

AN ELLIPTIC PROBLEM WITH DEGENERATE COERCIVITY AND A SINGULAR QUADRATIC GRADIENT LOWER ORDER TERM

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ABSTRACT. In this paper we study a Dirichlet problem for an elliptic equation with degenerate coercivity and a singular lower order term with natural growth with respect to the gradient. The model problem is

$$\begin{cases} -\operatorname{div} \left(\frac{\nabla u}{(1+|u|)^p} \right) + \frac{|\nabla u|^2}{|u|^\theta} = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open bounded set of \mathbb{R}^N , $N \geq 3$ and $p, \theta > 0$. The source f is a positive function belonging to some Lebesgue space. We will show that, even if the lower order term is singular, it has some regularizing effects on the solutions, when $p > \theta - 1$ and $\theta < 2$.

1. INTRODUCTION

In this paper we study the following problem:

$$(1) \quad \begin{cases} -\operatorname{div} \left(\frac{b(x)}{(1+|u|)^p} \nabla u \right) + B \frac{|\nabla u|^2}{|u|^\theta} = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open bounded set of \mathbb{R}^N , $N \geq 3$, $B, p > 0$ and $\theta > 0$. We assume that $b : \Omega \rightarrow \mathbb{R}$ is a measurable function such that for some positive constants α and β

$$(2) \quad \alpha \leq b(x) \leq \beta \quad \text{for a.e. } x \in \Omega.$$

Moreover f is a positive function belonging to some Lebesgue space $L^m(\Omega)$, with $m \geq 1$. We point out three characteristics of this problem: the operator $A(v) = -\operatorname{div} \left(\frac{b(x)}{(1+|v|)^p} \nabla v \right)$ is defined on $H_0^1(\Omega)$ but is not coercive on this space when v is large, as proved in [20]. The lower order term has a quadratic growth with respect to the gradient and is singular in the variable u . As we will see, existence and summability of solutions to problem (1) depend on these features.

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It is known that the degenerate coercivity has in some sense a bad effect on the summability of the solutions to problem

$$(3) \quad \begin{cases} -\operatorname{div}(a(x, u)\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

as proved in [9]. There $f \in L^m(\Omega)$ was not assumed to be positive, $a : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ was a Carathéodory function such that $\frac{\alpha}{(1+|s|)^p} \leq a(x, s) \leq \beta$, for $p \in (0, 1)$ and $\alpha, \beta > 0$. Apart from the case where $m > \frac{N}{2}$, the summability of the solutions is lower than the summability of the solutions to elliptic coercive problems. Indeed, in [9] it is shown that if $\frac{2N}{N+2-p(N-2)} < m < \frac{N}{2}$ there exists a $H_0^1(\Omega) \cap L^r(\Omega)$ distributional solution, with $r = \frac{Nm(1-p)}{N-2m}$; if $\frac{N}{N+1-p(N-1)} < m < \frac{2N}{N+2-p(N-2)}$, there exists a $W_0^{1,s}(\Omega)$ distributional solution, with $s = \frac{Nm(1-p)}{N-m(1+p)}$. For $p > 1$ the authors prove a non-existence result for constant sources f . Note that a bad effect on the regularity of the solutions appears even when the right hand side of (3) is an element of $H^{-1}(\Omega)$, such as $-\operatorname{div}(F)$, with $F \in L^2(\Omega)$. As a matter of fact, in this case the solutions are in general not in $H_0^1(\Omega)$ (see [16]).

The presence of lower order terms can have a regularizing effect on the solutions. In [7] and [14] three kinds of lower order terms are considered for elliptic problems with degenerate coercivity, with no restriction on p . In the first paper the author analyses a lower order term defined by a Carathéodory function $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ with the following properties. There exists $d \in L^1(\Omega)$, two positive constants $\mu_1, \mu_2 > 0$ and a continuous increasing real function h such that $g(x, s, \xi)s \geq 0$, $\mu_1|\xi|^2 \leq |g(x, s, \xi)|$ when $|s| \geq \mu_2$ and $|g(x, s, \xi)| \leq d(x)h(|s|)|\xi|^2$. It is proved that for a $L^1(\Omega)$ source there exists a $H_0^1(\Omega)$ distributional solution to

$$\begin{cases} -\operatorname{div}(a(x, u)\nabla u) + g(x, u, \nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

This proves that the summability of the gradient of the solutions is much larger than that one of the solutions of problem (3). It is even larger than the summability of the gradient of the solutions to elliptic coercive problems with $L^1(\Omega)$ sources, which is $L^s(\Omega)$ for every $s < \frac{N}{N-1}$ (see [10] for example). We remark moreover that the lower order term gives the existence of a solution for $p \geq 1$; for these values of p , (3) has no solution.

In a previous article [14] we consider two kinds of lower order terms $h(u)$. For $h(u) = |u|^{q-1}u$, with $q > p+1$, we establish the existence of a distributional solution $u \in W_0^{1,t}(\Omega) \cap L^q(\Omega)$, $t < \frac{2q}{p+1+q}$, for any $L^1(\Omega)$ source f . If $f \in L^m(\Omega)$, $m > 1$ and $q \geq \frac{p+1}{m-1}$ then there exists a distributional solution u in $H_0^1(\Omega) \cap L^{qm}(\Omega)$. If $\frac{p+1}{2m-1} < q < \frac{p+1}{m-1}$, there exists a distributional solution u in $W_0^{1, \frac{2qm}{p+1+q}}(\Omega)$ such that $|u|^{qm} \in L^1(\Omega)$. These results show that if q is sufficiently large, there

exists a distributional solution for any source; this is not the case for problem (3). The second lower order term analysed in [14] is $h(u)$, where $h : [0, s_0) \rightarrow \mathbb{R}$ is a continuous, increasing function such that $h(0) = 0$ and $\lim_{s \rightarrow s_0^-} h(s) = +\infty$ for some $s_0 > 0$. The regularizing effects of this lower order term are even better than the previous one. Indeed for a positive $L^1(\Omega)$ source, there exists a bounded $H_0^1(\Omega)$ solution.

In the literature we find several papers about elliptic coercive problems with lower order terms having a quadratic growth with respect to the gradient (see [6, 10, 11, 12, 8] for example and the references therein), that is, for problem

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) + g(u)|\nabla u|^2 = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In these works it is assumed that $M : \Omega \rightarrow \mathbb{R}^{N^2}$ is a bounded elliptic Carathéodory map, so that there exists $\alpha > 0$ such that $\alpha|\xi|^2 \leq M(x)\xi \cdot \xi$ for every $\xi \in \mathbb{R}^N$. Various assumptions are made on g . With no attempt of being exhaustive, we will describe some recent results where a singular g has been considered, namely $g(u) = \frac{1}{|u|^\theta}$. The case where $0 < \theta \leq 1$, introduced in [2, 3, 4], has been studied in [2, 3, 4, 8, 13, 15]. From this body of literature one can deduce that for a positive source $f \in L^m(\Omega)$, if $\frac{2N}{2N - \theta(N - 2)} \leq m < \frac{N}{2}$ there exists a strictly positive solution $u \in H_0^1(\Omega) \cap L^{(2-\theta)m^{**}}(\Omega)$; if $1 < m < \frac{2N}{2N - \theta(N - 2)}$ then the solution u belongs to $W_0^{1,q}(\Omega)$, with $q = \frac{Nm(2-\theta)}{N - m\theta}$. The authors of [5] consider the general case $\theta < 2$, assuming that f is a strictly positive function on every compactly contained subset of Ω . They prove that if $f \in L^{\frac{2N}{N+2}}(\Omega)$ there exists a positive $H_0^1(\Omega)$ solution. Finally, in [15] the lower order term is taken to be $\lambda u + \mu \frac{|\nabla u|^2}{|u|^\theta} \chi_{\{u>0\}}$, where $\chi_{\{u>0\}}$ denotes the characteristic function of the set $\{u > 0\}$, $\lambda > 0$ and $\mu \in \mathbb{R}$.

In this paper we consider the same lower order term as above in an elliptic problem defined by an operator with degenerate coercivity. We will see that if $0 < \theta < 2$, then $\frac{|\nabla u|^2}{|u|^\theta}$ has a regularizing effect, even if it is singular in u . We are going to state our results. We will distinguish the cases $0 < \theta < 1$ and $1 \leq \theta < 2$.

Theorem 1.1. *Let $0 < \theta < 1$. Assume that f is a positive function belonging to $L^m(\Omega)$, with $m \geq \frac{2N}{2N - \theta(N - 2)}$. Then there exists a function $u \in H_0^1(\Omega)$, strictly positive on Ω , such that $\frac{|\nabla u|^2}{u^\theta} \in L^1(\Omega)$ and*

$$(4) \quad \int_{\Omega} \frac{b(x)}{(1+u)^p} \nabla u \cdot \nabla \varphi + B \int_{\Omega} \frac{|\nabla u|^2}{u^\theta} \varphi = \int_{\Omega} f \varphi,$$

for every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

In the case where $m < \frac{2N}{2N - \theta(N - 2)} = \left(\frac{2^*}{\theta}\right)'$, we are able to prove the existence of an infinite energy solution, belonging to $W_0^{1,\sigma}(\Omega)$, with $\sigma = \frac{mN(2 - \theta)}{N - \theta m}$ (smaller than 2).

Theorem 1.2. *Let $0 < \theta < 1$. Assume that f is a positive function belonging to $L^m(\Omega)$, with $\frac{N}{2N - \theta(N - 1)} < m < \frac{2N}{2N - \theta(N - 2)}$. Then there exists a function $u \in W_0^{1,\sigma}(\Omega)$, strictly positive on Ω , such that $\frac{|\nabla u|^2}{u^\theta} \in L^1(\Omega)$ and*

$$(5) \quad \int_{\Omega} \frac{b(x)}{(1+u)^p} \nabla u \cdot \nabla \varphi + B \int_{\Omega} \frac{|\nabla u|^2}{u^\theta} \varphi = \int_{\Omega} f \varphi,$$

for every $\varphi \in C_0^1(\Omega)$.

In the case where $1 \leq \theta < 2$, we are able to prove the same results as in the case $0 < \theta < 1$, under a stronger hypothesis on f .

