EXISTENCE OF COMPACT SUPPORT SOLUTIONS FOR A QUASILINEAR AND SINGULAR PROBLEM

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Abstract. Let Ω be a $\mathcal{C}^2$ bounded domain of $\mathbb{R}^N$, $N \geq 2$. We consider the following quasilinear elliptic problem:

$$(\mathcal{P}_\lambda) \left\{ \begin{array}{l} -\Delta_p u = K(x)(\lambda u^q - u^r), \quad \text{in } \Omega, \\ \qquad u = 0 \quad \text{on } \partial\Omega, \quad u \geq 0 \quad \text{in } \Omega, \end{array} \right.$$

where $p > 1$ and $\Delta_p u \overset{\text{def}}{=} \text{div}\left(|\nabla u|^{p-2}\nabla u\right)$ denotes the p-Laplacian operator. In this paper, $\lambda > 0$ is a real parameter, the exponents q and r satisfy $-1 < r < q < p-1$ and $K : \Omega \longrightarrow \mathbb{R}$ is a positive function having a singular behaviour near the boundary $\partial\Omega$. Precisely, $K(x) = \delta(x)^{-k}L(\delta(x))$ in Ω, with $0 < k < p$, L a positive perturbation function and $\delta(x)$ the distance of $x \in \Omega$ to $\partial\Omega$.

By using a sub- and super-solution technique, we discuss the existence of positive solutions or compact support solutions of $(\mathcal{P}_\lambda)$ in respect to the blow-up rate k. Precisely, we prove that if $k < 1 + r$, $(\mathcal{P}_\lambda)$ has at least one positive solution for $\lambda > 0$ large enough, whereas it has only compact support solutions if $k \geq 1 + r$.

1. Introduction

Let Ω be a $\mathcal{C}^2$ bounded domain of $\mathbb{R}^N$, $N \geq 2$. We discuss the existence of weak solutions in $\mathrm{W}_0^{1,p}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ to

$$(\mathcal{P}_\lambda) \left\{ \begin{array}{l} -\Delta_p u = K(x)(\lambda u^q - u^r) \quad \text{in } \Omega, \\ \qquad u = 0 \quad \text{on } \partial\Omega, \quad u \geq 0 \quad \text{in } \Omega. \end{array} \right.$$

$u \in \mathrm{W}_0^{1,p}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ is a weak solution to $(\mathcal{P}_\lambda)$ if for all test functions $\varphi \in \mathcal{D}(\Omega)$,

$$\int_\Omega |\nabla u|^{p-2}\nabla u.\nabla\varphi \, dx = \int_\Omega K(x)\left(\lambda u^q - u^r\right)\varphi \, dx. \tag{1.1}$$

In the equation in $(\mathcal{P}_\lambda)$, $\lambda > 0$ is a positive parameter, $-1 < r < q < p-1$ and $K \in \mathcal{C}(\Omega)$ is a positive function having a singular behaviour near the boundary $\partial\Omega$. Precisely, $K(x) = \delta(x)^{-k}L(\delta(x))$ in Ω, with $0 < k < p$ and $L \in \mathcal{C}^2((0, 2d])$ a positive function, with $d \overset{\text{def}}{=} \text{diam}(\Omega)$, defined as follows:

$$L(t) = \exp\left(\int_t^{2d} \frac{z(s)}{s} ds\right), \tag{1.2}$$

with $z \in \mathcal{C}([0,2d]) \cap \mathcal{C}^1((0,2d])$ and $z(0)=0$. Let us note that (1.2) implies that

$$\lim_{t\to 0^+} \frac{tL'(t)}{L(t)} = 0 \tag{1.3}$$

and for all $\varepsilon > 0$,

$$\lim_{t\to 0^+} t^{\varepsilon} L(t) = 0 \tag{1.4}$$

and

$$\lim_{t\to 0^+} t^{-\varepsilon} L(t) = +\infty. \tag{1.5}$$

The above asymptotics of L force

$$\forall \xi > 0\,, \quad \lim_{t\to 0^+} \frac{L(\xi t)}{L(t)} = 1.$$

Then L belongs to the Karamata class [9]. Setting $\mathcal{K}$ the class of functions satisfying (1.2), we get the following properties: if $L_1, L_2 \in \mathcal{K}$ and if $p \in \mathbb{R}$, then $L_1.L_2 \in \mathcal{K}$ and ${L_1}^p \in \mathcal{K}$.

Example 1.1. *Let $m \in \mathbb{N}^*$ and $A > 0$ large enough. Let us define*

$$L(t) = \prod_{n=1}^{m} \left(\log_n\left(\frac{A}{t}\right)\right)^{\mu_n}, \quad t \in (0,2d]$$

with $\log_n \overset{\text{def}}{=} \log\circ\cdots\circ\log$ (n times) and $\mu_n > 0$. Then $L \in \mathcal{K}$.

In the present paper, we investigate first the following issues for the problem $(\mathcal{P}_\lambda)$:
existence of non-trivial weak solutions according to $\lambda > 0$, Hölder regularity of weak solutions. Next, we study further the properties of non-trivial solutions. Since the non-linearity in the right-hand side is a singular absorption term near the boundary, a non-trivial weak solution may not be positive everywhere in Ω and compact support (non-trivial) weak solutions or compactons (solutions with zero normal derivative at the boundary) may exist for stronger singularities, that is for large $k > 0$ whereas for small $k > 0$ any non-trivial weak solution is positive. Then, the natural question is to determine the borderline condition on the parameter k, which gives the strength of the singular potential K, between existence of positive weak solutions and existence of free boundary weak solutions. The existence of compact support solutions is important in the applications, in particular in biology models (population dynamics and epidemiology models for instance) and was investigated quite intensely for nonlinear reaction diffusion equations with absorption in the last decades. In particular, concerning the case where the equation involves a quasilinear and degenerate operator, we can refer to the result in Vázquez [14] where under a suitable condition about the behaviour of the non-linearity near the origin, a strong maximum principle is proved and consequently the positivity of solutions. The given condition is sharp in the sense that for different situations where this condition is not satisfied, the existence of free boundary solutions is shown. In Alvarez-Díaz [2] (see also Díaz [4] for related results on the subject),

the authors consider a class of nonhomogeneous reaction-diffusion equations with strong absorption and study the behaviour of the solution near the free boundary. In particular, a non degeneracy property (the solution grows faster than some function of the distance to the free boundary) is obtained when the growth of the reaction term near the boundary satisfies some estimate by below. In Il'yasov-Egorov [8], the authors consider a semilinear equation with a similar (and non singular) conflicting nonlinearity as in the equation in $(\mathcal{P}_\lambda)$ and the existence of compactons is proved using the fibering method. An interesting feature of this result is that the Hopf lemma is violated for such kind of equations. In the present work, we consider the extension case where the equation involves a p-Laplace operator and a singular potential in the right-hand side and show that a more complex situation occurs in respect to the non singular case.

In the next section, we give the main results proved in this paper. These results extend a previous work due to Haitao [7] in the semilinear case $(p = 2)$ and which involves a smaller class of nonlinearities.

2. Main results

The main results of our paper concerning the problem $(\mathcal{P}_\lambda)$ are stated below:

Theorem 2.1. *When $k < 1 + r$, there exists a constant $\Lambda_1 > 0$ such that:*

1. *For $\lambda > \Lambda_1$, $(\mathcal{P}_\lambda)$ admits a positive weak solution.*
2. *Any weak solution of $(\mathcal{P}_\lambda)$ is $\mathcal{C}^{1,\beta}\left(\overline{\Omega}\right)$ for some $\beta \in (0, 1)$.*
3. *For $\lambda < \Lambda_1$, $(\mathcal{P}_\lambda)$ has no positive solution.*

Theorem 2.2. *Let $r > 0$ and one of the two following conditions be satisfied:*

$$1 + r > q \quad \text{and} \quad k \in \left[1 + r, 1 + \frac{(p-1)(r+1)}{p-q+r}\right), \tag{2.1}$$

$$1 + r \geq q \quad \text{and} \quad k \in [1 + r, 2 + r). \tag{2.2}$$

Then, there exists $\Lambda_2 > 0$ such that:

1. *For $\lambda > \Lambda_2$, $(\mathcal{P}_\lambda)$ has a compact support weak solution u_λ.*
2. *Any weak solution of $(\mathcal{P}_\lambda)$ is $\mathcal{C}^{1,\beta}\left(\overline{\Omega}\right)$ for some $\beta \in (0, 1)$.*
3. *For $\lambda < \Lambda_2$, $(\mathcal{P}_\lambda)$ has no non-trivial solution.*

The outline of the paper is as follows. Before giving the proofs of those theorems, we establish some useful preliminary results in the next section. The proof of Theorem 2.1 is given in section 4 and the proof of Theorem 2.2 is given in section 5. The technical results stated in section 3 are finally proved in appendix A and B. The related regularity results are a consequense of the general regularity results stated in appendix C.

Throughout this paper, we will use the following notations:

(1) To $p \in (1,+\infty)$ we associate $p' \overset{\text{def}}{=} \frac{p}{p-1}$.

(2) For $x \in \Omega$, $\delta(x) \overset{\text{def}}{=} \text{dist}(x,\Omega) = \inf_{y\in\Omega} d(x,y)$.

(3) $d \overset{\text{def}}{=} \text{diam}(\Omega) = \sup_{x,y\in\Omega} d(x,y)$.

(4) Let ω be a non-empty set of Ω and $f,g : \omega \longrightarrow [0,+\infty]$. We write

$$f(x) \sim g(x) \text{ in } \omega$$

if there exist two positive constants C_1 and C_2 such that

$$\forall x \in \omega, \quad C_1 f(x) \le g(x) \le C_2 f(x).$$

(5) Let $\omega \subset \mathbb{R}^N$, $\mathcal{L}^N(\omega)$ denotes the $N-$dimentionnal Lebegue's mesure of ω.

(6) Let $\varepsilon > 0$, we define $\Omega_\varepsilon \overset{\text{def}}{=} \{x \in \Omega,\ \delta(x) < \varepsilon\}$.

(7) $\nu : \partial\Omega \longrightarrow \mathbb{R}^N$ denotes the outward normal associated to Ω.

(8) For $v \in \mathrm{W}_0^{1,p}(\Omega)$, we write $\|v\| \overset{\text{def}}{=} \|\nabla v\|_{\mathrm{L}^p(\Omega)} = \left(\int_\Omega |\nabla v|^p dx\right)^{\frac{1}{p}}$.

