

# On the strong convergence of the gradient in nonlinear parabolic equations

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## Abstract

We consider the Cauchy-Dirichlet Problem for a nonlinear parabolic equation with  $L^1$  data. We show how the concept of kinetic formulation for conservation laws [LPT94] can be used to give a new proof of the existence of renormalized solutions. To illustrate this approach, we also extend the method to the case where the equation involves an additional gradient term.

We consider the question of existence of solution to the nonlinear parabolic problem

$$u_t - \operatorname{div}(a(\nabla u)) = f \text{ in } \Omega \times (0, T), \quad (1a)$$

$$u = u_0 \text{ on } \Omega \times \{0\}, \quad (1b)$$

$$u = 0 \text{ on } \Sigma, \quad (1c)$$

where  $\Omega$  is a bounded subset of  $\mathbb{R}^N$ ,  $N \geq 1$ ,  $T$  is positive and  $\Sigma = \partial\Omega \times (0, T)$ . Let  $p > 1$  be given. In (1), the operator  $-\operatorname{div}(a(\nabla u))$  is assumed to be a Leray-Lions operator of exponent  $p$  (for example the  $p$ -Laplacian):

**Assumption 1** *The function  $a \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^N)$  satisfies: there exists  $\alpha > 0$ ,  $\beta \in C(\mathbb{R}_+, \mathbb{R}_+)$  such that*

$$a(X) \cdot X \geq \alpha |X|^p, \quad (2a)$$

$$|a(X)| \leq \beta |X|^{p-1}, \quad (2b)$$

$$(a(X) - a(Y)) \cdot (X - Y) > 0, \quad (2c)$$

for all distinct  $X, Y \in \mathbb{R}^N$ , where  $X \cdot Y$  is the canonical scalar product of two vectors of  $\mathbb{R}^N$  and  $|X|$  the associated euclidean norm of  $X$ .

The framework is  $L^1$ :

**Assumption 2** *The data  $u_0, f$  are  $L^1$  functions on  $\Omega$  and  $\Omega \times (0, T)$  respectively.*

**Remark 1** *The flux  $a$  may depend on  $x$  and  $u$ . More general problems also may be considered, with additional first-order terms  $\operatorname{div}(\Phi(u))$ ,  $\operatorname{div}(g)$  in Equation (1a), as in [BMR01] for example.*

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# 1 Introduction

The existence of solution (precisely, of renormalized solution, see Definition 1 below) to Problem (1) or quite more general problems has already been proved: we refer in particular to the paper by Blanchard, Murat, Redwane [BMR01]. Our purpose here is to give a new proof of this fact. The cornerstone in the proof of existence of solution (by means of a process of approximation) of such a nonlinear parabolic Problem as (1) is the proof of the strong convergence of the gradient. We give a new method (inspired from the kinetic formulation of conservation laws developed by Perthame and coauthors [LPT94, Per02, CP03]) to prove this result.

Let us briefly summarize how and in which context the question of strong convergence of the gradient occurs. First, as soon as the problem under consideration involves a nonlinear function of the gradient, for example (under the hypotheses above with  $p = 2$ ), the nonlinear elliptic Problem

$$-\operatorname{div}(a(\nabla u)) = g \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

for  $g \in L^2(\Omega)$ . Indeed, in order to prove existence of a solution (in  $H_0^1(\Omega)$ ), it is usual to prove existence by approximation (e.g. by Galerkin approximation), i.e. for a set of data  $g_n$  converging to  $g$ . Then weak convergence in  $H_0^1$  of (a subsequence of)  $u_n$ , the solution with datum  $g_n$ , although easily obtained by uniform estimate on  $\|u_n\|_{H^1(\Omega)}$ , is not enough to pass to the limit in the equation since  $a$  is nonlinear: one has to prove the *strong convergence* of the gradient  $\nabla u_n$ . This is done by use of monotonicity methods. We refer to [Min63, Bro63, LL65], and [Eva98] for a brief explanation of the technique.

Nonlinear expressions of the gradient also occur after renormalization of an elliptic or parabolic equation. Actually, they occur even if the original equation is linear. Nevertheless, renormalization for elliptic or parabolic equation has been introduced to deal with nonlinear equations with data of low regularity, so that the renormalized equation involves (at least) two nonlinear expressions of the gradient (see, e.g. Eq. (4) below). In any case, it will be necessary to prove the strong convergence of the (truncates of) the gradient in order to get existence of a solution by approximation.

We give a new proof of the strong convergence of the gradient by use of an equation on the characteristic function on the level sets of the unknown, similar to the kinetic formulation for conservation laws introduced in [LPT94] (see also [Per02] and [CP03] concerning the kinetic formulation of second-order conservation laws). We intend to use it to study certain systems of reaction-diffusion equations (a forthcoming paper).

Let us conclude this introduction by a few words about the concept of renormalized solutions. Introduced by DiPerna and Lions for the study of ordinary differential equations and Boltzmann Equation [DL89b, DL89a], it has been extended to nonlinear elliptic equations in [BGDM93] (in parallel with the (equivalent) notion of entropy solution [BBG<sup>+</sup>95]) and has been extended to nonlinear parabolic equations in [Bla93, BMR01, Lio96] (in parallel with the (equivalent) notion of entropy solution [Pri97]). It has also been extended to first-order conservation laws [BCW00, PV03].