Theorem 1.3. *Let $1 \leq \theta < 2$ and $p > \theta - 1$. Assume that $f \in L^m(\Omega)$, with $m \geq \frac{2N}{2N - \theta(N - 2)}$, and satisfies*

$$\text{ess inf}\{f(x) : x \in \omega\} > 0$$

for every $\omega \subset\subset \Omega$. Then there exists a function $u \in H_0^1(\Omega)$, strictly positive on Ω , such that $\frac{|\nabla u|^2}{u^\theta} \in L^1(\Omega)$ and

$$\int_{\Omega} \frac{b(x)}{(1+u)^p} \nabla u \cdot \nabla \varphi + B \int_{\Omega} \frac{|\nabla u|^2}{u^\theta} \varphi = \int_{\Omega} f \varphi$$

for every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

Theorem 1.4. *Let $1 \leq \theta < 2$ and $p > \theta - 1$. Assume that $f \in L^m(\Omega)$, with $\frac{N}{2N - \theta(N - 1)} < m < \frac{2N}{2N - \theta(N - 2)}$, and satisfies*

$$\text{ess inf}\{f(x) : x \in \omega\} > 0$$

for every $\omega \subset\subset \Omega$. Then there exists a function $u \in W_0^{1,\sigma}(\Omega)$, strictly positive on Ω , such that $\frac{|\nabla u|^2}{u^\theta} \in L^1(\Omega)$ and

$$\int_{\Omega} \frac{b(x)}{(1+u)^p} \nabla u \cdot \nabla \varphi + B \int_{\Omega} \frac{|\nabla u|^2}{u^\theta} \varphi = \int_{\Omega} f \varphi$$

for every $\varphi \in C_0^1(\Omega)$.

We remark that if $\theta < \frac{N}{N - 1}$ we are able to prove the existence of solutions when the source f belongs to $L^1(\Omega)$.

We would like to point out the regularizing effects of the lower order term, in the case where $p > \theta - 1$ and $0 < \theta < 2$. Our results furnish $H_0^1(\Omega)$ solutions for less summable sources than for problem (3), since $\frac{2N}{2N - \theta N + 2\theta} < \frac{2N}{N(1 - p) + 2(p + 1)}$.

Even in the case where the source f is less summable, we get a better regularity of solutions than for problem (3): indeed $\sigma = \frac{mN(2-\theta)}{N-\theta m} \geq \frac{Nm(1-p)}{N-m(1+p)}$, as $m \leq \frac{N}{2}$ and $p \leq \theta - 1$.

In the case where $0 < p \leq \theta - 1$, we are able to prove the existence of a solution to problem (1) with the same regularity as the solutions of problem (3).

Theorem 1.5. *Let $1 \leq \theta < 2$ and $0 < p \leq \theta - 1$. Assume that $f \in L^m(\Omega)$ and satisfies*

$$\text{ess inf}\{f(x) : x \in \omega\} > 0$$

for every $\omega \subset \subset \Omega$.

- (1) *If $m > \frac{N}{2}$, then there exists a strictly positive $H_0^1(\Omega) \cap L^\infty(\Omega)$ solution to problem (1).*
- (2) *If $\frac{2N}{N+2-p(N-2)} \leq m < \frac{N}{2}$, then there exists a strictly positive $H_0^1(\Omega) \cap L^r(\Omega)$ solution to problem (1), where $r = \frac{Nm(1-p)}{N-2m}$.*
- (3) *If $\frac{N}{N+1-p(N-1)} < m < \frac{2N}{N+2-p(N-2)}$, then there exists a strictly positive $W_0^{1,s}(\Omega)$ solution to problem (1), where $s = \frac{Nm(1-p)}{N-m(1+p)}$.*

Moreover $\frac{|\nabla u|^2}{u^\theta} \in L^1(\Omega)$.

In the case where $\theta \geq 2$, the situations changes. Indeed we will prove a non-existence result of finite energy solutions. Let $\lambda_1(f)$ denote the first positive eigenvalue of

$$\begin{cases} -\Delta u = \lambda f u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f \in L^q(\Omega)$, with $q > \frac{N}{2}$. Using a result of [5], it is quite easy to prove the following

Theorem 1.6. *Let $f \geq 0$, $f \not\equiv 0$, be a $L^q(\Omega)$ function, with $q > \frac{N}{2}$. If either $\theta > 2$, or $\theta = 2$ and $\lambda_1(f) > \frac{\beta}{B\alpha}$, then there is no $H_0^1(\Omega)$ solution to problem (1).*

2. A PRIORI ESTIMATES

To prove the existence of solutions to problem (1) we use the following approximating problems:

$$\begin{cases} -\text{div} \left(\frac{b(x)}{(1+|T_n(u_n)|)^p} \nabla u_n \right) + B \frac{u_n |\nabla u_n|^2}{(|u_n| + \frac{1}{n})^{\theta+1}} = T_n(f) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

where, for $n \in \mathbb{N}$ and $s \in \mathbb{R}$

$$T_n(s) = \max\{-n, \min\{n, s\}\}.$$

These problems are well-posed due to the following result proved in [6, 11, 12].

Theorem 2.1. *Let f be a bounded function. Let $M : \Omega \times \mathbb{R} \rightarrow \mathbb{R}^{N^2}$ be a Carathéodory function such that there exist two positive constants α_0 and β_0 such that*

$$M(x, s)\xi \cdot \xi \geq \alpha_0|\xi|^2, \quad |M(x, s)| \leq \beta_0$$

for a.e. $x \in \Omega$, for every $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$. Let $g(s)$ be a Carathéodory function such that $g(s)s \geq 0$, $|g(s)| \leq \gamma(s)$, where γ is a continuous, non-negative and increasing function. Then there exists a $H_0^1(\Omega)$ bounded solution to

$$\begin{cases} -\operatorname{div}(M(x, u)\nabla u) + g(u)|\nabla u|^2 = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

By Theorem 2.1 the solutions u_n of the above approximating problems are bounded $H_0^1(\Omega)$ non-negative functions, since f is assumed to be positive and the lower order term has the same sign as u_n . This implies that u_n satisfies

$$(6) \quad \begin{cases} -\operatorname{div}\left(\frac{b(x)}{(1+T_n(u_n))^p}\nabla u_n\right) + B\frac{u_n|\nabla u_n|^2}{(u_n + \frac{1}{n})^{\theta+1}} = T_n(f) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

We are now going to prove some a priori estimates. The next lemma gives a control of the lower order term.

Lemma 2.2. *Let u_n be the solutions to problems (6). Then it results*

$$(7) \quad B \int_{\Omega} \frac{u_n|\nabla u_n|^2}{(u_n + \frac{1}{n})^{\theta+1}} \leq \int_{\Omega} f.$$

Proof. Let us consider $\frac{T_h(u_n)}{h}$, $h > 0$, as a test function in (6). We have, dropping the non-negative operator term,

$$B \int_{\Omega} \frac{|\nabla u_n|^2 u_n}{(u_n + \frac{1}{n})^{\theta+1}} \frac{T_h(u_n)}{h} \leq \int_{\Omega} f \frac{T_h(u_n)}{h}.$$

It is now sufficient to pass to the limit as $h \rightarrow 0$, using Fatou's lemma and the fact that $\frac{T_h(u_n)}{h} \rightarrow 1$ as $h \rightarrow 0$. \square

We prove now two a priori estimates on u_n , which are true for every $p > 0$ and $\theta \in (0, 2)$. In the sequel C will denote a positive constant independent of n ; $\mu(E)$ will be the Lebesgue measure of a set $E \subset \mathbb{R}^N$.

Lemma 2.3. *Let $0 < \theta < 2$. Let f be a positive function belonging to $L^m(\Omega)$, with $m \geq \frac{2N}{2N - \theta(N - 2)}$. Then the solutions u_n to problems (6) are uniformly bounded in $H_0^1(\Omega)$. Thus there exists a function $u \in H_0^1(\Omega)$ such that, up to a subsequence, $u_n \rightarrow u$ weakly in $H_0^1(\Omega)$ and a.e. in Ω .*

Proof. The assertion follows by proving that the solutions u_n to problems (6) are uniformly bounded in $H_0^1(\Omega)$. If we take $(u_n + 1)^\theta - 1$ as a test function in problem (6) we obtain

$$B \int_{\Omega} \frac{|\nabla u_n|^2}{(u_n + \frac{1}{n})^{\theta+1}} u_n (u_n + 1)^\theta \leq B \int_{\Omega} \frac{|\nabla u_n|^2}{(u_n + \frac{1}{n})^{\theta+1}} u_n + \int_{\Omega} f u_n^\theta + C,$$

dropping the positive operator term. We can estimate the right hand side using (7) in order to get

$$B \int_{\Omega} \frac{|\nabla u_n|^2}{(u_n + \frac{1}{n})^{\theta+1}} u_n (u_n + 1)^{\theta} \leq \int_{\Omega} f u_n^{\theta} + C.$$

By working in $\{u_n \geq 1\}$, the previous inequality gives

$$\frac{B}{2} \int_{\{u_n \geq 1\}} |\nabla u_n|^2 \leq \int_{\Omega} f u_n^{\theta} + C \leq \int_{\{u_n \geq 1\}} f u_n^{\theta} + C \leq C \int_{\{u_n \geq 1\}} f (u_n - 1)^{\theta} + C.$$

We use the Sobolev inequality in the left hand side and the Hölder inequality with exponent $\frac{2^*}{\theta}$ in the last term, recalling that f belongs to $L^m(\Omega)$ with $m \geq$

$$\frac{2N}{2N - \theta(N - 2)} = \left(\frac{2^*}{\theta}\right)'. \quad \text{Thus}$$

(8)

$$\mathcal{S} \frac{B}{2} \left[\int_{\{u_n \geq 1\}} (u_n - 1)^{2^*} \right]^{\frac{2}{2^*}} \leq \frac{B}{2} \int_{\{u_n \geq 1\}} |\nabla u_n|^2 \leq C \left[\int_{\{u_n \geq 1\}} (u_n - 1)^{2^*} \right]^{\frac{\theta}{2^*}} + C.$$

Since we are assuming $\theta < 2$, we deduce that

$$\int_{\{u_n \geq 1\}} (u_n - 1)^{2^*} \leq C.$$

It follows from (8) that

$$(9) \quad \int_{\{u_n \geq 1\}} |\nabla u_n|^2 \leq C.$$

Let us search for the same kind of estimate in $\{u_n < 1\}$. Taking $T_1(u_n)$ as a test function in problem (6), we get

$$(10) \quad \frac{\alpha}{2^p} \int_{\{u_n < 1\}} |\nabla T_1(u_n)|^2 \leq \alpha \int_{\{u_n < 1\}} \frac{|\nabla T_1(u_n)|^2}{(1 + u_n)^p} \leq \int_{\Omega} f T_1(u_n) \leq \int_{\Omega} f$$

using hypothesis (2) and dropping the non-negative lower order term. As a consequence of estimates (9) and (10), u_n is uniformly bounded in $H_0^1(\Omega)$. By compactness, there exists a function $u \in H_0^1(\Omega)$ such that, up to a subsequence, $u_n \rightarrow u$ weakly in $H_0^1(\Omega)$ and a.e. in Ω . \square

