_\infty||| + |||\gamma^\infty I - B_\infty^{-1}A|||\ .$$

From this last inequality and using the proposition 11, we prove the second equality in the same way as in the first one. □

The following definition is justified by the previous result.

Definition 2. If, for any matrix $A(\bullet)$ belonging to $Q(\alpha,\beta)$, we have $J_\infty(B_\infty)=0$, the matrix $A(\bullet)$ is called constant as $||x||$ goes to infinity or shortly constant at infinity.

Theorem 13. [2] [4]. If the matrix $A(\bullet)$ is constant for any x, $||x||\geqslant R$, the solution of the problem (12) has the following asymptotic behavior :

$$u(x) \sim \frac{c}{||x||^{N-2}}$$

for $||x||$ large enough.

***Proof*.** In (12) we use the following change of variables :

$$x = B^{-1}y \quad \text{i.e.} \quad y = Bx \quad \text{with} \quad B = \sqrt{B_\infty^{-1}}\ .$$

We obtain the following equation

$$(18) \qquad \begin{cases} -Tr\big(\hat{A}(y)D^2\omega(y)\big) = f(B^{-1}y)\ , \\ u(x) = \omega(B.x)\ , \quad \lim_{||y||\to+\infty}\omega(y) = 0\ , \quad \omega(y) > 0\ , \end{cases}$$

where

$$\hat{A}(y) = B.A\big(B^{-1}y\big).B\ .$$

From proposition 11 and corollary 12 the matrix $\hat{A}(y)$ satisfies

$$\frac{Tr\big(\hat{A}(y)\big)}{\dfrac{2}{||y||^2}.\big(\hat{A}(y).y, y\big)} = \frac{N}{2}\ , \quad \forall x\ , \quad ||x|| \geqslant R\ .$$

We conclude by the result of [4], [2] and the fact that $u(\bullet)$ and $\omega(\bullet)$ have the same asymptotic behaviour. □

B - Second case : the amplitude $J_\infty(B_\infty)$ is positive. The matrix $A(\bullet)$ is not constant at infinity. We call it oscillating matrix. The amplitude of its oscillations is measured by the real number $J_\infty(B_\infty)$. We have two possibilities to examine.

i) the behavior of the spectral dispersion is radial that is to say there exists a radial function $x \longrightarrow \gamma\big(||x||^2\big)$ such that $R \longrightarrow \varphi(R) = \sup_{||x||\geqslant R}\big|\wedge(x) - \gamma\big(||x||^2\big)\big|$ verifies :

$$(18.1) \qquad \lim_R \varphi(R) = 0$$

with $x \longrightarrow \gamma\big(||x||^2\big)$ is oscillating at infinity. In this case, under some hypothesis about $\varphi(\bullet)$, the behavior of the solution $u(\bullet)$ of (12) is known [4] : there exists a real number $R > 0$ large enough such that for any x, $||x|| \geqslant R$ we have $u(x) \sim c.v(||x||^2)$. The function $v(\bullet)$ is given explicitly by

$$(18.2) \qquad v(||x||^2) = \int_{||x||^2}^{+\infty} \exp\Big(-\int_{R^2}^{s} \frac{\gamma(\sigma)}{\sigma}\,\mathrm{d}\sigma\Big)\,\mathrm{d}s$$

For more details we can see Theorem 3.1 and remark 4.1 hereafter.

ii) the function of spectral dispersion $x \longrightarrow \wedge(x)$ is anisotropically oscillating at infinity. In the sequel our goal is to study this case. In the sequel we consider the constructed matrix $\hat{A}$ and the associated equation (18). To simplify the notations we denote $\hat{A}(\bullet)$ by $A(\bullet)$ and we identify (18) to (12) with $f(B^{-1}\bullet)$ identified to $f(\bullet)$.

Remark 4. For any matrix $B \in Q(\alpha,\beta)$, we use the change of variables $y = \sqrt{B^{-1}}.x$. Thus the problem (12) and

$$(18.3) \qquad \begin{cases} -TrA_B(y)D^2\omega(y) = f\big(\sqrt{B}.y\big) \\ \omega(y) > 0\ , \quad \lim_{||y||\to+\infty}\omega(y) = 0\ , \quad A_B(y) = \sqrt{B^{-1}}A\big(\sqrt{B}.y\big).\sqrt{B^{-1}} \end{cases}$$

have the same asymptotic behaviour since $u(x) = \omega\big(\sqrt{B^{-1}}.x\big)$.

Theorem 13.1. Without any hypothesis about $\wedge(\bullet)$, for any $\varepsilon > 0$, there exist $R_0 := R_0(\varepsilon) > 0$, $c_1 := c_1(\varepsilon) > 0$, $c_2 := c_2(\varepsilon) > 0$ such that the solution $u(\bullet)$ of (12) satisfies : for any x, $||x|| \geqslant R_0$

$$\frac{c_1}{||x||^{2(\overline{\wedge}_\infty - 1 + \varepsilon)}} \leqslant u(x) \leqslant \frac{c_2}{||x||^{2(\underline{\triangle}_\infty - 1 - \varepsilon)}}$$

where

$$\overline{\wedge}_\infty = \inf_{B \in Q(\alpha,\beta)} \lim_{R \to +\infty} \sup_{||x|| \geqslant R} \wedge(A,B)(x)$$

$$\underline{\triangle}_\infty = \sup_{B \in Q(\alpha,\beta)} \lim_{R \to +\infty} \inf_{||x|| \geqslant R} \wedge(A,B)(x)$$

with $\wedge(A,B)(x) = \frac{Tr\left(A_B(x)\right)}{\frac{2}{||x||^2} \cdot \left(A_B(x).x, x\right)}$ where $A_B(x) = \sqrt{B^{-1}}.A\left(\sqrt{B}.x\right).\sqrt{B^{-1}}$.
Before to give the proof of this result, let us remark that from the very definition of $\wedge(A,B)$ we have $\underline{\triangle}_\infty > 1$ since $\frac{N}{2} > \frac{\beta}{\alpha}$.

Proof. Let us prove the right hand side inequality. From the very definition of $\underline{\triangle}_\infty$ we can see that for any $\varepsilon > 0$ there exist $B_0 = B_0(\varepsilon) \in Q(\alpha,\beta)$ and $r_0 = r_0(\varepsilon)$ such that : for any x, $||x|| \geqslant r_0$ we have $\underline{\triangle}_\infty - \varepsilon \leqslant \wedge(A, B_0)(x)$. After the change of variables $y = \sqrt{B_0^{-1}}.x$, we first prove our inequality for the function $\omega(\bullet)$ defined by

$$u(x) = \omega\left(\sqrt{B_0^{-1}}.x\right) \qquad \text{or} \qquad \omega(y) = u\left(\sqrt{B_0}.y\right) .$$

This function satisfies

$$(18.3.1) \quad \begin{cases} -T_r\left(A_{B_0}(x) D^2\omega(x)\right) = f\left(\sqrt{B_0}.x\right) \\ \omega(x) = \omega_\varepsilon(x) > 0 , \quad \omega(x) = u\left(\sqrt{B_0}.x\right) , \quad \lim_{||x|| \to +\infty} \omega(x) = 0 , \end{cases}$$

since problems (12) and (18.3) have the same asymptotic behaviour. For this let us set, for any $x \in \mathbb{R}^N$,

$$\overline{f}(x) = \sup \left\{ \frac{f(\sqrt{C}.x)}{4(A_C x, x)} \quad / \quad C \in Q(\alpha,\beta) \right\} .$$

There exists a radial positive function $h(\bullet)$, with compact support $[0, \rho_0^2]$ such that $h(||x||^2) \geqslant \overline{f}(x)$ $\forall x \in \mathbb{R}^N$. Increasing r_0 if necessary, we give the following positive, continuous and increasing function :

$$(18.3.2) \quad \wedge(r) = \begin{cases} \underline{\triangle}_\infty - \varepsilon , \quad \forall\, r \geqslant r_0^2 , \\ l(r) , \quad \underline{\triangle}_\infty - \varepsilon \geqslant l(r) \geqslant \frac{\alpha N}{2\beta} > 1 , \quad \forall\, r < r_0^2 \quad \text{with} \quad l(||x||^2) \leqslant \wedge(A, B_0)(x) \\ \qquad \forall\, x , \ ||x|| \leqslant r_0 , \end{cases}$$

and let us consider $\omega_{rad}(\bullet)$ solution of the following differential equation

$$(18.3.3) \quad \begin{cases} -\frac{1}{\sigma(r)}\left(\sigma(r)\omega'_{rad}(r)\right)' = h(r) , \\ \omega'_{rad}(0) = 0 , \quad \omega_{rad}(r) > 0 , \quad \lim_{r \to +\infty} \omega_{rad}(r) = 0 , \end{cases}$$

with

$$\sigma(r) = \exp\left(\int_1^r \frac{\wedge(\theta)}{\theta}\, \mathrm{d}\theta\right) .$$

From the very definition of $\wedge(\bullet)$ we have $\wedge(||x||^2) \leqslant \wedge(A, B_0)(x)\ \forall\, x$, and using the same methode as in section IV - B1 - first step or as in [4], we can prove that $\omega_r(x) := \omega_{rad}(||x||^2)$ is a super-solution of (18.3) that is to say :

$$(18.3.4)\quad \begin{cases} -\mathrm{div}\big(A_{B_0} D^2\omega_r(x)\big) \geqslant f(\sqrt{B_0}.x)\ , \\ \omega_r(x) > 0\ ,\quad \lim_{||x||\to+\infty} \omega_r(x) = 0\ . \end{cases}$$

And from the comparison principle [1], [3], we obtain :

$$(18.3.5)\quad \forall\, x \in \mathbb{R}^N\ ,\quad 0 \leqslant \omega(x) \leqslant \omega_r(x) = \int_{||x||^2}^{+\infty} \frac{1}{\sigma(\theta)} \int_0^\theta \sigma(t)h(t)\,\mathrm{d}t\,\mathrm{d}\theta\ .$$

