

A SELECTION CRITERION OF SOLUTIONS TO A SYSTEM OF EIKONAL EQUATIONS

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1. INTRODUCTION

In this article we deal with the following system of eikonal equations:

$$(1.1) \quad \begin{cases} \left| \frac{\partial u}{\partial x_i} \right| = 1, & i = 1, \dots, N, \text{ a.e. in } \Omega \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open bounded connected Lipschitz domain of \mathbb{R}^N . System (1.1) arises in several nonlinear models in mechanics and material science and the problem of the existence of solutions has been widely studied. With no attempt to be exhaustive here we recall some results that have motivated our study.

Examining the *pyramidal construction*, introduced in [5] and [13], one can easily see that there exist infinitely many $W_0^{1,\infty}(\Omega)$ solutions of (1.1). Indeed, if $Q \subset \mathbb{R}^N$ is a hyperrectangle oriented in such a way that the exterior normal to its faces is contained in the set

$$E = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : |x_1| = \dots = |x_N|\},$$

then the distance function in the l^1 -norm from the boundary of Q , $d_1(\cdot, \partial Q)$, solves problem (1.1) in Q . In a general domain Ω , Vitali's theorem allows us to cover Ω , up to a Lebesgue measure zero set, by a countable union of domains Q_i as before. Then the function defined by $d_1(\cdot, \partial Q_i)$ in Q_i , $i \in \mathbb{N}$, and 0 elsewhere, is a $W^{1,\infty}(\Omega)$ solution to system (1.1). Since there are infinitely many Vitali's coverings of Ω , problem (1.1) admits infinitely many $W^{1,\infty}(\Omega)$ solutions. As a consequence, it is an interesting question to select and characterize a particular class of solutions or better one single solution.

Let $\mathfrak{S}(\Omega)$ denote the set of the $W^{1,\infty}(\Omega)$ solutions to problem (1.1). One possible strategy to select a specific function in $\mathfrak{S}(\Omega)$ can be developed using the theory of viscosity solution, introduced by Crandall and Lions (see [7, 9]). In this approach one takes advantage of the various useful properties enjoyed by viscosity solutions, such as maximality, uniqueness and explicit formulas. To do that, note that system (1.1) is equivalent to

$$(1.2) \quad \begin{cases} F(Du) = 0, & \text{a.e. in } \Omega \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

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where $F : \mathbb{R}^N \rightarrow \mathbb{R}$ is any continuous function which is zero if and only if $|x_i| = 1$ for every $i = 1, \dots, N$. One therefore investigates the existence of a viscosity solution to problem (1.2) with any suitably chosen F . It is well known that for $N = 1$, $u(x) = d_1(x, \partial\Omega)$ is the unique viscosity solution to problem (1.2). In higher dimensions, some geometrical conditions on the domain come into play. Indeed in [4] and [14] it is proved that there exists a viscosity solution to (1.2) if and only if Ω is a hyperrectangle such that the normals to each face lie in E . In this case, $d_1(\cdot, \partial\Omega)$ is a viscosity solution. Although the previous result is in some sense negative, it was nonetheless put to use in [10] to select a special solution to problem (1.1) (and to more general problems). There the authors construct a Vitali covering of Ω made up of domains Ω_i admitting a viscosity solution and define the relative viscosity solution over each of these sets. The covering is built in a recursive way, with the idea of taking, at every step, the largest possible hyperrectangle.

Another strategy to characterize a class of functions in $\mathfrak{S}(\Omega)$ is to use a variational method, that is, to choose a meaningful functional over $\mathfrak{S}(\Omega)$ and to optimize it. The minimizers or maximizers, supposing they exist, would be selected solutions to problem (1.1). There is an evident difficulty to apply this method, since the set $\mathfrak{S}(\Omega)$ is not convex. For example the natural functionals

$$v \rightarrow \int_{\Omega} |v|^p, \quad p \geq 1,$$

have in general neither a minimizer nor a maximizer over $\mathfrak{S}(\Omega)$. Indeed any minimizing sequence converges to 0, which does not belong to $\mathfrak{S}(\Omega)$, and the limit function of maximizing sequences is plus or minus the distance from $\partial\Omega$ in the L^1 -norm, which usually do not verify (1.1).

In [8] we find the first attempt of selection through a variational method. The authors study numerically a variational problem over the set of non-negative solutions to problem (1.1): they obtain a maximizing sequence for the problem

$$(1.3) \quad \sup \left\{ \int_{\Omega} u, \quad u \geq 0, \quad u \in \mathfrak{S}(\Omega) \right\}$$

through the numerical minimization of

$$\mathcal{J}(u) := - \int_{\Omega} u + \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \frac{\varepsilon}{2} \int_{\Omega} |\Delta u|^2 + \frac{1}{2\varepsilon} \sum_{i=1}^N \int_{\Omega} \left(\left| \frac{\partial u}{\partial x_i} \right| - 1 \right)^2.$$

Unfortunately, as we said above, there is in general no optimal solution to the variational problem (1.3). Nevertheless, according to the seminal idea presented in [8] of selecting "regular solutions", it seems to be natural to minimize in some way the discontinuity set of the gradient of the solutions. With this aim in mind let us consider the functions $v \in \mathfrak{S}(\Omega)$ such that the distributional gradient of $\frac{\partial v}{\partial x_i}$, $i = 1, \dots, N$, has no Cantor part locally, that is, $\frac{\partial v}{\partial x_i}$ is a $SBV_{loc}(\Omega)$ function (see section 2 for further details). If $J_{\frac{\partial v}{\partial x_i}}$ denotes the jump set of $\frac{\partial v}{\partial x_i}$, $i = 1, \dots, N$, one could try to minimize the functional

$$\mathcal{J}_t(v) := \mathcal{H}^t \left(\bigcup_{i=1}^N J_{\frac{\partial v}{\partial x_i}} \right)$$

over

$$\mathcal{E}(\Omega) = \left\{ v \in \mathfrak{S}(\Omega) : \frac{\partial v}{\partial x_i} \in SBV_{loc}(\Omega), i = 1, \dots, N \right\},$$

for some $t \geq N - 1$. As shown in [6], for $t = N - 1$, this variational problem is not well-posed, since $\mathcal{J}_{N-1}(v)$ can be infinite for every $v \in \mathcal{E}(\Omega)$, in general. Indeed the jump set of Dv could have a fractal behaviour near the boundary of Ω . From the other hand, $\mathcal{H}^t \left(J_{\frac{\partial v}{\partial x_i}} \right) = 0$ for every $i = 1, \dots, N$, for every function in $\mathcal{E}(\Omega)$ and for every $t > N - 1$ (see Section 3). The above negative result on the variational problem

$$\inf \left\{ \mathcal{H}^{N-1} \left(\bigcup_{i=1}^N J_{\frac{\partial v}{\partial x_i}} \right), v \in \mathcal{E}(\Omega) \right\}$$

leads the authors of [6] to consider a weighted \mathcal{H}^{N-1} measure of the jumps. Using a given increasing sequence $\Omega_n \subset \Omega$ of polyhedra whose boundary is composed of a finite number of hyperplanes with normals lying in E , they define a C_0 function $h : \Omega \rightarrow \mathbb{R}^+$ and minimize

$$v \rightarrow \sum_{i=1}^N \int_{J_{\frac{\partial v}{\partial x_i}}} h(x) d\mathcal{H}^{N-1}(x)$$

over $\mathcal{E}(\Omega)$. It is obvious that this selection criterion depends on the definition of the function h and in turn, h depends on the particular sequence of sets Ω_n . At the same time it is clear that the candidate functional should depend on some geometric properties of the domain Ω .

Our aim in this paper is to find an "optimal" weighted measure of the jump set of Dv . We propose a family of weights which depends only on the distance from the boundary of Ω and then minimize the corresponding variational problem.

We assume that $\Omega \subset \mathbb{R}^2$ is a *compatible* domain, according to Definition 2.1. Observe that any convex, open, bounded, Lipschitz subset of \mathbb{R}^2 such that $\partial\Omega$ is composed of a finite number of C^1 curves is compatible. A polygon is a compatible domain, as well. This hypothesis is motivated in section 3 through some enlightening examples. Our main result is the following

Theorem 1.1. *Let Ω be a compatible domain of \mathbb{R}^2 . Let $H : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a continuous, increasing function such that*

$$\int_0^1 \frac{H(t)}{t} dt < +\infty.$$

Let

$$\mathcal{F}(v) = \sum_{i=1}^2 \int_{\Omega} H(d_1(x, \partial\Omega)) d \left| D \frac{\partial v}{\partial x_i} \right|.$$

Then the variational problem

$$(1.4) \quad \inf \{ \mathcal{F}(v), v \in \mathcal{E}(\Omega) \}$$

has a solution.

We point out that the most difficult part of the proof of Theorem 1.1 consists in showing that the variational problem (1.4) is well-posed, that is, defining a function $v \in \mathcal{E}(\Omega)$ such that $\mathcal{F}(v)$ is finite. This will be done in Proposition 4.4. In section 3 we sketch the proof of Proposition 4.4 in the special case $\Omega = (0, 1) \times (0, 1)$ where the computations are much easier than in the general case; we motivate there the hypothesis on H . After proving that \mathcal{F} is finite for at least one function in $\mathcal{E}(\Omega)$, we show the existence of a minimizer using the direct methods of the calculus of

variations, as in [6]. We finally remark that the generalization of our result to higher dimensions is not immediate, because in that case Proposition 4.4 is much more complicated. This is why we state our main result in \mathbb{R}^2 . However, we plan to address the problem in higher dimensions in a future work.