Lemma 2.4. *Let $0 < \theta < 2$. Let f be a positive function belonging to $L^m(\Omega)$, with $\frac{N}{2N - \theta(N - 1)} < m < \frac{2N}{2N - \theta(N - 2)}$. Then the solutions u_n to problems (6) are uniformly bounded in $W_0^{1,\sigma}(\Omega)$, $\sigma = \frac{mN(2 - \theta)}{N - \theta m}$. Thus there exists a function $u \in W_0^{1,\sigma}(\Omega)$ such that, up to a subsequence, $u_n \rightarrow u$ weakly in $W_0^{1,\sigma}(\Omega)$ and a.e. in Ω .*

Proof. The assertion follows by proving that the solutions u_n to problems (6) are uniformly bounded in $W_0^{1,\sigma}(\Omega)$. Take $(u_n + 1)^{\theta+2\gamma} - 1$, with $\gamma = \frac{2^* - \theta m'}{2m' - 2^*}$, as a test

function in problems (6). Note that $\gamma < 0$: indeed $2^* - \theta m' < 0$ and $2m' - 2^* > 0$, since $m < \frac{N}{2}$. Moreover, $\theta + 2\gamma = \frac{2^*(2-\theta)}{2m' - 2^*} > 0$, as $\theta < 2$. Dropping the non-negative operator term and using estimate (7), we get

$$B \int_{\Omega} \frac{|\nabla u_n|^2}{(u_n + \frac{1}{n})^{\theta+1}} u_n (u_n + 1)^{2\gamma+\theta} \leq \int_{\Omega} f(u_n + 1)^{2\gamma+\theta} + C.$$

By working in $\{u_n \geq 1\}$ the previous inequality gives

$$(11) \quad \begin{aligned} & \frac{B}{2(\gamma+1)^2} \int_{\{u_n \geq 1\}} |\nabla [(u_n + 1)^{\gamma+1} - 2^{\gamma+1}]|^2 \leq \frac{B}{2} \int_{\{u_n \geq 1\}} |\nabla u_n|^2 (u_n + 1)^{2\gamma} \\ & \leq \int_{\{u_n \geq 1\}} f(u_n + 1)^{2\gamma+\theta} + \int_{\{u_n \leq 1\}} f(u_n + 1)^{2\gamma+\theta} + C \leq \int_{\{u_n \geq 1\}} f(u_n + 1)^{2\gamma+\theta} + C. \end{aligned}$$

The Hölder inequality on the right hand side and the Sobolev inequality on the left one imply

$$(12) \quad \begin{aligned} \mathcal{S} \left[\int_{\{u_n \geq 1\}} [(u_n + 1)^{\gamma+1} - 2^{\gamma+1}]^{2^*} \right]^{\frac{2}{2^*}} & \leq C \int_{\{u_n \geq 1\}} |\nabla u_n|^2 (u_n + 1)^{2\gamma} \\ & \leq C + C \left[\int_{\{u_n \geq 1\}} (u_n + 1)^{(2\gamma+\theta)m'} \right]^{\frac{1}{m'}}. \end{aligned}$$

We remark that the choice of γ is equivalent to require $(\gamma+1)2^* = (2\gamma+\theta)m'$; moreover $\frac{2}{2^*} \geq \frac{1}{m'}$, due to the hypotheses on m and θ . Hence

$$(13) \quad \int_{\{u_n \geq 1\}} (u_n + 1)^{(\gamma+1)2^*} = \int_{\{u_n \geq 1\}} (u_n + 1)^{(2\gamma+\theta)m'} \leq C \quad \forall n \in \mathbb{N}.$$

Now, with $\sigma = \frac{mN(2-\theta)}{N-\theta m}$ as in the statement, and recalling that $\gamma < 0$, let us write

$$\int_{\{u_n \geq 1\}} |\nabla u_n|^\sigma = \int_{\{u_n \geq 1\}} \frac{|\nabla u_n|^\sigma}{(u_n + 1)^{-\gamma\sigma}} (u_n + 1)^{-\gamma\sigma}.$$

The Hölder inequality with exponent $\frac{2}{\sigma}$ and estimates (12) and (13) give

$$(14) \quad \int_{\{u_n \geq 1\}} |\nabla u_n|^\sigma \leq \left[\int_{\{u_n \geq 1\}} \frac{|\nabla u_n|^2}{(u_n + 1)^{-2\gamma}} \right]^{\frac{\sigma}{2}} \left[\int_{\{u_n \geq 1\}} (u_n + 1)^{-\gamma\sigma \frac{2}{2-\sigma}} \right]^{\frac{2-\sigma}{2}} \leq C$$

since $-\gamma \frac{2\sigma}{2-\sigma} = (\gamma+1)2^*$. It remains to analyse the behaviour of ∇u_n on $\{u_n \leq 1\}$. Taking $T_1(u_n)$ as a test function in (6) and dropping the non-negative the lower

order term, we get

$$\frac{\alpha}{2^p} \int_{\{u_n \leq 1\}} |\nabla T_1(u_n)|^2 \leq \alpha \int_{\{u_n \leq 1\}} \frac{|\nabla T_1(u_n)|^2}{(1+u_n)^p} \leq \int_{\Omega} f T_1(u_n) \leq \int_{\Omega} f$$

by hypothesis (2). This last estimate and (14) imply that u_n is uniformly bounded in $W_0^{1,\sigma}(\Omega)$. Since $\sigma > 1$, there exists a function $u \in W_0^{1,\sigma}(\Omega)$ such that, up to a subsequence, $u_n \rightarrow u$ weakly in $W_0^{1,\sigma}(\Omega)$ and a.e. in Ω . \square

In the following lemma, we will assume some hypotheses on p . This will give, in some cases, some better estimates than Lemmata 2.3 and 2.4.

Lemma 2.5. *Let $0 < p < 1$. Let $f \in L^m(\Omega)$, $r = \frac{Nm(1-p)}{N-2m}$ and $s = \frac{Nm(1-p)}{N-m(1+p)}$.*

(1) *If $m > \frac{N}{2}$, the solutions of (6) are uniformly bounded in $H_0^1(\Omega) \cap L^\infty(\Omega)$.*

Thus there exists a function $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that, up to a subsequence, $u_n \rightarrow u$ weakly in $H_0^1(\Omega)$ and a.e. in Ω .

(2) *If $\frac{N}{N+2-p(N-2)} \leq m < \frac{N}{2}$, the solutions of (6) are uniformly bounded in $H_0^1(\Omega) \cap L^r(\Omega)$. Thus there exists a function $u \in H_0^1(\Omega) \cap L^r(\Omega)$ such that, up to a subsequence, $u_n \rightarrow u$ weakly in $H_0^1(\Omega)$ and a.e. in Ω .*

(3) *If $\frac{N}{N+1-p(N-1)} < m < \frac{N}{N+2-p(N-2)}$, the solutions of (6) are uniformly bounded in $W_0^{1,s}(\Omega)$. Thus there exists a function $u \in W_0^{1,s}(\Omega)$ such that, up to a subsequence, $u_n \rightarrow u$ weakly in $W_0^{1,s}(\Omega)$ and a.e. in Ω .*

Proof. In problems (6) consider as a test function the same test functions as in [9]. With this choice, the lower order term is non-negative and we can take into account only the term given by the operator. Therefore one can follow the same proofs as in [9] to get the above estimates. \square

Remark 1. Let $p > \theta - 1$. Lemmata 2.3 and 2.4 give a further uniform estimate on u_n than Lemma 2.5. Indeed, if one chooses u_n as a test function in (6), then, by hypothesis (2)

$$\int_{\Omega} |\nabla u_n|^2 \left[\frac{\alpha}{(1+|u_n|)^p} + \frac{B u_n^2}{(u_n + \frac{1}{n})^{\theta+1}} \right] \leq \int_{\Omega} f u_n.$$

If $p > \theta - 1$, the lower order term has a leading role in the left hand side of the previous inequality.

We are going to prove the a.e. convergence of the gradients of u_n . We will follow the same technique as in [8]. Remark that a similar technique was used for elliptic degenerate problems in [1].

Lemma 2.6. *Let u_n be the solutions to problems (6) and u be the function found in Lemmata 2.3, or 2.4 or 2.5, according to the summability of f . Up to a subsequence, ∇u_n converges to ∇u a.e. in Ω .*

Proof. Let $h, k > 0$. In the sequel C will denote a constant independent of n, h, k . Let us consider $T_h(u_n - T_k(u))$ as a test function in problems (6). Then

$$\int_{\Omega} \frac{b(x)}{(1 + T_n(u_n))^p} \nabla u_n \cdot \nabla T_h(u_n - T_k(u)) \leq h \int_{\Omega} f + B \int_{\Omega} \frac{|\nabla u_n|^2 u_n}{(u_n + \frac{1}{n})^{\theta+1}} h.$$

By estimate (7) on the right hand side and by hypothesis (2) on the left one, we get

$$\int_{\Omega} \frac{\nabla u_n \cdot \nabla T_h(u_n - T_k(u))}{(1 + T_n(u_n))^p} \leq Ch.$$

Then we can write

$$\begin{aligned} \int_{\{|u_n - T_k(u)| \leq h\}} \frac{|\nabla(u_n - T_k(u))|^2}{(1 + u_n)^p} &\leq \int_{\Omega} \frac{\nabla(u_n - T_k(u)) \cdot \nabla T_h(u_n - T_k(u))}{(1 + T_n(u_n))^p} \\ &\leq Ch - \int_{\Omega} \frac{\nabla T_k(u) \cdot \nabla T_h(u_n - T_k(u))}{(1 + T_n(u_n))^p}. \end{aligned}$$

At the limit as $n \rightarrow \infty$ one has

$$\limsup_{n \rightarrow \infty} \int_{\{|u_n - T_k(u)| \leq h\}} \frac{|\nabla T_h(u_n - T_k(u))|^2}{(1 + u_n)^p} \leq Ch.$$