(9) The function $\varphi_1 \in \mathrm{W}_0^{1,p}(\Omega)$ denotes the positive and renormalized (*i.e.* $\|\varphi_1\|_{\mathrm{L}^p(\Omega)} = 1$) eigenfunction corresponding to the first eigenvalue of $-\Delta_p$,

$$\lambda_1 \overset{\text{def}}{=} \inf\left\{\int_\Omega |\nabla v|^p dx,\ v \in \mathrm{W}_0^{1,p}(\Omega),\ \int_\Omega |v|^p dx = 1\right\}.$$

It is a weak solution of the following eigenvalue problem:

$$\begin{cases} -\Delta_p u = \lambda_1 u^{p-1} & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \quad u \ge 0 & \text{in } \Omega. \end{cases}$$

Using Moser iterations and the regularity result in LIEBERMAN [10], $\varphi_1 \in \mathcal{C}^{1,\alpha}\left(\overline{\Omega}\right)$ for some $\alpha \in (0,1)$. Moreover the strong maximum and boundary principles from VÁSQUEZ [14], guarantee that φ_1 satisfies those two properties:

(a) There exist two positive constants K_1 and K_2 only depending on p and Ω such that:

$$\forall x \in \Omega, \qquad K_1\delta(x) \le \varphi_1(x) \le K_2\delta(x). \tag{2.3}$$

(b) There exist $\varepsilon^* > 0$ and $\delta^* > 0$ only depending on p and Ω such that:

$$\forall x \in \Omega_{\delta^*}, \qquad |\nabla\varphi_1(x)| > \varepsilon^*. \tag{2.4}$$

3. PRELIMINARY RESULTS

3.1. A non-existence lemma.

Lemma 3.1. *When $k < 1 + r$, there exists $\lambda_* > 0$ such that $(\mathcal{P}_\lambda)$ has no non-trivial solution for $\lambda \le \lambda_*$.*

Proof: Let us define

$$\lambda_{1,K} \overset{\text{def}}{=} \inf_{\substack{v\in \mathrm{W}_0^{1,p}(\Omega)\\ v\neq 0}} \frac{\displaystyle\int_\Omega |\nabla v|^p dx}{\displaystyle\int_\Omega K(x)|v|^p dx}.$$

From the Hardy's inequality, there exists a constant $C > 0$ only depending on Ω and p such that for all $v \in \mathrm{W}_0^{1,p}(\Omega)$,

$$\int_\Omega \frac{|v|^p}{\delta(x)^p} dx \leq C \int_\Omega |\nabla v|^p dx.$$

Since $k < p$, $\lambda_{1,K} > 0$. Let $u \in \mathrm{W}_0^{1,p}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ be a non-trivial solution of $(\mathcal{P}_\lambda)$, then from (1.1) taking $u \in \mathrm{W}_0^{1,p}(\Omega)$ as a test function we get,

$$0 < \lambda_{1,K} \int_\Omega K(x) u^p dx \leq \int_\Omega |\nabla u|^p dx = \int_\Omega K(x)\left(\lambda u^{q+1} - u^{r+1}\right) dx. \tag{3.1}$$

This inequality is impossible for $\lambda \leq \lambda_* \overset{\text{def}}{=} \min\{1, \lambda_{1,K}\}$. Indeed,

- if $u(x) \leq 1$, $\lambda u^{q+1} - u^{r+1} \leq 0$ as soon as $\lambda \leq 1$,

- if $u(x) > 1$, $K(x)\left(\lambda u^{q+1} - u^{r+1}\right) < \lambda_{1,K} K(x) u^{q+1}$ as soon as $\lambda \leq \lambda_{1,K}$.

Then, either $\mathcal{L}^N\left(\{x \in \Omega,\ u(x) > 1\}\right) = 0$ and we get

$$0 < \lambda_{1,K} \int_\Omega K(x) u^p dx \leq 0,$$

or

$$\lambda_{1,K} \int_\Omega \mathbb{1}_{\{u>1\}} K(x) u^p dx \leq \lambda_{1,K} \int_\Omega \mathbb{1}_{\{u>1\}} K(x) u^{q+1} dx,$$

which contradicts $q < p - 1$. ■

3.2. Construction of a sub-solution for $(\mathcal{P}_\lambda)$.

Lemma 3.2. *When $k < 1 + r$, there exist $M > 0$, $\lambda^* > 0$ and $\tau > 1$ such that $\underline{u}_\lambda \overset{\text{def}}{=} M{\varphi_1}^\tau$ is a sub-solution of $(\mathcal{P}_\lambda)$ in Ω, provided that $\lambda \geq \lambda^*$.*

Proof: Let $M > 0$ and $\tau > 1$, then we define $\underline{u}_\lambda = M{\varphi_1}^\tau$ in Ω. A straightforward computation yields

$$-\Delta_p \underline{u}_\lambda = -(M\tau)^{p-1}\left[(p-1)(\tau-1)|\nabla\varphi_1|^p {\varphi_1}^{(\tau-1)(p-1)-1} - \lambda_1 {\varphi_1}^{\tau(p-1)}\right]$$

and

$$K(x)\left(\lambda {\underline{u}_\lambda}^q - {\underline{u}_\lambda}^r\right) = -K(x)\left(M^r {\varphi_1}^{\tau r} - \lambda M^q {\varphi_1}^{\tau q}\right).$$

By properties (2.3) and (2.4) of the function φ_1, if we let

$$\delta_0 \overset{\text{def}}{=} \min\left\{\delta^*, \frac{\varepsilon^*}{K_2}\left(\frac{(\tau-1)(p-1)}{2\lambda_1}\right)^{\frac{1}{p}}, \frac{1}{K_2}\left(\frac{1}{2\lambda M^{q-r}}\right)^{\frac{1}{\tau(q-r)}}\right\},$$

both of the above expressions are negative on Ω_{δ_0}. Moreover,

$$\Delta_p \underline{u}_\lambda(x) \sim (M\tau)^{p-1}(\tau-1)\delta(x)^{(\tau-1)(p-1)-1} \quad \text{in } \Omega_{\delta_0}$$

and

$$K(x)\left(\underline{u}_\lambda{}^r - \lambda \underline{u}_\lambda{}^q\right) \sim M^r L(\delta(x))\delta(x)^{\tau r - k} \ \text{ in } \Omega_{\delta_0}.$$

Since $k < 1 + r$, we can choose a constant $\tau > 1$ satisfying

$$(\tau - 1)(p - 1) - 1 < \tau r - k.$$

Hence, for $M > 0$ large enough we get $-\Delta_p \underline{u}_\lambda \le K(x)\left(\lambda \underline{u}_\lambda^q - \underline{u}_\lambda^r\right)$ in Ω_{δ_0}. In $\Omega \setminus \Omega_{\delta_0}$, K, φ_1 and $|\nabla \varphi_1|$ are bounded, therefore there exists $\lambda^* > 0$ such that for $\lambda \ge \lambda^*$, $-\Delta_p \underline{u}_\lambda \le K(x)\left(\lambda \underline{u}_\lambda^q - \underline{u}_\lambda^r\right)$ in $\Omega \setminus \Omega_{\delta_0}$. Thus, $\underline{u}_\lambda$ is a sub-solution of $(\mathcal{P}_\lambda)$ in Ω for M large enough and $\lambda \ge \lambda^*$. ■

3.3. **Construction of a super-solution for** $(\mathcal{P}_\lambda)$.

We consider the following problem:

$$(\mathcal{Q}) \begin{cases} -\Delta_p v = K(x) v^q & \text{in } \Omega, \\ \quad v = 0 \quad \text{on } \partial\Omega, \quad v > 0 & \text{in } \Omega, \end{cases}$$

with q, p and K satisfying the above assumptions.

Lemma 3.3.

(1) *If* $k \in (0, 1+q)$, $(\mathcal{Q})$ *has a unique solution* $v \in \mathrm{W}_0^{1,p}(\Omega) \cap \mathcal{C}\left(\overline{\Omega}\right)$ *satisfying*

$$v(x) \sim \delta(x) \ \text{ in } \Omega.$$

(2) *If* $k = 1+q$, $(\mathcal{Q})$ *has a unique solution* $v \in \mathrm{W}_0^{1,p}(\Omega) \cap \mathcal{C}\left(\overline{\Omega}\right)$ *satisfying*

$$v(x) \sim \delta(x) \left(\int_{\delta(x)}^{2d} \frac{L(t)}{t}\, dt \right)^{\frac{1}{p-k}} \ \text{ in } \Omega.$$

(3) *If* $k \in \left(1+q, 1+q+\frac{p-(1+q)}{p}\right)$, $(\mathcal{Q})$ *has a unique solution* $v \in \mathrm{W}_0^{1,p}(\Omega) \cap \mathcal{C}\left(\overline{\Omega}\right)$ *satisfying*

$$v(x) \sim \delta(x)^{\frac{p-k}{p-(1+q)}} \Big(L\left(\delta(x)\right) \Big)^{\frac{1}{p-(1+q)}} \ \text{ in } \Omega.$$

(4) *If* $k \in \left[1+q+\frac{p-(1+q)}{p}, p\right)$, $(\mathcal{Q})$ *has a unique solution* $v \in \mathrm{W}_{loc}^{1,p}(\Omega) \cap \mathcal{C}_0\left(\overline{\Omega}\right)$ *satisfying*

$$v(x) \sim \delta(x)^{\frac{p-k}{p-(1+q)}} \Big(L\left(\delta(x)\right) \Big)^{\frac{1}{p-(1+q)}} \ \text{ in } \Omega.$$

(5) *If* $k = p$ *and if* L *satisfies the following condition:*

$$\int_0^{2d} t^{-1} L(t)^{\frac{1}{p-1}}\, dt < +\infty, \tag{3.2}$$

$(\mathcal{Q})$ *has a unique solution* $v \in \mathrm{W}_{loc}^{1,p}(\Omega) \cap \mathcal{C}_0\left(\overline{\Omega}\right)$ *satisfying*

$$v(x) \sim \left(\int_0^{\delta(x)} t^{-1} L(t)^{\frac{1}{p-1}}\, dt \right)^{\frac{p-1}{p-(1+q)}} \ \text{ in } \Omega.$$

Proof: See section A in appendix. ■

From a solution of $(\mathcal{Q})$, we can easily construct a super-solution of $(\mathcal{P}_\lambda)$. Indeed, let us consider $v \in \mathrm{W}^{1,p}_{loc}(\Omega) \cap \mathcal{C}_0\left(\overline{\Omega}\right)$ the solution of $(\mathcal{Q})$ given by lemma 3.3. Then, $\overline{u}_\lambda \overset{\text{def}}{=} \overline{M}v$ is a super-solution of $(\mathcal{P}_\lambda)$ in Ω as soon as $\overline{M} \geq \lambda^{\frac{1}{p-(1+q)}}$. Particularly when $k < 1 + r$ and $\lambda \geq \lambda^*$, chosing $\overline{M}$ sufficiently large $\overline{u}_\lambda \in \mathrm{W}^{1,p}_0(\Omega) \cap \mathcal{C}\left(\overline{\Omega}\right)$ and is a super-solution of $(\mathcal{P}_\lambda)$ in Ω satisfying

$$\underline{u}_\lambda \leq \overline{u}_\lambda \quad \text{and} \quad \overline{u}_\lambda(x) \sim \delta(x) \quad \text{in } \overline{\Omega}.$$

Now let us state a non-existence result for the problem $(\mathcal{Q})$.

Proposition 3.1. *Let $\underline{v} \in \mathrm{W}^{1,p}_0(\Omega) \cap \mathcal{C}\left(\overline{\Omega}\right)$ be a positive sub-solution of $(\mathcal{Q})$ in Ω and let us suppose there exists $\varepsilon > 0$ such that*

$$\int_\Omega K(x){\varphi_1}^{p-1+\varepsilon} dx = +\infty. \tag{3.3}$$

Therefore, for any $\eta > 0$, $(\mathcal{Q})$ has no weak solution $v \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega) \cap \mathcal{C}_0\left(\overline{\Omega}\right)$ such that $v \geq \eta \underline{v}$ in Ω.