The problem of strong convergence of the gradient (hence the question of existence of solution) has initially been solved by the method of Minty-Browder and Leray-Lions [Min63, Bro63, LL65], then extended to the case of nonlinear elliptic (then parabolic) equations with less and less regular data by several methods, see, e.g. [BG92a, BM92, BGM93, BGDM93, DMMOP97, BDGO99, DMMOP99, BMR01, BP05]. Notice that this list of references to some works in the field of renormalized solutions for elliptic and parabolic equations is far from being complete.

The paper is organized as follows : in Section 2.1, we introduce the notion of renormalized solution and state the equivalent formulation by the (so-called) level-set P.D.E. In Section 2.2, we analyze this formulation and explain how it can be relaxed (although still characterizing renormalized solutions), see Theorem 2 and Lemma 1. In Section 2.3, we apply our tools to prove the convergence of an approximation to Problem (1) and thus existence of a renormalized solution to (1) (of course, we focus on the strong

convergence of the gradient). In Section 3, we give the proofs of various results, which are reported at the end of the paper to let the main arguments of Section 2 stand out. Eventually, in Section 4, we extend the method to prove the existence of a renormalized solution to the Cauchy-Dirichlet Problem for a nonlinear parabolic equation with a term with natural growth.

**Notations :** We set  $Q_T := \Omega \times (-1, T)$  and  $U_T := Q_T \times \mathbb{R}$ . Any measurable function  $v: \Omega \times (0, T) \rightarrow \mathbb{R}^m$  is implicitly extended to a measurable function  $Q_T \rightarrow \mathbb{R}^m$  still denoted by  $v$ , defined by  $v \equiv 0$  on  $\Omega \times (-1, 0)$ .

If  $\nu$  is a Radon measure on  $U_T$ , we denote by  $\nu_*$  be the push-forward of  $\nu$  by projection on  $\mathbb{R}_\xi$ :

$$\nu_*(E) = \nu(Q_T \times E), \quad \forall E \in \mathcal{B}(\mathbb{R}),$$

where  $\mathcal{B}(\mathbb{R})$  is the Borel  $\sigma$ -algebra of  $\mathbb{R}$ . More generally, if  $E$  is a topological space,  $\mathcal{B}(E)$  denotes the  $\sigma$ -algebra of the Borel subsets of  $E$ .

If  $q \geq 1$  and  $V$  is an open subset of  $\mathbb{R}^q$  we denote by  $\mathcal{D}(V)$  the set of smooth ( $C^\infty$ ) functions on  $V$  compactly supported in  $V$  and we denote by  $\mathcal{D}'(V)$  the set of distributions on  $V$ .

## 2 Existence of a renormalized solution - strong convergence of the gradient

### 2.1 Renormalized solutions

#### 2.1.1 Renormalized solutions

For  $k > 0$ , we let  $T_k(u)$  be the truncate of a function  $u$  at level  $k$ :  $T_k(u) := \min(u, k)$  if  $u \geq 0$ ,  $T_k$  odd.

**Definition 1** A function  $u \in L^\infty(0, T; L^1(\Omega))$  is said to be a renormalized solution of the problem (1) if

1. (Regularity of the truncates)

$$T_k(u) \in L^p(0, T; W_0^{1,p}(\Omega)), \quad \forall k > 0. \quad (3)$$

2. (Renormalized equation) For every function  $S \in W^{2,\infty}(\mathbb{R})$  with  $S(0) = 0$  such that  $S'$  has compact support, the equation

$$S(u)_t - \operatorname{div}(S'(u)a(\nabla u)) = S(u_0) \otimes \delta_{t=0} + S'(u)f - S''(u)a(\nabla u) \cdot \nabla u \quad (4)$$

is satisfied in the sense of distributions in  $Q_T$ .

3. (Recovering at infinity)

$$\lim_{k \rightarrow +\infty} \int_{Q_T \cap \{k < |u| < k+1\}} a(\nabla u) \cdot \nabla u dx dt = 0. \quad (5)$$

#### 2.1.2 Level-set P.D.E.

For  $\alpha \in \mathbb{R}$ ,  $\xi \in \mathbb{R}$ , we set  $\chi_\alpha(\xi) = \mathbf{1}_{0 < \xi < \alpha} - \mathbf{1}_{\alpha < \xi < 0}$ . This is the ‘‘equilibrium function’’ in the kinetic formulation of conservation laws [LPT94]. Let  $u \in L^\infty(0, T; L^1(\Omega))$  satisfy (3). Then we define the (vector-valued) distribution

$$a(\nabla u)\delta_{u=\xi}$$

on  $U_T$  by its restriction to each space  $\mathcal{D}_K(U_T)^N$  (the set of smooth vector-valued functions with support in the compact subset  $K$  of  $U_T$ ) as

$$\langle a(\nabla u)\delta_{u=\xi}, \alpha \rangle = \int_Q a(\nabla T_k(u)) \cdot \alpha(x, t, T_k(u)) dx dt, \quad (6)$$

where  $\alpha \in \mathcal{D}_K(U_T)^N$ ,  $K \subset Q_T \times [-k, k]$ . Similarly, we define the distribution  $a(\nabla u) \cdot \nabla u \delta_{u=\xi}$  on  $U_T$  by