Since $u_n \leq h + k$ in $\{|u_n - T_k(u)| \leq h\}$, we get

$$(15) \quad \limsup_{n \rightarrow \infty} \int_{\{|u_n - T_k(u)| \leq h\}} |\nabla T_h(u_n - T_k(u))|^2 \leq Ch(1 + h + k)^p.$$

We recall that u_n is uniformly bounded in $W_0^{1,\eta}(\Omega)$, where η equals 2 or σ or s , according to the statements of Lemmata 2.3, 2.4 and 2.5. Let $q \in (1, \eta)$. We can write

$$\begin{aligned} \int_{\Omega} |\nabla(u_n - u)|^q &= \\ \int_{\{|u_n - u| \leq h, |u| \leq k\}} |\nabla(u_n - u)|^q &+ \int_{\{|u_n - u| \leq h, |u| > k\}} |\nabla(u_n - u)|^q + \int_{\{|u_n - u| > h\}} |\nabla(u_n - u)|^q. \end{aligned}$$

Using the Hölder inequality with exponent $\frac{2}{q}$ on the first term of the right hand side and exponent $\frac{q}{q-1}$ on the other ones, we have

$$\begin{aligned} \int_{\Omega} |\nabla(u_n - u)|^q &\leq \\ \leq C \left[\int_{\{|u_n - u| \leq h, |u| \leq k\}} |\nabla(u_n - u)|^2 \right]^{\frac{q}{2}} &+ C \left[\mu(\{|u| > k\})^{1-\frac{q}{\eta}} + \mu(\{|u_n - u| > h\})^{1-\frac{q}{\eta}} \right], \end{aligned}$$

where we have used that u_n is uniformly bounded in $W_0^{1,\eta}(\Omega)$ to estimate the last two terms. By (15) the limit as $n \rightarrow \infty$ gives

$$\limsup_{n \rightarrow \infty} \int_{\Omega} |\nabla(u_n - u)|^q \leq [Ch(1 + k + h)^p]^{\frac{q}{2}} + C\mu(\{|u| > k\})^{1-\frac{q}{\eta}}.$$

The limit as $h \rightarrow 0$ implies

$$\limsup_{n \rightarrow \infty} \int_{\Omega} |\nabla(u_n - u)|^q \leq C \mu(\{|u| > k\})^{1 - \frac{q}{n}}.$$

At the limit as $k \rightarrow +\infty$, $\mu(\{|u| > k\})$ converges to 0. Therefore $\nabla u_n \rightarrow \nabla u$ in $L^q(\Omega)$. Up to a subsequence, $\nabla u_n \rightarrow \nabla u$ a.e. in Ω . \square

3. EXISTENCE RESULTS IN THE CASE $0 < \theta < 1$

To prove the existence of solutions to problem (1), the key point is to prove that the function u found by compactness in the lemmata of Section 2 is strictly positive. In the case $0 < \theta < 1$, we use a technique similar to that in [8].

Proposition 1. *Let $0 < \theta < 1$. Let u_n and u be as in Lemma 2.6. Then $u > 0$.*

Proof. We define, for $s \geq 0$,

$$H_n(s) = \int_0^s \frac{t(1 + T_n(t))^p}{\alpha(t + \frac{1}{n})^{\theta+1}} dt, \quad H(s) = \int_0^s \frac{(1+t)^p}{\alpha t^\theta} dt.$$

Observe that H is well-defined, since $\theta < 1$. We choose $e^{-BH_n(u_n)}\phi$, where ϕ is a positive $C_0^\infty(\Omega)$ function, as a test function in (6). This gives

$$\begin{aligned} & \int_{\Omega} \frac{b(x)}{(1 + T_n(u_n))^p} \nabla u_n \cdot \nabla \phi e^{-BH_n(u_n)} - \int_{\Omega} T_n(f) e^{-BH_n(u_n)} \phi = \\ & = B \int_{\Omega} \frac{b(x)}{(1 + T_n(u_n))^p} e^{-BH_n(u_n)} |\nabla u_n|^2 \phi H'_n(u_n) - B \int_{\Omega} \frac{e^{-BH_n(u_n)} |\nabla u_n|^2 u_n}{(\frac{1}{n} + u_n)^{\theta+1}} \phi \\ & \geq B \int_{\Omega} \frac{\alpha}{(1 + T_n(u_n))^p} e^{-BH_n(u_n)} |\nabla u_n|^2 \phi H'_n(u_n) - B \int_{\Omega} \frac{e^{-BH_n(u_n)} |\nabla u_n|^2 u_n}{(\frac{1}{n} + u_n)^{\theta+1}} \phi \end{aligned}$$

by hypothesis (2). The last quantity is positive, due to the choice of H_n and ϕ . As a consequence

$$\int_{\Omega} \frac{b(x)}{(1 + T_n(u_n))^p} \nabla u_n \cdot \nabla \phi e^{-BH_n(u_n)} \geq \int_{\Omega} T_n(f) e^{-BH_n(u_n)} \phi \geq \int_{\Omega} T_1(f) e^{-BH_n(u_n)} \phi.$$

Now, we set

$$P_n(s) = \int_0^s \frac{e^{-BH_n(t)}}{(1 + T_n(t))^p} dt, \quad P(s) = \int_0^s \frac{e^{-BH(t)}}{(1+t)^p} dt.$$

With these definitions, we remark that we have just proved that the inequality

$$-\operatorname{div}(b(x)\nabla(P_n(u_n))) \geq T_1(f) e^{-BH_n(u_n)}$$

holds distributionally. Observe that for every $n \in \mathbb{N}$, $P_n(u_n) \in H_0^1(\Omega)$, since P'_n is bounded and $u_n \in H_0^1(\Omega)$. Let z_n be the $H_0^1(\Omega)$ solution to

$$-\operatorname{div}(b(x)\nabla z_n) = T_1(f) e^{-BH_n(u_n)};$$

let z be the $H_0^1(\Omega)$ solution to

$$-\operatorname{div}(b(x)\nabla z) = T_1(f) e^{-BH(u)}.$$

Then

$$-\operatorname{div}(b(x)\nabla(P_n(u_n))) \geq -\operatorname{div}(b(x)\nabla z_n).$$

The comparison principle in $H_0^1(\Omega)$ implies that $P_n(u_n(x)) \geq z_n(x)$ for a.e. $x \in \Omega$. Up to a subsequence, $z_n \rightarrow z$ weakly in $H_0^1(\Omega)$ and a.e. in Ω . At the limit a.e. in Ω , as $n \rightarrow +\infty$, we have $P(u) \geq z$. By the strong maximum principle $z > 0$ and so $P(u) > 0$. Since P is strictly increasing, $u > 0$ in Ω . \square

Corollary 1. *Let $0 < \theta < 1$. Let u_n and u be as in Lemma 2.6. Then $\frac{|\nabla u|^2}{u^\theta} \in L^1(\Omega)$.*

Proof. We pass to the limit in (7). The a.e. convergence of u_n to u (see Lemmata 2.3, 2.4 and 2.5), the a.e. convergence of ∇u_n to ∇u (see Lemma 2.6) and Proposition 1 imply

$$B \int_{\Omega} \frac{|\nabla u|^2}{u^\theta} \leq \int_{\Omega} f$$

by Fatou's lemma. \square

We are going to prove Theorem 1.1.

Proof. We are going to prove that the function u found in Lemma 2.3, and studied in Lemma 2.6, Proposition 1 and Corollary 1, is a weak solution to problem (1). We use the same technique as in [8].

We will prove that (4) holds true for every positive and bounded $\varphi \in H_0^1(\Omega)$. The general case follows from the fact that every such function φ can be written as $\varphi_+ - \varphi_-$ with φ_{\pm} bounded, positive and belonging to $H_0^1(\Omega)$.

We pass to the limit as $n \rightarrow \infty$ in

$$\int_{\Omega} \frac{b(x)}{(1 + T_n(u_n))^p} \nabla u_n \cdot \nabla \varphi + B \int_{\Omega} \frac{|\nabla u_n|^2 u_n}{(u_n + \frac{1}{n})^{1+\theta}} \varphi = \int_{\Omega} T_n(f) \varphi,$$

where φ is a positive bounded $H_0^1(\Omega)$ function. Regarding the first term we observe that $\frac{b(x)}{(1 + T_n(u_n))^p} \nabla \varphi$ strongly converges to $\frac{b(x)}{(1 + u)^p} \nabla \varphi$ in $L^2(\Omega)$ and ∇u_n weakly converges to ∇u in $L^2(\Omega)$. For the second one we use the a.e. convergence of ∇u_n , proved in Lemma 2.6. Fatou's lemma implies

$$(16) \quad \int_{\Omega} \frac{b(x)}{(1 + u)^p} \nabla u \cdot \nabla \varphi + B \int_{\Omega} \frac{|\nabla u|^2}{u^\theta} \varphi \leq \int_{\Omega} f \varphi.$$

The proof of the opposite inequality is more delicate. To this aim, we define, for $n \in \mathbb{N}$ and $s \geq 0$,

$$H_{\frac{1}{n}}(t) = \int_0^t \frac{B(1+s)^p}{\alpha(s + \frac{1}{n})^\theta} ds, \quad H_0(t) = \int_0^t \frac{B(1+s)^p}{\alpha s^\theta} ds.$$

H_0 is well-posed, since $\theta < 1$. Let us consider

$$v = e^{-H_{\frac{1}{n}}(u_n)} e^{H_{\frac{1}{j}}(T_j(u))} \varphi,$$

where $j \in \mathbb{N}$ and φ is a positive bounded $H_0^1(\Omega)$ function, as a test function in (6). Then

$$\int_{\Omega} \frac{b(x)}{(1 + T_n(u_n))^p} \nabla u_n \cdot \nabla \varphi e^{-H_{\frac{1}{n}}(u_n)} e^{H_{\frac{1}{j}}(T_j(u))}$$

$$\begin{aligned}
 & + \frac{B}{\alpha} \int_{\Omega} \frac{b(x)}{(1+T_n(u_n))^p} \varphi e^{-H_{\frac{1}{n}}(u_n)} e^{H_{\frac{1}{j}}(T_j(u))} \frac{\nabla u_n \cdot \nabla T_j(u)}{(T_j(u) + \frac{1}{j})^\theta} (T_j(u) + 1)^p \\
 = & \int_{\Omega} T_n(f) e^{-H_{\frac{1}{n}}(u_n)} e^{H_{\frac{1}{j}}(T_j(u))} \varphi + \frac{B}{\alpha} \int_{\Omega} \frac{b(x) |\nabla u_n|^2}{(1+T_n(u_n))^p} \frac{(1+u_n)^p}{(\frac{1}{n} + u_n)^\theta} \varphi e^{-H_{\frac{1}{n}}(u_n)} e^{H_{\frac{1}{j}}(T_j(u))} \\
 & - B \int_{\Omega} \frac{|\nabla u_n|^2 u_n}{(\frac{1}{n} + u_n)^{\theta+1}} e^{-H_{\frac{1}{n}}(u_n)} e^{H_{\frac{1}{j}}(T_j(u))} \varphi.
 \end{aligned}$$