Proof: See section B in appendix. ■

Corollary 3.1. *If $k > p$, there is no non-trivial weak solution of $(\mathcal{Q})$.*

4. PROOF OF THEOREM 2.1

4.1. Existence of a $\mathcal{C}^{1,\beta}$ positive solution when $\lambda \geq \lambda^*$.

Proposition 4.1. *When $k < 1 + r$, provided $\lambda \geq \lambda^*$, $(\mathcal{P}_\lambda)$ has a weak solution $u_\lambda \in \mathrm{W}^{1,p}_0(\Omega) \cap \mathrm{L}^\infty(\Omega)$. Furthermore, there exists $\beta \in (0, 1)$ such that $u_\lambda \in \mathcal{C}^{1,\beta}\left(\overline{\Omega}\right)$.*

Proof: In the equation of $(\mathcal{P}_\lambda)$, the expression $h_\lambda(x, v) \overset{\text{def}}{=} K(x)(\lambda v^q - v^r)$ involves a singular term $K(x)$ which blows up as $\delta(x) \to 0$, so we can not directly apply the sub- and super-solution method on Ω. To overcome this difficulty, we apply a sub- and super-solution method in a sequense of subdomains of Ω. Let us introduce $(\Omega_k)_{k\in\mathbb{N}^*} \subset \Omega$ an increasing sequence of smooth subdomains of Ω such that $\Omega_k \underset{k\to\infty}{\longrightarrow} \Omega$ in the Hausdorff topology with

$$\forall k \in \mathbb{N}^*, \quad \frac{1}{k+1} < \mathrm{dist}(\partial\Omega, \partial\Omega_k) < \frac{1}{k}.$$

Then, for all $k \in \mathbb{N}^*$ we consider the following problem:

$$(\mathcal{P}_k) \begin{cases} -\Delta_p u_k = K(x)(\lambda {u_k}^q - {u_k}^r) & \text{in } \Omega_k, \\ u_k = \underline{u}_\lambda \quad \text{on } \partial\Omega_k, \quad u_k \geq 0 & \text{in } \Omega_k. \end{cases}$$

By definition of Ω_k, there exists $C_k > 0$ such that

$$\forall v \in I_k \overset{\text{def}}{=} \left[\min_{\overline{\Omega_k}} \underline{u}_\lambda \, , \, \max_{\overline{\Omega_k}} \overline{u}_\lambda\right], \quad \sup_{x\in\Omega_k} \left|\frac{\partial h_\lambda}{\partial v}(x,v)\right| \leq C_k.$$

As a consequense, there exists $\mu_k > 0$ such that for all $x \in \Omega_k$, the function $v \longmapsto h_\lambda(x,v) + \mu_k v^{p-1}$ is increasing on I_k. Therefore by sub- and super-solution method, $(\mathcal{P}_k)$ has a solution $u_k \in \mathrm{W}^{1,p}(\Omega_k)$. Indeed, we can construct the following iterative monotone scheme: for all $n \in \mathbb{N}^*$, let $u_{k,n} \in \mathrm{W}^{1,p}(\Omega_k)$ be the weak solution of

$$(\mathcal{P}_{k,n}) \left\{ \begin{array}{ll} -\Delta_p u_{k,n} + \mu_k \left(u_{k,n}\right)^{p-1} = h_\lambda(x, u_{k,n-1}) + \mu_k \left(u_{k,n-1}\right)^{p-1} & \text{in } \Omega_k, \\ u_{k,n} = \underline{u}_\lambda \quad \text{on } \partial\Omega_k, \quad u_{k,n} \geq 0 \quad \text{in } \Omega_k \end{array} \right.$$

with $u_{k,0} = \underline{u}_\lambda$ in Ω_k. By induction on $n \in \mathbb{N}$, $(\mathcal{P}_{k,n})$ has a unique solution $u_{k,n} \in \mathrm{W}^{1,p}(\Omega_k)$. By using the weak comparison principle $\left(u_{k,n}\right)_{n\in\mathbb{N}}$ satisfies

$$\underline{u}_\lambda \leq u_{k,n} \leq u_{k,n+1} \leq \overline{u}_\lambda \text{ in } \Omega_k.$$

Consequently, for all $n \in \mathbb{N}^*$,

$$\left| h_\lambda(x, u_{k,n-1}) + \mu_k \left(\left(u_{k,n-1}\right)^{p-1} - \left(u_{k,n}\right)^{p-1} \right) \right| \in \mathrm{L}^\infty(\Omega_k)$$

and since $\underline{u}_\lambda$ is smooth in Ω, we can state by a regularity result due to Lieberman [10] (see Theorem 1) that $\left(u_{k,n}\right)_{n\in\mathbb{N}} \subset \mathcal{C}^{1,\gamma}\left(\overline{\Omega_k}\right)$ for some $\gamma \in (0,1)$. Moreover there exists a constant $C > 0$ only depending on γ, Ω_k, $\|\overline{u}_\lambda\|_{\mathrm{L}^\infty(\Omega_k)}$ and $\|\underline{u}_\lambda\|_{\mathrm{L}^\infty(\Omega_k)}$ such that $\|u_{k,n}\|_{\mathcal{C}^{1,\gamma}(\overline{\Omega_k})} \leq C$. From Ascoli-Arzelà theorem, there exist $u_k \in \mathcal{C}^1(\overline{\Omega_k})$ and a subsequence $\left(u_{k,m}\right)_{m\in\mathbb{N}}$ such that $u_{k,m} \underset{m\to\infty}{\longrightarrow} u_k$ in $\mathcal{C}^1\left(\overline{\Omega_k}\right)$. Passing to the limit when $n \to +\infty$ in $(\mathcal{P}_{k,n})$, u_k is a weak solution of $(\mathcal{P}_k)$.

For all $k \in \mathbb{N}$, we define $\tilde{u}_k \overset{\text{def}}{=} \mathbb{1}_{\Omega_k}.u_k$ in order to extend u_k on Ω by zero. We prove that $\left(\tilde{u}_k\right)_{k\in\mathbb{N}}$ is an increasing sequence in Ω. Indeed, since $\Omega_k \subset \Omega_{k+1}$, if we compare u_{k+1} with every term of $\left(u_{k,n}\right)_{n\in\mathbb{N}}$ in Ω_k, using the weak comparison principle we get

$$\forall n \in \mathbb{N}, \quad u_{k,n} \leq \tilde{u}_{k+1} \text{ in } \Omega_k.$$

Passing to the limit in the above inequality, $\left(\tilde{u}_k(x)\right)_{k\in\mathbb{N}}$ is non-decreasing for all $x \in \Omega$. Therefore there exists $u_\lambda \in \mathrm{L}^\infty(\Omega)$ such that $\tilde{u}_k \underset{k\to\infty}{\longrightarrow} u_\lambda$ *a.e.* in Ω and

$$\underline{u}_\lambda \leq u_\lambda \leq \overline{u}_\lambda \text{ in } \Omega. \tag{4.1}$$

It follows that $\tilde{u}_k \underset{k\to\infty}{\longrightarrow} u_\lambda$ in $\mathcal{D}'(\Omega)$ and u_λ satisfies (1.1). Using inequality (4.1) and Hardy's inequality, $K(x)\left[\lambda(u_\lambda)^q - (u_\lambda)^r\right] \in \mathrm{W}^{-1,p'}(\Omega)$, which implies that $u_\lambda \in \mathrm{W}_0^{1,p}(\Omega)$. Finally applying proposition C.1 of the appendix, we get the $\mathcal{C}^{1,\beta}\left(\overline{\Omega}\right)$ regularity of u_λ. ∎

4.2. Existence of $\mathbf{\Lambda_1}$.

Let us define

$$\Lambda_1 \overset{\text{def}}{=} \inf\{\lambda > 0,\ (\mathcal{P}_\lambda)\ \text{ has a positive solution}\}.$$

By Lemma 3.1 and the first step of this proof, $\lambda_* \le \Lambda_1 \le \lambda^* < +\infty$. By definition of Λ_1, for any $\lambda > \Lambda_1$ there exists $\mu \in (\Lambda_1, \lambda)$ such that $(\mathcal{P}_\mu)$ has a positive solution $u_\mu \in \mathrm{W}_0^{1,p}(\Omega) \cap \mathrm{L}^\infty(\Omega)$. Moreover using proposition C.1, $u_\mu \in \mathcal{C}^{1,\beta}\left(\overline{\Omega}\right)$. Since u_μ is a sub-solution to $(\mathcal{P}_\lambda)$, we prove that $u_\mu \le \overline{u}_\lambda$ in Ω. Indeed, $K(x) > 0$ in Ω, so there exists $\delta_0 > 0$ such that

$$-\Delta_p u_\mu \le 0 \le -\Delta_p\left(C_0\varphi_1\right) \text{ in } \Omega_{\delta_0},$$

with $C_0 > 0$ large enough to satisfy

$$u_\mu \le C_0\varphi_1 \text{ on } \partial\Omega_{\delta_0}.$$

By the weak comparison principle, $u_\mu \le C_0\varphi_1$ in Ω_{δ_0}. Morever u_μ and φ_1 are bounded in $\Omega \setminus \Omega_{\delta_0}$, thus $u_\mu \le C\varphi_1$ in Ω for some $C > 0$. Therefore chosing $\overline{M}$ sufficiently large in the definition of $\overline{u}_\lambda$, we get $u_\mu \le \overline{u}_\lambda$ in Ω. Finally, applying again sub- and super-solution technique as in step 1, we get a solution $u_\lambda \in \mathcal{C}^{1,\beta}\left(\overline{\Omega}\right)$ of $(\mathcal{P}_\lambda)$. ■

Proof of Theorem 2.1: The proof follows from proposition 4.1 and subsection 4.2. ■

5. PROOF OF THEOREM 2.2

5.1. Existence of a solution under condition (2.1) or (2.2).

Proposition 5.1. *Let* $k \in \left[1 + r, 1 + q + \frac{p-(1+q)}{p}\right)$. *Then, under condition*

$$\int_\Omega K(x)\left(\overline{u}_\lambda\right)^{r+1} dx < +\infty, \tag{5.1}$$

there exists $\lambda^{**} > 0$ *such that the problem* $(\mathcal{P}_\lambda)$ *has a non-trivial weak solution* $u_\lambda \in \mathrm{W}_0^{1,p}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ *as soon as* $\lambda > \lambda^{**}$.

Remark 5.1. *Since* $\overline{u}_\lambda \in \mathrm{W}_0^{1,p}(\Omega)$, *by Hardy's inequality* $\frac{\overline{u}_\lambda}{\delta(x)} \in \mathrm{L}^p(\Omega)$. *So using Hölder's inequality, assumption* (5.1) *in Theorem 5.1 is satisfied if*

$$L(\delta(x))\delta(x)^{r+1-k} \in \mathrm{L}^{\alpha'}(\Omega),$$

where $\alpha = \frac{p}{r+1} > 1$. *And this last condition is satisfied if*

$$k < 1 + r + \frac{p-(1+r)}{p}. \tag{5.2}$$

So (5.2) *implies* (5.1); *but this condition is not sharp and can be weakened by using the precise behaviour of* u_λ *given in lemma 3.3. Indeed,*

(1) *if* $k \in [1+r, 1+q)$, $\overline{u}_\lambda(x) \sim \delta(x)$ *in* Ω. *Therefore condition* (5.1) *is satisfied if*

$$k < 2 + r. \tag{5.3}$$

(2) *if* $k = 1+q$, *condition* (5.1) *is also satisfied if* $k < 2+r$.

(3) *if* $k \in \left(1+q, 1+q+\frac{p-(1+q)}{p}\right)$, $\overline{u}_\lambda \sim \delta(x)^{\frac{p-k}{p-(1+q)}}\left(L\left(\delta(x)\right)\right)^{\frac{1}{p-(1+q)}}$ *in* Ω. *Therefore, condition* (5.1) *is satisfied if*

$$k < 1 + \frac{(p-1)(r+1)}{p-q+r}. \tag{5.4}$$

Remark that if $1+r > q$, (5.3) *is always true for* $k \in [1+r, 1+q]$ *and since*

$$1 + q < 1 + \frac{(p-1)(r+1)}{p-q+r} \Longleftrightarrow r+1 > q, \tag{5.5}$$

condition (2.1) *implies* (5.1). *Similarly if* $1+r \leq q$, *by equivalence* (5.5), (5.4) *is never satisfied for* $k \in \left(1+q, 1+q+\frac{p-(1+q)}{p}\right)$ *and condition* (2.2) *implies* (5.1). *We can easily check that both conditions* (2.1) *and* (2.2) *are weaker than* (5.2). *Moreover, let us suppose one of the following conditions be satisfied:*

$$1+r > q \quad \text{and} \quad k \in \left(1 + \frac{(p-1)(r+1)}{p-q+r}, 1+q+\frac{p-(1+q)}{p}\right),$$

$$1+r \geq q \quad \text{and} \quad k \in \left(2+r, 1+q+\frac{p-(1+q)}{p}\right).$$

Then, using lemma 3.3 again, condition (5.1) *is not satisfied, which guarantees the "sharpness" of conditions* (2.1) *and* (2.2).

In the proof of Proposition 5.1, we will need the following well known lemma.