But we have :

$$\forall\,\theta\ ,\quad \theta \geqslant r_0(\varepsilon) > \rho_0\ ,\quad \int_0^\theta \sigma(t)h(t)\,\mathrm{d}t = \int_0^{\rho_0} \sigma(t)h(t)\,\mathrm{d}t \leqslant ||\sigma||_\infty.||h|||_\infty.\rho_0 \leqslant c_0(h)$$

where $c_0(h)$ is a constant independant of ε;

$$\forall\,\theta\ ,\quad \theta \geqslant r_0(\varepsilon) > \rho_0\ ,\quad \sigma(\theta) = \exp\Big(\int_1^\theta \frac{\wedge(r)}{r}\,\mathrm{d}r\Big) = \exp\Big(\int_1^{r_0} \frac{l(r)}{r}\,\mathrm{d}r\Big).\Big(\frac{\theta}{r_0}\Big)^{\triangle_\infty-\varepsilon}\ .$$

And thus for any x, $||x|| \geqslant r_0$ we obtain :

$$0 \leqslant \omega(x) \leqslant \frac{c_0(h).r_0^{\triangle_\infty-\varepsilon}}{\exp\Big(\int_1^{r_0} \frac{l(r)}{r}\,\mathrm{d}r\Big)}.\frac{1}{||x||^{2(\triangle_\infty-1-\varepsilon)}}\ ,$$

$$0 \leqslant \omega(x) \leqslant \frac{c_\varepsilon.c_0(h)}{||x||^{2(\triangle_\infty-1-\varepsilon)}}\ ,\quad \forall\, x\ ,\ ||x|| \geqslant r_0(\varepsilon)\ .$$

By the reverse change of variables $x = \sqrt{B_0^{-1}}t$ we have $u(t) = \omega\Big(\sqrt{B_0^{-1}}.t\Big)$; and we obtain

$$u(t) \leqslant \frac{c_\varepsilon.c_0(h)}{||\sqrt{B_0^{-1}}t||^{2(\triangle_\infty-1-\varepsilon)}}\ .$$

And since $\frac{\sqrt{\alpha}}{\beta^2}I \leqslant \sqrt{B_0^{-1}} \leqslant \frac{\sqrt{\beta}}{\alpha^2}I$, there exists $c(\alpha,\beta) > 0$ such that

$$(18.3.6)\quad u(t) \leqslant \frac{c_\varepsilon.c(\alpha,\beta).c_0(h)}{||t||^{2(\triangle_\infty-1-\varepsilon)}}$$

for any t, $||t||^2 \geqslant r_0^2.\frac{\alpha^2}{\sqrt{\beta}}$. In the same way we prove the left hand side inequality. □

Let us set $\wedge_B(r) = \inf_{||x||\geqslant r} \wedge(A,B)(x)$, $\forall\, r \geqslant 0$.

Corollary 13.2 Let us suppose that the matrix $A(\bullet)$ is such that the decreasing function $r \longrightarrow \varphi(r) = \sup_{B\in Q(\alpha,\beta)}[\triangle_\infty - \wedge_B(r)]$ satisfies the following : $\exp\Big(\int_1^{+\infty} \frac{\varphi(r)}{r}\,\mathrm{d}r\Big) < +\infty$. Then we obtain the following result : there exist some constants $c_1(\alpha,\beta,h) > 0$, $r_0 > 0$, such that

$$u(x) \leqslant \frac{c_1(\alpha,\beta,h)}{||x||^{2(\triangle_\infty-1)}}\quad \forall\, x\ ,\quad ||x|| \geqslant r_0\ .$$

Proof : From our hypothesis we have, for any $B \in Q(\alpha,\beta)$

$$\triangle_\infty - \varphi(r) \leqslant \wedge_B(r) \leqslant \wedge(A,B)(x)\ , \quad \forall\, x\ , \quad ||x|| \geqslant r\ .$$

In (18.3.1) we use B instead of B_0 and in (18.3.2) we set $\wedge(r) := \triangle_\infty - \varphi(r)\ \forall\, r \geqslant 0$. Then (18.3.5) becomes :

$$0 \leqslant \omega_B(x) := \omega(x) \leqslant c_0(h).\int_{||x||^2}^{+\infty} \frac{\mathrm{d}\theta}{\sigma(\theta)}\ , \quad \forall\, ||x|| \geqslant \rho_0\ ,$$

with

$$\sigma(\theta) = \theta^{\triangle_\infty}.\exp\Big(- \int_1^\theta \frac{\varphi(r)}{r}\,\mathrm{d}r \Big)\ ,$$

that is to say

$$0 \leqslant \omega(x) \leqslant \exp\Big(\int_1^{+\infty} \frac{\varphi(r)}{r}\,\mathrm{d}r \Big).\frac{c_0(h)}{||x||^{2(\triangle_\infty - 1)}}\ .$$

That is to say

$$0 \leqslant \omega(x) \leqslant \frac{c_1(h)}{||x||^{2(\triangle_\infty - 1)}}\ ,$$

where $c_1(h)$ is a constant independant on $B \in Q(\alpha,\beta)$. And as previously, (18.3.6) becomes

$$u(x) \leqslant \frac{c(h).c(\alpha,\beta)}{||x||^{2(\triangle_\infty - 1)}} \qquad \forall\, ||x|| > r_0\ ,$$

for some $r_0 > \rho_0$. □

We have a similar result to estimate $u(\bullet)$ from below :

$$u(x) \geqslant c_2(\alpha,\beta,h)\ /\ ||x||^{2(\overline{\wedge}_\infty - 1)}\ .$$

Remark 4.1 Contrary to what is stated in theorems 21 and 22 of [4], the result obtained in [4] is the following : for any $\varepsilon > 0$ there exist $R_0 := R_0(\varepsilon) > 0$, $c_1(\varepsilon) > 0$ and $c_2(\varepsilon) > 0$ such that

$$c_1(\varepsilon)v_1(||x||^2) \leqslant u(x) \leqslant c_2(\varepsilon)v_2(||x||^2) \qquad \forall\, x\ , \quad ||x|| \geqslant R_0$$

with

$$v_1(||x||^2) = \int_{||x||^2}^{+\infty} \exp\Big(- \int_{R^2}^{s} \frac{\gamma(\sigma)+\varepsilon}{\sigma}\,\mathrm{d}\sigma \Big)\,\mathrm{d}s$$

$$v_2(||x||^2) = \int_{||x||^2}^{+\infty} \exp\Big(- \int_{R^2}^{s} \frac{\gamma(\sigma)-\varepsilon}{\sigma}\,\mathrm{d}\sigma \Big)\,\mathrm{d}s\ .$$

The function $\gamma(\bullet)$ can play the role of $\underline{\theta}$, $\overline{\theta}$ or $\gamma(||x||^2)$ given in the introduction, for instance. Indead the conclusion in the last line of the proof - ([4] p 180, line 5 from the top) - is not correct because it is some what fast : the various constants depend to the variable ε. *To* overcome this difficulty we need to replace the hypothesis (H3) of [4] by the following : the decreasing function $r \longrightarrow \varphi(r) = \sup_{||x|| \geqslant r} \big| \wedge(x) - \gamma(||x||^2) \big|$ satisfies $\exp\Big(\int_1^{+\infty} \frac{\varphi(r)}{r}\,\mathrm{d}r \Big) < +\infty$. And in this case the result of [4] follows from the same idea of corollary 13.2 i. e.

$$u(x) \sim \int_{||x||^2}^{+\infty} \exp\Big(- \int_{R^2}^{s} \frac{\gamma(\sigma)}{\sigma}\,\mathrm{d}\sigma \Big)\,\mathrm{d}s$$

for $||x||$ large enough. □

IV - Anisotropic asymptotic behavior.
In the sequel we denote again by $A(x)$ the matrix $\hat{A}(x) = \sqrt{B_\infty^{-1}}.A\big(\sqrt{B_\infty}.x\big).\sqrt{B_\infty^{-1}}$, where B_∞ is given by proposition 10. From the very definition of $A(\bullet)$, the amplitude of its spectrum is positive. Let us set :

$$(18.4) \qquad \begin{cases} \overline{\wedge} = \lim_R \sup_{||x||\geqslant R} \dfrac{Tr\big(A(x)\big)}{\frac{2}{||x||^2}\big(A(x).x,x\big)} \\ \underline{\wedge} = \lim_R \inf_{||x||\geqslant R} \dfrac{Tr\big(A(x)\big)}{\frac{2}{||x||^2}\big(A(x).x,x\big)} \end{cases}$$

Let δ_0 be a positive real number close enough to $\overline{\wedge} - \underline{\wedge}$ such that $\delta_0 > \overline{\wedge} - \underline{\wedge}$. Let $2\eta_0$ be a positive real number very small with respect to $\delta_0 - (\overline{\wedge} - \underline{\wedge}) > 0$. And let us give $\eta > 0$, $0 < \eta < \eta_0$ - (arbitrary small) - . From (18.4) there exists $R_\eta = R_\infty$ such that for any x, $||x|| \geqslant R_\infty$ we have

$$(18.5) \qquad \underline{\wedge}(\eta) = \underline{\wedge} - \eta \leqslant \wedge(x) \leqslant \overline{\wedge} + \eta = \overline{\wedge}(\eta)$$

where

$$\wedge(x) = \frac{Tr\big(A(x)\big)}{\frac{2}{||x||^2}\big(A(x).x,x\big)} \ .$$

Let us consider the operator $\mathcal{A}(\bullet) = \sum_{ij} a_{ij}(x)\dfrac{\partial^2(\bullet)}{\partial x_i \partial x_j}$. From operator $\mathcal{A}(\bullet)$ we introduce a family of radial operators

$$x \in \mathbb{R}^N \longrightarrow \mathcal{A}_r(x)(\bullet) = \frac{\mathrm{d}^2(\bullet)}{\mathrm{d}r^2} + \frac{\wedge(x)}{r}\frac{\mathrm{d}}{\mathrm{d}r}(\bullet) = \frac{1}{r^{\wedge(x)}}\frac{\mathrm{d}}{\mathrm{d}r}\Big(r^{\wedge(x)}\frac{\mathrm{d}}{\mathrm{d}r}(\bullet)\Big) \ ,$$

where $r > 0$ and $x \longrightarrow \wedge(x)$ is the function of the spectral dispersion associated to the matrix $A(x)$. The link between $\mathcal{A}(\bullet)$ and $\mathcal{A}_r(x)(\bullet)$ is the following : for any regular and radial function $\Psi(x) = \varphi(||x||^2)$ we have

$$\frac{\mathcal{A}\Psi}{4\sum_{ij} a_{ij}x_i x_j} = \varphi''(||x||^2) + \frac{\wedge(x)}{||x||^2}\varphi'(||x||^2) = \mathcal{A}_\rho(x)\varphi(\rho)_{/\rho=||x||^2} \ .$$

Remark 5. Let us point out that for some technical reasons the variable r plays the role of $||x||^2$, instead of $||x||$ as usually.