Note that by hypothesis (2) and inequality

$$\left(\frac{u_n + 1}{1 + T_n(u_n)} \right)^p \geq 1 > \frac{u_n}{u_n + \frac{1}{n}},$$

the sum of the last two terms is non-negative. At the limit as $n \rightarrow \infty$ we have

$$\begin{aligned}
 & \int_{\Omega} \frac{b(x)}{(1+u)^p} \nabla u \cdot \nabla \varphi e^{-H_0(u)} e^{H_{\frac{1}{j}}(T_j(u))} \\
 & + \frac{B}{\alpha} \int_{\Omega} \frac{b(x)}{(1+u)^p} \varphi e^{-H_0(u)} e^{H_{\frac{1}{j}}(T_j(u))} \frac{\nabla u \cdot \nabla T_j(u)}{(T_j(u) + \frac{1}{j})^\theta} (T_j(u) + 1)^p \\
 \geq & \int_{\Omega} f e^{-H_0(u)} e^{H_{\frac{1}{j}}(T_j(u))} \varphi + \frac{B}{\alpha} \int_{\Omega} \frac{b(x) |\nabla u|^2}{u^\theta} \varphi e^{-H_0(u)} e^{H_{\frac{1}{j}}(T_j(u))} \\
 & - B \int_{\Omega} \frac{|\nabla u|^2}{u^\theta} e^{-H_0(u)} e^{H_{\frac{1}{j}}(T_j(u))} \varphi,
 \end{aligned}$$

using the weak convergence of u_n to u in $H_0^1(\Omega)$ in the left hand side and Fatou's lemma in the right one. Now we pass to the limit as $j \rightarrow \infty$, using that $e^{-H_0(u)} e^{H_{\frac{1}{j}}(T_j(u))} \leq 1$ and Corollary 1. We obtain

$$(17) \quad \int_{\Omega} \frac{b(x)}{(1+u)^p} \nabla u \cdot \nabla \varphi \geq \int_{\Omega} f \varphi - B \int_{\Omega} \varphi \frac{|\nabla u|^2}{u^\theta}.$$

Inequalities (16) and (17) imply that

$$\int_{\Omega} \frac{b(x)}{(1+u)^p} \nabla u \cdot \nabla \varphi + B \int_{\Omega} \varphi \frac{|\nabla u|^2}{u^\theta} = \int_{\Omega} f \varphi$$

for every positive and bounded $\varphi \in H_0^1(\Omega)$. □

We are going to prove Theorem 1.2.

Proof. We are going to prove that the function u found in Lemma 2.4 and studied in Lemma 2.6, Proposition 1 and Corollary 1, is a weak solution to problem (1). We use the same technique as in [11, 21].

We first prove (5) for every positive $C_0^1(\Omega)$ function φ . With the same argument as in the previous theorem (i.e., using Fatou's lemma) one can prove that

$$(18) \quad \int_{\Omega} \frac{b(x)}{(1+u)^p} \nabla u \cdot \nabla \varphi + B \int_{\Omega} \frac{|\nabla u|^2}{u^\theta} \varphi \leq \int_{\Omega} f \varphi.$$

To prove the opposite inequality, we slightly modify the previous proof, since we no longer have uniform estimates of u_n in $H_0^1(\Omega)$. Observe that, however, $T_k(u_n)$ is uniformly bounded in $H_0^1(\Omega)$. Indeed, it is sufficient to consider $T_k(u_n)$ as a test function in (6): we obtain

$$\int_{\{u_n \leq k\}} |\nabla T_k(u_n)|^2 \leq Ck(1+k)^p \quad \forall n \in \mathbb{N}$$

by hypothesis (2). We will use, for $k \in \mathbb{N}$ and $s \in \mathbb{R}$

$$R_k(s) = \begin{cases} 1, & s \leq k \\ k+1-s, & k \leq s \leq k+1 \\ 0, & s > k+1, \end{cases}$$

to define a test function. We set, for $t \geq 0$,

$$H_{\frac{1}{n}}(t) = \int_0^t \frac{B(1+s)^p}{\alpha(s+\frac{1}{n})^\theta} ds, \quad H_0(t) = \int_0^t \frac{B(1+s)^p}{\alpha s^\theta} ds.$$

This is possible, since $\theta < 1$. We consider

$$v = e^{-H_{\frac{1}{n}}(u_n)} e^{H_{\frac{1}{j}}(T_j(u))} R_k(u_n) \varphi,$$

where φ is a positive $C_0^1(\Omega)$ function and $j \in N$, as a test function in (6). Then

$$\begin{aligned} & \int_{\Omega} \frac{b(x)}{(1+T_n(u_n))^p} \nabla u_n \cdot \nabla \varphi e^{-H_{\frac{1}{n}}(u_n)} e^{H_{\frac{1}{j}}(T_j(u))} R_k(u_n) \\ & + \frac{B}{\alpha} \int_{\Omega} \frac{b(x)}{(1+T_n(u_n))^p} \varphi e^{-H_{\frac{1}{n}}(u_n)} e^{H_{\frac{1}{j}}(T_j(u))} \frac{\nabla u_n \cdot \nabla T_j(u)}{(T_j(u) + \frac{1}{j})^\theta} (T_j(u) + 1)^p R_k(u_n) \\ = & \int_{\Omega} T_n(f) e^{-H_{\frac{1}{n}}(u_n)} e^{H_{\frac{1}{j}}(T_j(u))} \varphi R_k(u_n) + \int_{\{k \leq u_n \leq k+1\}} \frac{b(x) |\nabla u_n|^2}{(1+T_n(u_n))^p} \varphi e^{-H_{\frac{1}{n}}(u_n)} e^{H_{\frac{1}{j}}(T_j(u))} \\ & + \frac{B}{\alpha} \int_{\Omega} \frac{b(x) |\nabla u_n|^2}{(1+T_n(u_n))^p} \frac{(1+u_n)^p}{(\frac{1}{n}+u_n)^\theta} \varphi e^{-H_{\frac{1}{n}}(u_n)} e^{H_{\frac{1}{j}}(T_j(u))} R_k(u_n) \\ & - B \int_{\Omega} \frac{|\nabla u_n|^2 u_n}{(\frac{1}{n}+u_n)^{\theta+1}} e^{-H_{\frac{1}{n}}(u_n)} e^{H_{\frac{1}{j}}(T_j(u))} R_k(u_n) \varphi. \end{aligned}$$

The sum of the last two terms is positive, since $b(x) \geq \alpha$ by hypothesis (2) and by inequality

$$\left(\frac{u_n + 1}{1 + T_n(u_n)} \right)^p \geq 1 \geq \frac{u_n}{u_n + \frac{1}{n}}.$$

Dropping the non-negative term $\int_{\{k \leq u_n \leq k+1\}} \frac{b(x) |\nabla u_n|^2}{(1+T_n(u_n))^p} \varphi e^{-H_{\frac{1}{n}}(u_n)} e^{H_{\frac{1}{j}}(T_j(u))}$,

at the limit as $n \rightarrow \infty$ we have, by Fatou's lemma, the weak convergence of u_n in $W_0^{1,\sigma}(\Omega)$ and the weak convergence of $T_k(u_n)$ in $H_0^1(\Omega)$,

$$\int_{\Omega} \frac{b(x)}{(1+u)^p} \nabla u \cdot \nabla \varphi e^{-H_0(u)} e^{H_{\frac{1}{j}}(T_j(u))} R_k(u) +$$

$$\begin{aligned}
 & + \frac{B}{\alpha} \int_{\Omega} \frac{b(x)}{(1+u)^p} \varphi e^{-H_0(u)} e^{H_{\frac{1}{j}}(T_j(u))} \frac{\nabla u \cdot \nabla T_j(u)}{(T_j(u) + \frac{1}{j})^\theta} (T_j(u) + 1)^p R_k(u) \\
 \geq & \int_{\Omega} f e^{-H_0(u)} e^{H_{\frac{1}{j}}(T_j(u))} \varphi R_k(u) + \frac{B}{\alpha} \int_{\Omega} \frac{b(x)|\nabla u|^2}{u^\theta} \varphi e^{-H_0(u)} e^{H_{\frac{1}{j}}(T_j(u))} R_k(u) \\
 & - B \int_{\Omega} \frac{|\nabla u|^2}{u^\theta} e^{-H_0(u)} e^{H_{\frac{1}{j}}(T_j(u))} R_k(u) \varphi.
 \end{aligned}$$

As in the previous proof, it is now sufficient to pass to the limit as $j \rightarrow \infty$ first, using that $e^{-H_0(u)} e^{H_{\frac{1}{j}}(T_j(u))} \leq 1$ and Corollary 1, and then to the limit as $k \rightarrow \infty$, using that $R_k(u)$ tends to 1. We thus obtain

$$(19) \quad \int_{\Omega} \frac{b(x)}{(1+u)^p} \nabla u \cdot \nabla \varphi \geq \int_{\Omega} f \varphi - B \int_{\Omega} \frac{|\nabla u|^2}{u^\theta} \varphi.$$

Inequalities (18) and (19) imply that

$$(20) \quad \int_{\Omega} \frac{b(x)}{(1+u)^p} \nabla u \cdot \nabla \varphi + B \int_{\Omega} \frac{|\nabla u|^2}{u^\theta} \varphi = \int_{\Omega} f \varphi$$

for every positive $\varphi \in C_0^1(\Omega)$. Now, let φ any $C_0^1(\Omega)$ function. We define $\varphi_{\pm}^\varepsilon = \rho^\varepsilon * \varphi_{\pm}$ as the convolution of a mollifier ρ^ε , for $\varepsilon > 0$, with φ_{\pm} . Then $\varphi_{\pm}^\varepsilon$ is a positive $C_0^1(\Omega)$ function, for ε sufficiently small. By (20) we have

$$\int_{\Omega} \frac{b(x)}{(1+u)^p} \nabla u \cdot \nabla (\varphi_-^\varepsilon - \varphi_+^\varepsilon) + B \int_{\Omega} \frac{|\nabla u|^2}{u^\theta} (\varphi_-^\varepsilon - \varphi_+^\varepsilon) = \int_{\Omega} f (\varphi_-^\varepsilon - \varphi_+^\varepsilon).$$

Since $\varphi_-^\varepsilon - \varphi_+^\varepsilon \rightarrow \varphi$ uniformly in Ω and in $W_0^{1,q}(\Omega)$ for every $q \geq 1$, as $\varepsilon \rightarrow 0$, the result follows. \square

4. EXISTENCE RESULTS IN THE CASE $1 \leq \theta < 2$

As in the above case, we need to prove that the function u found in Section 2 is not 0 in Ω . To this aim, we are going to prove that for every $\omega \subset\subset \Omega$ there exists a positive constant c_ω such that the solutions u_n to problems (6) satisfy $u_n \geq c_\omega$ in ω for every $n \in \mathbb{N}$. We will follow a similar technique to that one in [5]. The following theorem, proved in [18] (and in [5]), will be useful to us.