$$\langle a(\nabla u) \cdot \nabla u \delta_{u=\xi}, \alpha \rangle = \int_Q a(\nabla T_k(u)) \cdot \nabla T_k(u) \alpha(x, t, T_k(u)) dx dt, \quad (7)$$

for all  $\alpha \in \mathcal{D}_K(U_T)$ . By (3) and assumption (2b), we have

$$\begin{aligned} |\langle a(\nabla u) \delta_{u=\xi}, \alpha \rangle| &\leq \|a(\nabla T_k(u))\|_{L^{p'}(Q_T)} \|\alpha\|_{L^p(K)} \\ &\leq \beta C(K) \|T_k(u)\|_{L^p(0,T;W_0^{1,p}(\Omega))} \|\alpha\|_{L^\infty(K)}, \end{aligned}$$

and

$$|\langle a(\nabla u) \cdot \nabla u \delta_{u=\xi}, \alpha \rangle| \leq \beta \|T_k(u)\|_{L^p(0,T;W_0^{1,p}(\Omega))} \|\alpha\|_{L^\infty(K)}.$$

This shows that the right-hand sides of (6) and (7) are distributions on  $U_T$  of order 0. To prove that (6) and (7) makes sense, we must also show that their respective right-hand sides do not depend on the choice of  $k$ : suppose  $k < k'$  for example, with  $K \subset Q_T \times [-k, k]$ , then  $\alpha(x, t, T_k(u)) \neq 0$  for  $|u| \leq k$  only, in which case  $T_k(u) = T_{k'}(u)$ .

With this definitions at hand, we can give the “level-set” formulation of Definition 1.

**Theorem 1** *A function  $u \in L^\infty(0, T; L^1(\Omega))$  is a renormalized solution of the problem (1) if, and only if, it has the regularity of the truncates (3) and satisfies*

1. (Level-set P.D.E.) *The function  $(x, t, \xi) \mapsto \chi_{u(x,t)}(\xi)$ , denoted by  $\chi_u$ , is solution in  $\mathcal{D}'(U_T)$  of the equation*

$$\partial_t \chi_u - \operatorname{div}(a(\nabla u) \delta_{u=\xi}) = \chi_{u_0} \otimes \delta_{t=0} + f \delta_{u=\xi} + \partial_\xi \mu, \quad (8)$$

where  $\mu$  is defined by

$$\mu := a(\nabla u) \cdot \nabla u \delta_{u=\xi}, \quad (9)$$

2. (Recovering at infinity)

$$\lim_{k \rightarrow +\infty} \int_{Q_T \cap \{k < |u| < k+1\}} a(\nabla u) \cdot \nabla u dx dt = 0. \quad (10)$$

**Proof of Theorem 1 :** see Section 3.1.

## 2.2 Relaxation of the definition of renormalized solution - analysis of $\mu$

### 2.2.1 Analysis of $\mu$

Since  $\mu \geq 0$ ,  $\mu$  is represented by a nonnegative Radon measure on  $U_T$ . We study the properties of the push-forward  $\mu_*$  of  $\mu$ :  $\mu_*(E) = \mu(Q_T \times E)$ ,  $E \in \mathcal{B}(\mathbb{R})$ .

**Fact 1.** For every  $h \in C_c(\mathbb{R})$ ,

$$\int_{\mathbb{R}} h(\xi) d\mu_*(\xi) = \int_{U_T} h(\xi) d\mu(x, t, \xi). \quad (11)$$

**Proof:** by definition of  $\mu_*$ , (11) is satisfied if  $h = \mathbf{1}_E$  is the characteristic function of a Borel set  $E \subset \mathbb{R}$ , and therefore if  $h$  is a simple function. There exists a pointwise converging sequence of bounded simple functions with limit  $h$  with the same compact support as  $h$ : the Lebesgue dominated convergence Theorem gives the result. ■

**Fact 2.** For every  $h \in C_c(\mathbb{R})$  with, say,  $\operatorname{supp}(h) \subset [-k, k]$ ,

$$\int_{\mathbb{R}} h(\xi) d\mu_*(\xi) = \int_{Q_T} a(\nabla T_k(u)) \cdot \nabla T_k(u) h(u) dx dt. \quad (12)$$

**Proof:** let  $(\varphi_n)$  be a nonnegative sequence of  $C_c(Q_T)$  such that  $\varphi_n \uparrow 1$  everywhere on  $Q_T$ . By definition of  $\mu$ , we have

$$\int_{U_T} \varphi_n(x, t) h(\xi) d\mu(x, t, \xi) = \int_{Q_T} a(\nabla T_k(u)) \cdot \nabla T_k(u) \varphi_n(x, t) h(u) dx dt.$$

The Lebesgue dominated convergence Theorem then gives, at the limit  $[n \rightarrow +\infty]$ ,

$$\int_{U_T} h(\xi) d\mu(x, t, \xi) = \int_{Q_T} a(\nabla T_k(u)) \cdot \nabla T_k(u) h(u) dx dt.$$