A - Method of sub and supersolution.
Let us give a positive radial function $r \longrightarrow g(r)$ which will be choosen later in a suitable way. For any -(but fixed)- x belonging to $\mathbb{R}^N$, with $||x||$ large enough, we consider a family of radial problems :

$$(\mathcal{P}_x) \ : \ (19) \quad \begin{cases} -\mathcal{A}_r(x)h(x,r) = -h''(x,r) - \frac{\wedge(x)}{r}h'(x,r) = g(r)\,, \quad r > 0 \\ h'(x,0) = 0\,, \quad \lim_{r\to+\infty} h(x,r) = 0\,, \quad h(x,r) > 0 \end{cases}$$

where g satisfies $\int_1^{+\infty} \sigma^{\overline{\wedge}} g(\sigma)\,\mathrm{d}\sigma < +\infty$ with $\overline{\wedge}$ given by (18.4) section IV. The unique solution $r \longrightarrow h(x,r)$ of (19) is given by

$$(20) \quad h(x,r) := h_g(x,r) = \int_r^{+\infty} \frac{1}{s^{\wedge(x)}}\Big(\int_0^s \sigma^{\wedge(x)} g(\sigma)\,\mathrm{d}\sigma\Big)\,\mathrm{d}s \ .$$

If $\wedge(\bullet) = \text{constant}\ \forall\, x\,,\ ||x|| \geqslant R_0$, the function $h_g(x, ||x||^2)$ taks the form $h_g(||x||^2) = h(||x||^2)$. And let us point out that in this case $x \longrightarrow h(||x||^2)$ gives the behavior at infinity of the solution of (12) [4]. So roughly speaking, **our idea** is to look for a subsolution $\underline{\sigma}$, -(respectivement a supersolution $\overline{\sigma}$)- of (12) in the form

$$(21) \quad H(x) = H_g(x) = h_g(x, ||x||^2) = h(x, ||x||^2)\ .$$

How to construct these functions? For this it is sufficient to choose two functions g_1 and g_2 in a suitable way such that $\underline{\sigma}(\bullet) = H_{g_1}(\bullet)$ and $\overline{\sigma}(\bullet) = H_{g_2}(\bullet)$ have the same behavior as $||x||$ goes to infinity. Next we establish the result thanks the comparison principle. □

B- Construction of sub and supersolution of (12).

B0 - Preliminary results.
Let us introduce the following functions that we need later. For any $x \in \mathbb{R}^N$, any $r > 0$ and any suitable positive function $g(\bullet) : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, we set

$$\mathcal{K}_1 = \mathcal{K}_1(x, r, g) = \int_r^{+\infty} \frac{1}{s^{\wedge(x)}} \Big[\int_0^s \mathrm{Log}\Big(\frac{\sigma}{s}\Big).\sigma^{\wedge(x)} g(\sigma)\,\mathrm{d}\sigma \Big]\,\mathrm{d}s$$

$$\mathcal{K}_2 = \mathcal{K}_2(x, r, g) = \int_r^{+\infty} \frac{1}{s^{\wedge(x)}} \Big[\int_0^s \Big(\mathrm{Log}\Big(\frac{\sigma}{s}\Big)\Big)^2.\sigma^{\wedge(x)} g(\sigma)\,\mathrm{d}\sigma \Big]\,\mathrm{d}s$$

$$\mathcal{K}_3 = \mathcal{K}_3(x, r, g) = -\frac{1}{r^{\wedge(x)}} \int_0^s \mathrm{Log}\Big(\frac{\sigma}{r}\Big).\sigma^{\wedge(x)} g(\sigma)\,\mathrm{d}\sigma\ .$$

To simplify the presentation we use the short notation $\mathcal{K}_i,\ i = 1, 2, 3$. Since $g(\bullet) \geqslant 0$ we have $\mathcal{K}_1 \leqslant 0$, $\mathcal{K}_2 \geqslant 0$ and $\mathcal{K}_3 \geqslant 0$.

Proposition 14. The following estimates are true for $r > 0$ large enough :

$$1) \quad 0 \leqslant -\mathcal{K}_1 = k.\frac{r_1^{\wedge+1} - r_0^{\wedge+1}}{(\wedge+1)(\wedge-1)}.\frac{\mathrm{Log}\, r}{r^{\wedge-1}} + \bigcirc\Big(\frac{1}{r^{\wedge-1}}\Big)$$

$$2) \quad 0 \leqslant \mathcal{K}_2 = k.\frac{r_1^{\wedge+1} - r_0^{\wedge+1}}{(\wedge+1)(\wedge-1)}.\frac{(\mathrm{Log}\, r)^2}{r^{\wedge-1}} + \bigcirc\Big(\frac{1}{r^{\wedge-1}}\Big)$$

$$3) \quad 0 \leqslant \mathcal{K}_3 = k.\frac{r_1^{\wedge+1} - r_0^{\wedge+1}}{\wedge+1}.\frac{\mathrm{Log}\, r}{r^{\wedge-1}} + \bigcirc\Big(\frac{1}{r^{\wedge-1}}\Big)$$

where $\wedge = \wedge(x)$ and $g(r) = k.\chi_{[r_0,r_1]}(r)$, $k > 0$ and $\chi_{[r_0,r_1]}(r) = 1$ if $r_0 \leqslant r \leqslant r_1$ and $\chi_{[r_0,r_1]}(r) = 0$ if $r \notin [r_0, r_1]$. The proof is straightforward.

Proposition 15. Let us assume the spectral dispersion satisfies :

$$||\nabla \wedge||_{\mathrm{L}^\infty\left(\mathbb{R}^N \setminus \mathrm{B}(0,R_0)\right)} + ||D^2 \wedge||_{\mathrm{L}^\infty\left(\mathbb{R}^N \setminus \mathrm{B}(0,R_0)\right)} \leqslant c$$

for R_0 large enough. Then we have the following : for any $\varepsilon > 0$ there exist $R_\varepsilon \geqslant R_0$ and $c_\varepsilon > 0$ such that

$$(23) \quad \begin{aligned} &\Big|\mathcal{K}_2.\big(A(x)\nabla\wedge, \nabla\wedge\big) + \mathcal{K}_1.Tr\big(A(x)D^2\wedge\big) + 4\mathcal{K}_3.\big(A(x)\nabla\wedge, x\big)\Big| \\ &\leqslant c_\varepsilon.\frac{||g||_\infty\,[\rho_1^{\overline{\wedge}+1} - \rho_0^{\underline{\wedge}+1}]}{||x||^{2(\wedge(x)-1-\varepsilon)}} \end{aligned}$$

for any x, $||x|| \geqslant R_\varepsilon$, $r = ||x||^2$ and for any positive function $g(\bullet)$ such that support $g \subseteq \big[x \ / \ \rho_0 \leqslant ||x|| \leqslant \rho_1\big]$ and where $\underline{\wedge}$ and $\overline{\wedge}$ are given by (18.4) section IV.

Proof. It follows easily from the proposition 14. □

Remark 6. Without another assumption on $\wedge(\bullet)$ the left-hand side of (23) tends to zero as $||x||$ goes to infinity. But we will see that the behavior of (23) at infinity is not sufficient to prove our result. So we will assume that the left-hand side of (23) goes to zero faster than $\dfrac{1}{||x||^{2(\wedge(x)-1-\varepsilon)}}$ as $||x||$ goes to infinity. The results of section II allow us to show that the supposed assumptions are suitable for a large classe of matrices. □

From [4] and theorem 13.1 and for some fixed $\varepsilon > 0$ small enought, we have

$$\frac{a}{||x||^{2(\overline{\wedge}-1+\varepsilon)}} \leqslant u(x) \leqslant \frac{b}{||x||^{2(\underline{\wedge}-1-\varepsilon)}}$$

for $||x||$ sufficiently large and with $0 < a < b$. These inequalities prove that $u(\bullet)$ has a behavior of the form $u(x) = \dfrac{c(x)}{||x||^{\delta(x)}}$ where the function $\delta(\bullet)$ and $c(\bullet)$ satisfy :

$$0 < c_0 \leqslant c(x) \leqslant c_1 \ ,$$

$$2(\underline{\wedge} - 1) \leqslant \delta(x) \leqslant 2(\overline{\wedge} - 1) \ ,$$

$$\frac{\operatorname{Log} c_0 - \operatorname{Log} u(x)}{\operatorname{Log}(||x||)} \leqslant \delta(x) \leqslant \frac{\operatorname{Log} c_1 - \operatorname{Log} u(x)}{\operatorname{Log}(||x||)} \ .$$

That is to say $\delta(x) = -\dfrac{\operatorname{Log} u(x)}{\operatorname{Log}(||x||)}$ for $||x||$ large enough.

Proposition 16. Let $A(\bullet)$ be a matrix belonging to $Q(\alpha, \beta)$. Let us assume that, for any positive continuous function f with compact support, the solution $u(\bullet) = u_f(\bullet)$ of

$$\begin{cases} -Tr(AD^2 u_f) = f \quad \text{in} \quad \mathbb{R}^N \\ u_f(x) > 0 \quad \forall x \,, \quad \lim_{||x|| \to +\infty} u_f(x) = 0 \end{cases}$$

has the following asymptotic behavior :

$$\frac{a(f)}{||x||^{\delta_f(x)}} \leqslant u(x) \leqslant \frac{b(f)}{||x||^{\delta_f(x)}} \,, \quad \forall x \,, \quad ||x|| \geqslant R_f > 0$$

with

$$0 < a(f) \leqslant b(f) \,, \quad \frac{b(f)}{a(f)} \leqslant \gamma \,, \quad b(f) \geqslant \frac{||u||_\infty}{2}$$

Then $\delta_f(\bullet)$ is independant of f and satisfies

$$N.\frac{\alpha}{\beta} - 2 \leqslant \delta_f(\bullet) \leqslant N.\frac{\beta}{\alpha} - 2$$

Proof. We need two steps.