Theorem 4.1. *Let $B : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function such that for every $\omega \subset\subset \Omega$ there exists $m_\omega > 0$ such that $B(x, s) \geq m_\omega l(s)$ for a.e. $x \in \Omega$ and for every $s \geq 0$. Assume that $l : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous increasing function such that $l(s)/s$ is increasing for s sufficiently large and for some $t_0 > 0$*

$$(21) \quad \int_{t_0}^{+\infty} \frac{dt}{\sqrt{\int_0^t l(s) ds}} < +\infty.$$

Then for every $\omega \subset\subset \Omega$ there exists a constant $C_\omega > 0$ such that every sub-solution $v \in H_{loc}^1(\Omega)$ of $-\operatorname{div}(b(x)\nabla v) + B(x, v) = 0$ such that $v^+ \in L_{loc}^\infty(\Omega)$ and $B(x, v^+) \in L_{loc}^1(\Omega)$ satisfies $v \leq C_\omega$ in ω .

Remark 2. We recall that a sub-solution of $-\operatorname{div}(b(x)\nabla v) + l(v)g(x) = 0$ is a $W_{loc}^{1,1}(\Omega)$ function such that

$$\int_{\Omega} b(x)\nabla v \cdot \nabla \phi + \int_{\Omega} l(v)g(x)\phi \leq 0$$

for every $C_c^\infty(\Omega)$ positive function ϕ .

Remark 3. In the literature condition (21) is called the Keller-Osserman condition, due to the papers [17, 19] on semilinear equations.

Proposition 2. *Let $1 \leq \theta < 2$. Let u_n be the solutions of (6). Then for every $\omega \subset \subset \Omega$ there exists a strictly positive constant c_ω such that $u_n \geq c_\omega$ in ω for every $n \in \mathbb{N}$.*

Proof. Step 1. Let u_n be a $H_0^1(\Omega) \cap L^\infty(\Omega)$ solution to (6). We perform a change of variable in order to get a sub-solution of an elliptic semi-linear problem, as in Theorem 4.1.

We set $a_n(s) = \frac{1}{(1 + T_n(s))^p}$. Then u_n satisfies, distributionally,

$$-\operatorname{div}(b(x)a_n(u_n)\nabla u_n) + \frac{B}{u_n^\theta}|\nabla u_n|^2 \geq T_1(f),$$

that is,

$$(22) \quad -\operatorname{div}(b(x)\nabla u_n)a_n(u_n) - a_n'(u_n)b(x)|\nabla u_n|^2 + \frac{B}{u_n^\theta}|\nabla u_n|^2 \geq T_1(f).$$

Let $k_n(t) = \int_1^t \frac{B}{\alpha r^\theta a_n(r)} dr$ and $\psi_n(s) = \int_s^1 e^{-k_n(t)} a_n^{\frac{\beta}{\alpha}}(t) dt$. We remark that

$$(23) \quad \psi_n'(s) = -a_n^{\frac{\beta}{\alpha}}(s)e^{-k_n(s)}, \quad \frac{\psi_n''(s)}{\psi_n'(s)} = \frac{\beta a_n'(s)}{\alpha a_n(s)} - \frac{B}{\alpha s^\theta a_n(s)}.$$

We define $v_n = \psi_n(u_n)$. Then

$$\operatorname{div}(b(x)\nabla v_n) = \operatorname{div}(b(x)\psi_n'(u_n)\nabla u_n) = \psi_n'(u_n)\operatorname{div}(b(x)\nabla u_n) + b(x)\psi_n''(u_n)|\nabla u_n|^2$$

and therefore

$$-a_n(u_n)\operatorname{div}(b(x)\nabla u_n) = -a_n(u_n)\frac{\operatorname{div}(b(x)\nabla v_n)}{\psi_n'(u_n)} + a_n(u_n)b(x)\frac{\psi_n''(u_n)}{\psi_n'(u_n)}|\nabla u_n|^2.$$

By inequality (22) we have

$$T_1(f) \leq -a_n(u_n)\frac{\operatorname{div}(b(x)\nabla v_n)}{\psi_n'(u_n)} + a_n(u_n)b(x)\frac{\psi_n''(u_n)}{\psi_n'(u_n)}|\nabla u_n|^2 - a_n'(u_n)b(x)|\nabla u_n|^2 + \frac{B}{u_n^\theta}|\nabla u_n|^2.$$

Using that $a_n'(s) \leq 0$, $\frac{\psi_n''(s)}{\psi_n'(s)} \leq 0$ and hypothesis (2) we obtain

$$T_1(f) \leq -a_n(u_n)\frac{\operatorname{div}(b(x)\nabla v_n)}{\psi_n'(u_n)} + a_n(u_n)\alpha\frac{\psi_n''(u_n)}{\psi_n'(u_n)}|\nabla u_n|^2 - a_n'(u_n)\beta|\nabla u_n|^2 + \frac{B}{u_n^\theta}|\nabla u_n|^2.$$

Due to (23)

$$T_1(f) \leq -a_n(u_n)\frac{\operatorname{div}(b(x)\nabla v_n)}{\psi_n'(u_n)}.$$

Observing that $\psi'_n(s) = -a_n^{\frac{\beta}{\alpha}}(s)e^{-k_n(s)} \leq 0$, v_n satisfies

$$0 \geq -\operatorname{div}(b(x)\nabla v_n) + T_1(f)e^{-k_n(\psi_n^{-1}(v_n))}a_n^{\frac{\beta}{\alpha}-1}(\psi_n^{-1}(v_n)).$$

Step 2. We now study, for $s \geq 0$

$$s \rightarrow e^{-k_n(\psi_n^{-1}(s))}a_n^{\frac{\beta}{\alpha}-1}(\psi_n^{-1}(s)).$$

We remark that $\psi_n^{-1}(s) \leq 1$, since $s \geq 0 = \psi_n(1)$ and ψ_n is decreasing. Therefore

$$(24) \quad a_n^{\frac{\beta}{\alpha}-1}(\psi_n^{-1}(s)) \geq a_n^{\frac{\beta}{\alpha}-1}(1) = a_0,$$

as a_n is decreasing.

Recalling that

$$\psi_n(s) = \int_s^1 e^{-k_n(t)}a_n^{\frac{\beta}{\alpha}}(t)dt, \quad k_n(t) = \int_1^t \frac{B}{\alpha r^\theta a_n(r)}dr$$

and

$$\begin{cases} a_n(s) = a_1(s), & s \leq 1 \\ a_n(s) \leq a_1(s), & s > 1 \end{cases}$$

it is not difficult to prove that

$$(25) \quad \psi_n(s) \geq \psi_1(s),$$

distinguishing the cases $s \leq 1$ and $s > 1$. Now, inequality (25) and the fact that ψ_n is decreasing imply that $\psi_n^{-1}(s) \leq \psi_1^{-1}(s)$ for every $s \geq 0$. Recalling that $\psi_n^{-1}(s) \leq 1$ and $a_n(s) = a_1(s) \geq 0$ for $s \geq 1$, we deduce easily that

$$(26) \quad e^{-k_n(\psi_n^{-1}(s))} \geq e^{-k_1(\psi_1^{-1}(s))}.$$

Due to (24) and (26), v_n satisfies

$$0 \geq -\operatorname{div}(b(x)\nabla v_n) + B(x, v_n)$$

with

$$B(x, s) = \begin{cases} T_1(f)a_0 l(s), & s \geq 0 \\ 0, & s \leq 0, \end{cases}$$

where $l(s) = e^{-k_1(\psi_1^{-1}(s))} - 1$, $s \geq 0$.

Step 3. We are going to prove that l satisfies the hypotheses of Theorem 4.1.

We observe that l is continuous and increasing, since ψ_1^{-1} is decreasing and k_1 is increasing. We claim that $l(s)/s$ is increasing for s sufficiently large. This is equivalent to prove that $Y(t) = \frac{l(\psi_1(t))}{\psi_1(t)}$ is decreasing for small positive t . Now, $Y'(t) < 0$ if and only if

$$(27) \quad l'(\psi_1(t))\psi_1(t) - \int_1^t l'(\psi_1(s))\psi_1'(s)ds > 0.$$

We remark that $l'(\psi_1(s)) = \frac{B}{\alpha s^\theta a_1^{\frac{\beta}{\alpha}+1}(s)}$. Let $w_0 \in (0, 1)$ be such that $h(t) =$

$l'(\psi_1(t)) = \frac{B}{\alpha t^\theta a_1^{\frac{\beta}{\alpha}+1}(t)}$ is decreasing in $(0, w_0]$. Therefore

$$l'(\psi_1(t))\psi_1(t) - \int_1^t l'(\psi_1(s))\psi_1'(s)ds = \int_t^1 e^{-k_1(s)}a_1^{\frac{\beta}{\alpha}}(s) [h(t) - h(s)] ds$$

$$\geq \int_{w_0}^1 e^{-k_1(s)} a_1^{\frac{\beta}{\alpha}}(s) [h(t) - h(s)] ds$$

due to the choice of w_0 . Let

$$M_1 = \int_{w_0}^1 e^{-k_1(s)} a_1^{\frac{\beta}{\alpha}}(s) ds, \quad M_2 = \int_{w_0}^1 e^{-k_1(s)} a_1^{\frac{\beta}{\alpha}}(s) h(s) ds.$$

We have proved that

$$l'(\psi_1(t))\psi_1(t) - \int_1^t l'(\psi_1(s))\psi_1'(s) ds \geq M_1 h(t) - M_2.$$

If t is sufficiently small, the last quantity is positive, since h is decreasing for small positive t . Therefore (27) holds.