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2. NOTATIONS AND PRELIMINARIES

This section is divided into two parts. In the first one we will fix the hypotheses on Ω and the related notations. The second one is devoted to some preliminary results on functions of bounded variation.

Throughout, given a real number t we will use the notation $[t]$ to denote its integer part. Given a continuous function $f : [a, b] \rightarrow \mathbb{R}$, $f \in C^1((a, b))$ with $f'(t) < 0$ for every $t \in (a, b)$, we set

$$T_f := \{(s, t) \in \mathbb{R}^2 : a \leq s \leq b, f(b) \leq t \leq f(s)\}.$$

We will call T_f a *triangular domain*. The class of all triangular domains will be denoted by \mathcal{T} . We will write T instead of T_f if the function f will be clear by the context.

In the sequel a special role is played by the family of lines in the plane with slope ± 1 . We will denote by ℓ_{\pm} the line with slope ± 1 , passing through $(0, 0)$. Let \mathcal{L} be the set of lines parallel to ℓ_+ or ℓ_- . We define

$$\mathcal{S} := \{[x, y] : [x, y] \subset \ell, \ell \in \mathcal{L}\},$$

where $[x, y]$ stands for the segment joining the two points $x = (x_1, x_2)$, $y = (y_1, y_2)$. We will denote by \mathcal{P} the class of bounded Lipschitz domains whose boundary can be written as an at most countable union of segments lying in \mathcal{S} ; $\mathcal{P}^f \subset \mathcal{P}$ will be the class of domains whose boundary can be written as a finite union of segments in \mathcal{S} ; finally, $\mathcal{P}^\infty = \mathcal{P} \setminus \mathcal{P}^f$. For a given set $A \subset \mathbb{R}^2$, with $\overset{\circ}{A}$ we denote the relative interior of A with respect to the topology induced by the euclidean metric on \mathbb{R}^2 . Let

$$\mathcal{R} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \text{ and } \mathcal{V} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

be the $\frac{\pi}{4}$ counterclockwise rotation and the reflexion with respect to the vertical axis respectively. We define

$$\mathfrak{T} = \{\mathcal{R}^{2k+1}(T), \mathcal{V}(\mathcal{R}^{2k+1}(T)), k = 0, 1, 2, 3, T \in \mathcal{T}\}.$$

Let us consider the set $E := \{\nu \in S^1 : |\nu_1| = |\nu_2|\}$. For a given rectifiable curve of class C^1 , $\gamma \subset \mathbb{R}^2$, we denote by $N : \gamma \mapsto S^1$ the Gauss map.

Definition 2.1. *We say that γ is an admissible boundary curve if the set $N^{-1}(E) \subset \gamma$ has a finite number of connected components. We will denote by Γ the class of admissible boundary curves. A bounded connected Lipschitz set $\Omega \subset \mathbb{R}^2$ will be called compatible domain if its boundary $\partial\Omega$ is the union of a finite number of admissible boundary curves.*

We recall that an open bounded set $\Omega \subset \mathbb{R}^2$ is Lipschitz if for every $p \in \partial\Omega$ there exist a radius $r > 0$ and a map $h_p : B_r(p) \rightarrow B_1(0)$ such that h_p is a bijection, h_p, h_p^{-1} are Lipschitz functions, $h_p(\partial\Omega \cap B_r(p)) = Q_0$ and $h_p(\Omega \cap B_r(p)) = Q_+$, where $B_r(p) = \{x \in \mathbb{R}^2 : \|x - p\| \leq r\}$, $Q_0 = \{x = (x_1, x_2) \in B_1(0) : x_2 = 0\}$ and $Q_+ = \{x = (x_1, x_2) \in B_1(0) : x_2 > 0\}$. We remark that if Ω is Lipschitz, then $\mathcal{H}^1(\partial\Omega)$ is finite.

We observe that any domain in \mathcal{P}^f is compatible, since any segment $\sigma \in \mathcal{S}$ is an admissible boundary curve, being $N^{-1}(E) = \overset{\circ}{\sigma}$ connected. Any convex, open, bounded, Lipschitz subset of \mathbb{R}^2 such that $\partial\Omega$ is composed of a finite number of C^1 curves is compatible. A polygon is also a compatible domain.

Remark 2.2. If Ω is a compatible domain, then it can be covered by a polygonal set in \mathcal{P}^f and a finite number of domains in \mathfrak{T} with mutually disjoint interior, i.e.,

$$\bar{\Omega} = \bar{P} \cup \bigcup_{j=1}^N \mathcal{W}_j(T_j),$$

where $P \in \mathcal{P}^f$, $N \in \mathbb{N}$, $T_j \in \mathcal{T}$ and \mathcal{W}_j is a linear transformation of the form \mathcal{R}^{2k_j+1} or $\mathcal{V}(\mathcal{R}^{2k_j+1})$, with $k_j \in \{0, 1, 2, 3\}$.

As already explained in the introduction, in this article we minimize a weighted measure of the discontinuity set of the gradient of the solutions to problem (1.1).

To do that, we will use the definition and some results on the spaces $BV(\Omega)$ and $SBV(\Omega)$ of functions of bounded variation that we recall here. For simplicity we will restrict ourselves to the case where Ω is a subset of \mathbb{R}^2 . We refer to [1, 2, 11, 12] for further details.

Definition 2.3 (*BV, SBV function*). A $BV(\Omega)$ function is an $L^1(\Omega)$ function such that its distributional gradient is a finite Radon measure. A $SBV(\Omega)$ function is a $BV(\Omega)$ function such that its gradient can be decomposed in the following way:

$$\begin{aligned} Dw &= D^a w + D^j w \\ &= \nabla w \mathcal{L}^2 + (w^+ - w^-) \nu_w \mathcal{H}^1 \llcorner J_w = \nabla w \mathcal{L}^2 + [w] \nu_w \mathcal{H}^1 \llcorner J_w \end{aligned}$$

where $D^a w$ is absolutely continuous with respect to the Lebesgue measure \mathcal{L}^2 in \mathbb{R}^2 with density ∇w , $D^j w = [w] \nu_w \mathcal{H}^1 \llcorner J_w$ is the jump part of Dw , \mathcal{H}^1 is the one-dimensional Hausdorff measure, w^+ and w^- denote the upper and lower approximate limits of w , J_w the jump set of w and ν_w its generalized normal.

We shall handle the solutions v of problem (1.1) belonging to

$$\mathcal{E}(\Omega) = \{v \in \mathfrak{S}(\Omega) : v_{x_i} \in SBV_{loc}(\Omega), i = 1, 2\}$$

where v_{x_i} stands for $\frac{\partial v}{\partial x_i}$, $i = 1, 2$. Therefore

$$(2.1) \quad D(v_{x_i}) = 2 \nu_{v_{x_i}} \mathcal{H}^1 \llcorner J_{v_{x_i}} \quad \text{on } \omega \quad \forall i = 1, 2$$

for any open subset $\omega \subset \bar{\omega} \subset \Omega$.

We recall moreover the following result:

Lemma 2.4. *If $P : \mathbb{R}^2 \rightarrow \mathbb{R}$ is the projection $(x_1, x_2) \rightarrow x_2$, and E is a measurable set of \mathbb{R}^2 then*

$$\int_{\mathbb{R}} \mathcal{H}^0(E \cap P^{-1}\{y\}) dy \leq \mathcal{H}^1(E).$$

Proof. It is sufficient to apply Theorem 2.10.25 of [12] with $f = P$, $X = \mathbb{R}^2$, $Y = \mathbb{R}$, $k = 0$ and $m = 1$. \square

We end this section with some results on compactness and lower semi-continuity in the space of functions of bounded variation.

Definition 2.5 (*BV norm*). *The BV norm of a $BV(\Omega)$ function w is defined by*

$$\|w\|_{BV(\Omega)} = \|w\|_{L^1(\Omega)} + |Dw|(\Omega).$$

Definition 2.6 (*weak* convergence in BV*). *Let $(u_n)_n, u \in BV(\Omega)$. We say that $(u_n)_n$ weakly* converges to u in $BV(\Omega)$ if $u_n \rightarrow u$ in $L^1(\Omega)$ and the measures Du_n weakly* converge to the measure Du in $\mathcal{M}(\Omega, \mathbb{R}^2)$, that is,*

$$\lim_{n \rightarrow \infty} \int_{\Omega} \varphi dDu_n = \int_{\Omega} \varphi dDu \quad \forall \varphi \in C_0(\Omega).$$

Theorem 2.7 (*compactness for SBV functions*). *Let Ω be a bounded open subset of \mathbb{R}^2 with $\mathcal{H}^1(\partial\Omega) < +\infty$. Let $(u_n)_n$ be a sequence of functions in $SBV(\Omega)$ and assume that:*

- i) the functions u_n are uniformly bounded in $BV(\Omega)$;*
- ii) the gradients ∇u_n are equi-integrable;*
- iii) there exists a function $f : [0, \infty) \rightarrow [0, \infty]$ such that $f(t)/t \rightarrow \infty$ as $t \rightarrow 0^+$ and*

$$\int_{J_{u_n}} f([u_n]) d\mathcal{H}^1 \leq C < \infty \quad \forall n \in \mathbb{N}.$$

Then there exists a subsequence $(u_{n_k})_k$ and a function $u \in SBV(\Omega)$ such that $u_{n_k} \rightarrow u$ weakly in $BV(\Omega)$, with the Lebesgue and jump parts of the derivatives converging separately, i.e., $D_a u_{n_k} \rightarrow D_a u$ and $D_j u_{n_k} \rightarrow D_j u$ weakly* in $\mathcal{M}(\Omega, \mathbb{R}^2)$.*

Theorem 2.8 (*semicontinuity in BV*). *Let Ω be a bounded open subset of \mathbb{R}^2 . Let $(u_n)_n$ be a sequence of functions in $BV(\Omega)$ such that $u_n \rightarrow u$ weakly* in $BV(\Omega)$. Then*

$$\int_{\Omega} f(x) d|D_j u|(x) \leq \liminf_{n \rightarrow \infty} \int_{\Omega} f(x) d|D_j u_n|(x)$$

for any non-negative continuous function $f : \Omega \rightarrow [0, +\infty[$.