1) **First step : asymptotic behavior of the somme of two solutions.** Let us consider

$$(24) \quad \begin{cases} -Tr(AD^2u_i) = f_i \quad \text{in} \quad \mathbb{R}^N\,, \quad i=1,2 \\ u_i(x) > 0\,, \quad \lim_{||x||\to+\infty} u_i(x) = 0\;. \end{cases}$$

Let us set $\delta_i(\bullet) = \delta_{f_i}(\bullet)$, $\omega(\bullet) = u_1(\bullet) + u_2(\bullet)$, $g = f_1 + f_2$, $a_i = a(f_i)$, $b_i = b_i(f)$, $R_i = R_{f_i}$. For any x, $||x|| \geqslant \sup\big(R_1, R_2, R_g\big)$ we have :

$$\begin{cases} \dfrac{a(g)}{||x||^{\delta_g}} \leqslant \omega(x) \leqslant \dfrac{b(g)}{||x||^{\delta_g}} \\ \dfrac{a_i}{||x||^{\delta_i}} \leqslant u_i(x) \leqslant \dfrac{b_i}{||x||^{\delta_i}}\,, \quad i=1,2\;. \end{cases}$$

Since $\omega = u_1 + u_2$, we obtain

$$\frac{a_i}{||x||^{\delta_i}} \leqslant u_i(x) \leqslant \frac{b(g)}{||x||^{\delta_g}}\;.$$

There exists $R \geqslant \sup\big(R_1, R_2, R_g\big)$ such that $\delta_i(x) \geqslant \delta_g(x) \quad \forall\, x\,, \quad ||x|| \geqslant R$.

This entails that we have $\inf\big(\delta_1(x), \delta_2(x)\big) \geqslant \delta_g(x)$. As we have

$$(25) \quad \frac{a(g)}{||x||^{\delta_g}} \leqslant \omega = u_1 + u_2 \leqslant \frac{b_1}{||x||^{\delta_1}} + \frac{b_2}{||x||^{\delta_2}} \leqslant \frac{b_1+b_2}{||x||^{\inf(\delta_1,\delta_2)}}$$

we obtain $\delta_g(x) \geqslant \inf\big(\delta_1(x), \delta_2(x)\big)$ for $||x||$ sufficiently large. Thus we have

$$(26) \quad \delta_g(x) = \inf\big(\delta_1(x), \delta_2(x)\big)\;.$$

And since operator $-Tr(AD^2\bullet)$ is linear, we have :

$$(27) \quad \delta_{kf}(x) = \delta_f(x)\,, \quad \forall\, k > 0\;.$$

And thus from (26) and (27) we obtain

$$(28) \quad \begin{cases} \delta_{kf_2+f_1}(x) = \inf\big(\delta_1(x), \delta_2(x)\big) \\ \forall\, k > 0 \quad \text{and} \quad ||x|| \quad \text{large enough}\;. \end{cases}$$

2) **Second step : a perturbation result.**

For any $\varepsilon > 0$ we set $g_\varepsilon = f_1 + \varepsilon f_2$; and $u_2^\varepsilon(x) = \varepsilon u_2(x)$ is the solution of (24) with εf_2 instead of f_2. From (28) we obtain $\delta_{g_\varepsilon}(x) = \inf\big(\delta_1(x), \delta_2(x)\big) = \delta_g(x)$. From (25) we can write

$$(29) \quad \frac{a(g_\varepsilon)}{||x||^{\delta_1}} \leqslant \frac{a(g_\varepsilon)}{||x||^{\delta_g}} \leqslant u_1(x) + \varepsilon u_2(x) \leqslant \frac{b_1}{||x||^{\delta_1}} + \frac{\varepsilon b_2}{||x||^{\delta_2}}$$

for $||x||$ sufficiently large, since $\delta_1(x) \geqslant \delta_g(x)$. Consequently as

$$b(g_\varepsilon) \geqslant \frac{||u_1 + \varepsilon u_2||_\infty}{2} \geqslant \frac{||u_1||_\infty - \varepsilon||u_2||_\infty}{2} \geqslant \frac{||u_1||_\infty}{4}\,,$$

for ε sufficiently small, we have $\beta_0 a(g_\varepsilon) \geqslant b(g_\varepsilon) \geqslant \frac{||u_1||_\infty}{4}$ and thus $a(g_\varepsilon) \geqslant \frac{||u_1||_\infty}{4\beta_0}$. Let us pass to the limit in (29) as ε goes to zero. We obtain

$$\frac{1}{||x||^{\delta_g}} \frac{||u_1||_\infty}{4\beta_0} \leqslant u_1(x) \leqslant \frac{b_1}{||x||^{\delta_1}}\;.$$

This entails, for $||x||$ sufficiently large, $\delta_g(x) \geqslant \delta_1(x)$. In the same way we can prove that $\delta_g(x) \geqslant \delta_2(x)$, and thus we have

$$(30) \quad \delta_g(x) \geqslant \sup\big(\delta_1(x), \delta_2(x)\big) \ .$$

Finally (26) and (30) give the result :

$$\delta_g(x) = \delta_1(x) = \delta_2(x) \ .$$

The inequalities $-2 + N\frac{\alpha}{\beta} \leqslant \delta(x) \leqslant -2 + N\frac{\beta}{\alpha}$ come from [4] □

The behavior at infinity of matrices is important as shown by the following result.

Proposition 17. Let us give two matricies $A_1(\bullet)$ and $A_2(\bullet)$ belonging to $Q(\alpha,\beta)$ such that $A_1(x) = A_2(x)$ for any x, $||x|| \geqslant R_0 > 0$. Then the solutons $u_i(\bullet)$, $i = 1, 2$, of

$$(31) \quad \begin{cases} -Tr(A_i D^2 u_i) = f_i \quad \text{in} \quad \mathbb{R}^N \\ u_i(x) > 0\,, \quad \lim_{||x||\to+\infty} u_i(x) = 0 \end{cases}$$

where $f_i = f$, $(i = 1, 2)$, is a positive continuous function with compact support, have the same asymptotic behavior, that is to say there exist $0 < a_i < b_i$, $i = 1, 2$ and $x \longrightarrow \delta(x) \in [N\frac{\alpha}{\beta} - 2, N\frac{\beta}{\alpha} - 2]$ such that

$$\frac{a_i}{||x||^{\delta(x)}} \leqslant u_i(x) \leqslant \frac{b_i}{||x||^{\delta(x)}} \quad i = 1, 2$$

for $||x||$ sufficiently large.

Proof. From (31) it follows

$$(32) \quad -Tr\big(A_1 D^2(u_1 - u_2)\big) = g(x)$$

where $g(x) = -Tr\big((A_2 - A_1)D^2 u_2(x)\big)$ is a continuous function with compact support in $B(O, R_0)$ since $A_2(x) - A_1(x) = 0, \forall x, ||x|| > R_0$. Let us set $h(x) = |g(x)|$. From comparison principle -(cf [1] , [3])- we have

$$(33) \quad 0 \leqslant \big|u_1(x) - u_2(x)\big| \leqslant s(x)$$

where $s(\bullet)$ is the solution of (31) with $h(\bullet)$ and A_1 instead of $f_i = f$ and A_i, respectively. From proposition 16 we have

$$(33.1) \quad \begin{cases} \dfrac{a(h)}{||x||^{\delta_1(x)}} \leqslant s(x) \leqslant \dfrac{b(h)}{||x||^{\delta_1(x)}} \quad \forall x, \quad ||x|| \geqslant R \geqslant R_0\ , \\ \dfrac{a_i(f)}{||x||^{\delta_i(x)}} \leqslant u_i(x) \leqslant \dfrac{b_i(f)}{||x||^{\delta_i(x)}} \quad \forall x, \quad ||x|| \geqslant R \geqslant R_0\ , \quad i = 1, 2\ . \end{cases}$$

From (33) we have $u_2(x) \leqslant u_1(x) + s(x)$ and thus it follows

$$\frac{a_2(f)}{||x||^{\delta_2(x)}} \leqslant \frac{b_1(f) + b(h)}{||x||^{\delta_1(x)}}\ .$$

For $||x||$ large enough, this give $\delta_2(x) \geqslant \delta_1(x)$. In the same way, reversing the role of u_1 and u_2 we get $\delta_1(x) = \delta_2(x)$. □

Proposition 18. Consider the two functions

$$H_i(x) = \int_{||x||^2}^{+\infty} \frac{1}{s^{\wedge_i(x)}} \Big(\int_0^s \sigma^{\wedge_i(x)} g_i(\sigma)\,\mathrm{d}\sigma \Big)\,\mathrm{d}s \quad i = 1, 2\ .$$

Let us assume that

$$\wedge_1(x) = \wedge_2(x) = \wedge(x) \quad \forall\, x, \quad ||x|| \geqslant R\ ,$$

and $g_i(\bullet)$ is positive bounded function with compact support, $i = 1, 2$. Then the two functions $H_1(\bullet)$ and $H_2(\bullet)$ have the same behavior as the function $\dfrac{1}{||x||^{2(\wedge(x)-1)}}$ at infinity.

Proof. Let us recall that we have

$$0 < \int_0^{+\infty} \sigma^{\wedge_i(x)} g_i(\sigma) < \Gamma_i < +\infty$$

It is sufficient to prove the result for $H_1(\bullet)$, for instance. We have

$$0 \leqslant H_1(x) \leqslant \Gamma_1 . \int_{||x||^2}^{+\infty} \frac{\mathrm{d}s}{s^{\wedge_1(x)}} = \frac{\Gamma_1}{\big(\wedge_1(x) - 1\big)||x||^{2(\wedge_1 - 1)}}$$

that is to say

$$0 \leqslant H_1(x) \leqslant \frac{\Gamma_1}{\big(\wedge_1(x) - 1\big)||x||^{2(\wedge(x)-1)}} \quad \forall\, x\ , \quad ||x|| \geqslant R\ .$$

Now let us prove the reverse inequality. Since $g_1(\bullet)$ has a compact support, there exists $R_1 \geqslant R$ such that

$$\int_0^{+\infty} \sigma^{\wedge_1(x)} g_1(\sigma)\,\mathrm{d}\sigma = \int_0^{R_1} \sigma^{\wedge_1(x)} g_1(\sigma)\,\mathrm{d}\sigma \geqslant \int_1^{R_1} \sigma^{\tilde{\underline{\wedge}}_1} g_1(\sigma)\,\mathrm{d}\sigma + \int_0^1 \sigma^{\tilde{\overline{\wedge}}_1} g_1(\sigma)\,\mathrm{d}\sigma \geqslant \Gamma_0 > 0$$

where $\tilde{\underline{\wedge}}_1 = \inf_{||x|| \leqslant R_1} \wedge_1(x)$ and $\tilde{\overline{\wedge}}_1 = \sup_{||x|| \leqslant R_1} \wedge_1(x)$. This gives the following :

$$\forall\, x, \quad ||x|| \geqslant R_1\ , \quad H_1(x) \geqslant \Gamma_0 \int_{||x||^2}^{+\infty} \frac{\mathrm{d}s}{s^{\wedge_1(x)}} = \frac{\Gamma_0}{\big(\wedge_1(x) - 1\big)||x||^{2(\wedge(x)-1)}}$$

□

Remark 6.1. The previous result will enable us to prove that the solution $u(\bullet)$ of (12) behaves like a suitable function H associated to $\wedge(\bullet)$. This shows that only $\wedge(\bullet)$ controls the behaviour of $u(\bullet)$. The following result is easy to obtain.