Lemma 5.1. *Let* $x, y \in \mathbb{R}^N$ *and* $\langle \cdot\,, \cdot \rangle$ *the standard scalar product in* $\mathbb{R}^N$. *Then there exists a constant* $C_p > 0$ *such that*

$$\langle |x|^{p-2}x - |y|^{p-2}y, x-y \rangle \geq \begin{cases} C_p|x-y|^p & if \quad p \geq 2, \\ C_p\dfrac{|x-y|^2}{(|x|+|y|)^{2-p}} & if \quad 1 < p < 2. \end{cases}$$

Proof: See Lemma 4.2 in Lindqvist [11] or Lemma A.0.5 in Peral [12]. ■

Proof of proposition 5.1: Let us introduce the functionnal

$$I_\lambda(v) = \frac{1}{p}\int_\Omega |\nabla v|^p\, dx + \frac{1}{r+1}\int_\Omega K(x)|v|^{r+1}\, dx - \frac{\lambda}{q+1}\int_\Omega K(x)|v|^{q+1}\, dx,$$

with $v \in \mathrm{W}_0^{1,p}(\Omega)$. Let $\varphi_0 \neq 0 \in \mathcal{D}(\Omega)$ be a non-negative function. Therefore there exists $\lambda^{**} > 0$ such that $I_\lambda(\varphi_0) < 0$ for $\lambda > \lambda^{**}$. Let us fix a constant $M > 1$ such that $M\overline{u}_\lambda \geq \varphi_0$ in Ω and introduce the cut-off function f_λ defined in $\Omega \times \mathbb{R}$ by:

$$f_\lambda(x,v) = \begin{cases} K(x)\left[\lambda(M\overline{u}_\lambda)^q - (M\overline{u}_\lambda)^r\right] & \text{if } \ v > M\overline{u}_\lambda(x), \\ K(x)\left[\lambda|v|^q - |v|^r\right] & \text{if } \ v \in [0, M\overline{u}_\lambda(x)]\,, \\ 0 & \text{if } \ v < 0. \end{cases}$$

The function $v \longmapsto f_\lambda(x,v)$ is a Carathéodory function. For $(x,v) \in \Omega \times \mathbb{R}$, let us set $F_\lambda(x,v) = \displaystyle\int_0^v f_\lambda(x,t)\,dt$ and consider the functional E_λ defined as follows:

$$\forall v \in \mathrm{W}_0^{1,p}(\Omega), \qquad E_\lambda(v) = \frac{1}{p}\int_\Omega |\nabla v|^p\,dx - \int_\Omega F_\lambda(x, v(x))\,dx.$$

A straightforward computation gives us

$$\begin{aligned} E_\lambda(v) \quad = \quad & \frac{1}{p}\int_\Omega |\nabla v|^p\,dx - \frac{\lambda}{q+1}A(v,q) + \frac{1}{r+1}A(v,r) \\ & -\lambda B(v,q) + B(v,r) - \frac{r}{r+1}C(r) + \lambda\frac{q}{q+1}C(q), \end{aligned} \tag{5.6}$$

with

$$A(v,s) \overset{\text{def}}{=} \int_\Omega \mathbb{1}_{\{0 \leq v \leq M\overline{u}_\lambda\}} K(x)|v|^{s+1}dx, \ B(v,s) \overset{\text{def}}{=} \int_\Omega \mathbb{1}_{\{v \geq M\overline{u}_\lambda\}} K(x)\left(M\overline{u}_\lambda\right)^s v dx$$

and

$$C(s) \overset{\text{def}}{=} \int_\Omega \mathbb{1}_{\{v \geq M\overline{u}_\lambda\}} K(x)(M\overline{u}_\lambda)^{s+1}dx.$$

Let $\varepsilon > 0$ and $v \in \mathrm{W}_0^{1,p}(\Omega)$, then we split the integral $A(v,q)$ in $\Omega \setminus \Omega_\varepsilon$ and Ω_ε :

$$\begin{aligned} A(v,q) \quad & = \quad \int_{\Omega\setminus\Omega_\varepsilon} \mathbb{1}_{\{0 \leq v \leq M\overline{u}_\lambda\}} K(x)|v|^{q+1}dx + \int_{\Omega_\varepsilon} \mathbb{1}_{\{0 \leq v \leq M\overline{u}_\lambda\}} K(x)|v|^{q+1}dx \\ & \overset{\text{def}}{=} \quad A_{\Omega\setminus\Omega_\varepsilon}(v,q) + A_{\Omega_\varepsilon}(v,q). \end{aligned}$$

Since in $\Omega \setminus \Omega_\varepsilon$, K is bounded, from the embedding $\mathrm{W}_0^{1,p}(\Omega) \hookrightarrow \mathrm{L}^{q+1}(\Omega)$, there exists a constant C_1 such that

$$A_{\Omega\setminus\Omega_\varepsilon}(v,q) \leq C_1\|v\|^{q+1}. \tag{5.7}$$

In Ω_ε, by Hölder's inequality we have,

$$A_{\Omega_\varepsilon}(v,q) \leq A_{\Omega_\varepsilon}(v,r)^\tau \left(\int_{\Omega_\varepsilon} \mathbb{1}_{\{0 \leq v \leq M\overline{u}_\lambda\}} K(x)|v|^p dx\right)^{1-\tau},$$

with $\tau = \frac{p-(1+q)}{p-(1+r)} < 1$. Using inequality (1.4) and Hardy's Inequality, we finally obtain, for ε small enough

$$\begin{aligned} A_{\Omega_\varepsilon}(v,q) \quad & \leq \quad C_2\varepsilon^{\frac{1}{2}(p-k)(1-\tau)} A_{\Omega_\varepsilon}(v,r)^\tau \left(\int_{\Omega_\varepsilon} \frac{|v|^p}{\delta(x)^p}\,dx\right)^{1-\tau} \\ & \leq \quad C_2\varepsilon^{\frac{1}{2}(p-k)(1-\tau)}\left(\tau A(v,r) + C_3(1-\tau)\|v\|^p\right). \end{aligned} \tag{5.8}$$

From the above arguments and since

$$\begin{aligned} B(v,q) &= \int_{\Omega\setminus\Omega_\varepsilon} \mathbb{1}_{\{v\geq M\overline{u}_\lambda\}} K(x)\,(M\overline{u}_\lambda)^q\, v dx + \int_{\Omega_\varepsilon} \mathbb{1}_{\{v\geq M\overline{u}_\lambda\}} K(x)\,(M\overline{u}_\lambda)^q\, v dx \\ &\overset{\text{def}}{=} B_{\Omega\setminus\Omega_\varepsilon}(v,q) + B_{\Omega_\varepsilon}(v,q), \end{aligned}$$

we also get

$$B_{\Omega\setminus\Omega_\varepsilon}(v,q) \leq C_4\|v\| \tag{5.9}$$

and

$$B_{\Omega_\varepsilon}(v,q) \leq C_5 \varepsilon^{\frac{1}{2}(p-k)(1-\tau)} \left(\tau B(v,r) + C_6(1-\tau)\|v\|^p\right). \tag{5.10}$$

Using inequalities (5.7) to (5.10),

$$\begin{aligned} E_\lambda(v) \geq\ & \frac{1}{2p}\|v\|^p - \lambda\frac{C_1}{q+1}\|v\|^{q+1} - \lambda C_4\|v\| + \frac{1}{2}B(v,r) \\ & + \frac{1}{2(r+1)}A(v,r) - \frac{r}{r+1}C(r) + \lambda\frac{q}{q+1}C(q), \end{aligned} \tag{5.11}$$

for $\varepsilon > 0$ sufficiently small. With condition (5.1), this inequality proves that E_λ is coercive and bounded from below on $\mathrm{W}_0^{1,p}(\Omega)$. So let us define

$$c_\lambda \overset{\text{def}}{=} \inf_{v\in \mathrm{W}_0^{1,p}(\Omega)} E_\lambda(v)$$

and let $(v_n)_{n\in\mathbb{N}} \subset \mathrm{W}_0^{1,p}(\Omega)$ be a minimizing sequence of E_λ, that is to say $E_\lambda(v_n) \underset{n\to+\infty}{\longrightarrow} c_\lambda$. $\Big(E_\lambda(v_n)\Big)_{n\in\mathbb{N}}$ is bounded, therefore by inequality (5.11) $(v_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{W}_0^{1,p}(\Omega)$. Thus, there exist $u_\lambda \in \mathrm{W}_0^{1,p}(\Omega)$ and a subsequence $(v_m)_{m\in\mathbb{N}}$ such that $v_m \underset{m\to+\infty}{\longrightarrow} u_\lambda$ weakly in $\mathrm{W}_0^{1,p}(\Omega)$, strongly in $\mathrm{L}^{q+1}(\Omega)$ and in $L^1(\Omega)$ and *a.e.* in Ω. Then we get

$$\|u_\lambda\| \leq \liminf_{m\to+\infty} \|v_m\|. \tag{5.12}$$

Using Fatou's Lemma and inequality (5.1), it follows that

$$\frac{1}{r}A(u_\lambda,r) + B(u_\lambda,r) \leq \liminf_{m\to+\infty}\left(\frac{1}{r}A(v_m,r) + B(v_m,r)\right) < +\infty. \tag{5.13}$$

Again from Fatou's lemma and inequalities (5.8),(5.10) and (5.12),

$$\begin{aligned} \frac{\lambda}{q+1}A_{\Omega_\varepsilon}(u_\lambda,q) + \lambda B_{\Omega_\varepsilon}(u_\lambda,q) &\leq \liminf_{m\to+\infty}\left(\frac{\lambda}{q+1}A_{\Omega_\varepsilon}(v_m,q) + \lambda B_{\Omega_\varepsilon}(v_m,q)\right) \\ &\leq C_7\varepsilon^{\frac{1}{2}(p-k)(1-\tau)}. \end{aligned} \tag{5.14}$$

Since $v_m \underset{m\to+\infty}{\longrightarrow} u_\lambda$ in $\mathrm{L}^{q+1}(\Omega)$ and in $\mathrm{L}^1(\Omega)$,

$$A_{\Omega\setminus\Omega_\varepsilon}(v_m,q) \underset{m\to+\infty}{\longrightarrow} A_{\Omega\setminus\Omega_\varepsilon}(u_\lambda,q) \quad \text{and} \quad B_{\Omega\setminus\Omega_\varepsilon}(v_m,q) \underset{m\to+\infty}{\longrightarrow} B_{\Omega\setminus\Omega_\varepsilon}(u_\lambda,q). \tag{5.15}$$

Gathering the estimates (5.12) to (5.15) and using (5.6), we obtain:

$$c_\lambda = \liminf_{m\to+\infty} E(v_m) \geq E_\lambda(u_\lambda) - C_7\varepsilon^{\frac{1}{2}(p-k)(1-\tau)} \geq c_\lambda - C_7\varepsilon^{\frac{1}{2}(p-k)(1-\tau)}.$$

Passing to the limit as $\varepsilon \to 0$, we finally get $E_\lambda(u_\lambda) = c_\lambda$. By definition of c_λ, u_λ satisfies

$$E_\lambda(u_\lambda) = \min_{v\in \mathrm{W}_0^{1,p}(\Omega)} E_\lambda(v)$$

and since E_λ is Gâteaux differentiable, u_λ satisfies the Euler-Lagrange equation associated to E_λ:

$$\forall v \in \mathrm{W}_0^{1,p}(\Omega), \qquad \int_\Omega |\nabla u_\lambda|^{p-2}\nabla u_\lambda.\nabla v\, dx = \int_\Omega f_\lambda(x,u_\lambda)v\, dx.$$

In particular, setting $v = (u_\lambda)^- \in \mathrm{W}_0^{1,p}(\Omega)$, by weak maximum principle it follows that $u_\lambda \geq 0$ *a.e.* in Ω. Moreover, since $M\overline{u}_\lambda$ is a super-solution of $(\mathcal{P}_\lambda)$, for all non-negative $v \in \mathrm{W}_0^{1,p}(\Omega)$,

$$\int_\Omega |\nabla(M\overline{u}_\lambda)|^{p-2}\nabla(M\overline{u}_\lambda).\nabla v\, dx \geq \int_\Omega K(x)\left[\lambda(M\overline{u}_\lambda)^q - (M\overline{u}_\lambda)^r\right] v\, dx.$$

Setting $v = (u_\lambda - M\overline{u}_\lambda)^+ \in \mathrm{W}_0^{1,p}(\Omega)$, we obtain

$$\begin{aligned} 0 &= \int_\Omega \Big(f_\lambda(x,u_\lambda) - K(x)\left[\lambda(M\overline{u}_\lambda)^q - (M\overline{u}_\lambda)^r\right]\Big)(u_\lambda - M\overline{u}_\lambda)^+\, dx \\ &\geq \int_\Omega \Big(|\nabla u_\lambda|^{p-2}\nabla u_\lambda - |\nabla(M\overline{u}_\lambda)|^{p-2}\nabla(M\overline{u}_\lambda)\Big).\nabla\Big((u_\lambda - M\overline{u}_\lambda)^+\Big)\, dx. \end{aligned}$$

Using lemma 5.1, $\nabla\Big((u_\lambda - M\overline{u}_\lambda)^+\Big) = 0$ *a.e.* in Ω and by Poincaré's inequality $u_\lambda \leq M\overline{u}_\lambda$ *a.e.* in Ω. Finally

$$I_\lambda(u_\lambda) = E_\lambda(u_\lambda) = \min_{v\in \mathrm{W}_0^{1,p}(\Omega)} E_\lambda(v) \leq E_\lambda(\varphi_0) = I_\lambda(\varphi_0) < 0,$$

therefore u_λ is a non-trivial weak solution of $(\mathcal{P}_\lambda)$. ■

5.2. **Compacted support of the solution.**

In this section we define

$$g_\lambda(t) \overset{\text{def}}{=} t^r - \lambda t^q, \quad t\in[0,+\infty) \quad \text{and} \quad a^* \overset{\text{def}}{=} \left(\frac{r}{\lambda q}\right)^{\frac{1}{q-r}} \tag{5.16}$$

in such a way that g_λ is positive and increasing on the interval $(0, a^*)$. Let us start by stating a result which guarantees the existence of an appropriate super-solution of $(\mathcal{P}_\lambda)$ near the boundary.

