

EXISTENCE OF SOLUTIONS TO A NONLINEAR PROBLEM INVOLVING ISOTROPIC DEFORMATIONS

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ABSTRACT. In this paper we show, under suitable hypotheses on the boundary datum φ , existence of Lipschitz maps $u : \Omega \rightarrow \mathbb{R}^2$ satisfying the nonlinear differential inclusion

$$\begin{cases} Du \in E, & \text{a.e. in } \Omega \\ u = \varphi, & \text{on } \partial\Omega \end{cases}$$

where Ω is an open bounded subset of \mathbb{R}^2 and E is a compact subset of $\mathbb{R}^{2 \times 2}$, which is isotropic, that is to say, invariant under orthogonal transformations. Our result relies on an abstract existence theorem due to Müller and Šverák which requires the set E to satisfy a certain in-approximation property.

1. INTRODUCTION

A microstructure is a structure on a scale between the macroscopic and the atomic ones. Microstructures are abundant in nature, for example, they are present in molecular tissues or in biomaterials. Crystals such as igneous rocks or metal alloys (for example nickel-aluminium, zinc-lead) also develop microstructures. The microstructure of a material can strongly influence physical properties such as strength, toughness, ductility, hardness and corrosion resistance. This influence can vary as a function of the temperature of the material.

In the last twenty years successful models for studying the behaviour of crystal lattices undergoing solid-solid phase transitions have been studied. In such models it is assumed that the elements of crystal lattices have certain preferable affine deformations; this is true for example for martensite or for quartz crystals (see [1, 15]).

Denoting by E the set of matrices corresponding to the gradient of these deformations, the physical models motivate the mathematical question of the existence of solutions to Dirichlet problems related to systems of differential inclusions such as $Du \in E$ a.e. in Ω , $u = \varphi$ on $\partial\Omega$, where Ω is a domain of \mathbb{R}^n and $E \subset \mathbb{R}^{n \times n}$ is a compact set.

Two abstract theories to establish the existence of solutions of general differential inclusion problems are due to Dacorogna and Marcellini (see [8, 6]), whose result is based on Baire's category theorem, and Müller and Šverák [16, 17], who use ideas of convex integration by Gromov [12]. In these two theories certain convex hulls of the set E play an important role. We say that a set $E \subseteq \mathbb{R}^{n \times n}$ is rank one convex if for every $\xi, \eta \in E$ such that $\text{rank}(\xi - \eta) = 1$ and for every $t \in [0, 1]$ then

$$t\xi + (1 - t)\eta \in E.$$

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In the spirit of the usual definition of the convex hull, the rank one convex hull of a set E can be defined as the smallest rank one convex set that contains E . However, in order to solve differential inclusion problems, several authors, namely Müller and Šverák [16, 17], consider the following alternative notion of the rank one convex hull of a compact set $E \subset \mathbb{R}^{N \times n}$, denoted by E^{rc} :

$$E^{rc} = \left\{ \xi \in \mathbb{R}^{N \times n} : f(\xi) \leq 0, \text{ for every} \right. \\ \left. \text{rank one convex function } f \in \mathcal{F}^E \right\}$$

where

$$\mathcal{F}^E = \{ f : \mathbb{R}^{N \times n} \rightarrow \mathbb{R} : f|_E \leq 0 \}.$$

In each of the aforementioned theories, provided certain approximation properties hold, if the gradient of the boundary datum φ belongs to the interior of the appropriate convex hull of E , then there exists a solution $u \in \varphi + W_0^{1,\infty}(\Omega, \mathbb{R}^n)$ to $Du \in E$ a.e. in Ω .

Using these abstract theorems various interesting problems related to the existence of microstructures have been solved, such as the two-well problem, where $E = \mathcal{SO}(2)A \cup \mathcal{SO}(2)B$, where A and B are two fixed $\mathbb{R}^{2 \times 2}$ matrices and $\mathcal{SO}(2)$ stands for the special orthogonal group (see [7, 8, 11, 15, 16]).

In this article we study the case where the set E is an arbitrary $\mathbb{R}^{2 \times 2}$ isotropic set, that is, invariant under orthogonal transformations. More precisely, we assume that E is a compact subset of $\mathbb{R}^{2 \times 2}$ such that $RES \subseteq E$ for every R, S in the orthogonal group $\mathcal{O}(2)$. Let Ω be an open bounded subset of \mathbb{R}^2 . We investigate the existence of weakly differentiable maps $u : \Omega \rightarrow \mathbb{R}^2$ that satisfy

$$(1.1) \quad \begin{cases} Du \in E, & \text{a.e. in } \Omega \\ u = \varphi, & \text{on } \partial\Omega. \end{cases}$$

Note that this problem is fully nonlinear. Since E is isotropic, it can be written as

$$(1.2) \quad E = \{ \xi \in \mathbb{R}^{2 \times 2} : (\lambda_1(\xi), \lambda_2(\xi)) \in K \},$$

for some compact set $K \subset \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq y\}$, where we have denoted by $0 \leq \lambda_1(\xi) \leq \lambda_2(\xi)$ the singular values of the matrix ξ , that is, the eigenvalues of the matrix $\sqrt{\xi\xi^t}$.

Thanks to the properties of the singular values (see Section 3), problem (1.1) can be rewritten in the following equivalent way: for almost every $x \in \Omega$ there exists $(a, b) \in K$ such that

$$\begin{cases} \|Du(x)\|^2 = a^2 + b^2, \\ |\det Du(x)| = ab, \end{cases}$$

and $u(x) = \varphi(x)$ if $x \in \partial\Omega$. In the case where K consists of a unique point these two equations are the vectorial eikonal equation and the equation of prescribed absolute value of the Jacobian determinant.