We conclude by (11). ■

**Fact 3.** The measure  $\mu_*$  has no atom.

**Proof:** Given  $k > 0$ , set  $v = T_k(u)$ . For  $\xi_* \in (-k, k)$ , let  $(h_n)$  be a sequence of  $C_c(-k, k)$  converging monotonically to  $\mathbf{1}_{\{\xi_*\}}$  (take the  $h_n$  to be tent functions for example). For every  $n$ , we have, by (12),

$$\int_{\mathbb{R}} h_n(\xi) d\mu_*(\xi) = \int_{Q_T} a(\nabla v) \cdot \nabla v h_n(v) dx dt.$$

At the limit  $[n \rightarrow +\infty]$ , we obtain, by the Lebesgue dominated convergence Theorem,

$$\mu_*(\{\xi_*\}) = \int_{Q_T} a(\nabla v) \cdot \nabla v \mathbf{1}_{\{\xi_*\}}(v) dx dt. \quad (13)$$

For a.e.  $t$ ,  $v(t) \in W^{1,p}(\Omega)$ . For such  $t$ 's, we have  $\nabla v(t) = 0$  a.e. on  $\{x \in \Omega, v(x, t) = \xi_*\}$ . Indeed, we recall that, if  $w \in W^{1,1}(\Omega)$  and  $Z \subset \mathbb{R}$  is a Borel negligible set, then the set

$$\{x \in \Omega; w(x) \in Z, \nabla w(x) \neq 0\}$$

is negligible in  $\Omega$  (the proof goes back to Stampacchia and can be found in [BM84]). It follows therefore from (13) that  $\mu_*(\{\xi_*\}) = 0$ . ■

**Fact 4.** For every  $l > k$ ,

$$\int_{\mathbb{R}} \mathbf{1}_{(k,l)}(\xi) d\mu_*(\xi) = \int_{Q_T \cap \{k < u < l\}} a(\nabla u) \cdot \nabla u dx dt. \quad (14)$$

**Proof:** In the right hand-side of (14),  $u$  stands for  $T_m(u)$ ,  $m := \max(|k|, |l|)$ . Let  $(h_n)$  be a nonnegative sequence of  $C_c(k, l)$  such that  $h_n \uparrow \mathbf{1}_{(k,l)}$ . For each  $n$ , we have by (12),

$$\int_{\mathbb{R}} h_n(\xi) d\mu_*(\xi) = \int_{Q_T} a(\nabla u) \cdot \nabla u h_n(u) dx dt.$$

At the limit  $[n \rightarrow +\infty]$ , the dominated convergence Theorem gives the result. ■

**Fact 5.** For  $\varphi \in C_c(Q_T)$ ,  $\varphi \geq 0$ , define

$$\mu_\varphi(A) := \int_A \varphi(x, t) d\mu(x, t, \xi), \quad \forall A \text{ Borel subset of } U_T.$$

The measure  $\mu_\varphi$  has the same properties as  $\mu$  and its analysis follows the same lines. In particular,  $\mu_{\varphi,*}$  has no atoms and, for every  $k > 0$ ,

$$\mu_{\varphi,*}([-k, k]) = \mu_{\varphi,*}((-k, k)) = \int_{Q_T} a(\nabla T_k(u)) \cdot \nabla T_k(u) \varphi(x, t) dx dt. \quad (15)$$

**Remark 2** Note that the proof of the above **Facts** depends only on the property (3) of the truncates  $T_k(u)$ . Actually, we may even replace  $a(\nabla u)$  by any measurable  $\sigma: Q_T \rightarrow \mathbb{R}^N$ , such that  $\sigma \mathbf{1}_{|u| < k} \in L^p(Q_T)^N$  for all  $k > 0$ . This will be used in paragraph 2.3.3.

### 2.2.2 Relaxation of the definition of renormalized solution

According to the above **Facts** (paragraph 2.2.1), the condition (10) may be rewritten in terms of the push-forward  $\mu_*$  uniquely as

$$\lim_{k \rightarrow \pm\infty} \mu_*((k, k+1)) = 0, \quad (16)$$

where we recall that  $\mu$  is defined by (9). This simplifies the statement of Theorem 1 somewhat. However, what really makes plainer the characterization of renormalized solutions is the fact that, to some extent, it is not necessary to specify  $\mu$ . This characterization is as follows.

**Theorem 2** *Let  $u$  be a function of  $L^\infty(0, T; L^1(\Omega))$  which has the regularity of the truncates (3) and satisfies the condition at infinity (5). Then  $u$  is a renormalized solution of the problem (1) if, and only if, there exists a nonnegative Radon measure  $\mu$  on  $U_T$  satisfying (16) and such that*

$$\partial_t \chi_u - \operatorname{div}(a(\nabla u) \delta_{u=\xi}) = \chi_{u_0} \otimes \delta_{t=0} + f \delta_{u=\xi} + \partial_\xi \mu, \quad (17)$$

in the sense of distributions on  $U_T$ .

The proof of Theorem 2 consists in showing that  $\mu = a(\nabla u) \cdot \nabla u \delta_{u=\xi}$ . It is therefore a result of structure of  $\mu$ : under the hypotheses of Theorem 2 and (17),  $\mu$  has to be the measure  $a(\nabla u) \cdot \nabla u \delta_{u=\xi}$ . Theorem 2 has the virtue to give a plain characterization of renormalized solutions to (1). However, to prove the convergence of a sequence of approximate solutions to (1) and the existence of solution, we will need a slight generalization of Theorem 2 contained in the following lemma.