3. STATEMENT OF THE MAIN RESULT AND REMARKS

As already explained in the introduction, our strategy to select some particular or special solutions to (1.1) is based on a variational method, in other words we search for minimizers of a suitable and meaningful functional defined on $\mathcal{E}(\Omega)$. We will motivate the choice of the functional and the conditions imposed on the domain Ω with the aid of some enlightening examples. Our main result is the following

Theorem 3.1. *Let Ω be a compatible domain and $H : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be an increasing, continuous function such that*

$$(3.1) \quad \int_0^1 \frac{H(t)}{t} dt < \infty.$$

Let

$$\mathcal{F}(v) = \sum_{i=1}^2 \int_{\Omega} H(d_1(x, \partial\Omega)) d(|Dv_{x_i}|)(x).$$

Then the variational problem

$$(3.2) \quad \inf\{\mathcal{F}(v), v \in \mathcal{E}(\Omega)\}$$

has a solution.

One could ask why we do not minimize the simpler functionals

$$\mathcal{H}^t (J_{v_{x_1}} \cup J_{v_{x_2}}), \quad t \geq 1$$

over $\mathcal{E}(\Omega)$. To give the answer we will distinguish the cases $t = 1$ and $t > 1$. The first case was partially studied in [6], where it was proved the following theorem for the functional

$$\mathcal{G}_\Omega(v) := \sum_{i=1}^2 \int_{\Omega} d(|Dv_{x_i}|)(x).$$

Theorem 3.2. *Let $P \in \mathcal{P}^f$. Then the variational problem*

$$\inf\{\mathcal{G}_P(v), v \in \mathcal{E}(P)\}$$

has a solution.

If Ω does not belong to \mathcal{P}^f , the minimization of \mathcal{G}_Ω may be not well-posed. Indeed we recall that in [6] it was proved that if $\Omega = (0, 1) \times (0, 1)$ then $\mathcal{G}_\Omega(v)$ is infinite for every $v \in \mathcal{E}(\Omega)$. In Examples 3.3 and 3.4 we consider the case where Ω is a domain in \mathcal{P}^∞ .

Example 3.3. We define a set $\Omega \in \mathcal{P}^\infty$ such that for every $v \in \mathcal{E}(\Omega)$ $\mathcal{G}_\Omega(v) = \infty$. We start fixing some notations. For a function $f : [a, b] \rightarrow \mathbb{R}^+$ we will denote by S_f the set

$$S_f = \{(x_1, x_2) \in \mathbb{R}^2 : a \leq x_1 \leq b, 0 \leq x_2 \leq f(x_1)\}.$$

For a given $N \in \mathbb{N}$, with the symbol $g_{(a,b)}^N$ we will denote the step function

$$g_{(a,b)}^N = \sum_{j=0}^{N-1} (b-a) \frac{N-j}{N} \chi_{[a+j\frac{b-a}{N}, a+(j+1)\frac{b-a}{N})}.$$

Let us consider the sequence of real numbers $t_0 = 0, t_n = \sum_{i=1}^n \frac{1}{2^i} = \frac{2^n-1}{2^n}$, and define for $n \in \mathbb{N}$ the functions

$$h_n = \frac{1}{2^n} \chi_{[t_{n-1}, t_n)}, \quad g_n = g_{(t_{n-1}, t_n)}^{N_n},$$

$$h = \sum_{n=1}^{+\infty} h_n, \quad f = \sum_{n=1}^{+\infty} (h_n + g_n),$$

where N_n will be chosen later. Let Ω be $\mathcal{R}(S_f)$. We can easily see that, if we set $C_n = \mathcal{R}(S_{g_n + \frac{1}{2^n}})$, then $\overset{\circ}{C}_i \cap \overset{\circ}{C}_j = \emptyset$ and

$$\mathcal{R}(S_f) = \mathcal{R}(S_h) \cup \bigcup_{n=1}^{\infty} C_n.$$

Thus the claim will be proved if we show that for every $n \in \mathbb{N}$

$$\mathcal{H}^1 \left(J_{v_{x_1}} \cap \overset{\circ}{C}_n \right) + \mathcal{H}^1 \left(J_{v_{x_2}} \cap \overset{\circ}{C}_n \right) \geq 1.$$

Let σ_n be the third middle part of the segment $[\mathcal{R}((t_{n-1}, f(t_{n-1}))), \mathcal{R}((t_n, f(t_n)))]$; then $\mathcal{H}^1(\sigma_n) = \frac{\sqrt{2}}{3 \cdot 2^n}$. For any $0 < t < \frac{\mathcal{H}^1(\sigma_n)}{4}$ consider the segment $\sigma_n^t = \tau_{-t}(\sigma_n)$, where $\tau_w, w \in \mathbb{R}$, denotes the translation in the direction $(0, w)$. Clearly we have $\sigma_n^t \subset \overset{\circ}{C}_n$. Now, let $r_t(s)$ be a parametrization for σ_n^t . If $w_t(s)$ denotes the restriction of a function $v \in \mathcal{E}(\Omega)$ to the segment σ_n^t , we have that

$$|w_t(s)| \leq d_1(r_t(s), \partial\Omega) = d_1(r_t(s), \partial C_n) \leq t + h_n, \quad h_n = \frac{\sqrt{2}}{2^{n+1}N_n}.$$

This implies that w_t' has at least $\left\lfloor \frac{\mathcal{H}^1(\sigma_n)}{2(t+h_n)} \right\rfloor$ jumps. Therefore, by Lemma 2.4

$$\begin{aligned} \mathcal{H}^1(J_{v_{x_1}} \cap \overset{\circ}{C}_n) &\geq \int_0^{\frac{\mathcal{H}^1(\sigma_n)}{4}} \left\lfloor \frac{\mathcal{H}^1(\sigma_n)}{2(t+h_n)} \right\rfloor dt \\ &\geq \int_0^{\frac{\mathcal{H}^1(\sigma_n)}{4}} \left(\frac{\mathcal{H}^1(\sigma_n)}{2(t+h_n)} - 1 \right) dt \\ &= \frac{\mathcal{H}^1(\sigma_n)}{2} \ln \left(\frac{\mathcal{H}^1(\sigma_n)}{4h_n} + 1 \right) - \frac{\mathcal{H}^1(\sigma_n)}{4} \\ &= \frac{\sqrt{2}}{3 \cdot 2^{n+1}} \ln \left(\frac{\sqrt{2}}{12} N_n + 1 \right) - \frac{\sqrt{2}}{3 \cdot 2^{n+2}}. \end{aligned}$$

A suitable choice of N_n implies the claim.

Example 3.4. We are going to define a domain $\Omega \in \mathcal{P}^\infty$ and a function $v \in \mathcal{E}(\Omega)$ such that $\mathcal{G}_\Omega(v)$ is finite. Let t_n be defined as in the previous example. For any $n \in \mathbb{N}$ choose $h_n < \frac{1}{2^n}$ and consider the domain $\Omega = \mathcal{R}(\overset{\circ}{S}_f)$ with f defined by

$$f = \sum_{n=1}^{\infty} h_n \chi_{(t_{n-1}, t_n)}.$$

Let $R_n = [t_{n-1}, t_n] \times [0, h_n]$ and define the function $v(x) = d_1(x, \partial R_n)$ if $x \in R_n$. Clearly v is a solution to problem (1.1); moreover $\mathcal{R}(\overset{\circ}{S}_f) = \cup_{n=1}^{\infty} \mathcal{R}(R_n)$ and

$$\mathcal{H}^1(J_{v_{x_i}} \cap \mathcal{R}(R_n)) \leq 4 \left(h_n + \frac{1}{2^n} \right) \leq \frac{8}{2^n}.$$

It follows the following estimate:

$$\begin{aligned} \mathcal{G}_\Omega(v) &\leq \mathcal{H}^1(J_{v_{x_1}}) + \mathcal{H}^1(J_{v_{x_2}}) \\ &\leq \sum_{n=1}^{\infty} [\mathcal{H}^1(J_{v_{x_1}} \cap \mathcal{R}(R_n)) + \mathcal{H}^1(J_{v_{x_2}} \cap \mathcal{R}(R_n))] \\ &\leq C \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty, \end{aligned}$$

where C denotes a positive constant independent of n .