The main result of our article is the following

Theorem 1.1. *Let $E := \{ \xi \in \mathbb{R}^{2 \times 2} : (\lambda_1(\xi), \lambda_2(\xi)) \in K \}$ where $K \subset \{(x, y) \in \mathbb{R}^2 : 0 < x \leq y\}$ is a compact set. Let $\Omega \subset \mathbb{R}^2$ be a bounded open set and let $\varphi \in C_{\text{piec}}^1(\overline{\Omega}, \mathbb{R}^2)$ be such that $D\varphi \in E \cup \text{int}E^{rc}$ a.e. in Ω . Then there exists a map $u \in \varphi + W_0^{1,\infty}(\Omega, \mathbb{R}^2)$ such that $Du \in E$ a.e. in Ω .*

To prove this theorem we use the following characterization of E^{rc} which can be obtained using results of Cardaliaguet and Tahraoui [2] on isotropic sets (see Section 3 for more details):

$$E^{rc} = \left\{ \xi \in \mathbb{R}^{2 \times 2} : f_{\theta}(\lambda_1(\xi), \lambda_2(\xi)) \leq \max_{(a,b) \in K} f_{\theta}(a, b), \forall \theta \in [0, \max_{(a,b) \in K} b] \right\}$$

where

$$f_{\theta}(x, y) := xy + \theta(y - x), \quad 0 < x \leq y, \theta \geq 0.$$

This characterization is only known to hold in dimension 2×2 and this is the reason our analysis is restricted to this case.

The above existence theorem was first obtained by Croce in [4] using the theory developed by Dacorogna and Marcellini and a refinement due to Dacorogna and Pisante [9]. In this article we treat the same problem using the theory by Müller and Šverák, which leads to different technical difficulties, nevertheless we arrive at the same result. Notice that our only restriction on the compact, isotropic set E is that it contains no singular matrices. The reason for this is to be able to construct an in-approximation sequence for E (see Section 4). This restriction was already present in [4]. We point out that in the case where K consists of a unique point and $K \subset \mathbb{R}^n$, $n \geq 2$ the same existence result was obtained by Dacorogna and Marcellini in [8].

2. NOTIONS OF CONVEXITY

In this section we gather together some generalized convexity notions and properties which will be useful in the sequel. For more details on this matter we refer to [5] and [10].

Notation 2.1. For $\xi \in \mathbb{R}^{N \times n}$ we let

$$T(\xi) = (\xi, \text{adj}_2 \xi, \dots, \text{adj}_{N \wedge n} \xi) \in \mathbb{R}^{\tau(N, n)}$$

where $\text{adj}_s \xi$ stands for the matrix of all $s \times s$ subdeterminants of the matrix ξ , $1 \leq s \leq N \wedge n = \min\{N, n\}$ and where

$$\tau = \tau(N, n) = \sum_{s=1}^{m \wedge n} \binom{N}{s} \binom{n}{s} \text{ and } \binom{N}{s} = \frac{N!}{s!(N-s)!}.$$

In particular, if $N = n = 2$, then $T(\xi) = (\xi, \det \xi)$.

Definition 2.1. (i) A function $f : \mathbb{R}^{N \times n} \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to be *polyconvex* if there exists a convex function $g : \mathbb{R}^{\tau(N, n)} \rightarrow \mathbb{R} \cup \{+\infty\}$ such that

$$f(\xi) = g(T(\xi)).$$

(ii) A Borel measurable function $f : \mathbb{R}^{N \times n} \rightarrow \mathbb{R}$ is said to be *quasiconvex* if

$$f(\xi) \text{ meas}(U) \leq \int_U f(\xi + D\varphi(x)) dx$$

for every bounded open set $U \subset \mathbb{R}^n$, $\xi \in \mathbb{R}^{N \times n}$ and $\varphi \in W_0^{1, \infty}(U; \mathbb{R}^N)$.

(iii) A function $f : \mathbb{R}^{N \times n} \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to be *rank one convex* if

$$f(t\xi + (1-t)\eta) \leq tf(\xi) + (1-t)f(\eta)$$

for every $t \in [0, 1]$ and every $\xi, \eta \in \mathbb{R}^{N \times n}$ with $\text{rank}(\xi - \eta) = 1$.

It is well known that, if $f : \mathbb{R}^{N \times n} \rightarrow \mathbb{R}$, then

$$f \text{ polyconvex} \Rightarrow f \text{ quasiconvex} \Rightarrow f \text{ rank one convex.}$$

Definition 2.2. (i) We say that $E \subset \mathbb{R}^{N \times n}$ is *polyconvex* if

$$\left. \begin{array}{l} \sum_{i=1}^{\tau+1} t_i T(\xi_i) = T \left(\sum_{i=1}^{\tau+1} t_i \xi_i \right) \\ \xi_i \in E, t_i \geq 0, \sum_{i=1}^{\tau+1} t_i = 1 \end{array} \right\} \Rightarrow \sum_{i=1}^{\tau+1} t_i \xi_i \in E.$$

In the case $N = n = 2$ we recall that $\tau + 1 = 5$ and E is polyconvex if for all $t_i \geq 0$ with $\sum_{i=1}^5 t_i = 1$ and for all $\xi_i \in E$ with

$$\sum_{i=1}^5 t_i \det \xi_i = \det \left(\sum_{i=1}^5 t_i \xi_i \right)$$

then $\sum_{i=1}^5 t_i \xi_i \in E$.

(ii) Let $E \subset \mathbb{R}^{N \times n}$. We say that E is *rank one convex* if for every $\xi, \eta \in E$ such that $\text{rank}(\xi - \eta) = 1$ and for every $t \in [0, 1]$ then

$$t\xi + (1-t)\eta \in E.$$

In the following theorem we mention some properties of polyconvex and rank one convex sets that can be found in the literature. Properties (i) and (iii) were proved in [10], see also [5], whereas (iv) is well known. The characterization given in (ii) corresponds to the definition of polyconvex set used by Cardaliaguet and Tahraoui in [2].

Theorem 2.1. Let $E \subset \mathbb{R}^{N \times n}$.

(i) The set E is polyconvex if and only if

$$E = \{\xi \in \mathbb{R}^{N \times n} : T(\xi) \in \text{co}T(E)\},$$

where $\text{co}T(E)$ denotes the convex hull of $T(E)$.

(ii) If E is compact, then it is polyconvex if and only if there exists a polyconvex function $f : \mathbb{R}^{N \times n} \rightarrow \mathbb{R}$ such that

$$E = \{\xi \in \mathbb{R}^{N \times n} : f(\xi) \leq 0\}.$$

(iii) If E is polyconvex (respectively, rank one convex) then $\text{int} E$ is also polyconvex (respectively, rank one convex). However, even if $N = n = 2$, \overline{E} is not necessarily polyconvex (respectively, rank one convex).

(iv) If E is polyconvex then it is rank one convex.

Remark 2.1. The converse of (iv) is, in general, false.