Proposition 19. Let us set $H(x) = h_g(x, ||x||^2)$ where $r \longrightarrow h_g(x, r)$ is defined by (19) and (20). Then function $H(\bullet)$ satisfies the following

$$\begin{aligned} Tr\big(A(x)D^2 H\big) = \quad & -4\big(A(x).x, x\big).g\big(||x||^2\big) + \mathcal{K}_1(x, ||x||^2, g).Tr\big(A(x)D^2 \wedge\big) + \\ & \mathcal{K}_2(x, ||x||^2, g).\big(A(x)\nabla\wedge, \nabla\wedge\big) + \\ & 4\mathcal{K}_3(x, ||x||^2, g).\big(A(x)\nabla\wedge, x\big) \end{aligned}$$

Proof. For any $x \in \mathbb{R}^N$, let us consider the solution $r \longrightarrow h(x, r)$ of (19) given by (20). To conclude it is sufficient to see that

$$\big(\mathcal{A}_r(x)h(x, \bullet)\big)\big(||x||^2\big) = -4\big(A(x).x, x\big)g\big(||x||^2\big)$$

$\square$

Let us introduce our essential assumption $(\mathcal{H})$: there exists $R_0(\mathcal{H}) := R_0 > 0$ such that for any $R > 0$ and any positive, bounded function g with compact support in $\mathrm{B}(O, R)$, the spectral dispersion $\wedge(\bullet)$ satisfies the following :

$$\begin{array}{l} -\mathcal{K}_1(x, ||x||^2, g).Tr\big(A(x)D^2 \wedge\big) - \mathcal{K}_2(x, ||x||^2, g).\big(A(x)\nabla\wedge, \nabla \wedge\big) \\ -4\mathcal{K}_3(x, ||x||^2, g).\big(A(x)\nabla\wedge, x\big) \in \\ \Big[-\dfrac{c_0(R)||g||_\infty}{1+||x||^{2(\wedge(x)+\delta(x))}}, \dfrac{c_0(R)||g||_\infty}{1+||x||^{2(\wedge(x)+\delta(x))}} \Big] \quad \forall x \ , \quad ||x|| \geqslant R_0 \ , \end{array}$$

with $\delta(x) \geqslant \delta_0 > 0$ and $\underline{\wedge} - \overline{\wedge} + \delta_0 > 0$, where $\underline{\wedge}$ and $\overline{\wedge}$ are defined in (18.4).

Remark 5.1. if $\underline{\wedge} = \overline{\wedge} = \wedge_\infty$ then we have $\lim_{R\to+\infty} \sup_{||x||\geqslant R} \big| \wedge(x) - \wedge_\infty \big| = 0$, and the result of [4] can apply.

Comments about the assumption $(\mathcal{H})$: The differential inequality $(\mathcal{H})$ is suitable as shown by the following remarks

1) if $\wedge(\bullet)$ is constant for $||x|| \geqslant R$, $(\mathcal{H})$ is satisfied. And in this case our result already follows from [4]. And from proposition 3 section II this result does not obtained by [2] since in this work [2] $\lim_{||x||\to+\infty} A(x) = A_0$, a constant matrix.

2) Let us consider the matrix $M(\wedge)$ defined by :

$$M(\wedge) = \mathcal{K}_1.D^2 \wedge + \mathcal{K}_2.\nabla \wedge \otimes \nabla \wedge + \mathcal{K}_3.\frac{\wedge(x)}{||x||^2}.N(\wedge)$$

where

$$\mathcal{K}_i = \mathcal{K}_i(x, ||x||^2, g) \quad \text{and} \quad N(\wedge)_{ij} = x_i.\frac{\partial\wedge}{\partial x_j} + x_j.\frac{\partial\wedge}{\partial x_i} \quad \forall\, i, j \ .$$

If $M(\wedge)$ satisfies the inequalities (4) for any x, $||x|| \geqslant R$, then from corollary 7 there exists a matrix $x \longrightarrow A(x) \in Q(\alpha, \beta)$ such that

$$(34) \quad \mathcal{K}_1.Tr\big(A(x)D^2 \wedge\big) + \mathcal{K}_2\big(A(x)\nabla\wedge, \nabla \wedge\big) + \mathcal{K}_3\big(A(x)\nabla\wedge, x\big) = 0 \ , \ \forall x, ||x|| \geqslant R.$$

And thus $(\mathcal{H})$ is satisfied by $A(\bullet)$ and $\wedge(\bullet)$.

3) If in addition $\wedge(\bullet)$ is the spectral dispersion of the matrix that we seek, then we use the remark 2 section II with the two following matrices $V = M(\wedge)$ and $R = I - \dfrac{2\wedge}{||x||^2}.x \otimes x$. And we find $A(\bullet) \in Q(\alpha, \beta)$ satisfying $(\mathcal{H})$ and such that $\wedge(\bullet)$ is its spectral dispersion.

4) Let us write the equation (34) in the form

$$(35) \quad Tr\big(A(x)D^2 \wedge\big) + \frac{\mathcal{K}_2}{\mathcal{K}_1}.\big(A(x)\nabla\wedge, \nabla \wedge\big) + \frac{\mathcal{K}_3}{\mathcal{K}_1}\big(A(x)\nabla\wedge, x\big) = 0 \ , \ \forall x, ||x|| \geqslant R.$$

Then for $R > 0$ large enough and $A = I$, the equation (35) is "very close" to the radial form :

$$(36) \quad \begin{cases} -\triangle \wedge + \frac{1}{2}\mathrm{Log}\big(||x||^2\big).||\nabla \wedge||^2 + \wedge(x)\dfrac{x.\nabla\wedge}{||x||^2} = 0 \ . \\ \forall x \ , \quad ||x|| \geqslant R \ . \end{cases}$$

The equation (36) is obtained by "approximation" using the proposition 14 section IV. The proposition 8 section II shows that the equation (36) has many non constant positive and bounded solutions. This gives a partial answer to the question posed in the remark 3 section II.

5) an elementary example satisfying $(\mathcal{H})$: Let k be a positive real number such that

$$\varphi(x) = k + \sin\Big(\frac{2\pi}{1+||x||^{2m}}\Big) \in \Big[\frac{\alpha}{\beta}N, \frac{\beta}{\alpha}N\Big]$$

for any $x \in \mathbb{R}^N$. After some elementary calculations we obtain that there exist some positive constant c such that

$$|||D^2\varphi(x)||| + ||\nabla\varphi(x)|| \leqslant \frac{c}{1+||x||^{2m-1}} \quad \forall\, x \in \mathbb{R}^N$$

and thus

$$\begin{aligned}\Big|c_1(x)TrA(x)D^2\varphi(x) + c_2(x)a\big(\nabla\varphi(x), \nabla\varphi(x)\big) + c_3(x)a\big(\nabla\varphi(x), x\big)\Big|\\ \leqslant \frac{c}{1+||x||^{2m-1}}\end{aligned}$$

for any c_1, c_2, c_3 bounded functions in $\mathbb{R}^N$, and any matrix $A(\bullet)$ belonging to $Q(\alpha,\beta)$. The positive real number m is arbitrary.

B1 - Construction of a super-solution of (12) or (18)

Thanks to the proposition 16 we can suppose that the function $f(\bullet)$ has its support in the open ball $\mathrm{B}(O,\rho_1)$. To choose a suitable function $r \longrightarrow g(r)$ for which the function $H(\bullet) = H_g(\bullet)$ is a super-solution of (12), we introduce the following radial function

$$\overline{f}(r) = \sup\big\{f(x)\ /\ x \in \mathbb{R}^N\,, \quad ||x||^2 = r\big\}\ .$$

Regularizing $\overline{f}(\bullet)$ if necessary, thanks to proposition 16, we can suppose that $\overline{f}(\bullet)$ is continuous and satisfies $\overline{f}(||x||^2) \geqslant f(x)$ for any x. And we can suppose $\sup\overline{f} \subset [0,\rho_1^2[$. Let k be a positive real number such that

$$(37) \qquad k.\alpha.||x||^2\chi_K\big(||x||^2\big) > \overline{f}\big(||x||^2\big)$$

where $K = \big\{x\ /\ ||x||^2 < \rho_1^2\big\} = \mathrm{B}(O,\rho_1)$, $\chi_K\big(||x||^2\big) = 1$ if $x \in K$, $\chi_K\big(||x||^2\big) = 0$ if $x \notin K$. Let us set

$$(38) \qquad G(x) := g\big(||x||^2\big) := k\chi_{K_r}\big(||x||^2\big)$$

where $\chi_{K_r}\big(||x||^2\big) := \chi_{[0,\rho_1^2]}\big(||x||^2\big)$. We need to introduce the following function $\sigma(\bullet)$ defined by :

$$(38.1) \quad \begin{cases} \sigma'(r)/\sigma(r) = \frac{\tilde{\overline{\wedge}}(r)}{r}\ , \\ r \geqslant 0\ ,\ \sigma(1) = 1 \quad \text{with} \quad \tilde{\overline{\wedge}}(r) = \overline{\wedge} + \eta \quad \text{for any} \quad r \geqslant R_0 \quad (\mathcal{H}) = R_0 \\ \text{and such that} \quad \tilde{\overline{\wedge}}(||x||^2) \geqslant \wedge(x) \quad \forall\, x \in \mathbb{R}^N. \end{cases}$$

We have :

$$(38.2) \quad \sigma(r) = \exp\Big(\int_1^r \frac{\tilde{\overline{\wedge}}(\theta)}{\theta}\,\mathrm{d}\theta\Big)\ .$$

Let ε be a positive and arbitrary small real number. After increasing $R_0 = R_0(\mathcal{H})$, if necessary, -(and in this case $R_0 = R_0(\varepsilon)$)- we can assume that we have the following :

1)

$$(39)\quad \begin{cases} R_0 > \rho_1 \\ \dfrac{3\alpha}{2}.\displaystyle\int_0^{\rho_1^2} \sigma(\theta)\,\mathrm{d}\theta > c_0(\rho_1).\int_{R_0^2}^{+\infty} \frac{\sigma(\theta)}{(1+\theta^{\triangle(\eta)+\delta_0})\theta}\mathrm{d}\theta \end{cases}$$

since we have $\underline{\triangle}(\eta) - \overline{\wedge}(\eta) + \delta_0 = \underline{\triangle} - \overline{\wedge} + \delta_0 - 2\eta$ and $\sigma(\theta) \sim c.\theta^{\overline{\wedge}(\eta)}$ for θ large enough.