Lemma 5.2. *Let $u_\lambda \in \mathrm{W}_0^{1,p}(\Omega)\cap \mathrm{L}^\infty(\Omega)$ be a weak solution of $(\mathcal{P}_\lambda)$. Then $u_\lambda \in \mathcal{C}\left(\overline{\Omega}\right)$ and there exist $\delta_* > 0$, $M > 0$ and $\alpha \in (1, p')$ such that*

$$u_\lambda(x) \leq M\varphi_1(x)^\alpha \quad \text{in } \Omega_{\delta_*}.$$

In the proof of this lemma, we will use the following weak comparison principle:

Proposition 5.2. *Let us consider the Dirichlet problems:*

$$\begin{cases} -\Delta_p u - b(x,u) = f & \text{in } \Omega, \\ u = f' & \text{on } \partial\Omega \end{cases} \tag{5.17}$$

and

$$\begin{cases} -\Delta_p v - b(x,v) = g & \text{in } \Omega, \\ v = g' & \text{on } \partial\Omega. \end{cases} \tag{5.18}$$

Assume that $f \leq g$ *in* $\mathrm{L}^{p'}(\Omega)$, $f' \leq g'$ *in* $\mathrm{W}^{\frac{1}{p'},p}(\partial\Omega)$, $u, v \in \mathrm{W}^{1,p}(\Omega)$ *are any weak solutions of the Dirichlet problems* (5.17) *and* (5.18), *respectively and* $b(x,\cdot) : \mathbb{R} \to \mathbb{R}$ *is non-increasing for a.e.* $x \in \Omega$. *Then,* $u \leq v$ *in* Ω.

Proof: *See proposition 2.3 in* Cuesta-Takáč [3]. ■

Proof of Lemma 5.2: With the previous notations, the set $\omega^* \overset{\text{def}}{=} \{x \in \Omega, \quad u_\lambda(x) \leq a^*\}$ contains a neigbourhood of $\partial\Omega$ and there exists $\delta_0 > 0$ such that $\Omega_{\delta_0} \subset \omega^*$. Since u_λ is bounded, there exists $C^* > 0$ large enough such that $u_\lambda \leq C^*\varphi_1$ on $\partial\Omega_{\delta_0}$. Hence, u_λ and $C^*\varphi_1$ satisfy

$$\begin{cases} -\Delta_p u_\lambda \leq -\Delta_p \left(C^*\varphi_1\right) & \text{in } \Omega_{\delta_0}, \\ u_\lambda \leq C^*\varphi_1 & \text{on } \partial\Omega_{\delta_0}. \end{cases} \tag{5.19}$$

Therefore, by the weak comparison principle $u_\lambda \leq C^*\varphi_1$ in Ω_{δ_0}. From this estimate and the interior regularity result of Serrin [13], $u_\lambda \in \mathcal{C}\left(\overline{\Omega}\right)$.

Let $M > 0$ and $\alpha \in (1, p')$, we want to construct a super-solution v to $(\mathcal{P}_\lambda)$ near the boundary such that $v \overset{\text{def}}{=} M{\varphi_1}^\alpha$. Similarly to the proof of Lemma 3.2, there exists a $\delta_1 > 0$ only depending on Ω, p, M and α such that:

$$\Delta_p v \sim (M\alpha)^{p-1}(\alpha-1)(p-1)\delta(x)^{(\alpha-1)(p-1)-1} \quad \text{in } \Omega_{\delta_1} \tag{5.20}$$

and

$$K(x)(v^r - \lambda v^q) \sim M^r L(\delta(x))\delta(x)^{\alpha r - k} \quad \text{in } \Omega_{\delta_1}. \tag{5.21}$$

Precisely,

$$\delta_1 \overset{\text{def}}{=} \min\left\{\delta^*, \frac{\varepsilon^*}{K_2}\left(\frac{(\alpha-1)(p-1)}{2\lambda_1}\right)^{\frac{1}{p}}, \frac{1}{K_2}\left(\frac{1}{2\lambda}\right)^{\frac{1}{\alpha(q-r)}}\left(\frac{1}{M}\right)^{\frac{1}{\alpha}}\right\},$$

where ε^* and δ^* are defined in (2.3). By definition of δ_1, choosing $\alpha > 1$ small enough, $\delta_1 = \frac{\varepsilon^*}{K_2}\left(\frac{(\alpha-1)(p-1)}{2\lambda_1}\right)^{\frac{1}{p}}$ and we can impose

$$M \leq \left[\frac{\inf\limits_{\delta \leq \delta_1} L(\delta)\delta^{-(\alpha(p-r-1)-(p-k))}}{\alpha^{p-1}(\alpha-1)}\right]^{\frac{1}{p-(1+r)}}. \tag{5.22}$$

Then, by estimates (5.20) and (5.21), v is a super-solution of $(\mathcal{P}_\lambda)$ in Ω_{δ_1}. Moreover, if we set

$$\delta_2 \overset{\text{def}}{=} \min\left\{\delta_0, \frac{a^*}{C^*K_2}, \left(\frac{a^*\alpha^{p-1}(\alpha-1)}{C_1 \inf\limits_{\delta<\delta_1} L(\delta)}\right)^{\frac{p-(1+r)}{p-k}}\right\},$$

$u_\lambda \le a^*$ and $v \le a^*$ in Ω_{δ_2}. Finally, setting $\delta_* \overset{\text{def}}{=} \min\{\delta_1, \delta_2\}$ and choosing α close enough to 1, u_λ and v satisfy:

$$\begin{cases} -\Delta_p v - K(x)g_\lambda(v) \ge 0 & \text{in } \Omega_{\delta_*}, \\ -\Delta_p u_\lambda - K(x)g_\lambda(u_\lambda) = 0 & \text{in } \Omega_{\delta_*}, \\ v \ge u_\lambda & \text{on } \partial\Omega_{\delta_*}. \end{cases}$$

Note that the third assertion is a consequense of (5.19) and (5.22). Since $v \mapsto -K(x)g_\lambda(v)$ is non-increasing in $(0, a^*)$ for all $x \in \Omega_{\delta_*}$, applying the weak comparison principle of proposition 5.2, it follows that $u_\lambda \le v$ in Ω_{δ_*}. ■

Proposition 5.3. *Let $k \in [1+r, p)$ and let $u_\lambda \in \mathrm{W}_0^{1,p}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ be a weak solution of $(\mathcal{P}_\lambda)$, therefore u_λ has a compact support.*

proof: For $s \in [0, a^*]$, we define

$$G_\lambda(s) \overset{\text{def}}{=} \int_0^s g_\lambda(t)\,dt.$$

Since $r < p-1$, we have

$$\int_0^{a^*} G_\lambda(s)^{-\frac{1}{p}}\,ds < +\infty. \tag{5.23}$$

Note that this above equation is close to condition (2) in VÁSQUEZ [14] and implies that u_λ may be not positive everywhere in Ω. Precisely, let us fix $\varepsilon < a^*$ (small) and $\delta_\varepsilon \overset{\text{def}}{=} \left(\frac{\varepsilon}{M}\right)^{\frac{1}{\alpha}}$ with M and α defined in Lemma 5.2 in such a way that $u_\lambda < \varepsilon$ in $\Omega_{\delta_\varepsilon}$. Let us define for $t \in [0, a^*]$,

$$h(t) \overset{\text{def}}{=} \int_t^\varepsilon G_\lambda(s)^{-\frac{1}{p}}\,ds.$$

h is a $\mathcal{C}^2$ bijection from $[0, a^*]$ to $[h(a^*), h(0)]$ and

$$h'(t) = -G_\lambda(t)^{-\frac{1}{p}} < 0, \quad \text{for } t \in (0, a^*).$$

Then h^{-1} is also twice differentiable on $(h(a^*), h(0))$ and we get,

$$(h^{-1})'(y) = -G_\lambda(h^{-1}(y))^{\frac{1}{p}}$$

and

$$(h^{-1})''(y) = \frac{1}{p} g_\lambda\left(h^{-1}(y)\right) G_\lambda\left(h^{-1}(y)\right)^{\frac{2-p}{p}} \quad \text{for } y \in (h(a^*), h(0)).$$

Now let us define

$$j(x) \overset{\text{def}}{=} \int_{\varphi_1(x)}^{\inf_{\partial\Omega_{\delta_\varepsilon}} \varphi_1} \left(g_\lambda\left(\frac{s}{2}\right)\frac{s}{2}\right)^{-\frac{1}{p}} ds, \quad \text{for } x \in \Omega_{\delta_\varepsilon}$$

and

$$J(x) \overset{\text{def}}{=} \min\left\{j(x), h(0)\right\}, \quad \text{for } x \in \Omega_{\delta_\varepsilon}.$$

Remark that $j(x) > h(a^*)$ for $x \in \Omega_{\delta_\varepsilon}$ provided ε is sufficiently small. Indeed,

$$h(a^*) \leq C_1\left[\varepsilon^{\frac{p-(r+1)}{p}} - (a^*)^{\frac{p-(r+1)}{p}}\right] < -C_2\varepsilon^{\frac{p-(r+1)}{\alpha p}} \leq j(x), \;\; \text{for } x \in \Omega_{\delta_\varepsilon},$$

with C_1 and C_2 two positive constants independant of ε.

With all this notations, we finally define the function w in $\Omega_{\delta_\varepsilon}$ by

$$w(x) \overset{\text{def}}{=} h^{-1}\left(J(x)\right), \quad \text{for } x \in \Omega_{\delta_\varepsilon}.$$

In other words,

$$\int_{w(x)}^{\varepsilon} G_\lambda(s)^{-\frac{1}{p}}\, ds = J(x), \quad \text{for } x \in \Omega_{\delta_\varepsilon}.$$

Using this last caracterisation, w is non-negative in $\Omega_{\delta_\varepsilon}$ and $w \leq a^*$ in $\Omega_{\delta_\varepsilon}$. Moreover, w vanishes when $\delta(x)$ is small. Indeed, for $s \in (0, a^*)$,

$$g_\lambda(s)s > G_\lambda(s) > \int_{\frac{s}{2}}^{s} g_\lambda(t)\, dt > \frac{s}{2} g_\lambda\left(\frac{s}{2}\right).$$

Then for $\varepsilon \in (0, a^*)$,

$$\int_0^\varepsilon G_\lambda(s)^{-\frac{1}{p}}\, ds < \int_0^\varepsilon \left(\frac{s}{2} g_\lambda\left(\frac{s}{2}\right)\right)^{-\frac{1}{p}} ds.$$

So,

$$j(x) \xrightarrow[\delta(x)\to 0]{} \int_0^{\inf_{\partial\Omega_{\delta_\varepsilon}} \varphi_1} \left(g_\lambda\left(\frac{s}{2}\right)\frac{s}{2}\right)^{-\frac{1}{p}} ds \geq \int_0^{K_1\delta_\varepsilon} \left(g_\lambda\left(\frac{s}{2}\right)\frac{s}{2}\right)^{-\frac{1}{p}} ds > \int_0^{\varepsilon} G_\lambda(s)^{-\frac{1}{p}} ds,$$

for $\varepsilon > 0$ small, from which, together with the definitions of J and w, w has a compact support in Ω.