The concepts of polyconvexity and rank one convexity for sets were introduced as a tool for solving differential inclusion problems through the notion of convex hull in these generalized senses. This lead to different definitions of these hulls that can be found in the literature. The ones we give in Definition 2.3 are the natural ones in the spirit of the classical notion of convex hull. These hulls were considered by

Dacorogna and Marcellini to establish their abstract existence theorem for differential inclusions of the type we are considering. This was the theory used by Croce in [4]. In Definition 2.4 we recall the notions of polyconvex and rank one convex hulls of a given set used by Müller and Šverák [17] in their convex integration method. These are the notions we will use in our existence result in Sections 4 and 5.

Definition 2.3. The *polyconvex* and the *rank one convex* hull of a set $E \subset \mathbb{R}^{N \times n}$ are, respectively, the smallest polyconvex and rank one convex sets containing E and are respectively denoted by $\text{Pco } E$ and $\text{Rco } E$.

From Theorem 2.1, the following inclusions hold

$$E \subset \text{Rco } E \subset \text{Pco } E \subset \text{co } E,$$

where $\text{co } E$ denotes the convex hull of E .

It was proved by Dacorogna and Marcellini in [8] that

$$(2.1) \quad \text{Pco } E = \left\{ \xi \in \mathbb{R}^{N \times n} : T(\xi) = \sum_{i=1}^{\tau+1} t_i T(\xi_i), \right. \\ \left. \xi_i \in E, t_i \geq 0, \sum_{i=1}^{\tau+1} t_i = 1 \right\}.$$

One has (see [10]) that $\text{Pco } E$ and $\text{Rco } E$ are open if E is open, and $\text{Pco } E$ is compact if E is compact. However, it isn't true that $\text{Rco } E$ is compact if E is compact (see [14]).

It is well known that, for $E \subset \mathbb{R}^{N \times n}$,

$$(2.2) \quad \text{co } E = \{ \xi \in \mathbb{R}^{N \times n} : f(\xi) \leq 0, \\ \text{for every convex function } f \in \mathcal{F}_\infty^E \}$$

$$(2.3) \quad \overline{\text{co } E} = \{ \xi \in \mathbb{R}^{N \times n} : f(\xi) \leq 0, \\ \text{for every convex function } f \in \mathcal{F}^E \}$$

where $\overline{\text{co } E}$ denotes the closure of the convex hull of E and

$$\mathcal{F}_\infty^E = \{ f : \mathbb{R}^{N \times n} \rightarrow \mathbb{R} \cup \{+\infty\} : f|_E \leq 0 \} \\ \mathcal{F}^E = \{ f : \mathbb{R}^{N \times n} \rightarrow \mathbb{R} : f|_E \leq 0 \}.$$

Analogous representations to (2.2) can be obtained in the polyconvex and rank one convex cases:

$$\text{Pco } E = \{ \xi \in \mathbb{R}^{N \times n} : f(\xi) \leq 0, \\ \text{for every polyconvex function } f \in \mathcal{F}_\infty^E \}, \\ \text{Rco } E = \{ \xi \in \mathbb{R}^{N \times n} : f(\xi) \leq 0, \\ \text{for every rank one convex function } f \in \mathcal{F}_\infty^E \}.$$

However, (2.3) can only be generalized to the polyconvex case if the sets are compact, in the rank one convex case (2.3) is not true, even if compact sets are considered (see (2.4) and (2.5)). This shows that the hulls considered in Definitions 2.3 and 2.4 are, in fact, different.

Definition 2.4. If $E \subset \mathbb{R}^{N \times n}$ is a compact set,

$$\begin{aligned} E^{pc} &= \{ \xi \in \mathbb{R}^{N \times n} : f(\xi) \leq 0, \\ &\quad \text{for every polyconvex function } f \in \mathcal{F}^E \}, \\ E^{rc} &= \{ \xi \in \mathbb{R}^{N \times n} : f(\xi) \leq 0, \\ &\quad \text{for every rank one convex function } f \in \mathcal{F}^E \}. \end{aligned}$$

If $U \subset \mathbb{R}^{N \times n}$ is an open set,

$$\begin{aligned} U^{pc} &= \bigcup_{\substack{E \subset U \\ E \text{ compact}}} E^{pc} \\ U^{rc} &= \bigcup_{\substack{E \subset U \\ E \text{ compact}}} E^{rc}. \end{aligned}$$

Remark 2.2. Clearly $E^{rc} \subseteq E^{pc}$ and $U^{rc} \subseteq U^{pc}$. Notice that E^{pc} and E^{rc} are closed sets and U^{pc} and U^{rc} are open sets. Also, for compact sets E , these hulls coincide with what is denoted by $\text{Pco}_f E$ and $\text{Rco}_f E$ in [5, 10].

Both in the open and the compact cases the above sets are, respectively, polyconvex and rank one convex.

We next point out the relations between the closures of the convex hulls and the sets introduced in the above definition.

If E is a compact set, then

$$(2.4) \quad \text{Pco } E = \overline{\text{Pco } E} = E^{pc},$$

but, in general,

$$(2.5) \quad \text{Rco } E \subsetneq \overline{\text{Rco } E} \subsetneq E^{rc}.$$

However, in some cases these sets coincide (see the following section for more details).

If U is an open set, then

$$(2.6) \quad U^{pc} = \text{Pco } U = \bigcup_{\substack{E \subset U \\ E \text{ compact}}} \text{Pco } E \subset \overline{\text{Pco } U}$$

but, in general,

$$U^{rc} \supsetneq \text{Rco } U = \bigcup_{\substack{E \subset U \\ E \text{ compact}}} \text{Rco } E.$$

3. PROPERTIES OF ISOTROPIC SETS

Our aim in this article is to study a differential inclusion problem involving a compact set E which is isotropic, that is, invariant under orthogonal transformations. Therefore in this section we mention some results on isotropic subsets of $\mathbb{R}^{n \times n}$. However Theorems 3.2 and 3.3 are established for $n = 2$, since they rely on results of Cardaliaguet and Tahraoui which are known only in dimension 2×2 . These results are crucial to obtain our existence theorem which holds, thus, only in $\mathbb{R}^{2 \times 2}$.

Definition 3.1. Let E be a subset of $\mathbb{R}^{n \times n}$. We say E is isotropic if $RES \subseteq E$ for every R, S in the orthogonal group $\mathcal{O}(n)$.

We denote by $0 \leq \lambda_1(\xi) \leq \dots \leq \lambda_n(\xi)$ the singular values of the matrix ξ , that is, the eigenvalues of the matrix $\sqrt{\xi\xi^t}$.