We now pass to the case $t > 1$. The reason why we do not use the functionals $\mathcal{H}^t(J_{v_{x_1}} \cup J_{v_{x_2}})$ for some $t > 1$, to select a solution to problem (1.1) is clear: for every $v \in \mathcal{E}(\Omega)$,

$$\mathcal{H}^t(J_{v_{x_1}} \cup J_{v_{x_2}}) = 0, \quad \forall t > 1.$$

Indeed, for a given $v \in \mathcal{E}(\Omega)$

$$\mathcal{H}^1(J_{v_{x_i}} \cap \omega) < \infty,$$

for every $\omega \subset \bar{\omega} \subset \Omega$ and for $i = 1, 2$. Recall now that Hausdorff measures have the property that if E is any measurable set such that $\mathcal{H}^r(E) < +\infty$, then $\mathcal{H}^{r+\varepsilon}(E) = 0$ for every $\varepsilon > 0$ (see [11]). This implies that $\mathcal{H}^t(J_{v_{x_i}} \cap \omega) = 0$ for every $t > 1$. By definition of measure $\mathcal{H}^t(J_{v_{x_i}}) = 0$ for every $t > 1$. We remark that this argument can be applied also if we work in dimension greater than 2.

The previous observations suggest that, if one wants to isolate the most "regular" functions in $\mathcal{E}(\Omega)$, he is in some sense obliged to use a weighted \mathcal{H}^1 measure. In this article we define a weight depending only on the distance to the boundary.

In the following example we are going to motivate hypothesis (3.1) on H and to illustrate the ideas behind the proof of Theorem 3.1, in the case where Ω is a square. Due to the simple geometry of the domain, we can perform several steps of the proof avoiding almost all the technical difficulties that we need to deal with in the general case and that are addressed in the next section.

Example 3.5. Let $\Omega = (-1, 1) \times (-1, 1)$. We are going to define a "reasonable" function $v \in \mathcal{E}(\Omega)$ and impose that $\mathcal{F}(v)$ is finite. Since the weight H in the definition of \mathcal{F} depends only on the distance from the boundary, our aim is to investigate how fast the discontinuity of $\frac{\partial v}{\partial x_i}$, $i = 1, 2$ develops near the boundary. With this information we will be able to define an appropriate weight H .

We define v as follows. Let R be the triangle with vertices in $(-1, 1)$, $(1, 1)$ and $(0, 0)$. Then Ω is union of the counter-clockwise $\pm \frac{\pi}{2}$ and π -rotations of R . Therefore it is sufficient to define v in R . We fill R with squares belonging to \mathcal{P}^f ; consequently they can be identified once we know the position of the upper vertex and the length of the diagonal. Let Ω_0 be the square in \mathcal{P}^f with upper vertex in $(0, 1)$ and length of the diagonal equal to 1. For $n \in \mathbb{N}$ and $i = 1, \dots, 2^{n-1}$ let Q_n^i be the square in \mathcal{P}^f with upper vertex in $(\frac{2i-1}{2^n}, 1)$ and length of the diagonal equal to $\frac{1}{2^n}$. We set $Q_n^{-i} = \mathcal{V}(Q_n^i)$. With this notation we can define the following covering of R made up by squares with mutually disjoint interior:

$$\bar{R} = \bigcup_{n=0}^{\infty} \Omega_n, \quad \Omega_n := \bigcup_{i=1}^{2^{n-1}} Q_n^i \cup Q_n^{-i}.$$

Observe that the north corner of each square belongs to $\gamma = [(-1, 1), (1, 1)]$. Now we define the solution v as

$$v_0(x) = d_1(x, \partial\Omega_0)\chi_{\Omega_0}, \quad v_n^i(x) = d_1(x, \partial Q_n^i)\chi_{Q_n^i},$$

$$v_n^{-i}(x) = d_1(x, \partial Q_n^{-i})\chi_{Q_n^{-i}}, \quad v_n(x) = \sum_{i=1}^{2^{n-1}} [v_n^i(x) + v_n^{-i}(x)];$$

$$v(x) = \sum_{n=0}^{\infty} v_n(x).$$

It is clear from the definition that $v \in \mathcal{E}(\Omega)$ and that the distributional gradient of Dv is supported on the sides and the diagonals of the squares we used to fill R . In

order to estimate them, let us fix some notations. Let

$$R_m = \left\{ (x_1, x_2) \in R : \frac{1}{m+1} < x_2 \leq \frac{1}{m} \right\}.$$

We denote by:

- \mathcal{S} the union of the sides of the squares $Q_n^{\pm i}$,
- \mathcal{D}_v the union of the vertical diagonals of the squares $Q_n^{\pm i}$,
- \mathcal{D}_h the union of the horizontal diagonals of the squares $Q_n^{\pm i}$,

for $n \in \mathbb{N}$ and $i = 0, \dots, 2^{n-1}$. Therefore $\mathcal{F}(v)$ is finite if the three following quantities

$$\int_{\mathcal{D}_h} H(d_1(x, \gamma)) d\mathcal{H}^1, \quad \int_{\mathcal{D}_v} H(d_1(x, \gamma)) d\mathcal{H}^1, \quad \int_{\mathcal{S}} H(d_1(x, \gamma)) d\mathcal{H}^1$$

are finite. The first quantity is very simple to estimate, if one remarks that the horizontal diagonals appear only at height $\frac{1}{2^n}$, $n \in \mathbb{N}$, and their total length is 1 for every $n \in \mathbb{N}$. Therefore

$$\int_{\mathcal{D}_h} H(d_1(x, \gamma)) d\mathcal{H}^1 \leq \sum_{n=1}^{\infty} H\left(\frac{1}{2^n}\right).$$

Using the Cauchy's condensation criterion, the last series is finite if and only if

$$(3.3) \quad \sum_{n=1}^{\infty} H\left(\frac{1}{n}\right) \frac{1}{n} < \infty.$$

To estimate the sides and the vertical diagonals we will use a different strategy. We observe that the number N_m of squares which intersect R_m is bounded by $m+1$. Indeed if x_0 is the north corner of a square intersecting R_m , the distance from x_0 to the next north corner of a square intersecting R_m is at least $\frac{2}{m+1}$. Therefore

$$N_m = \frac{\mathcal{H}^1(\gamma)}{\frac{2}{m+1}} = m+1.$$

Now, let Q be a square intersecting R_m . The the \mathcal{H}^1 measure of the intersection of the vertical diagonal of Q with R_m is less or equal to $\frac{1}{m} - \frac{1}{m+1} = \frac{1}{m(m+1)}$ and the \mathcal{H}^1 measure of the intersection of one of the sides of Q with R_m is bounded by $\sqrt{2} \left(\frac{1}{m} - \frac{1}{m+1} \right) = \frac{\sqrt{2}}{m(m+1)}$. Therefore

$$\int_{\mathcal{D}_v} H(d_1(x, \gamma)) d\mathcal{H}^1 \leq \sum_{m=1}^{\infty} H\left(\frac{1}{m}\right) N_m \frac{1}{m(m+1)} \leq \sum_{m=1}^{\infty} H\left(\frac{1}{m}\right) \frac{1}{m}$$

and

$$\int_{\mathcal{S}} H(d_1(x, \gamma)) d\mathcal{H}^1 \leq 2 \sum_{m=1}^{\infty} H\left(\frac{1}{m}\right) N_m \frac{\sqrt{2}}{m(m+1)} \leq 2\sqrt{2} \sum_{m=1}^{\infty} H\left(\frac{1}{m}\right) \frac{1}{m}.$$

Hence we find the same condition on H as in (3.3):

$$\sum_{m=1}^{\infty} H\left(\frac{1}{m}\right) \frac{1}{m} < \infty.$$

4. PROOF OF THE MAIN RESULT

In this section we are going to prove Theorem 3.1. The proof can be divided into two parts: in the first we show that the variational problem (3.2) is well-posed (see Proposition 4.4); in the second one we prove that there exists a minimizer of the functional \mathcal{F} in $\mathcal{E}(\Omega)$.

We are going to concentrate on the first step, that is, we are going to construct a function $v \in \mathcal{E}(\Omega)$ such that $\mathcal{F}(v)$ is finite. For this purpose, recalling that Ω is a compatible domain, according to Remark 2.2, we have

$$(4.1) \quad \bar{\Omega} = \bar{P} \cup \bigcup_{j=1}^N \mathcal{W}_j(T_j)$$

where $P \in \mathcal{P}^f$, $T_j \in \mathcal{T}$ and $\mathcal{W}_j(T_j) = \mathcal{R}^{2k_j+1}(T_j)$ or $\mathcal{V}(\mathcal{R}^{2k_j+1}(T_j))$ for some $k_j \in \{0, 1, 2, 3\}$.

With the aim of setting v in each domain $\mathcal{W}_j(T_j)$ we will define a special countable covering of the interior of a triangular domain made up by squares. We start by introducing three operators defined in \mathcal{T} . For a given

$$T = \{(x_1, x_2) : a \leq x_1 \leq b, h(b) \leq x_2 \leq h(x_1)\} \in \mathcal{T},$$

let x_1^0 be such that $h(x_1^0) = x_1^0 + h(b) - a$ and define

$$\begin{aligned} q(T) &:= (a, x_1^0) \times (h(b), h(b) + x_1^0 - a); \\ u(T) &:= \{(x_1, x_2) \in T : a < x_1 < x_1^0, h(b) + x_1^0 < x_2 < h(x_1)\}; \\ r(T) &:= \{(x_1, x_2) \in T : a + x_1^0 < x_1 < b, h(b) < x_2 < h(x_1)\}. \end{aligned}$$

We explicitly observe that u and r have values in \mathcal{T} while q maps any triangular domain to a square in $\mathcal{R}(\mathcal{P}^f)$.