Then, to complete the proof, it's enough to show that $u_\lambda \leq w$ in $\Omega_{\delta_\varepsilon}$. Because of the compacted support of w, $J \in \mathrm{W}^{1,p}(\Omega_{\delta_\varepsilon})$, thus $w = h^{-1} \circ J \in \mathrm{W}^{1,p}(\Omega_{\delta_\varepsilon})$ and satisfies

$$\nabla w = -G_\lambda\left(w\right)^{\frac{1}{p}} \nabla j \qquad \text{in } \mathcal{D}'\left(\Omega_{\delta_\varepsilon}\right).$$

Then,

$$\Delta_p w + G_\lambda(w)^{\frac{1}{p'}}\Delta_p j = \frac{1}{p'}\, g_\lambda(w)|\nabla j|^p \qquad \text{in } \mathcal{D}'\left(\Omega_{\delta_\varepsilon}\right).$$

In this equation, provided ε is sufficiently small

$$\frac{1}{p'}|\nabla j|^p = \frac{1}{p'}|\nabla \varphi_1|^p \left[\frac{\varphi_1}{2} g_\lambda\left(\frac{\varphi_1}{2}\right)\right]^{-1} \leqslant K(x) \quad \text{in } \Omega_{\delta_\varepsilon}$$

and

$$\begin{aligned}\Delta_p j = {} & \frac{1}{p'}|\nabla \varphi_1|^p \left[\frac{\varphi_1}{2} g_\lambda\left(\frac{\varphi_1}{2}\right)\right]^{-\frac{1}{p'}} \left[\frac{1}{2} g_\lambda\left(\frac{\varphi_1}{2}\right) + \frac{\varphi_1}{4}(g_\lambda)'\left(\frac{\varphi_1}{2}\right)\right] \\ & + \lambda_1 {\varphi_1}^{p-1}\left[\frac{\varphi_1}{2} g_\lambda\left(\frac{\varphi_1}{2}\right)\right]^{\frac{1}{p'}} \geq 0 \qquad \text{in } \Omega_{\delta_\varepsilon}.\end{aligned}$$

Hence, $\Delta_p w \leq K(x) g_\lambda(w)$ in $\Omega_{\delta_\varepsilon}$. Moreover, since $g_\lambda \geq 0$ on $\partial\Omega_{\delta_\varepsilon}$, we have $u_\lambda(x) \leq \varepsilon \leq w(x)$ on $\partial\Omega_{\delta_\varepsilon}$. Therefore, by the weak comparison principle of proposition 5.2, $u_\lambda(x) \leq w(x)$ in $\Omega_{\delta_\varepsilon}$. ■

Proof of Theorem 2.2: Since u_λ is compactly supported in Ω, inequality (3.1) is also satisfied when $k \geq 1 + r$, which implies the existence of a critical parameter $\lambda_{**} > 0$ such that $(\mathcal{P}_\lambda)$ has no non-trivial solution for $\lambda < \lambda_{**}$. Thanks to above propositions 5.1 and 5.3 and remark 5.1, using the $\mathcal{C}^{1,\beta}$ regularity result of Lieberman [10] and the same arguments as in paragraph 4.2, Theorem 2.2 follows.

Appendix A. Proof of Lemma 3.3

A.1. **When $\mathbf{0 \leq k < 1 + q}$.** By (1.5), $\underline{v} \overset{\text{def}}{=} m\varphi_1$ is a sub-solution of $(\mathcal{Q})$ in Ω for $m > 0$ small enough. Now let us define

$$f(x) \overset{\text{def}}{=} M\delta(x)^{-(k-q)} L\left(\delta(x)\right) \quad \text{in } \Omega,$$

with $M > 0$. Let $(k-q)^+ < \varepsilon < 1$, therefore $0 < f(x) \leq C_1 \delta(x)^{-\varepsilon}$ in Ω. Thus if we consider the problem

$$\left(\overline{\mathcal{Q}}\right) \begin{cases} -\Delta_p v = f \quad \text{in } \Omega, \\ v = 0 \quad \text{on } \partial\Omega, \quad v > 0 \quad \text{in } \Omega, \end{cases}$$

by a result of Giacomoni, Schindler and Takáč [6] $\left(\overline{\mathcal{Q}}\right)$ has a unique solution $\overline{v} \in \mathcal{C}^{1,\alpha}\left(\overline{\Omega}\right)$, with $\alpha \in (0,1)$ and $\overline{v} \sim \delta(x)$ in Ω. Therefore, $-\Delta_p \overline{v} \geq K(x)\overline{v}^q$ in Ω for $M > 0$ sufficiently large. Hence we get both sub- and super-solution of the problem $\left(\overline{\mathcal{Q}}\right)$ which behave like the distance function $\delta(x)$ in Ω. Using the same sub- and super-solution method as section 4, we get a solution $v \in \mathrm{W}_0^{1,p}(\Omega) \cap \mathcal{C}\left(\overline{\Omega}\right)$ satisfying

$$v(x) \sim \delta(x) \quad \text{in } \Omega.$$

Now let $w \in \mathrm{W}_0^{1,p}(\Omega) \cap \mathcal{C}\left(\overline{\Omega}\right)$ be a solution to $\left(\overline{\mathcal{Q}}\right)$ satisfying $w(x) \sim \delta(x)$ in Ω. Then we can define

$$C^* \overset{\text{def}}{=} \sup\left\{C > 0, \quad Cw \leq v \text{ in } \Omega\right\} \in \mathbb{R}.$$

It's easy to see that $C^* > 0$ and $C^* w \leq v$ in Ω, so for all $x \in \Omega$ we get

$$-\Delta_p\left((C^*)^{\frac{q}{p-1}} w(x)\right) = K(x)\left(C^* w(x)\right)^q \leq K(x) v(x)^q = -\Delta_p(v(x)).$$

If we suppose $C^* < 1$, the weak maximum principle implies $(C^*)^{\frac{q}{p-1}} w \leq v$ in Ω. But $(C^*)^{\frac{q}{p-1}} > C^*$ because $C^* < 1$ and $q < p-1$, therefore

$$C^* w < (C^*)^{\frac{q}{p-1}}\, w \leqslant v \text{ in } \Omega,$$

wich contradicts the definition of C^*. So $C^* \geq 1$ and $w \leq C^* w \leq v$ in Ω. Interchanging the role of w and v, we finally get that $w = v$ and this proves the uniqueness of the solution of $\left(\overline{\mathcal{Q}}\right)$ in the convex set

$$\Lambda_1 \overset{\text{def}}{=} \left\{v \in \mathrm{W}_0^{1,p}(\Omega) \cap \mathcal{C}\left(\overline{\Omega}\right), \quad v(x) \sim \delta(x) \text{ in } \Omega\right\}.$$

A.2. **When $\mathbf{1 + q \leq k \leq p}$.** For $t \in (0, d]$ we define

$$\Theta(t) \overset{\text{def}}{=} \exp\left(\int_t^{2d} \frac{y(s)}{s} ds\right),$$

with $y \in \mathcal{C}\left([0, 2d]\right) \cap \mathcal{C}^1\left((0, 2d]\right)$ such that $y(0) = 0$ and $\lim\limits_{t \to 0^+} \frac{t y'(t)}{y(t)} = 0$ in order to satisfy

$$\lim_{t \to 0^+} \frac{t\Theta'(t)}{\Theta(t)} = 0 \quad \text{and} \quad \lim_{t \to 0^+} \frac{t\Theta''(t)}{\Theta'(t)} = -1. \tag{A.1}$$

Let $\beta \in [0, 1]$, for $x \in \Omega$ we also define

$$w(x) \overset{\text{def}}{=} \varphi_1(x)^\beta \Theta\left(\varphi_1(x)\right) \text{ in } \Omega.$$

A direct computation of $-\Delta_p w$ in Ω gives us

$$-\Delta_p w = \Big(\Theta(\varphi_1)\Big)^{p-1} {\varphi_1}^{(\beta-1)(p-1)-1}\left(\beta + \frac{\varphi_1 \Theta'(\varphi_1)}{\Theta(\varphi_1)}\right)^{p-2} \times$$
$$\left[\left(\beta + \tfrac{\varphi_1 \Theta'(\varphi_1)}{\Theta(\varphi_1)}\right)\lambda_1 {\varphi_1}^p + (p-1)|\nabla \varphi_1|^p \left(\beta(1-\beta) - 2\beta \tfrac{\varphi_1 \Theta'(\varphi_1)}{\Theta(\varphi_1)} - \tfrac{\varphi_1^2 \Theta''(\varphi_1)}{\Theta(\varphi_1)}\right)\right].$$

From now, we will distinguish the following cases.

A.2.1. *Case 1:* $0 < \beta < 1$.

There exists $\varepsilon > 0$ sufficiently small such that for $x \in \Omega_\varepsilon$,

$$\frac{\beta}{2} \leq \beta + \frac{\varphi_1(x)\Theta'(\varphi_1(x))}{\Theta(\varphi_1(x))} \leq \frac{3\beta}{2}$$

and

$$\frac{\beta(1-\beta)}{2} \leq \beta(1-\beta) - 2\beta\frac{\varphi_1(x)\Theta'(\varphi_1(x))}{\Theta(\varphi_1(x))} - \frac{\varphi_1(x)^2\Theta''(\varphi_1(x))}{\Theta(\varphi_1(x))} \leq \frac{3}{2}\beta(1-\beta).$$

Therefore we get

$$-\Delta_p w(x) \sim \Theta(\varphi_1(x))^{p-1}\varphi_1(x)^{(\beta-1)(p-1)-1} \text{ in } \Omega,$$

which implies

$$\Big(-\Delta_p w(x)\Big)w(x)^{-q} \sim \Theta(\varphi_1(x))^{p-(1+q)}\varphi_1(x)^{(\beta-1)(p-1)-1-q\beta} \text{ in } \Omega.$$

When $1+q < k < p$, if we choose $\beta = \frac{p-k}{p-(1+q)} \in (0,1)$ and $y(t) = \frac{z(t)}{p-(1+q)}$ for $t \in [0, 2d]$, w satisfies

$$\begin{cases} \Big(-\Delta_p w(x)\Big)w(x)^{-q} \sim K(x) \quad \text{in } \Omega, \\ \qquad w = 0 \quad \text{on } \partial\Omega, \quad w > 0 \text{ in } \Omega. \end{cases}$$

Therefore there exist $C_1, C_2 > 0$ such that $C_1 w$ and $C_2 w$ are respectively sub- and super-solutions of the problem $(\mathcal{Q})$. Thus, $(\mathcal{Q})$ has a solution $v \in \mathrm{W}^{1,p}_{loc}(\Omega) \cap \mathcal{C}_0\left(\overline{\Omega}\right)$ satisfying

$$v(x) \sim \delta(x)^{\frac{p-k}{p-(1+q)}} L\left(\delta(x)\right)^{\frac{1}{p-(1+q)}} \quad \text{in } \Omega. \tag{A.2}$$

Using the same arguments as section A.1, we get the uniqueness of the solution in the set

$$\Lambda_2 \overset{\text{def}}{=} \left\{ v \in \mathrm{W}^{1,p}_{loc}(\Omega) \cap \mathcal{C}_0\left(\overline{\Omega}\right), \quad v(x) \sim \delta(x)^{\frac{p-k}{p-(1+q)}} L\left(\delta(x)\right)^{\frac{1}{p-(1+q)}} \text{ in } \Omega \right\}.$$

Moreover, $\overline{u}_\lambda \in \mathrm{W}^{1,p}_0(\Omega)$ if and only if the right hand term in the equation of problem $(\mathcal{Q})$ is $\mathrm{W}^{-1,p'}(\Omega)$, *i.e.* if and only if there exists a constant $C > 0$ such that

$$\forall v \in \mathrm{W}^{1,p}_0(\Omega), \qquad \left| \int_\Omega K(x)\overline{u}_\lambda(x)^q v(x) dx \right| \leq C\|v\|.$$

Using estimate (A.2), Hardy's and Hölder's inequalities and property (1.4) of the perturbation L, this condition is satisfied if $k < 1+q+\frac{p-(1+q)}{p}$. Moreover, by (A.2), we have

$$\int_\Omega K(x)\left(\overline{u}_\lambda(x)\right)^{q+1} dx < +\infty$$

only if $k \leq 1+q+\frac{p-(1+q)}{p}$. Then, as soon as $k > 1+q+\frac{p-(1+q)}{p}$, $\overline{u}_\lambda \notin \mathrm{W}^{1,p}_0(\Omega)$.