We recall the following properties on the singular values:

$$\prod_{i=1}^n \lambda_i(\xi) = |\det \xi|$$

$$\sum_{i=1}^n (\lambda_i(\xi))^2 = \|\xi\|^2.$$

From these properties it follows that, in the 2×2 case, λ_1 and λ_2 are given by

$$\lambda_1(\xi) = \frac{1}{2} \left[\sqrt{\|\xi\|^2 + 2|\det \xi|} - \sqrt{\|\xi\|^2 - 2|\det \xi|} \right]$$

$$\lambda_2(\xi) = \frac{1}{2} \left[\sqrt{\|\xi\|^2 + 2|\det \xi|} + \sqrt{\|\xi\|^2 - 2|\det \xi|} \right].$$

In addition, for every $\xi \in \mathbb{R}^{n \times n}$, $R, S \in \mathcal{O}(n)$

$$\lambda_i(\xi) = \lambda_i(R\xi S),$$

λ_i are continuous functions, $\prod_{i=k}^n \lambda_i(\xi)$ is polyconvex for any $1 \leq k \leq n$ and λ_n is a norm. Moreover the following decomposition holds (see [13]): for every matrix ξ there exist $R, S \in \mathcal{O}(n)$ such that

$$\xi = R \begin{pmatrix} \lambda_1(\xi) & & \\ & \ddots & \\ & & \lambda_n(\xi) \end{pmatrix} S$$

$$= R \operatorname{diag}(\lambda_1(\xi), \dots, \lambda_n(\xi)) S.$$

Due to these properties, any isotropic set E may be written in the form

$$(3.1) \quad \{\xi \in \mathbb{R}^{n \times n} : (\lambda_1(\xi), \dots, \lambda_n(\xi)) \in \Gamma\}$$

where Γ is a set contained in $\{(x_1, \dots, x_n) \in \mathbb{R}^n : 0 \leq x_1 \leq \dots \leq x_n\}$. Clearly if Γ is compact (respectively, open) then E is also compact (respectively, open). On the other hand, if E is compact the set Γ can be taken to be compact and if E is open (3.1) holds for an open set $\Gamma \subset \mathbb{R}^n$.

In the following theorems we establish some results on isotropic sets.

Theorem 3.1. (i) *If $E \subseteq \mathbb{R}^{n \times n}$ is isotropic then $\operatorname{Pco} E$ is isotropic.*

(ii) *If $U \subseteq \mathbb{R}^{n \times n}$ is open, bounded and isotropic then*

$$U^{pc} = \bigcup_{\substack{E \subset U \\ E \text{ compact} \\ E \text{ isotropic}}} E^{pc}$$

and

$$U^{rc} = \bigcup_{\substack{E \subset U \\ E \text{ compact} \\ E \text{ isotropic}}} E^{rc}.$$

(iii) *If $E \subseteq \mathbb{R}^{n \times n}$ is compact and isotropic then E^{rc} is compact and isotropic.*

Proof. (i) Let ξ be a matrix belonging to $\text{Pco } E$ and let R, S be two orthogonal matrices. By (2.1), and using the fact that E is isotropic, it is easy to see that $R\xi S \in \text{Pco } E$ so $\text{Pco } E$ is isotropic.

(ii) To prove the first equality it suffices to show that

$$\bigcup_{\substack{E \subset U \\ E \text{ compact}}} E^{pc} \subseteq \bigcup_{\substack{E \subset U \\ E \text{ compact} \\ E \text{ isotropic}}} E^{pc},$$

since the reverse inclusion is clear.

Let $E \subset U$ be a compact set. Then $\bigcup_{R, S \in \mathcal{O}(n)} RES$ is an isotropic, bounded subset of U . It is also easy to see that this set is closed. Therefore we conclude that

$$\begin{aligned} \bigcup_{\substack{E \subset U \\ E \text{ compact}}} E^{pc} &\subseteq \bigcup_{\substack{E \subset U \\ E \text{ compact}}} \left(\bigcup_{R, S \in \mathcal{O}(n)} RES \right)^{pc} \\ &\subseteq \bigcup_{\substack{V \subset U \\ V \text{ compact} \\ V \text{ isotropic}}} V^{pc}. \end{aligned}$$

A similar argument proves the second equality.

(iii) As E is compact, $\text{Pco } E$ is also compact and, by definition, E^{rc} is closed. Since by Remark 2.2 and (2.4) we have

$$E^{rc} \subseteq E^{pc} = \text{Pco } E,$$

we conclude that E^{rc} is bounded and hence compact.

To prove that E^{rc} is isotropic we will show that if $\xi \notin E^{rc}$ then for every $R, S \in \mathcal{O}(n)$ one has $R\xi S \notin E^{rc}$. Let $\xi \notin E^{rc}$, then there exists a rank one convex function $f : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ such that $f|_E \leq 0$ and $f(\xi) > 0$. Let $R, S \in \mathcal{O}(n)$ and define

$$f_1(\eta) = f(R^{-1}\eta S^{-1}).$$

Then f_1 is rank one convex and for all $\eta \in E$,

$$f_1(\eta) = f(R^{-1}\eta S^{-1}) \leq 0,$$

as $R^{-1}\eta S^{-1} \in E$. However $f_1(R\xi S) = f(\xi) > 0$ and so $R\xi S$ doesn't belong to E^{rc} . \square

For the purposes of our existence theorem we will restrict our attention to compact, isotropic sets E of the form

$$(3.2) \quad E = \{\xi \in \mathbb{R}^{2 \times 2} : (\lambda_1(\xi), \lambda_2(\xi)) \in K\}$$

for which

$$(3.3) \quad K \subset \{(x, y) \in \mathbb{R}^2 : 0 < x \leq y\}.$$

The proof of the next theorem relies on results of [2] and [4].