We are going to study the last condition on l , that is, the existence of a positive t_0 such that

$$(28) \quad \int_{t_0}^{+\infty} \frac{dt}{\sqrt{\int_0^t l(s) ds}} < \infty.$$

Using the change of variable $\tau = \psi_1^{-1}(s)$ we get

$$\int_0^t l(s) ds = \int_0^{\psi_1^{-1}(t)} [e^{-k_1(\psi_1^{-1}(s))} - 1] ds = \int_{\psi_1^{-1}(t)}^1 [e^{-k_1(\tau)} - 1] a_1^{\frac{\beta}{\alpha}}(\tau) e^{-k_1(\tau)} d\tau.$$

It is easy to see that $e^{-k_1(\tau)} - 1 \geq \frac{1}{2}e^{-k_1(\tau)}$ for $\tau \leq \tau_0$ sufficiently small. Moreover $a_1(\tau) \geq \frac{1}{2}$, for $\tau \leq 1$. Therefore it suffices to find t_0 sufficiently large ($t_0 > \psi_1(\tau_0)$) such that

$$\int_{t_0}^{+\infty} \frac{dt}{\sqrt{\int_{\psi_1^{-1}(t)}^1 e^{-2k_1(\tau)} d\tau}} < \infty.$$

The last integral can be estimated, using the change $w = \psi_1^{-1}(t)$ and the fact that $a_1(s) \leq 1$, in the following way:

$$\int_{\psi_1^{-1}(t_0)}^0 \frac{\psi_1'(w) dw}{\sqrt{\int_w^1 e^{-2k_1(\tau)} d\tau}} = \int_0^{\psi_1^{-1}(t_0)} \frac{e^{-k_1(w)} a_1^{\frac{\beta}{\alpha}}(w) dw}{\sqrt{\int_w^1 e^{-2k_1(\tau)} d\tau}} \leq \int_0^{\psi_1^{-1}(t_0)} \frac{dw}{\sqrt{\int_w^{w_0} e^{2[k_1(w) - k_1(\tau)]} d\tau}}$$

where w_0 is chosen in such a way that k_1' is decreasing in $(0, w_0]$. We observe that $\int_0^1 \sqrt{k_1'(t)} dt < \infty$, as $\theta < 2$. Hence it suffices to prove that there exists a strictly positive constant c such that

$$k_1'(w) \int_w^{w_0} e^{2[k_1(w) - k_1(\tau)]} d\tau \geq c.$$

Now, since $k_1'(\tau)$ is decreasing in $(0, w_0]$,

$$k_1'(w) \int_w^{w_0} e^{2[k_1(w) - k_1(\tau)]} d\tau \geq \int_w^{w_0} k_1'(\tau) e^{2[k_1(w) - k_1(\tau)]} d\tau = \frac{1}{2} - \frac{1}{2} e^{2[k_1(w) - k_1(w_0)]}.$$

Observe that $e^{2k_1(w)} \rightarrow 0$ as $w \rightarrow 0$, since $k_1(w) = \int_1^w \frac{B}{at^\theta a_1(t)} dt \rightarrow -\infty$ as $w \rightarrow 0$, by hypothesis $\theta \geq 1$. Therefore (28) is proved.

Step 4. Theorem 4.1 applies and gives, for every $\omega \subset\subset \Omega$, the existence of a constant $C_\omega > 0$ such that $v_n \leq C_\omega$. Recalling that $\psi_n(s) \geq \psi_1(s)$ by (25), we have

$C_\omega \geq v_n = \psi_n(u_n) \geq \psi_1(u_n)$. Since ψ_1 is decreasing, $u_n \geq \psi_1^{-1}(C_\omega) = c_\omega > 0$ in every $\omega \subset\subset \Omega$. \square

Corollary 2. *Let $1 \leq \theta < 2$. Let u_n and u be as in Lemma 2.6. Then $\frac{|\nabla u|^2}{u^\theta} \in L^1(\Omega)$.*

Proof. As in the proof of Corollary 1, we pass to the limit in (7) using the a.e. convergence of u_n to u (see Lemmata 2.3, 2.4 and 2.5), the a.e. convergence of ∇u_n to ∇u (see Lemma 2.6) and Proposition 2. \square

Corollary 3. *For every $\omega \subset\subset \Omega$ there exists a positive constant \tilde{c}_ω such that*

$$\frac{u_n}{(u_n + \frac{1}{n})^{1+\theta}} \leq \tilde{c}_\omega \quad \forall x \in \omega.$$

Proof. It is sufficient to observe that in every subset $\omega \subset\subset \Omega$

$$\frac{u_n}{(u_n + \frac{1}{n})^{1+\theta}} \leq \frac{1}{u_n^\theta} \leq \frac{1}{c_\omega^\theta} = \tilde{c}_\omega,$$

since $u_n \geq c_\omega > 0$ in ω by Proposition 2. \square

As in [5] we prove the strong convergence of $T_k(u_n)$ in $H^1_{loc}(\Omega)$. This will be used to compute the limit of the lower order term in problems (6).

Lemma 4.2. *Let u_n be the solutions to problems (6) and u be the function found in Lemmata 2.3, 2.4, 2.5. Then, up to a subsequence, $T_k(u_n) \rightarrow T_k(u)$ in $H^1_{loc}(\Omega)$.*

Proof. We are going to prove that

$$\lim_{n \rightarrow \infty} \int_{\Omega} |\nabla(T_k(u_n) - T_k(u))|^2 \phi = 0$$

for all positive $\phi \in C_c^\infty(\Omega)$. Let $\varphi_\lambda(s) = se^{\lambda s^2}$, $\lambda > 0$. As in [11], we will consider as a test function $\varphi_\lambda(T_k(u_n) - T_k(u))\phi$, where λ will be chosen later. In the sequel $\varepsilon(n)$ will denote any quantity converging to 0, as $n \rightarrow \infty$. From (6) we get

$$\begin{aligned} & \int_{\Omega} \frac{b(x)}{(1 + T_n(u_n))^p} \nabla u_n \cdot \nabla(T_k(u_n) - T_k(u)) \varphi'_\lambda(T_k(u_n) - T_k(u)) \phi \\ & \quad + B \int_{\Omega} \frac{u_n |\nabla u_n|^2}{(u_n + \frac{1}{n})^{\theta+1}} \varphi_\lambda(T_k(u_n) - T_k(u)) \phi \\ & = - \int_{\Omega} \frac{b(x)}{(1 + T_n(u_n))^p} \nabla u_n \cdot \nabla \phi \varphi_\lambda(T_k(u_n) - T_k(u)) + \int_{\Omega} T_n(f) \varphi_\lambda(T_k(u_n) - T_k(u)) \phi. \end{aligned}$$

It is not difficult to prove that

$$\int_{\Omega} T_n(f) \varphi_\lambda(T_k(u_n) - T_k(u)) \phi \rightarrow 0, \quad \int_{\Omega} \frac{b(x)}{(1 + T_n(u_n))^p} \nabla u_n \cdot \nabla \phi \varphi_\lambda(T_k(u_n) - T_k(u)) \rightarrow 0,$$

as $n \rightarrow \infty$. Indeed for the first limit one can use the Lebesgue Theorem. For the second one it is sufficient to observe that ∇u_n converges weakly in some Sobolev space given by the statements of Lemmata 2.3, 2.4 and 2.5 and $\frac{b(x)}{(1 + T_n(u_n))^p} \nabla \phi \varphi_\lambda(T_k(u_n) - T_k(u))$ is uniformly bounded with respect to n .

We are going to treat the left hand side of (29). We choose $\omega_\phi \subset\subset \Omega$, with $\text{supp}\phi \subset \omega_\phi$. Then

$$B \int_{\Omega} \frac{u_n |\nabla u_n|^2}{(u_n + \frac{1}{n})^{\theta+1}} \varphi_\lambda(T_k(u_n) - T_k(u)) \phi \geq -B\tilde{c}_{\omega_\phi} \int_{\Omega} |\nabla T_k(u_n)|^2 |\varphi_\lambda(T_k(u_n) - T_k(u))| \phi$$

by Corollary 3. We deduce from (29) that

$$(30) \quad \int_{\Omega} \frac{b(x)}{(1 + T_n(u_n))^p} \nabla u_n \cdot \nabla(T_k(u_n) - T_k(u)) \varphi'_\lambda(T_k(u_n) - T_k(u)) \phi - B\tilde{c}_{\omega_\phi} \int_{\Omega} |\nabla T_k(u_n)|^2 |\varphi_\lambda(T_k(u_n) - T_k(u))| \phi \leq \varepsilon(n).$$

We remark that

$$\int_{\{u_n \geq k\}} \frac{b(x)}{(1 + T_n(u_n))^p} \nabla u_n \cdot \nabla(T_k(u_n) - T_k(u)) \varphi'_\lambda(T_k(u_n) - T_k(u)) \phi = \varepsilon(n).$$

Hence inequality (30) is equivalent to

$$(31) \quad \int_{\{u_n \leq k\}} \frac{b(x)}{(1 + T_n(u_n))^p} \nabla u_n \cdot \nabla(T_k(u_n) - T_k(u)) \varphi'_\lambda(T_k(u_n) - T_k(u)) \phi - B\tilde{c}_{\omega_\phi} \int_{\Omega} |\nabla T_k(u_n)|^2 |\varphi_\lambda(T_k(u_n) - T_k(u))| \phi \leq \varepsilon(n).$$

Remark that

$$\int_{\{u_n \leq k\}} \frac{b(x)}{(1 + T_n(u_n))^p} \nabla T_k(u) \cdot \nabla(T_k(u_n) - T_k(u)) \varphi'_\lambda(T_k(u_n) - T_k(u)) \phi \rightarrow 0, \quad n \rightarrow \infty.$$