Definition 4.1 (of the covering of T). *Let for $m \in \mathbb{N}$*

$$S_m := \{\sigma = (\alpha_1, \dots, \alpha_m) : \alpha_i \in \{u, r\}\},$$

be the set of all the m -permutations of the two letters u and r . For $\sigma \in S_m$, using the notation

$$\sigma(T) = \alpha_1 \circ \alpha_2 \circ \dots \circ \alpha_m(T),$$

we set

$$Q_T^{m,\sigma} = q(\sigma(T)) \ ; \ \sigma \in S_m.$$

We finally define the following family of squares contained in T :

$$\mathcal{Q}(T) := \{Q_T^{m,\sigma} : m \in \mathbb{N} \cup \{0\}, \sigma \in S_m\}.$$

Remark 4.2. We remark that $\mathcal{Q}(T)$ is a covering of $\overset{\circ}{T}$ composed of squares with mutually disjoint interiors belonging to $\mathcal{R}(\mathcal{P}^f)$. It may be useful to think of $\mathcal{Q}(T)$ as being constructed in steps, starting from $m = 1$ and adding at step m the squares $Q_T^{m,\sigma}$ with $\sigma \in S_m$. Since the cardinality of S_m , $\sharp(S_m)$, is equal to 2^m , we add 2^m squares at the m -th step. Therefore the first steps of the construction are:

- Step 0.* We start with $Q_T^0 = q(T)$;
- Step 1.* we add $Q_T^{1,u} = q(u(T))$ and $Q_T^{1,r} = q(r(T))$;
- Step 2.* we add $Q_T^{2,(u,u)} = q(u(u(T)))$, $Q_T^{2,(u,r)} = q(u(r(T)))$,
 $Q_T^{2,(r,u)} = q(r(u(T)))$, $Q_T^{2,(r,r)} = q(r(r(T)))$
- Step 3.* we add $Q_T^{3,(u,u,u)} = q(u(u(u(T))))$, $Q_T^{3,(u,u,r)} = q(u(u(r(T))))$,
 \dots \dots

In the sequel, when $\#(S_m)$ will play a role, we will use for $Q_T^{m,\sigma}$ the notation $Q_T^{m,k}$, $k \in \{1, \dots, 2^m\}$.

We are now in a position to define the candidate function $v : \Omega \rightarrow \mathbb{R}^+$ such that $\mathcal{F}(v)$ is finite, which is the natural generalization of the construction made in Example 3.5.

Definition 4.3 (of the candidate function). *Let w be a minimizer of the functional \mathcal{G}_P . For a fixed $j \in \{1, \dots, N\}$, let*

$$v_m^j(x) = \sum_{\sigma \in S_m} d_1 \left(x, \partial \left(\mathcal{W}_j(Q_{T_j}^{m,\sigma}) \right) \right) \chi_{Q_{T_j}^{m,\sigma}}(x);$$

$$v_j(x) = \sum_{m \in \mathbb{N}} v_m^j(x).$$

We define

$$v(x) = w(x)\chi_P(x) + \sum_{j=1}^N v_j(x).$$

In the following we will prove that problem (3.2) is well-posed. We remark that in $\Omega \setminus P$, Dv_{x_1} and Dv_{x_2} are supported on the sides of each square $\mathcal{W}_j(Q_{T_j}^{m,\sigma})$ and on its vertical and horizontal diagonals respectively. For this reason, we denote by \mathcal{D}_T^+ and \mathcal{D}_T^- the union of all the diagonals of the squares in $\mathcal{Q}(T)$ parallel to ℓ_+ and ℓ_- respectively, and by \mathcal{S}_T the union of all the sides of the squares in $\mathcal{Q}(T)$.

Proposition 4.4. *The function $v \in \mathcal{E}(\Omega)$ defined in 4.3 satisfies $\mathcal{F}(v) < +\infty$.*

Proof. First we note that according to (4.1) we can estimate $\mathcal{F}(v)$ as follows:

$$\begin{aligned} \mathcal{F}(v) &= \sum_{i=1}^2 \int_{\overline{P}} H(d_1(x, \partial\Omega)) d|D w_{x_i}|(x) + \sum_{i=1}^2 \int_{\Omega \setminus P} H(d_1(x, \partial\Omega)) d|D v_{x_i}|(x) \\ &\leq \sum_{i=1}^2 \int_{\overline{P}} C d|D w_{x_i}|(x) + \sum_{i=1}^2 \int_{\bigcup_{j=1}^N \mathcal{W}_j(T_j)} H(d_1(x, \partial\Omega)) d|D v_{x_i}|(x) \end{aligned}$$

where we have used that for some positive constant C , $H(d_1(x, \partial\Omega)) \leq C$ for all $x \in \overline{\Omega}$.

The first term of the last estimate is finite, due to the choice of w . Therefore it is sufficient to estimate

$$\sum_{i=1}^2 \sum_{j=1}^N \int_{\mathcal{W}_j(T_j)} H(d_1(x, \partial\Omega)) d|D v_{x_i}|(x).$$

The construction of v , the properties of the rigid motion \mathcal{W}_j and the duality between the metrics d_1 and d_∞ imply that we can restrict ourselves to prove that there exists a positive constant M such that

$$(4.2) \quad \int_{\mathcal{S}_{T_j} \cup \mathcal{D}_{T_j}^- \cup \mathcal{D}_{T_j}^+} H(d_\infty(x, \mathcal{W}_j^{-1}(\partial\Omega))) d\mathcal{H}^1 \leq M \quad \forall j \in \{1, \dots, N\}.$$

The last estimate is a consequence of Lemmata 4.8, 4.6 and 4.7. □

In the next lemmata we aim to prove (4.2). We will estimate separately the integral on the sets \mathcal{S}_{T_j} , $\mathcal{D}_{T_j}^-$ and $\mathcal{D}_{T_j}^+$. From now on we will work on a single triangular domain T_j ; since the proofs are independent of j , we will simply write T instead of T_j . Up to a dilatation and a rigid motion, we can assume in the sequel that

$$T = \{(x_1, x_2) : 0 \leq x_1 \leq 1, 0 \leq x_2 \leq h(x_1)\}.$$

with $h(1) = 0$, $h(0) \leq 1$ and $h'(t) < 0$ for every $t \in (0, 1)$. We will denote by γ the curve $\mathcal{W}_j^{-1}(\partial\Omega)$ and we observe explicitly that the graph of the function h is a proper subset of γ . We define

$$L_n = \left\{ x \in T : \frac{1}{n+1} < d_\infty(x, \gamma) \leq \frac{1}{n} \right\},$$

for $n \in \mathbb{N}$.

Let $Q \in \mathcal{Q}(T)$ having a not empty intersection with L_n and assume that Q touches γ in $(x_1, h(x_1))$. Let $\mathbf{e}_n(Q)$, $\mathbf{n}_n(Q)$, $\mathbf{d}_n(Q)$ denote respectively the length of the intersection of the east side, north side, diagonal parallel to ℓ_+ of Q with L_n . Then it is easily to verify that

$$(4.3) \quad \mathbf{e}_n(Q) \leq h\left(x_1 + \frac{1}{n+1}\right) - h\left(x_1 + \frac{1}{n}\right) + \frac{1}{n(n+1)},$$

$$(4.4) \quad \mathbf{n}_n(Q) \leq h^{-1}\left(h(x_1) + \frac{1}{n+1}\right) - h^{-1}\left(h(x_1) + \frac{1}{n}\right) + \frac{1}{n(n+1)}$$

and

$$(4.5) \quad \mathbf{d}_n(Q) \leq \frac{1}{n(n+1)}.$$

Lemma 4.5. *Let N_n be the number of squares in $\mathcal{Q}(T)$ which intersect L_n . Then there exists a positive constant c depending only on γ such that*

$$(4.6) \quad N_n \leq c(n+1).$$

Proof. Let (x_1^0, x_1^0) be the north-east corner of $q(T)$. Define

$$x_1^j = h^{-1}\left(x_1^0 - \frac{j}{1+n}\right), \quad j = 1, \dots, m \leq (n+1).$$

Fix $i \in \{0, \dots, m\}$ and note that if on the portion of γ lying between $(x_1^i, h(x_1^i))$ and $(x_1^{i+1}, h(x_1^{i+1}))$ there are two points, say p and q , which are north-east vertex of squares, Q_p and Q_q in $\mathcal{Q}(T)$, then $L_n \cap Q_p$ or $L_n \cap Q_q$ is empty.