Theorem 3.2. (i) *If E is a compact, isotropic set of the form (3.2) and (3.3) then*

$$E^{rc} = E^{pc} = \text{Pco } E.$$

(ii) If U is an open, isotropic and bounded subset of $\mathbb{R}^{2 \times 2}$ then

$$U^{rc} = U^{pc} = \text{Pco} U$$

and

$$(U^{rc})^{rc} = U^{rc}.$$

Proof. (i) It was shown by [2, Theorem 5.1] that any compact, isotropic, rank one convex subset of $\mathbb{R}^{2 \times 2}$ is polyconvex. This, together with Theorem 3.1 (iii) and Remark 2.2, shows that if E is a compact, isotropic subset of $\mathbb{R}^{2 \times 2}$ then

$$E^{rc} = E^{pc} = \text{Pco} E.$$

(ii) By equation (2.6), Theorem 3.1 (ii) and part (i) we have

$$(3.4) \quad \text{Pco} U = U^{pc} = \bigcup_{\substack{E \subset U \\ E \text{ compact} \\ E \text{ isotropic}}} E^{pc} = \bigcup_{\substack{E \subset U \\ E \text{ compact} \\ E \text{ isotropic}}} E^{rc} = U^{rc}.$$

Using this result we obtain

$$(U^{rc})^{rc} = (U^{pc})^{rc}.$$

By Remark 2.2 U^{pc} is open. Moreover, by Theorem 3.1 (i) and equation (2.6) U^{pc} is isotropic. Thus we may use equation (3.4) once again to conclude that

$$(U^{pc})^{rc} = (U^{pc})^{pc} = \text{Pco}(\text{Pco} U) = \text{Pco} U = U^{rc}.$$

Thus

$$(U^{rc})^{rc} = U^{rc}.$$

□

Letting

$$(3.5) \quad f_{\theta}(x, y) := xy + \theta(y - x), \quad 0 < x \leq y, \theta \geq 0$$

the following result is a consequence of Proposition 6.7 in [2], Propositions 2.4 and 2.5 in [4] and (2.4).

Theorem 3.3. *Let E be a compact, isotropic set of the form (3.2) and (3.3). Then*

$$\begin{aligned} E^{pc} = \text{Pco} E &= \left\{ \xi \in \mathbb{R}^{2 \times 2} : f_{\theta}(\lambda_1(\xi), \lambda_2(\xi)) \right. \\ &\leq \left. \max_{(a,b) \in K} f_{\theta}(a, b), \forall \theta \in [0, \max_{(a,b) \in K} b] \right\}, \\ \text{int} E^{pc} = \text{int} \text{Pco} E &= \left\{ \xi \in \mathbb{R}^{2 \times 2} : f_{\theta}(\lambda_1(\xi), \lambda_2(\xi)) \right. \\ &< \left. \max_{(a,b) \in K} f_{\theta}(a, b), \forall \theta \in [0, \max_{(a,b) \in K} b] \right\}. \end{aligned}$$

Remark 3.1. (i) Taking into account Theorem 3.2 (i) it follows that if E is a compact, isotropic set of the form (3.2) and (3.3) then

$$\begin{aligned} E^{rc} &= \left\{ \xi \in \mathbb{R}^{2 \times 2} : f_\theta(\lambda_1(\xi), \lambda_2(\xi)) \right. \\ &\quad \left. \leq \max_{(a,b) \in K} f_\theta(a, b), \forall \theta \in [0, \max_{(a,b) \in K} b] \right\}, \\ \text{int} E^{rc} &= \left\{ \xi \in \mathbb{R}^{2 \times 2} : f_\theta(\lambda_1(\xi), \lambda_2(\xi)) \right. \\ &\quad \left. < \max_{(a,b) \in K} f_\theta(a, b), \forall \theta \in [0, \max_{(a,b) \in K} b] \right\}. \end{aligned}$$

(ii) It was shown by Croce in [4, Theorem 3.1] that the set representing E^{pc} also coincides with $\text{Rco } E$. Therefore, in this case, $E^{rc} = \text{Rco } E$ but our existence result is independent of this fact. We also point out that this characterization of $\text{Rco } E$ does not follow from Theorem 5.1 in [2] since it is not known, a priori, that $\text{Rco } E$ is compact.

(iii) In the particular case where K is composed of a unique point (a, b) we obtain

$$E^{pc} = E^{rc} = \left\{ \xi \in \mathbb{R}^{2 \times 2} : \lambda_1(\xi) \cdot \lambda_2(\xi) \leq ab, \lambda_2(\xi) \leq b \right\}.$$

This result was first obtained by Dacorogna and Marcellini in [8] who also showed that this set coincides with $\text{Rco } E$ in the more general framework of $\mathbb{R}^{n \times n}$ matrices.

4. IN-APPROXIMATION

To show Theorem 1.1 we will use an existence result due to Müller and Šverák [16] which requires the following in-approximation property.

Definition 4.1. (*In-approximation*) Let E be a compact subset of $\mathbb{R}^{N \times n}$. We say that a sequence of open sets $U_i \subseteq \mathbb{R}^{N \times n}$ is an in-approximation of E if

- (1) $U_i \subseteq U_{i+1}^{rc}$;
- (2) $\sup_{\xi \in U_i} \text{dist}(\xi, E) \rightarrow 0$ as $i \rightarrow +\infty$.

In this section we will show that the set E , defined by (3.2) and (3.3), admits an in-approximation.

Definition 4.2. Let $0 < \varepsilon_n < \frac{a_0}{2}$, where $a_0 = \min_{(a,b) \in K} a$, be a decreasing sequence such that $\varepsilon_n \rightarrow 0^+$. For $(a, b) \in K$ we define the open sets,

$$R_{(a,b)}^n = \left\{ (x, y) \in \mathbb{R}^2 : a - 2\varepsilon_n < x < a - \varepsilon_n, \right. \\ \left. b - \varepsilon_n < y < b - \frac{\varepsilon_n}{2} \right\}$$

and $U_n := \left\{ \xi \in \mathbb{R}^{2 \times 2} : (\lambda_1(\xi), \lambda_2(\xi)) \in O_n \right\}$, where

$$O_n = \bigcup_{(a,b) \in K} R_{(a,b)}^n.$$

Remark 4.1. Notice that, given the choice of ε_n , the sets $R_{(a,b)}^n$ lie in the region $\{(x, y) \in \mathbb{R}^2 : 0 < x \leq y\}$. Also, the set U_n is open since the functions λ_1 and λ_2 are continuous.

Theorem 4.1. *Let E be given by (3.2) and (3.3). Then E admits an in-approximation.*

Proof. We will show that the sequence U_n of the above definition is an in-approximation for E .