**Lemma 1** *Let  $u$  be a function of  $L^\infty(0, T; L^1(\Omega))$  which has the regularity of the truncates (3). Let  $\sigma$  be a measurable function  $\Omega \times (0, T) \rightarrow \mathbb{R}^N$  such that*

$$\forall k > 0, \sigma \mathbf{1}_{|u| < k} \in L^{p'}(Q_T)^N.$$

Suppose that there exists a nonnegative Radon measure  $\mu$  on  $U_T$  such that

$$\lim_{k \rightarrow \pm\infty} \mu_*((k, k+1)) = 0, \quad (18)$$

and such that the following equation is satisfied in  $\mathcal{D}'(U_T)$

$$\partial_t \chi_u - \operatorname{div}(\sigma \delta_{u=\xi}) = \chi_{u_0} \otimes \delta_{t=0} + f \delta_{u=\xi} + \partial_\xi \mu. \quad (19)$$

Suppose also that: either

$$u \geq 0 \text{ a.e. and } \operatorname{supp}(\mu) \subset \overline{Q_T} \times [0, +\infty), \quad (20)$$

or the distribution  $\sigma \cdot \nabla u \delta_{u=\xi}$  satisfies the (sided) condition at infinity

$$\limsup_{k \rightarrow +\infty} \langle \sigma \cdot \nabla u \delta_{u=\xi}, \varphi \otimes \mathbf{1}_{(k, k+1)} \rangle \leq 0, \quad \forall \varphi \in C(\overline{Q_T}), \quad \varphi \geq 0. \quad (21)$$

Then  $\mu = \sigma \cdot \nabla u \delta_{u=\xi}$ .

**Proof of Lemma 1** : see Section 3.2.

In Lemma 1 the definition of the distribution  $\sigma \cdot \nabla u \delta_{u=\xi}$  is comparable to the definition of the distribution  $a(\nabla u) \cdot \nabla u \delta_{u=\xi}$  by (7):

$$\langle \sigma \cdot \nabla u \delta_{u=\xi}, \alpha \rangle = \int_Q (\sigma \mathbf{1}_{|u| < k}) \cdot \nabla T_k(u) \alpha(x, t, T_k(u)) dx dt, \quad (22)$$

for all  $\alpha \in \mathcal{D}_K(U_T)$ ,  $K$  compact subset of  $Q_T \times [-k, k]$ .

Equation (19) appears naturally when one considers limits of renormalized solutions, in particular of solutions of approximate equations  $u_t^n - \operatorname{div}(a(\nabla u^n)) = f^n$ : see Section 2.3.2 and Section 2.3.3.

In the situation of Lemma 1, once the equality  $\mu = \sigma \cdot \nabla u \delta_{u=\xi}$  has been proved, and thanks to Remark 2, we deduce the following corollary.

**Corollary 1** Under the hypotheses of Lemma 1, and given  $\varphi \in C_c(Q_T)$ ,  $\varphi \geq 0$ , the measure  $\mu_{\varphi,*}$  has no atom and

$$\mu_{\varphi,*}([-k, k]) = \mu_{\varphi,*}((-k, k)) = \int_{Q_T} \sigma \cdot \nabla T_k(u) \varphi(x, t) dx dt, \quad (23)$$

for all  $k > 0$ .

## 2.3 Existence of a renormalized solution - Strong convergence of the gradient

### 2.3.1 Approximation

Let  $(u_0^n)$  and  $(f^n)$  be some approximating sequences of, respectively,  $u_0$  and  $f$  in, respectively,  $L^1(\Omega)$  and  $L^1(\Omega \times (0, T))$  such that  $u_0^n \in L^p(\Omega)$ ,  $f^n \in L^{p'}(\Omega \times (0, T))$ . For each  $n$ , the problem

$$u_t^n - \operatorname{div}(a(\nabla u^n)) = f^n \text{ in } \Omega \times (0, T), \quad (24a)$$

$$u^n = u_0^n \text{ on } \Omega \times \{0\}, \quad (24b)$$

$$u^n = 0 \text{ on } \Sigma, \quad (24c)$$

has a unique solution  $u^n$  in the space

$$\left\{ v \in L^p(0, T; W_0^{1,p}(\Omega)); v_t \in L^{p'}(0, T; W^{-1,p}(\Omega)) \right\}.$$

(We refer to [Lio69] for example). The function  $u^n$  is a weak solution to (24), hence a renormalized solution to (24), and therefore satisfies the equation

$$\partial_t \chi_{u^n} - \operatorname{div}(a(\nabla u^n) \delta_{u^n=\xi}) = \chi_{u_0^n} \otimes \delta_{t=0} + f^n \delta_{u^n=\xi} + \partial_\xi \mu^n, \quad (25)$$

where  $\mu^n$  is defined by

$$\mu^n := a(\nabla u^n) \cdot \nabla u^n \delta_{u^n=\xi}. \quad (26)$$

### 2.3.2 Estimates and limit equation

There are bounds independent on  $n$  on  $u^n$  in  $L^\infty(0, T; L^1(\Omega))$ , on  $a(\nabla T_k(u^n)) \cdot \nabla T_k(u^n)$  in  $L^1(Q_T)$ , on  $\nabla T_k(u^n)$  in  $L^p(Q_T)$ , on  $T_k(u^n)_t$  in  $L^{p'}(0, T; W^{-1,p}(\Omega)) + L^1(\Omega \times (0, T))$ . They are obtained by multiplying the equation by  $T_k(u^n)$  (see, e.g., [BMR01]).