A.2.2. *Case 2:* $\beta = 1$.

The computation of $-\Delta_p w$ becomes

$$-\Delta_p w = \Theta'(\varphi_1)\left(\Theta(\varphi_1)\right)^{p-2}\left(1+\frac{\varphi_1\Theta'(\varphi_1)}{\Theta(\varphi_1)}\right)^{p-2}\times$$
$$\left[\left(1+\frac{\Theta(\varphi_1)}{\varphi_1\Theta'(\varphi_1)}\right)\lambda_1{\varphi_1}^p+(p-1)|\nabla\varphi_1|^p\left(-2-\frac{\varphi_1\Theta''(\varphi_1)}{\Theta'(\varphi_1)}\right)\right].$$

We choose Θ such that

$$C_1{\varphi_1}^p \leq -\frac{\Theta(\varphi_1){\varphi_1}^{p-1}}{\Theta'(\varphi_1)} \leq C_2{\varphi_1}^{p-1}$$

near the boundary, that is equivalent to require

$$C_1 t \leq -\frac{\Theta(t)}{\Theta'(t)} \leq C_2,\ \text{for } t>0 \text{ small enough.} \tag{A.3}$$

Hence,

$$\Big(-\Delta_p w(x)\Big)w(x)^{-q} \sim -\Theta'\left(\varphi_1(x)\right)\Theta\left(\varphi_1(x)\right)^{p-2-q}\varphi_1(x)^{-q} \text{ in } \Omega.$$

To get

$$\Big(-\Delta_p w(x)\Big)w(x)^{-q} \sim \varphi_1(x)^{-k}L\left(\varphi_1(x)\right) \text{ in } \Omega,$$

we require

$$t^{-(1+q)}y(t)\Theta(t)^{p-(q+1)} \sim t^{-k}\left(\int_t^{2d}\frac{z(s)}{s}ds\right) \text{ in } (0,d].$$

This condition can be satisfied only if $k = 1+q$. Then taking

$$\Theta(t) = \left(\int_t^{2d} s^{-1}L(s)ds\right)^{\frac{1}{p-(1+q)}},\ 0<t\leq d,$$

Θ satisfies conditions (A.1) and (A.3). Thus, if $k = 1+q$ and

$$w(x) = \varphi_1(x)\left(\int_{\varphi_1(x)}^{2d} t^{-1}L(t)dt\right)^{\frac{1}{p-(1+q)}} \quad \text{in } \Omega,$$

there exist $C_1, C_2 > 0$ such that $C_1 w$ and $C_2 w$ are respectively sub- and super-solutions of $(\mathcal{Q})$. Thus, there exists a solution $v \in \mathrm{W}_0^{1,p}(\Omega)\cap\mathcal{C}\left(\overline{\Omega}\right)$ of $(\mathcal{Q})$ satisfying

$$v(x) \sim \delta(x)\left(\int_{\delta(x)}^{2d} t^{-1}L(t)dt\right)^{\frac{1}{p-(1+q)}} \quad \text{in } \Omega.$$

Using the same argument as section A.1 we get the uniqueness of the solution in the set

$$\Lambda_3 \overset{\text{def}}{=} \left\{ v \in \mathrm{W}_0^{1,p}(\Omega)\cap\mathcal{C}\left(\overline{\Omega}\right),\quad v(x) \sim \delta(x)\left(\int_{\delta(x)}^{2d} t^{-1}L(t)dt\right)^{\frac{1}{p-(1+q)}} \quad \text{in } \Omega \right\}.$$

A.2.3. *Case 3:* $\beta = 0$.

In this case, we get

$$-\Delta_p w = \varphi(x)^{-1}\left(\Theta'(\varphi_1)\right)^{p-1}\left[\lambda_1 {\varphi_1}^p - (p-1)|\nabla\varphi_1|^p \frac{\varphi_1\Theta''(\varphi_1)}{\Theta(\varphi_1)}\right].$$

Hence,

$$\Big(-\Delta_p w(x)\Big) w(x)^{-q} \sim \varphi_1(x)^{-1}\Theta'(\varphi_1(x))^{p-1}\Theta(\varphi_1(x))^{-q} \text{ in } \Omega.$$

Similarly as the previous case, to get

$$\varphi_1(x)^{-1}\Theta'(\varphi_1(x))^{p-1}\Theta(\varphi_1(x))^{-q} \sim \varphi_1(x)^{-k}L(\varphi(x)) \text{ in } \Omega,$$

we require

$$t^{-p}\Theta(t)^{p-(1+q)}\Big(-y(t)\Big)^{p-1} \sim t^{-k}\exp\left(\int_t^{2d}\frac{z(s)}{s}dt\right) \text{ in } (0,t].$$

This condition can be satisfied only if $k = p$. Then if condition (3.2) holds and if we choose

$$\Theta(t) = \exp\left(\int_t^{2d}\frac{y(s)}{s}ds\right) = C\left(\int_0^t s^{-1}L(s)^{\frac{1}{p-1}}ds\right)^{\frac{p-1}{p-(1+q)}},\ 0 < t \leq d,$$

we get that Θ satisfies conditions (A.1) and (A.3). Thus if $k = p$ and

$$w(x) = C\left(\int_0^{\varphi_1(x)} t^{-1}L(t)^{\frac{1}{p-1}}dt\right)^{\frac{p-1}{p-(1+q)}},$$

there exists $C_1, C_2 > 0$ such that $C_1 w$ and $C_2 w$ are respectively sub- and super-solutions of $(\mathcal{Q})$ and there exists a solution $v \in \mathrm{W}^{1,p}_{loc}(\Omega) \cap \mathcal{C}_0\left(\overline{\Omega}\right)$ of $(\mathcal{Q})$ satisfying

$$v(x) \sim \left(\int_0^{\delta(x)} s^{-1}L(s)^{\frac{1}{p-1}}ds\right)^{\frac{p-1}{p-(1+q)}} \text{ in } \Omega.$$

Using the same argument as section A.1, we get the uniqueness of the solution in the set

$$\Lambda_4 \overset{\text{def}}{=} \left\{ v \in \mathrm{W}^{1,p}_{loc}(\Omega) \cap \mathcal{C}_0\left(\overline{\Omega}\right),\quad v(x) \sim \left(\int_0^{\delta(x)} s^{-1}L(s)^{\frac{1}{p-1}}ds\right)^{\frac{p-1}{p-(1+q)}} \text{ in } \Omega \right\}.$$

APPENDIX B. PROOF OF PROPOSITION 3.1

To prove this proposition, we need the two following lemmas:

Lemma B.1. *(Picone's Identity)*

Let $u, v \in \mathcal{C}^1\left(\overline{\Omega}\right)$ two positive functions satisfying the Hopf's lemma. Then,

$$L(u,v) \overset{\text{def}}{=} |\nabla u|^p + (p-1)\left(\frac{u}{v}\right)^p |\nabla v|^p - p\left(\frac{u}{v}\right)^{p-1} |\nabla v|^{p-2}\nabla v.\nabla u$$

satisfies $L(u,v) \geq 0$ in Ω and $L(u,v) = R(u,v)$ where

$$R(u,v) \overset{\text{def}}{=} |\nabla u|^p - |\nabla v|^{p-2}\nabla v.\nabla\left(\frac{u^p}{v^{p-1}}\right).$$

Moreover, $L(u,v) = 0$ in Ω if and only if there exists $C > 0$ such that $u = Cv$ in Ω.

Proof: See Theorem 1.1 in ALLEGRETTO-HUANG [1]. ■

Lemma B.2. *(Díaz-Saa inequality)*

For $i = 1, 2$ let $w_i \in \mathrm{L}^\infty(\Omega)$ such that $w_i \geq 0$ a.e. in Ω, ${w_i}^{\frac{1}{p}} \in W^{1,p}(\Omega)$, $\Delta_p\left({w_i}^{\frac{1}{p}}\right) \in \mathrm{L}^\infty(\Omega)$ and $w_1 = w_2$ on $\partial\Omega$. Moreover if $\frac{w_1}{w_2}, \frac{w_2}{w_1} \in \mathrm{L}^\infty(\Omega)$, we have the inequality

$$\int_\Omega \left(\frac{-\Delta_p\left({w_1}^{\frac{1}{p}}\right)}{{w_1}^{\frac{p-1}{p}}} + \frac{\Delta_p\left({w_2}^{\frac{1}{p}}\right)}{{w_2}^{\frac{p-1}{p}}}\right)(w_1 - w_2)\,dx \geq 0. \tag{B.1}$$

Futhermore, (B.1) *becomes an equality if and only if there exists $C > 0$ such that $w_1 = Cw_2$ a.e. in Ω.*

Proof: See Lemme 2 in DÍAZ-SAA [5]. ■

Proof of proposition 3.1: We argue by contradiction. If proposition 3.1 does not hold, there exist $\overline{v} \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega) \cap \mathcal{C}_0\left(\overline{\Omega}\right)$ weak solution of $(\mathcal{Q})$, $\eta > 0$ and $\varepsilon > 0$ satisfying $\overline{v} \geq \eta\underline{u}$ *a.e.* in Ω and (3.3).

B.1. **Step 1: when $\mathbf{q \geq 0}$.** We consider the following perturbated problem:

$$(\mathcal{Q}_n)\begin{cases} -\Delta_p v = K_n(x)v^q, \quad v > 0 \quad \text{in } \Omega, \\ \quad v = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $(K_n)_{n\in\mathbb{N}} \subset \mathrm{L}^\infty(\Omega)$ is increasing sequense satisfying $K_n \underset{n\to+\infty}{\longrightarrow} K$ *a.e.* in Ω. We will prove there exists a unique solution of $(\mathcal{Q}_n)$ in $\mathrm{W}^{1,p}_0(\Omega)$ and show that this solution is $\mathcal{C}^{1,\alpha}\left(\overline{\Omega}\right)$ for some $\alpha \in (0,1)$. Let us consider the functional,

$$I_n(u) \overset{\text{def}}{=} \int_\Omega |\nabla u|^p dx,\ u \in V \overset{\text{def}}{=} \left\{w \in \mathrm{W}^{1,p}_0(\Omega),\ \int_\Omega K_n(x)w^{q+1}dx = 1\right\}.$$

Since V is a compact subset of $\mathrm{W}_0^{1,p}(\Omega)$, there exists a non-negative and non-trivial $\tilde{v}_n \in \mathrm{W}_0^{1,p}(\Omega)$ satisfying

$$I_n(\tilde{v}_n) = \min_{u \in V} I_n(u).$$

Therefore, from the Lagrange multiplier rule, there exists a Lagrange multiplier $\lambda_n > 0$ such that

$$\begin{cases} -\Delta_p \tilde{v}_n = \lambda_n K_n(x)\,(\tilde{v}_n)^q & \text{in } \Omega, \\ \quad \tilde{v}_n = 0 \quad \text{on } \partial\Omega. \end{cases}$$

By homogeneity of the p-Laplacian operator, if we define

$$v_n \overset{\text{def}}{=} (\lambda_n)^{\frac{1}{p-(1+q)}}\, \tilde{v}_n \in \mathrm{W}_0^{1,p}(\Omega),$$

v_n satisfies

$$\begin{cases} -\Delta_p v_n = K_n(x) {v_n}^q, & \text{in } \Omega, \\ \quad v_n = 0 \quad \text{on } \partial\Omega, \end{cases}$$

Since $q < p-1$ and $K_n \in \mathrm{L}^\infty(\Omega)$, using Moser iterations we prove that $v_n \in \mathrm{L}^\infty(\Omega)$ and due to the well known regularity result in Liebermann [10], $v_n \in \mathcal{C}^{1,\alpha}\left(\overline{\Omega}\right)$ for a certain $\alpha \in (0,1)$. The positivity of v_n comes from the strong maximum principle in Vásquez [14] and v_n is a solution of $(\mathcal{Q}_n)$.