Step 1. We begin by showing the first condition of the definition of in-approximation, that is, $U_n \subseteq U_{n+1}^{rc}$. To this end it suffices to show that, for every n ,

$$(4.1) \quad V_{(a,b)}^n \subseteq (V_{(a,b)}^{n+1})^{rc},$$

where

$$V_{(a,b)}^n = \left\{ \xi \in \mathbb{R}^{2 \times 2} : (\lambda_1(\xi), \lambda_2(\xi)) \in R_{(a,b)}^n \right\}.$$

Let $\xi \in V_{(a,b)}^n$. We will show there exists a compact set $C \subset V_{(a,b)}^{n+1}$ such that $\xi \in C^{rc}$. Since the sequence ε_n is decreasing and $\xi \in V_{(a,b)}^n$ we may choose $0 < \delta < \varepsilon_{n+1}$ such that $\lambda_1(\xi) \leq a - \varepsilon_{n+1} - \delta$ and $\lambda_2(\xi) \leq b - \frac{\varepsilon_{n+1}}{2} - \frac{\delta}{2}$. Letting

$$C = \left\{ \eta \in \mathbb{R}^{2 \times 2} : \begin{aligned} \lambda_1(\eta) &= a - \varepsilon_{n+1} - \delta, \\ \lambda_2(\eta) &= b - \frac{\varepsilon_{n+1}}{2} - \frac{\delta}{2} \end{aligned} \right\}$$

by the choice of δ and Remark 3.1 (iii) it follows that C is a non-empty compact subset of $V_{(a,b)}^{n+1}$ and $\xi \in C^{rc}$.

Step 2. We now proceed with the proof of the second property of the in-approximation.

Let $\xi \in U_n$. Then there exists $(a, b) \in K$ such that $(\lambda_1(\xi), \lambda_2(\xi)) \in R_{(a,b)}^n$. Let $\eta =$

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \text{ and let } R, S \text{ be orthogonal matrices such that } R\xi S = \begin{pmatrix} \lambda_1(\xi) & 0 \\ 0 & \lambda_2(\xi) \end{pmatrix}.$$

Then $R^{-1}\eta S^{-1} \in E$ so

$$\begin{aligned} \text{dist}(\xi, E) &\leq \|\xi - R^{-1}\eta S^{-1}\| = \|R\xi S - \eta\| \\ &\leq \sqrt{2}(|\lambda_1(\xi) - a| + |\lambda_2(\xi) - b|) \leq 3\sqrt{2}\varepsilon_n. \end{aligned}$$

Thus we conclude that

$$\sup_{\xi \in U_n} \text{dist}(\xi, E) \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

□

In fact, given a compact set E satisfying the hypothesis of the previous theorem, it is possible to obtain an in-approximation sequence for E such that if $\xi \in \text{int}E^{rc}$ then ξ belongs to the first set of this sequence. In order to prove this stronger result, which will be used in the next section, we will need the following definitions.

Definition 4.3. Let ε_n be the sequence in Definition 4.2 and let δ_n be such that $0 < \delta_n < \frac{\varepsilon_n}{4}$. For $(a, b) \in K$ we define the compact sets,

$$\begin{aligned} \tilde{R}_{(a,b)}^n &= \{(x, y) \in \mathbb{R}^2 : a - 2\varepsilon_n + \delta_n \leq x \leq a - \varepsilon_n - \delta_n, \\ &\quad b - \varepsilon_n + \delta_n \leq y \leq b - \frac{\varepsilon_n}{2} - \delta_n\} \end{aligned}$$

and $C_n := \{\xi \in \mathbb{R}^{2 \times 2} : (\lambda_1(\xi), \lambda_2(\xi)) \in K_n\}$, where

$$K_n = \bigcup_{(a,b) \in K} \tilde{R}_{(a,b)}^n.$$

Remark 4.2. Notice that, given the choice of ε_n and δ_n , the sets $\tilde{R}_{(a,b)}^n$ are non-empty and lie in the region $\{(x, y) \in \mathbb{R}^2 : 0 < x \leq y\}$.

Proposition 4.1. *The function $f_\theta(x, y)$ defined in (3.5) satisfies the following properties:*

- i) f_θ is strictly increasing in y , for every $x > 0$ and $\theta \geq 0$;
- ii) f_θ is strictly increasing in x , for every $y > \theta$ and is strictly decreasing in x , for every $y < \theta$;
- iii) $f_\theta(\cdot, \theta)$ is constant, for every $\theta \geq 0$;
- iv) setting

$$\begin{aligned}\alpha_{(a,b)}^n(\theta) &= f_\theta(a - 2\varepsilon_n + \delta_n, b - \frac{\varepsilon_n}{2} - \delta_n) \\ \beta_{(a,b)}^n(\theta) &= f_\theta(a - \varepsilon_n - \delta_n, b - \frac{\varepsilon_n}{2} - \delta_n)\end{aligned}$$

with ε_n and δ_n as in Definitions 4.2 and 4.3, one has

$$\begin{aligned}\max_{(x,y) \in \tilde{R}_{(a,b)}^n} f_\theta(x, y) &= \max \left\{ \alpha_{(a,b)}^n(\theta), \beta_{(a,b)}^n(\theta) \right\} \\ &= \begin{cases} \beta_{(a,b)}^n(\theta), & \text{if } \theta \in [0, \max_{(x,y) \in \tilde{R}_{(a,b)}^n} y] \\ \alpha_{(a,b)}^n(\theta), & \text{if } \theta \geq \max_{(x,y) \in \tilde{R}_{(a,b)}^n} y. \end{cases}\end{aligned}$$

Proof. The first three properties are clear and the fourth one follows from i), ii) and iii). \square

We are now in position to prove the stronger in-approximation property which is required to obtain our existence result.

Theorem 4.2. *Let E be the set defined by (3.2) and (3.3) and let $\xi \in \text{int} E^{rc}$. Then there exists an in-approximation sequence U_n for E such that $\xi \in U_1$.*

Proof. Consider the sequence of sets U_n given in Definition 4.2 and let $\xi \in \text{int} E^{rc}$. We will show that there exists $N = N(\xi) \in \mathbb{N}$ such that

$$(4.2) \quad \xi \in (U_N)^{rc}.$$

Given that U_N is open, it suffices to show that ξ belongs to C^{rc} for a certain compact subset C of U_N .