Aubin-Simon's compactness Theorem shows that there exists a function  $u \in L^\infty(0, T; L^1(\Omega))$  such that, up to a subsequence,  $u^n \rightarrow u$  a.e. and in  $L^1(Q_T)$ . By weak compactness of  $L^p$  and  $L^{p'}$ , we can suppose that  $T_k(u^n) \rightarrow T_k(u)$  in  $L^p(0, T; W_0^{1,p}(\Omega))$ -weak and  $a(\nabla T_k(u^n)) \rightarrow \sigma_k$  in  $L^{p'}(Q_T)$ -weak. Consider some common subsequences, still denoted  $(u^n)$ , such that

$$a(\nabla T_k(u^n)) \rightarrow \sigma_k \text{ and } a(\nabla T_{k+1}(u^n)) \rightarrow \sigma_{k+1}$$

in  $L^{p'}(Q_T)$ -weak. Since  $\nabla T_k(u^n) = \nabla(T_k \circ T_{k+1})(u^n) = T'_k \circ T_{k+1}(u^n) \nabla T_{k+1}(u^n)$ , and since  $a(0) = 0$ , we have

$$a(\nabla T_k(u^n)) = \mathbf{1}_{|u^n| < k} a(\nabla T_{k+1}(u^n)). \quad (27)$$

As  $u^n \rightarrow u$  a.e., hence  $\mathbf{1}_{|u^n| < k} \rightarrow \mathbf{1}_{|u| < k}$  a.e. while being bounded, both sides of (27) converge in  $L^{p'}(Q_T)$ -weak and, at the limit, we obtain  $\sigma_k = \mathbf{1}_{|u| < k} \sigma_{k+1}$ . This shows (using a diagonal process) that there exists an additional subsequence still denoted  $(u^n)$  and a measurable function  $\sigma: \Omega \times (0, T) \rightarrow \mathbb{R}^n$  such that

$$\forall k > 0, \sigma \mathbf{1}_{|u| < k} \in L^{p'}(Q_T), \quad a(\nabla T_k(u^n)) \rightarrow \sigma \mathbf{1}_{|u| < k} \text{ in } L^{p'}(Q_T) \text{ weak.}$$

The bound on  $a(\nabla T_k(u^n)) \cdot \nabla T_k(u^n)$  in  $L^1(Q_T)$  gives a uniform bound on  $\mu^n(K)$  for each compact subset  $K$  of  $U_T$ . We can therefore suppose that  $(\mu^n)$  converges weakly to a Radon measure  $\mu$  in  $U_T$ . Note that we have then

$$\mu_*(E) \leq \liminf_{n \rightarrow +\infty} \mu_*^n(E), \quad (28)$$

for each  $E \subset \mathbb{R}$  open. Indeed,

$$\mu_*(E) = \mu(Q_T \times E) \leq \liminf_{n \rightarrow +\infty} \mu^n(Q_T \times E) = \liminf_{n \rightarrow +\infty} \mu_*^n(E),$$

since  $Q_T \times E$  is open in  $U_T$ .

With these results of convergence at hand, we let  $n \rightarrow +\infty$  in (25) to obtain the limit equation

$$\partial_t \chi_u - \operatorname{div}(\sigma \delta_{u=\xi}) = \chi_{u_0} \otimes \delta_{t=0} + f \delta_{u^n=\xi} + \partial_\xi \mu. \quad (29)$$

By multiplying (24a) by the function  $(T_{k+1} - T_k)(u^n)$  ( $k > 0$ ) we obtain the estimate

$$\int_{Q_T \cap \{k < |u^n| < k+1\}} a(\nabla u^n) \cdot \nabla u^n dx dt \leq \int_{\Omega \cap \{|u_0^n| > k\}} |u_0^n| dx + \int_{Q_T \cap \{|u^n| > k\}} |f^n| dx dt.$$

This is

$$\mu_*^n((k, k+1)) + \mu_*^n((-k-1, -k)) \leq \int_{\Omega \cap \{|u_0^n| > k\}} |u_0^n| dx + \int_{Q_T \cap \{|u^n| > k\}} |f^n| dx dt.$$

Up to a subsequence (and as a consequence of the strong convergence in  $L^1$ ), there exists some functions  $\bar{u}_0, \bar{u}, \bar{f}$  in  $L^1(\Omega)$  and  $L^1(Q_T)$  respectively such that  $|u_0^n| \leq \bar{u}_0$ ,  $|u^n| \leq \bar{u}$ ,  $|f^n| \leq \bar{f}$  a.e. This implies the uniform estimates

$$\mu_*^n((k, k+1)) + \mu_*^n((-k-1, -k)) \leq \int_{\Omega \cap \{\bar{u}_0 > k\}} \bar{u}_0 dx + \int_{Q_T \cap \{\bar{u} > k\}} \bar{f} dx dt, \quad (30)$$

from which we deduce by (28):

$$\mu_*((k, k+1)) + \mu_*((-k-1, -k)) \leq \int_{\Omega \cap \{\bar{u}_0 > k\}} \bar{u}_0 dx + \int_{Q_T \cap \{\bar{u} > k\}} \bar{f} dx dt.$$