Adding the above quantity in both sides of (31) we get

$$\int_{\{u_n \leq k\}} \frac{b(x)}{(1 + T_n(u_n))^p} \nabla(u_n - T_k(u)) \cdot \nabla(T_k(u_n) - T_k(u)) \varphi'_\lambda(T_k(u_n) - T_k(u)) \phi - B\tilde{c}_{\omega_\phi} \int_{\Omega} |\nabla T_k(u_n)|^2 |\varphi_\lambda(T_k(u_n) - T_k(u))| \phi \leq \varepsilon(n).$$

By hypothesis (2) on b , we obtain

$$(32) \quad \int_{\{u_n \leq k\}} \frac{\alpha}{(1+k)^p} |\nabla(T_k(u_n) - T_k(u))|^2 \varphi'_\lambda(T_k(u_n) - T_k(u)) \phi - B\tilde{c}_{\omega_\phi} \int_{\Omega} |\nabla T_k(u_n)|^2 |\varphi_\lambda(T_k(u_n) - T_k(u))| \phi \leq \varepsilon(n).$$

It is easy to prove that

$$\int_{\Omega} |\nabla T_k(u_n)|^2 |\varphi_\lambda(T_k(u_n) - T_k(u))| \phi \leq \int_{\{u_n \leq k\}} 2|\nabla(T_k(u_n) - T_k(u))|^2 |\varphi_\lambda(T_k(u_n) - T_k(u))| \phi + \varepsilon(n).$$

We deduce from (32) that the quantity

$$(33) \quad \int_{\{u_n \leq k\}} \left[\frac{\alpha}{(1+k)^p} \varphi'_\lambda(T_k(u_n) - T_k(u)) - 2B\tilde{c}_{\omega_\phi} |\varphi_\lambda(T_k(u_n) - T_k(u))| \right] |\nabla(T_k(u_n) - T_k(u))|^2 \phi$$

tends to 0. Now, φ_λ has the following property: for every $a, b > 0$, $a\varphi'_\lambda(s) - b|\varphi_\lambda(s)| \geq \frac{a}{2}$ if $\lambda > \frac{b^2}{4a^2}$. Therefore there exists $\lambda > 0$ such that

$$\frac{\alpha}{(1+k)^p} \varphi'_\lambda(s) - 2B\tilde{c}_{\omega_\phi} |\varphi_\lambda(s)| \geq \frac{\alpha}{2(1+k)^p} \quad \forall s \in \mathbb{R}.$$

Applying this inequality to the quantity (33), the statement of the theorem is proved. \square

We are now going to prove Theorems 1.3 and 1.4 in a unique proof. As we will see the only difference is the choice of the test functions φ . Theorem 1.5 can be proved with the same technique.

Proof. By Lemmata 2.3 and 2.4 the solutions u_n to (6) are uniformly bounded in $H_0^1(\Omega)$ and $W_0^{1,\sigma}(\Omega)$ respectively; moreover ∇u_n converges to ∇u a.e. in Ω up to a subsequence, by Lemma 2.6. The solutions u_n satisfy

$$\int_{\Omega} \frac{b(x)}{(1+T_n(u_n))^p} \nabla u_n \cdot \nabla \varphi + B \int_{\Omega} \frac{|\nabla u_n|^2 u_n}{(u_n + \frac{1}{n})^{1+\theta}} \varphi = \int_{\Omega} T_n(f) \varphi.$$

For the proof of Theorem 1.3 we consider for φ a bounded $H_0^1(\Omega)$ function. For the proof of Theorem 1.4, φ is a $C_0^1(\Omega)$ function. To compute the limit of the first term in the case where u_n weakly converges to u in $H_0^1(\Omega)$ (Theorem 1.3) it is sufficient to use that $\frac{b(x)}{(1+T_n(u_n))^p} \nabla \varphi$ strongly converges to $\frac{b(x)}{(1+u)^p} \nabla \varphi$ in $(L^2(\Omega))^N$ for every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. In the case where u_n weakly converges to u in $W_0^{1,\sigma}(\Omega)$, with $\sigma < 2$ (Theorem 1.4), one uses that $\frac{b(x)}{(1+T_n(u_n))^p} \nabla \varphi$ strongly converges to $\frac{b(x)}{(1+u)^p} \nabla \varphi$ in $(L^r(\Omega))^N$ for every $r \geq 1$ and for every $\varphi \in C_0^1(\Omega)$.

To compute the limit of $\int_{\Omega} \frac{|\nabla u_n|^2 u_n}{(u_n + \frac{1}{n})^{1+\theta}} \varphi$ we will use the same technique as in

[5]. We are going to prove that $\frac{|\nabla u_n|^2 u_n}{(u_n + \frac{1}{n})^{1+\theta}}$ is equi-integrable. Let $E \subset\subset \omega \subset\subset \Omega$.

Then

$$\begin{aligned} \int_E \frac{|\nabla u_n|^2 u_n}{(u_n + \frac{1}{n})^{1+\theta}} &\leq \int_{E \cap \{u_n \leq k\}} \frac{|\nabla u_n|^2 u_n}{(u_n + \frac{1}{n})^{1+\theta}} + \int_{E \cap \{u_n \geq k\}} \frac{|\nabla u_n|^2 u_n}{(u_n + \frac{1}{n})^{1+\theta}} \\ &\leq \tilde{c}_\omega \int_{E \cap \{u_n \leq k\}} |\nabla T_k(u_n)|^2 + \int_{\{u_n \geq k\}} \frac{|\nabla u_n|^2 u_n}{(u_n + \frac{1}{n})^{1+\theta}}, \end{aligned}$$

where we have used Corollary 3 to estimate the first term. Now, if we choose $T_1(u_n - T_{k-1}(u_n))$ in problems (6) we have, dropping the non-negative operator

term,

$$(34) \quad B \int_{\{u_n \geq k\}} \frac{|\nabla u_n|^2 u_n}{(u_n + \frac{1}{n})^{\theta+1}} \leq \int_{\{u_n \geq k-1\}} f.$$

Observe that there exists a constant $C > 0$ such that $\mu(\{u_n \geq k-1\}) \leq \frac{C}{k-1}$, as u_n are uniformly bounded in $L^1(\Omega)$. This implies that the right hand side of (34) converges to 0 as $k \rightarrow \infty$, uniformly with respect to n . We deduce that there exists $k_0 > 1$ such that

$$(35) \quad \int_{\{u_n \geq k\}} \frac{|\nabla u_n|^2 u_n}{(u_n + \frac{1}{n})^{\theta+1}} \leq \frac{\varepsilon}{2} \quad \forall k \geq k_0, \quad \forall n \in \mathbb{N}.$$

Moreover, since $T_k(u_n) \rightarrow T_k(u)$ in $H_{loc}^1(\Omega)$ by Lemma 4.2, there exist $n_\varepsilon, \delta_\varepsilon$ such that for every $E \subset \subset \Omega$ with $\mu(E) < \delta_\varepsilon$ we have

$$\int_{E \cap \{u_n \leq k\}} |\nabla T_k(u_n)|^2 = \int_E |\nabla T_k(u_n)|^2 \leq \frac{\varepsilon}{2\tilde{c}_\omega} \quad \forall n \geq n_\varepsilon.$$

This and (35) imply that $\frac{|\nabla u_n|^2 u_n}{(u_n + \frac{1}{n})^{1+\theta}}$ is equi-integrable. Now, recall that $\frac{|\nabla u_n|^2 u_n}{(u_n + \frac{1}{n})^{1+\theta}}$ converges a.e. to $\frac{|\nabla u|^2}{u^\theta}$, belonging to $L^1(\Omega)$ by Corollary 2. By Vitali's theorem we have the result. \square

5. A NON-EXISTENCE RESULT IN THE CASE $\theta \geq 2$

We are going to prove Theorem 1.6 about the non-existence of finite energy solutions to problem (1) when $\theta \geq 2$. We will use the following result of [5]:

Theorem 5.1. *Let $M(x, s)$ be a $N \times N$ matrix whose entries are Carathéodory functions $m_{ij} : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$, for every $i, j = 1, \dots, N$. Assume that there exist two positive constants α_1, β_1 such that $M(x, s)\xi \cdot \xi \geq \alpha_1 |\xi|^2$ and $|M(x, s)| \leq \beta_1$ for a.e. $x \in \Omega$, and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$. Let $g : \Omega \times (0, +\infty) \rightarrow \mathbb{R}^+$ be a Carathéodory function such that for some constants $s_0, \Lambda > 0$ and $\theta \geq 2$ it holds*

$$g(x, s) \geq \frac{\Lambda}{s^\theta} \quad \forall s \in (0, s_0].$$

Let $f \geq 0$, $f \not\equiv 0$, be a $L^q(\Omega)$ function, with $q > \frac{N}{2}$. If one of the following conditions holds:

- (1) $\theta > 2$
- (2) $\theta = 2$ and $\lambda_1(f) > \frac{\beta_1}{\Lambda \alpha_1}$,

then there is no $H_0^1(\Omega)$ solution to problem

$$\begin{cases} -\operatorname{div}(M(x, u)\nabla u) + g(x, u)|\nabla u|^2 = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Proof. (of Theorem 1.6) By the change of variables

$$v = \begin{cases} \frac{1 - (1+u)^{1-p}}{p-1}, & p \neq 1 \\ \ln(1+u), & p = 1, \end{cases}$$

problem (1) is equivalent to

$$(36) \quad \begin{cases} -\operatorname{div}(b(x)\nabla v) + Bg(v)|\nabla v|^2 = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

with

$$g(s) = \begin{cases} \frac{[1 - (p-1)s]^{\frac{2p}{1-p}}}{([1 - (p-1)s]^{\frac{1}{1-p}} - 1)^\theta}, & p \neq 1 \\ \frac{e^{2s}}{(e^s - 1)^\theta}, & p = 1. \end{cases}$$

It is easy to prove that $g(s)s^\theta \rightarrow 1$, as $s \rightarrow 0^+$. Hence for every fixed $0 < \varepsilon < B$ there exists $s_\varepsilon > 0$ such that $Bg(s) \geq \frac{B-\varepsilon}{s^\theta}$ for every $s \in (0, s_\varepsilon]$. Theorem 5.1 therefore applies to problem (36). We deduce that there is no $H_0^1(\Omega)$ solution to problem (36) if either $\theta > 2$, or $\theta = 2$ and $\lambda_1(f) > \frac{\beta}{(B-\varepsilon)\alpha}$, for every $0 < \varepsilon < B$. As a consequence there is no $H_0^1(\Omega)$ solution to problem (1) if either $\theta > 2$, or $\theta = 2$ and $\lambda_1(f) > \frac{\beta}{B\alpha}$. \square

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