Let C_n be the sets defined in Definition 4.3. Clearly $C_n \subseteq U_n$ and C_n are bounded since λ_2 is a norm and K is compact. To prove that each C_n is closed we consider a sequence $\xi_m \in C_n$ such that $\xi_m \rightarrow \xi$ as $m \rightarrow +\infty$. Then, for every m there exists $(a_m, b_m) \in K$ such that $(\lambda_1(\xi_m), \lambda_2(\xi_m)) \in \tilde{R}_{(a_m, b_m)}^n$. As K is compact, there exists $(a, b) \in K$ such that, up to a subsequence, $(a_m, b_m) \rightarrow (a, b)$ as $m \rightarrow +\infty$. By the inequalities that define $\tilde{R}_{(a_m, b_m)}^n$ and the continuity of λ_1 and λ_2 we conclude that $(\lambda_1(\xi), \lambda_2(\xi)) \in \tilde{R}_{(a,b)}^n$ and thus $\xi \in C_n$. Thus C_n is compact.

In order to choose an appropriate N to satisfy (4.2) we begin by showing that

$$(4.3) \quad \max_{(a,b) \in K_n} f_\theta(a, b) \rightarrow \max_{(a,b) \in K} f_\theta(a, b), \quad n \rightarrow +\infty,$$

uniformly with respect to $\theta \in [0, \max_{(a,b) \in K} b]$. By Proposition 4.1, iv)

$$\max_{(a,b) \in K_n} f_\theta(a, b) = \sup_{(a,b) \in K} \max \{ \alpha_{(a,b)}^n(\theta), \beta_{(a,b)}^n(\theta) \}$$

and

$$\left| \sup_{(a,b) \in K} \max \{ \alpha_{(a,b)}^n(\theta), \beta_{(a,b)}^n(\theta) \} - \max_{(a,b) \in K} f_\theta(a, b) \right|$$

$$\leq \sup_{(a,b) \in K} |\max\{\alpha_{(a,b)}^n(\theta), \beta_{(a,b)}^n(\theta)\} - f_\theta(a, b)|.$$

Therefore we must show that, as $n \rightarrow +\infty$,

$$|\alpha_{(a,b)}^n(\theta) - f_\theta(a, b)| \rightarrow 0, \quad |\beta_{(a,b)}^n(\theta) - f_\theta(a, b)| \rightarrow 0,$$

uniformly with respect to θ and to (a, b) . We start with the first limit. Letting

$$m_n = b - \frac{\varepsilon_n}{2} - 2\delta_n - a + 2\varepsilon_n,$$

and

$$q_n = (a - 2\varepsilon_n + \delta_n) \left(b - \frac{\varepsilon_n}{2} - \delta_n \right)$$

we have $\alpha_{(a,b)}^n(\theta) = m_n\theta + q_n$. Notice that $q_n - ab \rightarrow 0$ and $m_n - b + a \rightarrow 0$ uniformly with respect to (a, b) . This implies the result. The same reasoning applies to the second limit.

As $\xi \in \text{int } E^{rc}$ we know that

$$f_\theta(\lambda_1(\xi), \lambda_2(\xi)) < \max_{(a,b) \in K} f_\theta(a, b), \quad \forall \theta \in [0, \max_{(a,b) \in K} b]$$

thus, there exists $\tau > 0$ such that

$$f_\theta(\lambda_1(\xi), \lambda_2(\xi)) < \max_{(a,b) \in K} f_\theta(a, b) - \tau, \quad \forall \theta \in [0, \max_{(a,b) \in K} b].$$

By the uniform convergence shown in (4.3), for this τ there exists $N \in \mathbb{N}$ such that

$$\max_{(a,b) \in K} f_\theta(a, b) - \tau < \max_{(a,b) \in K_N} f_\theta(a, b), \quad \forall \theta \in [0, \max_{(a,b) \in K} b]$$

and thus

$$f_\theta(\lambda_1(\xi), \lambda_2(\xi)) < \max_{(a,b) \in K_N} f_\theta(a, b), \quad \forall \theta \in [0, \max_{(a,b) \in K} b].$$

In particular, as $\max_{(a,b) \in K_N} b < \max_{(a,b) \in K} b$, we obtain

$$f_\theta(\lambda_1(\xi), \lambda_2(\xi)) < \max_{(a,b) \in K_N} f_\theta(a, b), \quad \forall \theta \in [0, \max_{(a,b) \in K_N} b]$$

which proves that $\xi \in (C_N)^{rc}$. As C_N is a compact subset of U_N we have, therefore, shown (4.2).

To complete the proof we notice that the sequence

$$(U_N)^{rc}, U_{N+1}, U_{N+2}, \dots$$

is an in-approximation of E . Indeed, as U_N is open $(U_N)^{rc}$ is also open so all the sets of the above sequence are open. Since, by construction (cf. Theorem 4.1), $U_N \subseteq (U_{N+1})^{rc}$ we conclude that $(U_N)^{rc} \subseteq ((U_{N+1})^{rc})^{rc} = (U_{N+1})^{rc}$, by Theorem 3.2, (ii). Moreover, if $\xi \in \text{int } E^{rc}$ then $\xi \in (U_N)^{rc}$, by (4.2). \square

5. EXISTENCE THEOREM

In this section we will prove Theorem 1.1. We will assume that the boundary datum φ is $C^1_{piec}(\overline{\Omega}, \mathbb{R}^2)$, that is to say, $\varphi \in W^{1,\infty}(\Omega, \mathbb{R}^2)$, there exist open sets $\omega_i \subset \Omega$ such that $\varphi \in C^1(\overline{\omega_i}, \mathbb{R}^2)$ and $\Omega \setminus \bigcup_i \omega_i$ is a set of Lebesgue measure zero.

We recall the notion of fine C^0 -approximation which can be found in [16].

Definition 5.1. Let $\mathcal{F}(\Omega, \mathbb{R}^N)$ be a family of continuous mappings of Ω into \mathbb{R}^N . We say that a given continuous mapping $v_0 : \Omega \rightarrow \mathbb{R}^N$ admits a fine C^0 -approximation by the family $\mathcal{F}(\Omega, \mathbb{R}^N)$ if there exists, for every continuous function $\varepsilon : \Omega \rightarrow (0, +\infty)$, an element v of the family $\mathcal{F}(\Omega, \mathbb{R}^N)$ such that $|v(x) - v_0(x)| < \varepsilon(x)$ for each $x \in \Omega$.

The proof of Theorem 1.1 is based on the following abstract theorem of Müller and Šverák [16].

Theorem 5.1. Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set and let $E \subseteq \mathbb{R}^{N \times n}$ be a compact set which admits an in-approximation by the open sets U_i . Let $\varphi : \Omega \rightarrow \mathbb{R}^N$ be a C^1 function such that $D\varphi \in U_1$. Then φ admits a fine C^0 -approximation by Lipschitz mappings $u : \Omega \rightarrow \mathbb{R}^N$ such that $Du \in E$ a.e. in Ω .