Consequently the number of squares in $r(T)$ which intersect L_n can be at most $n+1$. With the same argument one can prove that the same estimate holds for the number of squares in $u(T)$ which intersect L_n . □

Lemma 4.6. *There exists a positive constant M such that*

$$\int_{\mathcal{D}_T^+} H(d_\infty(x, \gamma)) d\mathcal{H}^1 \leq M.$$

Proof. The proof easily follows from (4.5), (4.6) and the hypotheses on H . Indeed one has

$$\int_{\mathcal{D}_T^+} H(d_\infty(x, \gamma)) d\mathcal{H}^1 \leq C \sum_{n=1}^{\infty} H\left(\frac{1}{n}\right) \frac{1}{n} < +\infty,$$

for some positive constant C independent of n . \square

Lemma 4.7. *There exists a positive constant M such that*

$$(4.7) \quad \int_{\mathcal{D}_T^-} H(d_\infty(x, \gamma)) d\mathcal{H}^1 \leq M.$$

Proof. If L_i denotes the total length of the diagonals parallel to ℓ_- of the squares added at the i -th step of the construction of v , then

$$(4.8) \quad L_i \leq 2\mathcal{H}^1(\gamma).$$

Indeed L_i is nothing but the length of the non flat parts of the graph of a piecewise affine function defined on the interval $(0, 1)$ and whose derivative is 0 or -1 everywhere but on a finite number of points.

The proof is divided into several steps. We will denote by \mathbf{r}_n the length of the side of $Q_T^{n,\sigma}$ with $\sigma = \underbrace{(r, \dots, r)}_n$ and $\mathbf{R}_n = \sum_{i=1}^n \mathbf{r}_i$; \mathbf{u}_n will denote the length of the side of $Q_T^{n,\sigma}$ with $\sigma = \underbrace{(u, \dots, u)}_n$.

Step 1. Let $h \in C^1((0, 1))$. Assume that there exists $0 < \varepsilon < 1$ such that $-1 + \varepsilon < h'(t) < 0$ for every $t \in (0, 1)$. Let d_n^- be the intersection of L_n with the diagonal parallel to ℓ_- of a square $Q \in \mathcal{Q}(T)$. Let $x_1^Q = \min_{(x_1, x_2) \in d_n^-} x_1$. Assume that $(x_1^Q, x_2^Q) \in d_n^-$. Then

$$h\left(x_1^Q + \frac{1}{n}\right) - \frac{1}{n} \leq x_2^Q \leq h\left(x_1^Q + \frac{1}{n+1}\right) - \frac{1}{n+1},$$

since $(x_1^Q, x_2^Q) \in L_n$. The intersection between d_n^- , belonging to the line $x_2 = -x_1 + x_1^Q + x_2^Q$, and the lower boundary of L_n , that is, $x_2 = h\left(x_1 + \frac{1}{n}\right) - \frac{1}{n}$, gives

$$\begin{aligned} x_1 - x_1^Q &= x_2^Q + \frac{1}{n} - h\left(x_1 + \frac{1}{n}\right) \\ &\leq h\left(x_1^Q + \frac{1}{n+1}\right) - \frac{1}{n+1} + \frac{1}{n} - h\left(x_1 + \frac{1}{n}\right). \end{aligned}$$

Lagrange's theorem implies that

$$x_1 - x_1^Q \leq |h'(\xi)| \left[x_1 - x_1^Q + \frac{1}{n} - \frac{1}{n+1} \right] + \frac{1}{n} - \frac{1}{n+1}$$

for some $\xi \in \left[x_1^Q + \frac{1}{n+1}, x_1 + \frac{1}{n} \right]$. Using the hypothesis on h we get

$$x_1 - x_1^Q \leq (1 - \varepsilon) \left(x_1 - x_1^Q + \frac{1}{n^2} \right) + \frac{1}{n^2},$$

that is,

$$x_1 - x_1^Q \leq \frac{2 - \varepsilon}{\varepsilon} \frac{1}{n^2}.$$

This implies that

$$\mathcal{H}^1(d_n^-) \leq \sqrt{2} \frac{2 - \varepsilon}{\varepsilon} \frac{1}{n^2}$$

for a given square Q such that $Q \cap L_n \neq \emptyset$. Using estimate (4.6) on the number of squares intersecting L_n , the previous inequality gives

$$\int_{\mathcal{D}_T^-} H(d_\infty(x, \gamma)) d\mathcal{H}^1 \leq c_1(\varepsilon, \gamma) \sum_{n=1}^{\infty} H\left(\frac{1}{n}\right) \frac{1}{n^2} (n+1) \leq c_2(\varepsilon, \gamma) \sum_{n=1}^{\infty} H\left(\frac{1}{n}\right) \frac{1}{n}.$$

where $c_1(\varepsilon, \gamma)$ and $c_2(\varepsilon, \gamma)$ denote two positive constants depending only on γ and ε . Hypothesis (3.1) on H implies that the last sum is finite.

The case where $-1 + \varepsilon < h'(t) < 0$ for every $t \in (0, 1)$ can be handled in a similar way.

Step 2. Assume that there exists $0 < \varepsilon < 1$ such that $|h'(t) + 1| \leq \varepsilon$ for every $t \in [0, 1]$. We are going to prove by induction that the length of the side $l_{i,k}$, $k = 1, \dots, 2^i$ of any square $Q_T^{i,k}$ added at the i -th step of the covering of Definition 4.1 satisfies:

$$(4.9) \quad \frac{1}{(2 + \varepsilon)^i} \leq l_{i,k} \leq \frac{1}{(2 - \varepsilon)^i} \quad \forall k = 1, \dots, 2^i.$$

To to that, observe that the length of the side \tilde{l} of $q(\tilde{T})$, for a given domain $\tilde{T} = \{(x_1, x_2) : a < x_1 < b, c < x_2 < h(x_1)\}$, can be estimated by

$$(4.10) \quad \frac{\max\{b - a, h(a) - c\}}{2 + \varepsilon} \leq \tilde{l} \leq \frac{\min\{b - a, h(a) - c\}}{2 - \varepsilon}.$$

Indeed, it is sufficient to compute the intersections between the lines $x_2 = x_1 + c - a$ and $x_2 = (-1 \pm \varepsilon)(x_1 - b) + c$ or $x_2 = (-1 \pm \varepsilon)(x_1 - a) + h(a)$. This implies that the length $l_{0,1}$ of the side of $q(T)$ satisfies

$$(4.11) \quad \frac{1}{2 + \varepsilon} \leq l_{0,1} \leq \frac{1}{2 - \varepsilon}.$$

Now, suppose that estimate (4.9) holds for $i - 1$. At step i we add 2^i squares and any of these is confined in a domain belonging to \mathcal{T} with one of his sides that coincides with the side of one of the squares added in the previous step. Thus, to prove that estimate (4.9) holds for i , it suffices to use (4.10) with $\frac{1}{(2 + \varepsilon)^{i-1}} \leq \tilde{l} \leq \frac{1}{(2 - \varepsilon)^{i-1}}$.

We need to estimate the l^∞ -distance from γ of the diagonal d^- parallel to ℓ_- of a square $Q \in \mathcal{Q}(T)$. Assume that the length of the side of Q is l . Let $x^Q = (x_1^Q, x_2^Q) \in \gamma$ be the north-east corner of Q . Then the l^∞ -distance from γ of d^- is smaller than the l^∞ -distance of d^- from the piecewise affine function

$$x_2(x_1) = \begin{cases} (-1 + \varepsilon)(x_1 - x_1^Q) + x_2^Q, & \text{if } x_1 > x_1^Q \\ (-1 + \varepsilon)(x_1 - x_1^Q) + x_2^Q, & \text{if } x_1 \leq x_1^Q \end{cases}.$$

This gives

$$(4.12) \quad d_\infty(d^-, \gamma) \leq \frac{1 + \varepsilon}{2 - \varepsilon} l.$$

Using the previous estimates and (4.8) we obtain

$$\begin{aligned} \int_{\mathcal{D}_T^-} H(d_\infty(x, \gamma)) d\mathcal{H}^1 &\leq \sum_{n=1}^{\infty} L_n H\left(\frac{1+\varepsilon}{(2-\varepsilon)^{n+1}}\right) \\ &\leq 2\mathcal{H}^1(\gamma) \sum_{n=1}^{\infty} H\left(\frac{1+\varepsilon}{(2-\varepsilon)^{n+1}}\right). \end{aligned}$$

In order to prove that the last sum is finite, we start by observing that, by the monotonicity of H ,

$$H\left(\frac{1+\varepsilon}{(2-\varepsilon)^{n+1}}\right) \leq H\left(\frac{2}{[(2-\varepsilon)^n]}\right).$$

We now let

$$u_n = [(2-\varepsilon)^n] \quad \text{and} \quad a_n = H\left(\frac{2}{n}\right) \frac{1}{n}$$

and we apply the Schlömilch's generalization of the condensation criterion for series to deduce the desired convergence. We recall that, if a_n is a positive non increasing sequence of real numbers and u_n a strictly increasing sequence of natural numbers such that for some positive constant C

$$(4.13) \quad \frac{u_{n+1} - u_n}{u_n - u_{n-1}} \leq C \quad \forall n \in \mathbb{N}$$

then $\sum_{n=0}^{\infty} a_n$ is finite if and only if $\sum_{n=0}^{\infty} (u_{n+1} - u_n) a_{u_n}$ is finite. In our case we

observe that the convergence of the series $\sum_{n=0}^{\infty} a_n$ is assured by hypothesis (3.1) on