2) there exists $R_1 = R_1(\varepsilon)$ such that

$$(40)\quad \begin{cases} \rho_1 < R_1 < R_0 \\ \dfrac{3\alpha}{2}.\displaystyle\int_0^{\rho_1^2} \sigma^{\overline{\wedge}+1}\,\mathrm{d}\theta > 4\int_{R_1^2}^{R_0^2} \frac{c_1(\rho_1)\sigma^{\triangle(\eta)}}{(1+\theta^{(\triangle-1+\varepsilon)})\theta}\mathrm{d}\theta \end{cases}$$

with R_1 close enough to R_0. Thanks to the proposition 17, we can modify our matrix $A(\bullet)$ in a suitable ball. This procedure does not affect the assumption $(\mathcal{H})$ since this hypothesis is supposed true at infinity. For this, let $\theta_\varepsilon = \theta > 0$, be small enough with respect to $R_0 - R_1$ and ε. And let us consider the following matrix

$$\tilde{A}_\varepsilon(x) = \tilde{A}(x) = \begin{cases} A(x) & \forall x\,, \quad ||x|| \geqslant R_0 - \theta \\ \tau.I & \forall x\,, \quad ||x|| < R_0 - \theta \end{cases}$$

where τ is a real number such that $\alpha < \tau < \beta$, and I the identity matrix. From the approximation lemma giving the approximation of the matrix $\tilde{A}$, we obtain the regular matrix $A_1(x)$ which satisfies :

$$(38.3)\quad A_1(x) = \begin{cases} A(x) & \forall x\,, \quad ||x|| \geqslant R_0 \\ \tau.I & \forall x\,, \quad ||x|| \leqslant R_1 \end{cases}$$

The matrix $A_1(\bullet)$ depends to ε, R_0, R_1, η. From the approximation lemma the matrix $A_1(\bullet)$ belongs to $Q(\alpha,\beta)$. This is now the regularized matrix $A_1(\bullet)$ that we use in the equation (18), (or equivalently in (12)), to construct a super-solution. Hence the equation (18) becomes

$$\begin{cases} -Tr\big(A_1(x)D^2u_1\big) = f \quad \text{in} \quad \mathbb{R}^N\,, \\ u_1(x) > 0\,, \quad \lim_{||x||\to+\infty} u_1(x) = 0\,. \end{cases}$$

The spectral dispersion of $A_1(\bullet)$ is $\wedge(\bullet)$ if $||x|| \geqslant R_0$ and $\dfrac{N}{2}$ if $||x|| \leqslant R_1$. Thus $A_1(\bullet)$ satisfies $(\mathcal{H})$ for x belonging to $\{x \,/\, ||x|| \geqslant R_0\} \cup \{x \,/\, ||x|| \leqslant R_1\}$. To simplify the notation we denote again $\wedge(\bullet)$ the spectral dispersion of $A_1(\bullet)$.

Theorem 20. Let $g(\bullet)$ be a radical function satisfying (37) and (38). For any x let us consider the function $r \longrightarrow h(x,r) = h_g(x,r)$, solution of (19) associated to the matrix $A_1(\bullet)$. Then the function

$$x \longrightarrow H_g(x) := H_1(x) = h_g(x, ||x||^2) = h(x, ||x||^2)\,,$$

is a super-solution of (18).

Remark 5. In the sequel we use the notation $H_1(\bullet)$ instead of $H_g(\bullet)$ because the operator $-Tr(A_1D^2\bullet)$ is associated to the matrix $A_1(\bullet)$.

***Proof* of theorem 20.** From $(\mathcal{H})$ and the definition of $H_1(\bullet)$ given in proposition 18, and using the proposition 19, we obtain :

$$(41)\quad \begin{cases} -Tr(A_1 D^2 H_1) \geqslant & -\dfrac{kc_0(\rho_1).\chi_{[y\ /\ ||y||\geqslant R_0]}(x)}{1+||x||^{2(\wedge(x)+\delta(x))}} \\ & -\dfrac{c_1(\rho_1).k.\chi_{[R_1<||y||<R_0]}(x)}{1+||x||^{2(\wedge(x)-1-\varepsilon)}} \\ & +4\big(A_1(x).x,x\big).G(x) := p(x) \quad \text{in} \quad \mathbb{R}^N \end{cases}$$

with $\lim_{||x||\to+\infty} H_1(x)=0$ and where $c_1(\rho_1)=c_\varepsilon.[\rho_1^{\overline{\wedge}+1}]$. Let us consider the solution $\overline{\omega}$ of the following :

$$(42)\quad \begin{cases} -Tr(A_1 D^2\overline{\omega}) = \overline{f}(||x||^2) \quad \text{in} \quad \mathbb{R}^N, \\ \overline{\omega}(x)\geqslant 0, \quad \lim_{||x||\to+\infty}\overline{\omega}(x)=0\ . \end{cases}$$

Writing (41) - (42) we obtain :

$$(43)\quad \begin{cases} -Tr\big(A_1D^2(H_1-\overline{\omega})\big) \geqslant p(x)-\overline{f}(||x||^2) := \varphi(x) \\ \lim_{||x||\to+\infty}(H_1-\overline{\omega})(x)=0. \end{cases}$$

In the sequel we propose to estimate from below $H_1-\overline{\omega}$ by a suitable radial function $\omega_{\text{rad}}(\bullet)$.

First step : introduction of $\omega_{\text{rad}}(\bullet)$.

This function will be a solution of an ordinary differential equation. For this let us estimate from below the function $\varphi(\bullet)$ in (43) by a radial one. By (37) and (38) we have :

$$\big(A_1(x).x,x\big).G(x) = k\chi_{K_r}\big(||x||^2\big).\big(A_1(x).x,x\big) \geqslant k\alpha||x||^2\chi_{K_r}\big(||x||^2\big) \geqslant \overline{f}\big(||x||^2\big)$$

since $A_1(\bullet)\in Q(\alpha,\beta)$. From the definitions (18.4) and (18.5) we have $||x||^{2\wedge(x)} \geqslant ||x||^{2\triangle(\eta)}$ for any x, $||x||\geqslant R_1$ and it follows that the function $\varphi(\bullet)$ is estimated from below by the following radial function

$$r \longrightarrow \Psi(r) = \begin{array}{l} -\dfrac{kc_0(\rho_1).\chi_{[\rho\ /\ \rho>R_0^2]}(r)}{1+r^{\triangle(\eta)+\delta_0}} \\ -\dfrac{c_1(\rho_1).k.\chi_{[\rho\ /\ R_1^2<\rho<R_0^2]}(r)}{1+r^{(\triangle(\eta)-1-\varepsilon)}} \\ +3k\alpha.r.\chi_{K_r}(r) \end{array}$$

where $r=||x||^2$ that is to say

$$(44)\quad \Psi\big(||x||^2\big) \leqslant \varphi(x) \quad \forall\, x\ .$$

Then we define $\omega_{\text{rad}}(\bullet)$ as the solution of the following ordinary differential equation :

$$(45)\quad \begin{cases} -\omega''_{\text{rad}}(r) - \dfrac{\tilde{\wedge}(r)}{r}\omega'_{\text{rad}}(r) = \dfrac{\Psi(r)}{4\beta.r} := \tilde{\Psi}(r)\,, \quad r>0 \\ \omega'_{\text{rad}}(0)=0\,, \quad \lim_{r\to+\infty}\omega_{\text{rad}}(r)=0\,, \quad \omega_{\text{rad}}(r)\geqslant 0\,, \quad \omega'_{\text{rad}}(r)\leqslant 0\ . \end{cases}$$

Let us set

$$\omega_{\text{rad}}(r) := \int_r^{+\infty} \frac{1}{\sigma(\theta)}\Big(\int_0^s \sigma(\theta)\tilde{\Psi}(\theta)\,\mathrm{d}\theta\Big)\,\mathrm{d}s\ ,$$

where $\sigma(\bullet)$ and $\tilde{\wedge}(\bullet)$ are given by (38.1). It is easy to see that this function satisfies the condition $\omega'_{\text{rad}}(0) = 0$ and the differential equation in (45). It remains to prove the others conditions. For this it is sufficient to verify that we have :

$$(46) \quad \forall s \geqslant 0\,, \quad 0 \leqslant \int_0^s \sigma(\theta)\tilde{\Psi}(\theta)\,\mathrm{d}\theta \leqslant \int_0^{+\infty} \sigma(\theta)\tilde{\Psi}(\theta)\,\mathrm{d}\theta = \Gamma_\infty < +\infty\ .$$

We have four cases to distinguish.