In the following result we begin by considering the case where φ is an affine function.

Theorem 5.2. Let Ω be an open, bounded subset of \mathbb{R}^2 and let E be the set defined by (3.2) and (3.3). Let $\xi \in \mathbb{R}^{2 \times 2}$ be such that $\xi \in \text{int } E^{rc}$ and let $\varphi : \Omega \rightarrow \mathbb{R}^2$ satisfy $D\varphi = \xi$ in Ω . Then there exists $u \in \varphi + W_0^{1,\infty}(\Omega, \mathbb{R}^2)$ such that $Du \in E$ a.e. in Ω .

Proof. Given $\xi \in \text{int } E^{rc}$ by Theorem 4.2 there exists an in-approximation sequence U_n for E such that $\xi \in U_1$. Thus, if $\varphi : \Omega \rightarrow \mathbb{R}^2$ is a C^1 mapping such that $D\varphi = \xi$, by Theorem 5.1, φ admits a fine C^0 -approximation by Lipschitz mappings $u : \Omega \rightarrow \mathbb{R}^2$ satisfying $Du \in E$ a.e. in Ω . Hence for every continuous function $\varepsilon : \Omega \rightarrow (0, +\infty)$ there exists $u \in W^{1,\infty}(\Omega, \mathbb{R}^2)$ such that $Du \in E$ a.e. in Ω and

$$|u(x) - \varphi(x)| < \varepsilon(x), \quad \forall x \in \Omega.$$

Let $\varepsilon : \overline{\Omega} \rightarrow [0, +\infty)$ be a continuous function such that $\varepsilon(x) = 0 \Leftrightarrow x \in \partial\Omega$ and extend φ as a C^1 mapping to $\overline{\Omega}$. Let $x_0 \in \partial\Omega$ and $x_n \in \Omega$ be a sequence such that $x_n \rightarrow x_0$. Passing to the limit the inequality

$$|u(x_n) - \varphi(x_n)| < \varepsilon(x_n)$$

we conclude that $u \in \varphi + W_0^{1,\infty}(\Omega, \mathbb{R}^2)$. \square

To obtain our existence result in the general case we will once again make use of Theorem 4.2 together with the following result, proved by Dacorogna and Marcellini in [8] (Corollary 10.15).

Theorem 5.3. Let Ω be an open subset of \mathbb{R}^n and A be an open subset of $\mathbb{R}^{N \times n}$. Let $\varphi \in C^1(\Omega, \mathbb{R}^N) \cap W^{1,\infty}(\Omega, \mathbb{R}^N)$ be such that

$$D\varphi(x) \in A, \quad \forall x \in \Omega.$$

Then there exists a function $v \in W^{1,\infty}(\Omega, \mathbb{R}^N)$ such that v is piecewise affine in Ω , $v = \varphi$ on $\partial\Omega$ and $Dv \in A$ a.e. in Ω .

We now proceed with the proof of Theorem 1.1.

Proof. Assume first that $\varphi \in C^1(\overline{\Omega}, \mathbb{R}^2)$. Let

$$\Omega_0 = \{x \in \Omega : D\varphi(x) \in E\}$$

and $\Omega_1 = \Omega \setminus \Omega_0$. Since E is closed and φ is C^1 , the set Ω_0 is closed and thus Ω_1 is open. In Ω_1 we apply the previous theorem to φ and to the open set $\text{int } E^{rc}$ in order to obtain a map $v \in W^{1,\infty}(\Omega_1, \mathbb{R}^2)$ such that $v = \varphi$ on $\partial\Omega_1$, $Dv = c_i$ in Ω_1^i

for some constant $c_i \in \text{int } E^{rc}$ and $\bigcup_i \Omega_1^i = \Omega_1$. Due to Theorem 5.2 we can solve the problem

$$\begin{cases} Du \in E, & \text{a.e. in } \Omega_1^i \\ u(x) = v(x), & x \in \partial\Omega_1^i \end{cases}$$

in each set Ω_1^i . Denoting by u_i the solution in Ω_1^i , the map defined by

$$u = \begin{cases} u_i, & \text{in } \Omega_1^i \\ \varphi, & \text{in } \Omega_0 \end{cases}$$

belongs to $\varphi + W_0^{1,\infty}(\Omega, \mathbb{R}^2)$ and satisfies $Du \in E$.

Now suppose that $\varphi \in C_{\text{piec}}^1(\bar{\Omega}, \mathbb{R}^2)$. This means that there exist open sets $\omega_i \subset \Omega$ such that $\varphi \in C^1(\bar{\omega}_i, \mathbb{R}^2)$ and $\Omega \setminus \bigcup_i \omega_i$ is a set of Lebesgue measure zero.

By the first case, for each i , there exists $w_i \in \varphi + W_0^{1,\infty}(\omega_i, \mathbb{R}^2)$ such that $Dw_i \in E$ a.e. in ω_i . Thus, the function u defined as w_i in ω_i belongs to $\varphi + W_0^{1,\infty}(\Omega, \mathbb{R}^2)$ and satisfies $Du \in E$, a.e. in Ω . \square

We conclude this article by pointing out that Theorem 1.1 is not far from being optimal in the case where the boundary datum φ is affine.

Indeed, suppose that u is a solution of

$$\begin{cases} Du \in E, & \text{a.e. in } \Omega \\ u = u_{\xi_0}, & \text{on } \partial\Omega \end{cases}$$

where u_{ξ_0} is an affine function with $Du_{\xi_0} = \xi_0$. Then there exists a map $\psi \in W_0^{1,\infty}(\Omega, \mathbb{R}^2)$ such that $u = u_{\xi_0} + \psi$. Let $f \in \mathcal{F}^E$ be a polyconvex function. Then f is also quasiconvex and thus

$$f(\xi_0) \leq \frac{1}{|\Omega|} \int_{\Omega} f(\xi_0 + D\psi) dx = \frac{1}{|\Omega|} \int_{\Omega} f(Du) dx \leq 0$$

since $f|_{E} \leq 0$. This implies that $\xi_0 \in E^{pc}$. As already mentioned at the end of Section 3, in the case where E is an isotropic compact subset of $\mathbb{R}^{2 \times 2}$, results of [4] and [2] imply that $\text{Rco } E = E^{rc} = E^{pc}$. Therefore $\xi_0 \in E^{rc}$.

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