In particular,

$$\lim_{k \rightarrow \pm\infty} \mu_*((k, k+1)) = 0. \quad (31)$$

In the nonnegative case, i.e.  $u_0, u_0^n \geq 0$  a.e.,  $f, f^n \geq 0$  a.e., the approximate solutions are nonnegative, and therefore  $u \geq 0$  a.e. and  $\mu$  is supported in  $\overline{Q_T} \times [0, +\infty)$ : Hypothesis (20) in Lemma 1 is satisfied. Let us show that, independently on any sign condition, Hypothesis (21) is satisfied: let  $\varphi \in C(\overline{Q_T})$ ,  $\varphi \geq 0$ . For  $k > 0$ ,  $n, m \in \mathbb{N}$ , and by monotonicity of  $a$ , we have

$$0 \leq \int_{Q_T} \langle a(\nabla v_k^n) - a(\nabla v_k^m), \nabla v_k^n - \nabla v_k^m \rangle \varphi dx dt, \quad v_k^n := (T_{k+1} - T_k)^+(u^n),$$

i.e.

$$\begin{aligned} & \int_{Q_T} (a(\nabla u^n) \cdot \nabla u^m + a(\nabla u^m) \cdot \nabla u^n) \mathbf{1}_{(k, k+1)}(u^n) \mathbf{1}_{(k, k+1)}(u^m) \varphi dx dt \\ & \leq \int_{Q_T} (a(\nabla u^n) \cdot \nabla u^n \mathbf{1}_{(k, k+1)}(u^n) + a(\nabla u^m) \cdot \nabla u^m \mathbf{1}_{(k, k+1)}(u^m)) \varphi dx dt. \end{aligned}$$

Denoting by  $\varepsilon_k$  the right hand-side of (30), we deduce

$$\int_{Q_T} (a(\nabla u^n) \cdot \nabla u^m + a(\nabla u^m) \cdot \nabla u^n) \mathbf{1}_{(k, k+1)}(u^n) \mathbf{1}_{(k, k+1)}(u^m) \varphi dx dt \leq 2\varepsilon_k \|\varphi\|_\infty.$$

Taking the limit  $[n \rightarrow +\infty]$ , then  $[m \rightarrow +\infty]$  and  $[k \rightarrow \infty]$ , we obtain

$$\limsup_{k \rightarrow +\infty} \langle \sigma \cdot \nabla u \delta_{u=\xi}, \varphi \otimes \mathbf{1}_{(k, k+1)} \rangle \leq 0, \quad (32)$$

which shows that Hypothesis (21) is satisfied.

### 2.3.3 Strong convergence of the gradient

Taking into account Eq. (29) and the estimates (31)-(32), we are in position to apply Lemma 1, which gives

$$\mu = \sigma \cdot \nabla u \delta_{u=\xi}.$$

We want to examine the weak convergence of the push-forward  $\mu_*^n$  to  $\mu_*$ . We fix a test-function  $\varphi \in C_c(Q_T)$ ,  $\varphi \geq 0$ . We use the notations of Section 2.2.1, in particular

$$\mu_\varphi(A) := \int_A \varphi(x, t) d\mu(x, t, \xi), \quad \forall A \in \mathcal{B}(U_T).$$

Then, if  $\psi \in C_c(\mathbb{R})$ , we have

$$\int_{\mathbb{R}} \psi d\mu_{\varphi,*}^n = \int_{U_T} \varphi \otimes \psi d\mu^n,$$

where  $\varphi \otimes \psi(x, t, \xi) = \varphi(x, t)\psi(\xi) \in C_c(U_T)$ , hence

$$\int_{\mathbb{R}} \psi d\mu_{\varphi,*}^n \rightarrow \int_{U_T} \varphi \otimes \psi d\mu = \int_{\mathbb{R}} \psi d\mu_{\varphi,*},$$

and we conclude that  $(\mu_{\varphi,*}^n)$  converges weakly to  $\mu_\varphi$  on  $\mathbb{R}$ . Let  $k > 0$ . By (23) the  $\mu_{\varphi,*}$ -measure of the boundary of  $[-k, k]$  is zero and, by weak convergence, we obtain

$$\mu_{\varphi,*}^n([-k, k]) \rightarrow \mu_{\varphi,*}([-k, k]). \quad (33)$$

This identity (33) is the central result in the proof of the strong convergence of the gradient. Indeed, by (15) and (23), (33) reads

$$\int_{Q_T} a(\nabla T_k(u^n)) \cdot \nabla T_k(u^n) \varphi dx dt \rightarrow \int_{Q_T} \sigma \cdot \nabla T_k(u) \varphi dx dt \quad (34)$$

and from (34) follows the strong convergence of the gradient

$$\nabla T_k(u^n) \rightarrow \nabla T_k(u) \text{ a.e.} \quad (35)$$

Although the argument is classical, we give the proof of the implication (34)  $\Rightarrow$  (35) in Section 3.4. By (35), we have in particular  $\sigma = a(\nabla u)$  a.e. on  $Q_T$ : therefore  $u$  is solution to the level-set p.d.e. associated to Problem (1):

$$\partial_t \chi_u - \operatorname{div}(a(\nabla u) \delta_{u=\xi}) = \chi_{u_0} \otimes \delta_{t=0} + f \delta_{u=\xi} + \partial_\xi (a(\nabla u) \cdot \nabla u \delta_{u=\xi}).$$

By Theorem 1,  $(u^n)$  converges to  $u$ , which is a renormalized solution to Problem (1).