First case : for $0 \leqslant s \leqslant \rho_1^2$ we have

$$\int_0^s \sigma(\theta)\tilde{\Psi}(\theta)\,\mathrm{d}\theta = \frac{3k\alpha}{4\beta}.\int_0^s \sigma(\theta)\,\mathrm{d}\theta \geqslant 0\ .$$

Second case : for $\rho_1^2 \leqslant s \leqslant R_1^2$ we have

$$\int_0^s \sigma(\theta)\tilde{\Psi}(\theta)\,\mathrm{d}\theta = \frac{3k\alpha}{4\beta}.\int_0^{\rho_1^2} \sigma(\theta)\,\mathrm{d}\theta > 0\ .$$

Third case : for $R_1^2 \leqslant s \leqslant R_0^2$ we have

$$\int_0^s \sigma(\theta)\tilde{\Psi}(\theta)\,\mathrm{d}\theta \geqslant \frac{3k\alpha}{4\beta}.\int_0^{\rho_1^2} \sigma(\theta)\,\mathrm{d}\theta - \frac{k}{4\beta}\int_{R_1^2}^{R_0^2} \frac{c_1(\rho_1)\sigma(\theta)}{(1+\theta^{(\triangle(\eta)-1-\varepsilon)})\theta}\,\mathrm{d}\theta > 0\ .$$

by the choice of R_1 and R_0 given by (40).

Fourth case : for $s \geqslant R_0^2$, we have

$$\begin{aligned}\int_0^s \sigma(\theta)\tilde{\Psi}(\theta)\,\mathrm{d}\theta \geqslant\ & \frac{3k\alpha}{4\beta}.\int_0^{\rho_1^2} \sigma(\theta)\,\mathrm{d}\theta \\ & -\frac{k}{4\beta}\int_{R_1^2}^{R_0^2} \frac{c_1(\rho_1)\sigma(\theta)}{(1+\theta^{(\triangle(\eta)-1-\varepsilon)})\theta}\mathrm{d}\theta \\ & -k\int_{R_0^2}^{+\infty} \frac{c_0(\rho_1)\sigma(\theta)}{(1+\theta^{(\triangle(\eta)+\delta)})\theta}\mathrm{d}\theta > 0\end{aligned}$$

from (39) and (40). Thus we conclude that for any $s \geqslant 0$ we have :

$$0 \leqslant \int_0^s \sigma(\theta)\tilde{\Psi}(\theta)\,\mathrm{d}\theta \leqslant \frac{3k\alpha}{4\beta}.\int_0^{\rho_1^2} \sigma(\theta)\,\mathrm{d}\theta = \Gamma_\infty < +\infty\ .$$

And thus $\omega_{\text{rad}}(\bullet)$ is the solution of (45) and behaves like $\dfrac{1}{r^{\overline{\wedge}(\eta)-1}}$ as r goes to infinity. From (46) and since $\wedge(x) \leqslant \tilde{\wedge}(||x||^2)$ for any x, we have by (45) :

$$-\omega''_{\text{rad}}(r) - \omega'_{\text{rad}}(r)\frac{\wedge(x)}{r} \leqslant -\omega''_{\text{rad}}(r) - \omega'_{\text{rad}}(r)\frac{\tilde{\wedge}(r)}{r} = \tilde{\Psi}(r)\ ,$$

for any $x \in \mathbb{R}^N$ and $r = ||x||^2$. We obtain :

$$-4\big(A_1(x).x,x\big)\omega''_{\text{rad}}(||x||^2) - 4\omega'_{\text{rad}}(||x||^2) \wedge (x).\frac{(A_1(x).x,x)}{||x||^2} \leqslant 4\big(A_1(x).x,x\big)\tilde{\Psi}(||x||^2)$$

that is to say, after some computations and using (44,) the function $\omega(x) = \omega_{\rm rad}(||x||^2)$ satisfies

$$(47) \quad \begin{cases} -Tr(A_1 D^2\omega) \leqslant \varphi(x) \quad \text{in} \quad \mathbb{R}^N \ , \\ \omega(x) > 0\,, \quad \lim_{||x||\to+\infty} \omega(x) = 0\ . \end{cases}$$

□

Second step : Let us prove that we have :

$$H_1(x) - \overline{\omega}(x) \geqslant \omega_{\rm rad}(||x||^2) > 0\,, \quad \forall\, x \in \mathbb{R}^N\ .$$

It follows from (47) that we have

$$(48) \quad \begin{cases} -Tr\big(A_1 D^2(H_1 - \overline{\omega})\big) + Tr(A_1 D^2\omega) \geqslant \varphi(x) - \Psi(||x||^2) \geqslant 0 \quad \text{in} \quad \mathbb{R}^N \\ \lim_{||x||\to+\infty} \omega(x) = \lim_{||x||\to+\infty} (H_1 - \overline{\omega})(x) = 0\ . \end{cases}$$

Since $x \longrightarrow \varphi(x) - \Psi(||x||^2)$ is non negative, with compact support we proceed as in [4] to prove that $H_1(x) - \overline{\omega}(x) \geqslant \omega(x) = \omega_{\rm rad}(||x||^2) > 0$ for any $x \in \mathbb{R}^N$: we consider (48) in a ball $\mathrm{B}(O, n)$ with the Dirichlet boundary condition. Let us denote by $H_{1,n}$, $\overline{\omega}_n$, $\omega_{{\rm rad},n}$ the corresponding approximations of H_1, $\overline{\omega}$ and $\omega_{\rm rad}$ respectively. The maximum principle, [1] - [3], gives that we have

$$(H_{1,n} - \overline{\omega}_n)(x) - \omega_{{\rm rad},n}(||x||^2) \geqslant 0\ , \quad \forall\, x \in \mathrm{B}(O, n)\ .$$

After establishing some suitable estimates, we pass to the limit as n goes to infinity and we conclude that we have :

$$H_1(x) - \overline{\omega}(x) \geqslant \omega_{\rm rad}(||x||^2) > 0\ , \quad \forall\, x \in \mathbb{R}^N\ .$$

□

Third step : Since by construction $\overline{f}(||x||^2) \geqslant f(x)\ \forall\, x \in \mathbb{R}^N$, we prove as previously [4] that we have $\overline{\omega}(x) \geqslant u_1(x)\ \forall\, x \in \mathbb{R}^N$. And the final result follows :

$$H_1(x) \geqslant u_1(x) \quad \forall\, x \in \mathbb{R}^N$$

that is to say $H_1(\bullet)$ is a super-solution of (18) and recalling that $u_1(\bullet)$ is the solution of (18) associated to the operator $-Tr(A_1 D^2 \bullet)$. □

B2 - Construction of a sub-solution of (18) or (12).

We proceed exactly as before. To avoid repetition we only indicate the key points that differ from the section B1. Before to choose the suitable function g wich allows us to construct a sub-solution $H_g(\bullet)$ of (18), let us introduce, as previously, the following function

$$\underline{f}(r) = \inf\big\{f(x)\ /\ x, \quad ||x||^2 = r\big\}\ .$$

Thanks to the proposition 16 we can assume that the non negative and regular function $f(\bullet)$, with compact support, is such that :

$$(49) \quad \rho \longrightarrow \underline{f}(\rho) \not\equiv 0\,, \quad \text{support}\underline{f} = \{\rho\ /\ 0 \leqslant \rho < \rho_1^2\}\ .$$

Let us introduce the function

$$(50) \qquad G(x) := g(||x||^2) = k.\chi_{K_r}\big(||x||^2\big)$$

where $\chi_{K_r}(\bullet) = \chi_{[0,\rho_2^2]}(\bullet)$ with $\rho_2 = \dfrac{\rho_1}{2}$. The real number $k > 0$ is choosen such that

$$(51)\quad \begin{cases} \dfrac{1}{2}\underline{f}(||x||^2) - 4k\beta.\chi_{K_r}(||x||^2).||x||^2 \geqslant 0\,, & \forall\, x \in \mathbb{R}^n \\ \dfrac{1}{2}\underline{f}(||x||^2) - 4k\beta.\chi_{K_r}(||x||^2).||x||^2 > 0\,, & \forall\, x \in \mathrm{B}(0,\rho_2)\ . \end{cases}$$

There exist two positive real numbers R_0, R_1, $0 < R_1 < R_0$, large enough and satisfying

$$(52)\quad \begin{cases} \dfrac{\beta}{2}\displaystyle\int_0^{\rho_2^2} \frac{\sigma(\theta)}{\theta}\mathrm{d}\theta - 4c_1(\rho_2)\int_{R_1^2}^{R_0^2} \frac{\sigma(\theta)}{(1+\theta^{\triangle(\eta)-1-\varepsilon}).\theta}\mathrm{d}\theta > 0 \\ \dfrac{\beta}{2}\displaystyle\int_0^{\rho_2^2} \frac{\sigma(\theta)}{\theta}\mathrm{d}\theta - 4c_0(\rho_2)\int_{R_0^2}^{+\infty} \frac{\sigma(\theta)}{(1+\theta^{\triangle(\eta)+\underline{\delta_0}(\theta)}).\theta}\mathrm{d}\theta > 0\ . \end{cases}$$

As previously in (38.3), from (52) we construct a regular matrix $A_2(\bullet)$ such that $A_2(x) = A(x)$ $\forall\, x$, $||x|| \geqslant R_0$ and $A_2(x) = \text{constant}\ \forall\, x$, $||x|| \leqslant R_1$. We have the following result :

Theorem 21. Let $g(\bullet)$ be a function satisfying (49) to (51). For any $x \in \mathbb{R}^N$ we denote by $r \longrightarrow h(r,x) = h_g(r,x)$ the solution of (18) associated to $A_2(\bullet)$. Then the function $x \longrightarrow H_g(x) = H_2(x) = h_g(x,||x||^2) = h(x,||x||^2)$ is a sub-solution of (18) associated to $-Tr(A_2D^2\bullet)$.

Proof. We need several steps. From $(\mathcal{H})$, propositions 18 and 19 we obtain :

$$(53)\quad \begin{cases} Tr(A_2D^2H_2)(x) \geqslant \Omega(x) \quad \text{in} \quad \mathbb{R}^N \\ \lim_{||x||\to+\infty} H_2(x) = 0 \end{cases}$$

where

$$\begin{aligned}\Omega(x) = \quad & -\frac{k\chi_{[y\ /\ ||y||\geqslant R_0]}(x).c_0(\rho_2)}{1+||x||^{2(\wedge(x)+\delta)}} \\ & -c_1(\rho_2).\frac{k\chi_{[y\ /\ R_1<||y||<R_0]}(x)}{1+||x||^{2(\wedge(x)-1-\varepsilon)}} \\ & -4\big(A_2(x).x,x\big).G(x)\ .\end{aligned}$$

Let $\underline{\omega}(\bullet)$ be the solution of

$$(54)\quad \begin{cases} -Tr(A_2D^2\underline{\omega}) = \underline{f}(||x||^2) \quad \text{in} \quad \mathbb{R}^N \\ \underline{\omega}(x) \geqslant 0\,, \quad \lim_{||x||\to+\infty} \underline{\omega}(x) = 0\ . \end{cases}$$

Writing (53) + (54), it follows :

$$(55)\quad -TrA_2D^2(\underline{\omega} - H_2) \geqslant \Omega(x) + \underline{f}(||x||^2) := \varphi(x)\ .$$

Now, our goal is to prove that we have $\underline{\omega}(x) - H_2(x) \geqslant 0$ for any $x \in \mathbb{R}^N$. To do this we use the same way like the section B1. In a first step we construct a positive radial function $\omega_{\text{rad}}(\bullet)$ which estimates $\underline{\omega} - H_2$ from below.