Now, let us prove the uniqueness of a such solution. Therefore, for that, we use the Díaz-Saa inequality (B.1). So let $u_n \in \mathcal{C}^{1,\alpha}\left(\overline{\Omega}\right)$ be an other solution of $(\mathcal{Q}_n)$, then

$$\int_\Omega \left(\frac{-\Delta_p u_n}{{u_n}^{p-1}} + \frac{\Delta_p v_n}{{v_n}^{p-1}} \right) ({u_n}^p - {v_n}^p)\, dx \geq 0, \tag{B.2}$$

which implies

$$\int_\Omega K_n(x) \left(\frac{1}{{u_n}^{p-(1+q)}} - \frac{1}{{v_n}^{p-(1+q)}} \right) ({u_n}^p - {v_n}^p)\, dx = 0.$$

Then inequality (B.2) becomes an equality, therefore by lemma B.1 there exits $C > 0$ such that $u_n = C v_n$ in Ω. Futhermore, by homogeneity arguments, $-\Delta_p(Cv_n) \neq K_n(x)(Cv_n)^q$ in Ω if $C \neq 1$, so $u_n = v_n$ in Ω and we get the uniqueness.

Now, we will prove that for all $n \in \mathbb{N}$ $v_n \leq \overline{v}$. For that, we apply a sub- and super-solution method in a compact subset of Ω. So let us fix $n \in \mathbb{N}$ and define $(\Omega_m)_{m \in \mathbb{N}^*}$ an increasing sequense of smooth subdomains of Ω such that $\Omega_m \underset{m\to\infty}{\longrightarrow} \Omega$ in the Hausdorff topology with

$$\forall m \in \mathbb{N}^*, \qquad \frac{1}{m+1} < \operatorname{dist}(\partial\Omega, \partial\Omega_m) < \frac{1}{m}.$$

Then we consider the following sequense of problems:

$$(\mathcal{Q}_{n,m}) \begin{cases} -\Delta_p u = K_n(x) v^q & \text{in } \Omega_m, \\ \quad v = \eta \underline{v} \quad \text{on } \partial\Omega_m, \quad v > 0 & \text{in } \Omega_m, \end{cases}$$

with $\underline{v} \in \mathrm{W}_0^{1,p}(\Omega) \cap \mathcal{C}\left(\overline{\Omega}\right)$ the sub-solution of $(\mathcal{P})$. Since $\overline{v} \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ and $\overline{v} \geq \eta\underline{v}$ in Ω by hypothesis, using the same arguments as in the proof of Proposition 4.1, for all $m \in \mathbb{N}$ there exists $v_{n,m} \in W^{1,p}(\Omega_m) \cap \mathcal{C}\left(\overline{\Omega_m}\right)$ unique solution of $(\mathcal{Q}_{n,m})$. Movereover, $v_{n,m}$ satisfies

$$\eta\underline{v} \leq v_{n,m} \leq \overline{v} \quad \text{in } \Omega.$$

Now if we define $\tilde{v}_{n,m} \overset{\text{def}}{=} \mathbb{1}_{\Omega_m}.v_{n,m}$ in Ω in order to extend v_n in Ω by zero, the sequense $\left(\tilde{v}_{n,m}\right)_{m\in\mathbb{N}^*}$ is an increasing sequense which converges pointwise to an element $u_n \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega) \cap \mathcal{C}_0\left(\overline{\Omega}\right)$ solution of $(\mathcal{Q}_n)$, by similar arguments as in the proof of Proposition 4.1. Then, the uniqueness argument implies $u_n = v_n$ in Ω and then

$$\forall n \in \mathbb{N}, \quad v_n \leq \overline{v} \quad \text{in } \Omega.$$

B.2. **Step 2: when q < 0.** Let us define the following problem:

$$(\mathcal{Q}'_n) \begin{cases} -\Delta_p v = K_n(x)\left(v + \frac{1}{n}\right)^q, \quad v > 0 \quad \text{in } \Omega, \\ \qquad v = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Using a similar method as step 1, we get the existence and the uniqueness of a sequense of weak solutions of $(\mathcal{Q}'_n)$ in $\mathrm{W}_0^{1,p}(\Omega)$.

B.3. **Step 3:** Applying Picone's Identity with $u = {\varphi_1}^\beta \in \mathcal{C}^1\left(\overline{\Omega}\right)$, where $\beta = \frac{p-1+\varepsilon}{p}$ and $v = u_n \in \mathcal{C}^1\left(\overline{\Omega}\right)$, we get

$$0 \leq \int_\Omega |\nabla u|^p - |\nabla v|^{p-2}\nabla v.\nabla\left(\frac{u^p}{v^{p-1}}\right)dx. \tag{B.3}$$

(1) When $q \geq 0$ this inequality becomes

$$\begin{aligned} \beta^p \int_\Omega |\nabla\varphi_1|^p {\varphi_1}^{(\beta-1)p}dx &= \int_\Omega \left|\nabla{\varphi_1}^\beta\right|^p dx \\ &\geq \int_\Omega |\nabla v_n|^{p-2}\nabla v_n.\nabla\left(\frac{{\varphi_1}^{\beta p}}{{v_n}^{p-1}}\right)dx \\ &= \int_\Omega K_n(x)\frac{{\varphi_1}^{\beta p}}{{v_n}^{p-(q+1)}}dx. \end{aligned}$$

Therefore, passing to the limit when $n \to +\infty$, there exists $C > 0$ such that

$$\inf_{y\in\omega} \frac{1}{\overline{v}(y)^{p-(q+1)}} \int_\omega K(x){\varphi_1}^{p-1+\varepsilon}dx \leq C, \quad \forall\omega \subset\subset \Omega.$$

This inequality does not hold for ω close enough to Ω, *i.e* when $\mathrm{dist}(\Omega,\omega)$ is sufficiently small, because

$$\int_\Omega K(x){\varphi_1}^{p-1+\varepsilon}dx = +\infty,$$

by hypothesis.

(2) When $q < 0$ arguing similarly as in the first case, we get

$$\beta^p \int_\Omega |\nabla \varphi_1|^p {\varphi_1}^{(\beta-1)p} dx \geq \int_\Omega K_n(x) \frac{{\varphi_1}^{\beta p}}{\left(v_n + \frac{1}{n}\right)^{p-(q+1)}} dx.$$

Therefore passing to the limit when $n \to +\infty$,

$$\inf_{y \in \omega} \frac{1}{(\overline{v}(y) + 1)^{p-(q+1)}} \int_\omega K(x) {\varphi_1}^{p-1+\varepsilon} dx \leq C, \quad \forall \omega \subset\subset \Omega.$$

And we conclude as above. ■

APPENDIX C. $\mathcal{C}^{1,\beta}$ REGULARITY

We consider the following quasilinear elliptic boundary value problem,

$$(\mathcal{P}) \left\{ \begin{array}{r} -\mathrm{div}(\mathbf{a}(x, \nabla u)) = f(x) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \Omega. \end{array} \right.$$

In this equation, $f \in \mathrm{L}^\infty_{\mathrm{loc}}(\Omega)$ and

$$\mathrm{div}(\mathbf{a}(x, \nabla u)) \overset{\mathrm{def}}{=} \sum_{i=1}^N \frac{\partial}{\partial x_i} a_i(x, \nabla u(x)), \quad \text{for } x \in \Omega \text{ and } u \in \mathrm{W}^{1,p}_0(\Omega) \tag{C.1}$$

with values in $\mathrm{W}^{-1,p'}(\Omega)$. Moreover, the components a_i of the vector field $\mathbf{a} : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$, a $= (a_1, \cdots, a_N)$, are functions of x and $\eta \in \mathbb{R}^N$, such that for $i, j \in \{1, \cdots, N\}$, $a_i \in \mathcal{C}(\Omega \times \mathbb{R}^N)$ and $\frac{\partial a_i}{\partial \eta_j} \in \mathcal{C}(\Omega \times (\mathbb{R}^N \setminus \{0\}))$. We assume that $\mathbf{a}$ satisfies the following ellipticity and growth conditions:

(H1) There exist some constants $\kappa \in [0, 1]$, $\gamma, \Gamma \in (0, +\infty)$ and $\alpha \in (0, 1)$, such that for all $x, y \in \Omega$, all $\eta \in \mathbb{R}^N \setminus \{0\}$ and $\xi \in \mathbb{R}^N$,

$$a_i(x, 0) = 0, \quad \text{for } i = 1, \cdots, N, \tag{C.2}$$

$$\sum_{i,j=1}^N \frac{\partial a_i}{\partial \eta_j}(x, \eta) \xi_i \xi_j \geq \gamma(\kappa + |\eta|)^{p-2} |\xi|^2, \tag{C.3}$$

$$\sum_{i,j=1}^N \left| \frac{\partial a_i}{\partial \eta_j}(x, \eta) \right| \leq \Gamma(\kappa + |\eta|)^{p-2}, \tag{C.4}$$

$$\sum_{i=1}^N |a_i(x, \eta) - a_i(y, \eta)| \leq \Gamma(1 + |\eta|)^p |x - y|^\alpha. \tag{C.5}$$

We remark that condition (C.2) through (C.5) are motivated by the elliptic boundary value problem,

$$(\mathcal{P}) \left\{ \begin{array}{r} -\Delta_p u = f(x) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \Omega. \end{array} \right.$$

Finally, we impose the following growth condition on the function f:

(H2) $f \in \mathrm{L}^\infty_{\mathrm{loc}}(\Omega)$ and there exist some constants $c > 0$ and $\varepsilon \in (0,1)$ such that, for almost all $x \in \Omega$,

$$|f(x)| \leq c\delta(x)^{-\varepsilon}. \tag{C.6}$$

Proposition C.1. *Assume that* $\mathbf{a}(x,\eta)$ *satisfies the structural hypotheses* (C.2) *through* (C.5) *and* $f(x)$ *satisfies the growth hypothesis* (C.6)*. Let* $u \in \mathrm{W}^{1,p}_0(\Omega)$ *be a weak solution of the problem* $(\mathcal{P})$*. In addition assume*

$$0 \leq u(x) \leq C\delta(x) \quad \textit{for almost all } x \in \Omega, \tag{C.7}$$

where $C > 0$ *and* $\overline{u} \in \mathrm{W}^{1,p}_0(\Omega)$ *is a weak solution of*

$$(\overline{\mathcal{P}}) \left\{ \begin{array}{r} -\mathrm{div}(\mathbf{a}(x,\nabla u)) = f(x) \quad \textit{in } \Omega, \\ u = 0 \quad \textit{on } \Omega. \end{array} \right.$$

Then there exist constants $\beta \in (0,\alpha)$ *and* $M \geq 0$ *only depending on* $\Omega, N, p,$ *on* γ, Γ, α *in* (C.2) *through* (C.5)*, on the constants* c, ε *in* (C.6) *and on the constant* C *in* (C.7)*, but not on* $\kappa \in [0,1]$*, such that* u *satisfies* $u \in \mathcal{C}^{1,\beta}\left(\overline{\Omega}\right)$ *and* $\|u\|_{\mathcal{C}^{1,\beta}\left(\overline{\Omega}\right)} \leq M$*.*

proof: The proof is very simlar to the proof of Therorem B.1 in Giacomoni, Schindler and Takáč [6]. In particular, condition (C.6) replacing the growth condition (B.9) imply the estimate (B.17) in [6]. ■

The $\mathcal{C}^{1,\beta}$ regularity of u_λ is directy proved applying this proposition with $\mathbf{a}(x,\eta) \overset{\text{def}}{=} |\eta|^{p-2}\eta$ and $f(x) \overset{\text{def}}{=} K(x)\left[\lambda u_\lambda(x)^q - u_\lambda(x)^r\right]$ for $x \in \Omega$ and $\eta \in \mathbb{R}^N$.

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