H . Moreover for every $n \geq n(\varepsilon)$

$$\begin{aligned} c(\varepsilon)[(2-\varepsilon)^n] &\leq u_{n+1} - u_n \leq (2-\varepsilon)^{n+1} + 1 - (2-\varepsilon)^n \\ &\leq ([(2-\varepsilon)^n] + 1)(1-\varepsilon) + 1 \\ &\leq [(2-\varepsilon)^n] + 2 - \varepsilon \end{aligned}$$

for some positive constant $c(\varepsilon)$ independent of n . This implies that hypothesis (4.13) is satisfied. Therefore there exists a positive constant C such that

$$\begin{aligned} \sum_{n=0}^{\infty} H\left(\frac{2}{[(2-\varepsilon)^n]}\right) &= \sum_{n=0}^{\infty} H\left(\frac{2}{[(2-\varepsilon)^n]}\right) \frac{1}{[(2-\varepsilon)^n]} [(2-\varepsilon)^n] \\ &\leq C + \frac{1}{c(\varepsilon)} \sum_{n=n(\varepsilon)}^{\infty} (u_{n+1} - u_n) a_{u_n} < \infty. \end{aligned}$$

Step 3. Assume that $h \in C^1([0, 1])$ and fix $0 < \varepsilon < 1$. The uniform continuity of h' implies that there exists $\delta > 0$ such that if $|t - s| < \delta$ then $|h'(t) - h'(s)| < \varepsilon/4$. Let $n' \in \mathbb{N}$ be such that the length of the side of $Q_T^{n', \sigma}$ is less than δ for every $\sigma \in S_{n'}$. Let (x_1^σ, x_2^σ) be the north-east corner of $Q_T^{n', \sigma}$. For any $\sigma \in S_{n'}$, if $|h'(x_1^\sigma) + 1| \leq \varepsilon/2$, then

$$|h'(x_1) + 1| \leq |h'(x_1) - h'(x_1^\sigma)| + |h'(x_1^\sigma) + 1| < \varepsilon$$

and if $|h'(x_1^\sigma) + 1| > \varepsilon/2$, then

$$|h'(x_1) + 1| \geq |h'(x_1^\sigma) + 1| - |h'(x_1) - h'(x_1^\sigma)| > \frac{\varepsilon}{4}.$$

Let $N(n') = 1 + \sum_{i=0}^{n'} 2^i = 2^{n'+1}$. Using (4.8) we have

$$\int_{\mathcal{D}_T^-} H(d_\infty(x, \gamma)) d\mathcal{H}^1 \leq n' \mathcal{H}^1(\gamma) + \sum_{i=1}^{N(n')} \int_{\mathcal{D}_{T_i}^-} H(d_\infty(x, \gamma))$$

where, up to dilatations, T_i , with $i = 1 \dots N(n')$, are triangular domains satisfying the hypotheses of *Step 1* or *2*. This implies that $\int_{\mathcal{D}_T^-} H(d_\infty(x, \gamma)) d\mathcal{H}^1$ is finite.

Step 4. Assume that $h \in C^1((0, 1))$ and $\lim_{t \rightarrow 0} h'(t) = -\infty$ or $\lim_{t \rightarrow 1} h'(t) = -\infty$. Surely there exists $n'' \in \mathbb{N}$ such that the intervals $[0, \mathbf{u}_{n''}]$, $[\mathbf{R}_{n''}, 1]$ do not contain any x_1 such that $h'(x_1) = -1$. Let $N(n'') = \sum_{i=0}^{n''} 2^i - 2 = 2^{n''+1} - 3$. Using (4.8) we get the following estimate:

$$\int_{\mathcal{D}_T^-} H(d_\infty(x, \gamma)) d\mathcal{H}^1 \leq n'' \mathcal{H}^1(\gamma) + \sum_{i=1}^{N(n'')} \int_{\mathcal{D}_{T_i}^-} H(d_\infty(x, \gamma))$$

where, up to dilatations, T_i , with $i = 1 \dots N(n'')$, are triangular domains satisfying the hypotheses of *Step 3*. This implies that $\int_{\mathcal{D}_T^-} H(d_\infty(x, \gamma)) d\mathcal{H}^1$ is finite. □

Lemma 4.8. *There exists a positive constant M such that*

$$\int_{S_T} H(d_\infty(x, \gamma)) d\mathcal{H}^1 \leq M.$$

Proof. First we observe that, since T is a triangular domain, up to inverting the coordinate axes, we can assume that we are in one of the following cases:

Case 1. There exist two constants $c_1, c_2 > 0$ such that $-c_1 \leq h'(t) \leq -c_2 < 0$ for every $t \in [0, 1]$.

Case 2. There exists a constant $c_1 < 0$ such that $c_1 \leq h'(t) < 0$ for every $t \in (0, 1)$ and $h'(1) = 0$.

Case 3. There exists a constant $c_1 < 0$ such that $c_1 \leq h'(t) < 0$ for every $t \in (0, 1)$ and $\lim_{t \rightarrow 1} h'(t) = -\infty$.

Step 1. Assume that we are in the hypotheses of *Case 1*. We observe that h and h^{-1} are Lipschitz functions, say with Lipschitz constant \tilde{c} ; then (4.3) and (4.4) together with estimate (4.6) imply that

$$\int_{S_T} H(d_\infty(x, \gamma)) d\mathcal{H}^1 \leq \sum_{n=1}^{\infty} H\left(\frac{1}{n}\right) \frac{\tilde{c}}{n(n+1)} c(n+1)$$

where c depends only on γ . The last sum is finite due to hypothesis (3.1) on H .

Step 2. Assume that we are in the hypotheses of *Case 2*. It is sufficient to estimate

$$\int_{S_T \cap r(T)} H(d_\infty(x, \gamma)) d\mathcal{H}^1,$$

since *Step 1* implies that

$$\int_{S_T \cap u(T)} H(d_\infty(x, \gamma)) d\mathcal{H}^1$$

is bounded. To this purpose, we define the following sequence of points. Let $x_1^0 \in (0, 1)$ be such that $h(x_1^0) = x_1^0$. We set

$$x_1^j = h^{-1} \left(x_1^0 - \frac{j}{n} \right), \quad j = 1, \dots, [x_1^0 n] =: M_n.$$

There exists at most one point $(\tilde{x}_1^j, h(\tilde{x}_1^j))$ with $x_1^j \leq \tilde{x}_1^j < x_1^{j+1}$, $j = 1, \dots, M_n - 1$, which is the north-east vertex of a square $Q^j \in \mathcal{Q}(T)$ such that $L_n \cap Q^j$ is not empty. For such a square, since $|h'|$ is bounded, one has

$$(4.14) \quad \mathbf{e}_n(Q^j) \leq \left[1 + \sup_{x_1 \in [0, 1]} |h'(x_1)| \right] \frac{1}{n^2} \leq C_1 \frac{1}{n^2}$$

for some positive constant C_1 . As well

$$\begin{aligned} \mathbf{n}_n(Q^j) &\leq \frac{1}{n(n+1)} \left[1 + \sup_{x_2 \in [h(\tilde{x}_1^j) + \frac{1}{n+1}, h(\tilde{x}_1^j) + \frac{1}{n}]} |(h^{-1})'(x_2)| \right] \\ &\leq \frac{1}{n^2} \left[1 + \sup_{x_2 \in [h(x_1^{j+1}), h(x_1^{j-1})]} \frac{1}{|h'(h^{-1}(x_2))|} \right] \\ &\leq \frac{1}{n^2} \left[1 + \sup_{x_2 \in [h(x_1^{j+1}), h(x_1^j)]} \frac{1}{|h'(h^{-1}(x_2))|} + \sup_{x_2 \in [h(x_1^j), h(x_1^{j-1})]} \frac{1}{|h'(h^{-1}(x_2))|} \right]. \end{aligned}$$

We remark that

$$\sum_{j=1}^{M_n} \frac{1}{n} \left[\sup_{x_2 \in [h(x_1^j), h(x_1^{j-1})]} \frac{1}{|h'(h^{-1}(x_2))|} \right]$$

is a particular Riemann sum for

$$\int_{\frac{1}{n}}^{x_1^0 - \frac{1}{n}} \frac{1}{|h'(h^{-1}(x_2))|} dx_2$$

which is finite. Therefore there exists $n_0 \in \mathbb{N}$ such that

$$(4.15) \quad \sum_{j=1}^{M_n} \frac{1}{n} \left[\sup_{x_2 \in [h(x_1^j), h(x_1^{j-1})]} \frac{1}{|h'(h^{-1}(x_2))|} \right] \leq \int_0^{x_1^0} \frac{1}{|h'(h^{-1}(x_2))|} dx_2 + 1 \leq C_2$$

for every $n \geq n_0$, for some positive constant C_2 independent of n . Estimates (4.14) and (4.15) give

$$\begin{aligned} & \int_{\mathcal{S} \cap r(T)} H(d_\infty(x, \gamma)) d\mathcal{H}^1 \leq \\ & \leq \sum_{n=n_0}^{\infty} H\left(\frac{1}{n}\right) \sum_{j=1}^{M_n} \left\{ \frac{2}{n^2} \sup_{x_2 \in [h(x_1^j), h(x_1^{j-1})]} \frac{1}{|h'(h^{-1}(x_2))|} + \frac{C_1 + 1}{n^2} \right\} + \\ & + \sum_{n=1}^{n_0} H\left(\frac{1}{n}\right) \mathcal{H}^1(\mathcal{S} \cap r(T) \cap \overline{L_n}) \\ & \leq C_2 \sum_{n=1}^{\infty} H\left(\frac{1}{n}\right) \frac{1}{n} + (C_1 + 1) \sum_{n=1}^{\infty} H\left(\frac{1}{n}\right) \frac{1}{n} + C_3, \end{aligned}$$

where we have used that for $n \leq n_0$, $H\left(\frac{1}{n}\right) \leq H\left(\frac{1}{n_0}\right)$ and $\mathcal{H}^1(\mathcal{S} \cap r(T) \cap \overline{L_n})$ is finite, since $v_{x_i}, i = 1, 2$ is $SBV_{loc}(T)$. The last sum is finite due to hypothesis (3.1) on H .