First step : we estimate $\varphi(\bullet)$ from below by a radial function $\Psi(\bullet)$. Using the same idea as in section B1, we obtain

$$\varphi(x) \geqslant \Psi(||x||^2) \quad \forall\, x \in \mathbb{R}^N\ ,$$

where

$$\begin{aligned}\Psi(r) := \quad & -\frac{k\chi_{[\rho\ /\ \rho>R_0^2]}(r).c_0(\rho_2)}{1+r^{(\triangle(\eta)-1-\varepsilon)}} \\ & -\frac{k\chi_{[\rho\ /\ R_1^2<\rho<R_0^2]}(r).c_1(\rho_2)}{1+r^{(\triangle(\eta)-1-\varepsilon)}} \\ & +4k\beta\chi_{K_r}(r).r\ .\end{aligned}$$

As previously we remark that the radial function

$$\omega_{\mathrm{rad}}(r) = \int_r^{+\infty} \frac{1}{\sigma(s)}\Big(\int_0^s \sigma(\theta)\tilde{\Psi}(\theta)\,\mathrm{d}\theta\Big)\,\mathrm{d}s$$

is the solution of the following ordinary differential equation

$$(56)\qquad \begin{cases} -\omega''(r) - \frac{\tilde{\wedge}(r)}{r}\omega'(r) = \dfrac{\Psi(r)}{4\beta r} := \tilde{\Psi}(r) \\ \omega'(0) = 0\,, \quad \lim_{r\to+\infty}\omega(r) = 0\,, \quad \omega'(r) \leqslant 0\ , \end{cases}$$

with

$$\sigma(r) = \exp\Big(\int_1^r \frac{\tilde{\wedge}(s)}{s}\,\mathrm{d}s\Big)\ .$$

Using (52) and the definition of $\Psi(\bullet)$ we can prove that

$$(57)\quad \forall\, s \geqslant 0\,, \quad 0 \leqslant \int_0^s \sigma(\theta)\tilde{\Psi}(\theta)\,\mathrm{d}\theta \leqslant \Gamma_\infty < +\infty$$

and thus $\omega'_{\mathrm{rad}}(r) \leqslant 0$ for any $r > 0$. This entails that $\omega_{\mathrm{rad}}(\bullet)$ behaves like the function $r \longrightarrow \int_r^{+\infty} \dfrac{\mathrm{d}\theta}{\sigma(\theta)}$ as r goes to infinity. Now in a second step we acheave the proof of our result.

Second step : Let us set $\omega(x) := \omega_{\mathrm{rad}}(||x||^2)$. We can prove that $\omega(\bullet)$ satisfies

$$(58)\quad -Tr(A_2D^2\omega) \leqslant \Psi(||x||^2) \leqslant \varphi(x) \leqslant -Tr\big(A_2D^2(\underline{\omega} - H_2)\big)\ .$$

Since $\omega_{\mathrm{rad}}(\bullet)$ is non negative, from the Maximum Principle we obtain, using (58)

$$0 \leqslant \omega_{\mathrm{rad}}(||x||^2) \leqslant \underline{\omega}(x) - H_2(x) \quad \forall\, x \in \mathbb{R}^N\ ,$$

that is to say $H_2(x) \leqslant \underline{\omega}(x)\ \forall\, x \in \mathbb{R}^N$. Denoting by $u_2(\bullet)$ the solution of (18) associated to $A_2(\bullet)$, it follows $\underline{\omega}(x) \leqslant u_2(x)$ for any x, since $f(x) \geqslant \underline{f}(x)\ \forall\, x$. And thus $H_2(x) \leqslant u_2(x)$ $\forall\, x \in \mathbb{R}^N$ □

C - Asymptotic behavior of $u(\bullet)$.

Corollary 22 : asymptotic behavior of u. For any non negative bounded function $f(\bullet)$ with compact support, and for any matrix $A(\bullet)$ belonging to $Q(\alpha,\beta)$ such that $(\mathcal{H})$ is satisfied, the solution $u(\bullet)$ of the equation

$$\begin{cases} -Tr(AD^2u) = f \quad \text{in} \quad \mathbb{R}^N \\ u(x) > 0\,, \quad \lim_{||x||\to+\infty} u(x) = 0 \end{cases}$$

behaves like the function $x \longrightarrow \dfrac{1}{||x||^{2(\wedge(x)-1)}}$, when $||x||$ goes to infinity, where

$\wedge(x) = \dfrac{TrA(x)}{\frac{2}{||x||^2}.(A(x).x,x)}$.

Proof. Let us consider the previous functions $u_1(\bullet)$ and $u_2(\bullet)$ associated, respectively, to the matrices $A_1(\bullet)$ and $A_2(\bullet)$. By the propositions 16 and 17 the three functions $u_1(\bullet)$, $u_2(\bullet)$ and $u(\bullet)$ has the same asymptotic behavior since $A_1(x) = A_2(x) = A(x)$ for any x, $||x|| \geqslant R_0$. Applying the proposition 18 we can claim that $H_1(\bullet)$ and $H_2(\bullet)$ tend to zero like $x \longrightarrow \frac{1}{||x||^{2(\wedge(x)-1)}}$ as $||x||$ goes to infinity, since $\wedge_1(x) = \wedge_2(x) = \wedge(x)$ for any x, $||x|| \geqslant R_0$. From the Maximum Principle we have $u_i(x) > 0\,,\ \forall\, x \in \mathrm{R}^N\,,\ i = 1, 2$. Then we can introduce the following functions

$r \longrightarrow q_1(r) = \inf \left[\dfrac{u_1(y)}{u_2(y)}\ /\ y, ||y|| \geqslant r \right]$ which is non decreasing,

$r \longrightarrow q_2(r) = \sup \left[\dfrac{u_1(y)}{u_2(y)}\ /\ y, ||y|| \geqslant r \right]$ which is non increasing.

Consequently there exists $r_0 > 0$ large enough such that

$$(59) \quad 0 < q_1(r_0) \leqslant \frac{u_1(x)}{u_2(x)} \leqslant q_2(r_0) \quad \forall\, x\,, ||x|| \geqslant r_0\ ,$$

since $u_1(\bullet)$ and $u_2(\bullet)$ have the same asymptotic behavior. Using (59), theorems (20) and (21) and proposition 12 it follows :

$$q_1(r_0).H_2(x) \leqslant q_1(r_0).u_2(x) \leqslant u_1(x) \leqslant H_1(x)\,, \quad \forall\, ||x|| \geqslant r_0$$

that it to say $u_1(\bullet)$ goes to zero as $x \longrightarrow \dfrac{1}{||x||^{2(\wedge(x)-1)}}$, thanks the proposition 18. And consequently the final result follows for $u_2(\bullet)$ and $u(\bullet)$. □

D - Application. We can apply the previous corollary to find the asymptotic behavior in a nonlinear equation :

$$\begin{cases} -Tr\Big(A\big(x,u(x)\big)D^2u\Big) = f(x)\,, & x \in \mathbb{R}^N\ , \\ u(x) > 0\,, \quad \lim_{||x||\to+\infty} u(x) = 0\ . \end{cases}$$

where $(x,\eta) \to A(x,\eta) \in Q(\alpha,\beta)$. This problem is in the form of (12) with $B(x) = A\big(x,u(x)\big) \in Q(\alpha,\beta)$. We assume that there exist two positive real constants R_0 and η_0 such that for any $x \in \mathbb{R}^N$, $||x|| \geqslant R_0$, any $\eta \in \mathbb{R}$, $|\eta| \leqslant \eta_0$ we have :

$$(60) \quad A(x,\eta) = A(x,0)\ .$$

Let us set $A_0(x) = A(x,0)$. From [4] there exist $c > 0$ and $R \geqslant R_0$ such that

$$|u(x)| \leqslant \frac{c}{||x||^{\underline{\wedge}}} \quad \forall\, x\,, \quad ||x|| \geqslant R\ ,$$

where $\underline{\wedge} = \lim_{r\to+\infty} \inf_{||x||\geqslant r} \wedge_B(x)$. This gives

$$(61) \quad B(x) = A\big(x,u(x)\big) = A(x,0) = A_0(x) \quad \forall x\,, \quad ||x|| \geqslant R\ .$$

From (61) and proposition 17 $u(\bullet)$ behaves like the solution $u_0(\bullet)$ of

$$\begin{cases} -Tr\big(A_0(x)D^2u_0(x)\big) = f(x)\,, \quad x \in \mathbb{R}^N \ , \\ u_0(x) > 0\,, \quad \lim_{||x||\to+\infty} u_0(x) = 0 \ . \end{cases}$$

In addition if

$$\wedge_0(x) = \frac{Tr\big(A_0(x)\big)}{\frac{2}{||x||^2}\big(A_0(x).x, x\big)}$$

satisfies hypothetis $(\mathcal{H})$, then $u(\bullet)$ behaves like $x \longrightarrow \frac{1}{||x||^{2(\wedge_0(x)-1)}}$ $\square$

References

[1] M.G. Crandall, H. Ishii and P.L. Lions
User's guide to viscosity solutions of second order partial differential equation
Bull. Amer. Math. Soc. (N.S.) 27(1), 1992, p 1-67.

[2] N. Meyyers and J. Serrin
The exterior Dirichlet problem for second order elliptic partial differential equations
J. Math. Mech. 9, 1960, p 513-538.

[3] M. Protter and H. Weinberger
Principles in Differential Equations
Prentice-Hall 1967.

[4] R. Tahraoui
Comparison principle for second order elliptic operators and applications.
Ann. Inst. H. Poincaré Anal. non Linéaire. , AN23, 2006, 159-183.