Step 3. Assume that we are in the hypotheses of *Case 3*. As in the previous step it is sufficient to estimate

$$\int_{\mathcal{S}_T \cap r(T)} H(d_\infty(x, \gamma)) d\mathcal{H}^1.$$

Let $x_1^0 \in (0, 1)$ be such that $h(x_1^0) = x_1^0$. Let us consider the following sequence of points:

$$x_1^j = x_1^0 + \frac{j}{n}, \quad j = 1, \dots, M_n = [n(1 - x_1^0)].$$

There exists at most one point $(\tilde{x}_1^j, h(\tilde{x}_1^j))$ with $x_1^j \leq \tilde{x}_1^j < x_1^{j+1}$, $j = 1, \dots, M_n - 1$, which is the north-east vertex of a square $Q^j \in \mathcal{Q}(T)$ such that $L_n \cap Q^j$ is not empty.

For such a square, since $|(h^{-1})'|$ is bounded, one has

$$(4.16) \quad \mathbf{n}_n(Q^j) \leq \left[1 + \sup_{x_1 \in [0, h(0)]} |(h^{-1})'(x_1)| \right] \frac{1}{n^2} \leq C_1 \frac{1}{n^2}$$

for some positive constant C_1 . As well

$$\begin{aligned} \mathbf{e}_n(Q^j) & \leq \frac{1}{n(n+1)} \left[1 + \sup_{x_1 \in [\tilde{x}_1^j + \frac{1}{n+1}, \tilde{x}_1^j + \frac{1}{n}]} |h'(x_1)| \right] \\ & \leq \frac{1}{n^2} \left[1 + \sup_{x_1 \in [x_1^j, x_1^{j+2}]} |h'(x_1)| \right] \\ & \leq \frac{1}{n^2} \left[1 + \sup_{x_2 \in [x_1^j, x_1^{j+1}]} |h'(x_1)| + \sup_{x_2 \in [x_1^{j+1}, x_1^{j+2}]} \frac{1}{|h'(x_1)|} \right]. \end{aligned}$$

We remark that

$$\sum_{j=1}^{M_n} \sup_{x_1 \in [x_1^j, x_1^{j+1}]} |h'(x_1)| \frac{1}{n}$$

is a Riemann sum for

$$\int_{x_1^0}^{1-\frac{1}{n}} |h'(t)| dt$$

which is finite. Then there exists $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n} \sum_{j=1}^{M_n} \sup_{x_1 \in [x_1^j, x_1^{j+1}]} |h'(y)| < \int_{x_1^0}^1 |h'(t)| dt + 1 \leq C_2$$

for every $n \geq n_0$, for some positive constant C_2 independent of n . Therefore, arguing as in the previous step, we have

$$\begin{aligned} & \int_{\mathcal{S} \cap r(T)} H(d_\infty(x, \gamma)) d\mathcal{H}^1 \\ & \leq \sum_{n=n_0}^{\infty} H\left(\frac{1}{n}\right) \sum_{j=1}^{M_n} \left\{ \frac{2}{n^2} \sup_{x_1 \in [x_1^j, x_1^{j+1}]} |h'(x_1)| + \frac{C_1 + 1}{n^2} \right\} \\ & \quad + \sum_{n=1}^{n_0} H\left(\frac{1}{n}\right) \mathcal{H}^1(\mathcal{S} \cap r(T) \cap \overline{L_n}) \\ & \leq C_2 \sum_{n=1}^{\infty} H\left(\frac{1}{n}\right) \frac{1}{n} + (C_1 + 1) \sum_{n=1}^{\infty} H\left(\frac{1}{n}\right) \frac{1}{n} + C_3 < \infty, \end{aligned}$$

where C_3 denotes a positive constant independent of n . \square

We have therefore shown that the functional \mathcal{F} is well-defined. We are now in position to show Theorem 3.1. The proof follows quite easily from the direct methods of the calculus of variations. Note that this technique was already used in [6]. We start recalling a lemma proved in [6].

Lemma 4.9. *Let Ω be an open bounded connected subset of \mathbb{R}^N with Lipschitz boundary. Then*

$$-d_1(\cdot, \partial\Omega) \leq v \leq d_1(\cdot, \partial\Omega) \quad \text{on } \overline{\Omega}$$

for every function $v \in \mathfrak{S}(\Omega)$.

Proof. (of Theorem 3.1) Let $(v^n)_n \subset \mathcal{E}(\Omega)$ be a minimizing sequence. Lemma 4.9 assures that $(v^n)_n$ is uniformly bounded in $L^\infty(\Omega)$. Moreover v_n is uniformly Lipschitz in $\overline{\Omega}$ since Ω is Lipschitz and $|\nabla v^n| \leq \sqrt{2}$ a.e. in Ω for every n . Therefore, up to a subsequence, $v^n \rightarrow v^\infty$ in $C_0(\Omega)$ and $v^n \rightarrow v^\infty$ weakly* in $W^{1,\infty}(\Omega)$ for some $v^\infty \in W_0^{1,\infty}(\Omega)$.

We are now going to show that v^∞ belongs to $\mathcal{E}(\Omega)$. Since $(v^n)_n$ is a minimizing sequence, there exists $C > 0$ such that $\mathcal{F}(v^n) \leq C$ for every $n \in \mathbb{N}$. For a fixed $m \in \mathbb{N}$ we can say that

$$C \geq \mathcal{F}(v^n) \geq \alpha(m) \sum_{i=1}^2 \int_{B_m} d(|Dv_{x_i}^n|)(x), \quad \forall n \in \mathbb{N}$$

where

$$B_m = \left\{ x \in \Omega : d_1(x, \partial\Omega) > \frac{1}{m} \right\}.$$

Let us fix $i \in \{1, 2\}$. Note that $\frac{\partial v^n}{\partial x_i} \in SBV(B_m)$ and takes only the two values ± 1 for every n . This implies that

$$\left\| \frac{\partial v^n}{\partial x_i} \right\|_{BV(B_m)} = \mathcal{L}^2(B_m) + 2\mathcal{H}^1(J_{v^n_{x_i}} \cap B_m) \leq \mathcal{L}^2(B_m) + 2C.$$

We can apply Theorem 2.7 to the sequence $\left(\frac{\partial v^n}{\partial x_i}\right)_n$:

- hypothesis *i*) has been verified in the previous estimate;
- hypothesis *ii*) is verified as $\nabla \frac{\partial v^n}{\partial x_i} = 0$ a.e. in Ω ;
- hypothesis *iii*) can be verified choosing $f \equiv 1$: in this way

$$\int_{J_{v^n_{x_i}} \cap B_m} f \left(\left[\frac{\partial v^n}{\partial x_i} \right] \right) d\mathcal{H}^1(x) = \mathcal{H}^1(J_{v^n_{x_i}} \cap B_m) \leq C.$$

Consequently $\frac{\partial v^n}{\partial x_i} \rightarrow g_i$ weak* in $BV(B_m)$ for some $g_i \in SBV(B_m)$. Since $\frac{\partial v^n}{\partial x_i} \rightarrow \frac{\partial v^\infty}{\partial x_i}$ weak* in $L^\infty(\Omega)$, we infer that $\frac{\partial v^\infty}{\partial x_i} \in SBV(B_m)$. Moreover $\frac{\partial v^n}{\partial x_i} \rightarrow \frac{\partial v^\infty}{\partial x_i}$ in $L^1(B_m)$ and so $\left| \frac{\partial v^\infty}{\partial x_i} \right| = 1$ a.e. in B_m . This being true for every i and for every B_m , we deduce that v^∞ belongs to $\mathcal{E}(\Omega)$.

To show that v^∞ is a minimizer of \mathcal{F} , we remark that

$$\liminf_{n \rightarrow \infty} \int_{B_m} H(d_1(x, \partial\Omega)) d(|Dv^n_{x_i}|)(x) \geq \int_{B_m} H(d_1(x, \partial\Omega)) d(|Dv^\infty_{x_i}|)(x)$$

for every B_m , due to Theorem 2.8. This implies that

$$\begin{aligned} \liminf_{n \rightarrow \infty} \int_{\Omega} H(d_1(x, \partial\Omega)) d(|Dv^n_{x_i}|)(x) &\geq \sup_{B_m} \int_{B_m} H(d_1(x, \partial\Omega)) d(|Dv^\infty_{x_i}|)(x) \\ &= \int_{\Omega} H(d_1(x, \partial\Omega)) d(|Dv^\infty_{x_i}|)(x). \end{aligned}$$

Therefore $\liminf_{n \rightarrow \infty} \mathcal{F}(v^n) \geq \mathcal{F}(v^\infty)$, i.e., v^∞ minimizes \mathcal{F} . □

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