## 3 Missing proofs

### 3.1 Proof of Theorem 1

Since  $\{\varphi \otimes \theta; \varphi \in \mathcal{D}(Q_T), \theta \in \mathcal{D}(\mathbb{R})\}$  is dense in  $\mathcal{D}(U_T)$ , (8) is equivalent to: for all  $\theta \in \mathcal{D}(\mathbb{R})$ ,

$$\langle \partial_t \chi_u - \operatorname{div}(a(\nabla u) \delta_{u=\xi}), \theta \rangle_{\mathcal{D}'(\mathbb{R}_\xi), \mathcal{D}(\mathbb{R}_\xi)} = \langle \chi_{u_0} \otimes \delta_{t=0} + f \delta_{u=\xi} + \partial_\xi \mu, \theta \rangle_{\mathcal{D}'(\mathbb{R}_\xi), \mathcal{D}(\mathbb{R}_\xi)}$$

in  $\mathcal{D}'(Q_T)$ . By definition of  $\mu$ , this is equivalent to: for all  $\theta \in \mathcal{D}(\mathbb{R})$ ,

$$\partial_t \int_{\mathbb{R}} \chi_u \theta d\xi - \operatorname{div}(\theta(u) a(\nabla u)) = \left( \int_{\mathbb{R}} \chi_{u_0} \theta d\xi \right) \otimes \delta_{t=0} + \theta(u) f - \theta'(u) a(\nabla u) \cdot \nabla u \quad (36)$$

in  $\mathcal{D}'(Q_T)$ . The correspondence between (4) and (8) is obtained by taking  $\theta = S'$  in (36), by the identity

$$\int_{\mathbb{R}} \chi_u(\xi) S'(\xi) d\xi = S(u),$$

satisfied for all  $S \in W^{2,\infty}(\mathbb{R})$  such that  $S(0) = 0$ , and by a standard argument of density.  $\blacksquare$

### 3.2 Proof of Lemma 1

Set  $\nu := \sigma \cdot \nabla u \delta_{u=\xi}$  (see (22) for the definition of  $\nu$ ). We have to check that  $\langle \mu, \varphi \otimes \psi \rangle = \langle \nu, \varphi \otimes \psi \rangle$  for all  $\varphi \in \mathcal{D}(Q_T)$ ,  $\psi \in \mathcal{D}(\mathbb{R})$ . We first suppose that  $\psi = \partial_\xi \theta$  with  $\theta \in \mathcal{D}(\mathbb{R})$ , so that  $\langle \mu, \varphi \otimes \psi \rangle = -\langle \partial_\xi \mu, \varphi \otimes \theta \rangle$ . By (19),  $\langle \mu, \varphi \otimes \psi \rangle = \langle \nu, \varphi \otimes \psi \rangle$  is then equivalent to the following identity

$$-\int_0^T \int_\Omega \left( \int_{u_0}^u \theta(\xi) d\xi \right) \varphi_t + \int_0^T \int_\Omega (\sigma \cdot \nabla \varphi) \theta(u) - \int_0^T \int_\Omega f \varphi \theta(u) = -\int_0^T \int_\Omega (\sigma \cdot \nabla u) \varphi \theta'(u). \quad (37)$$

By use of the rule of derivation of a product of functions in  $W^{1,p} \cap L^\infty$ , we obtain the equivalent, more compact form of (37):

$$-\int_0^T \int_\Omega \left( \int_{u_0}^u \theta(\xi) d\xi \right) \varphi_t + \int_0^T \int_\Omega \sigma \cdot \nabla (\varphi \theta(u)) - \int_0^T \int_\Omega f \varphi \theta(u) = 0. \quad (38)$$

Eq. (38) can be *formally* deduced from the chain-rule formula and from the equation

$$0 = \partial_t u - \operatorname{div}(\sigma) - u_0 \otimes \delta_{t=0} - f. \quad (39)$$

Let us also remark that, formally, the equation (39) can be deduced from Eq. (19) by integrating with respect to  $\xi \in \mathbb{R}$ . Indeed, that  $\mu(\xi) \rightarrow 0$  when  $\xi \rightarrow \pm\infty$  is, still at the formal level, a consequence of the condition  $\mu_*((k, k+1)) \rightarrow 0$  when  $k \rightarrow \pm\infty$ . Therefore, we begin with the derivation of an approximate form of Eq. (39): fix  $k > 0$ , let  $(\rho_n)_n$  be an approximation of the unit on  $\mathbb{R}$  ( $\rho_n$  having compact support in  $[-1/n, 1/n]$ ), set  $\alpha_k := \rho_k * \mathbf{1}_{[k, k+1]}$ , and define

$$r^k = r^k(u) = \int_{|u|}^\infty \alpha_k, \quad v^k := \int_{\mathbb{R}} \chi_u(\xi) r^k(\xi) d\xi, \quad v_0^k := \int_{\mathbb{R}} \chi_{u_0}(\xi) r^k(\xi) d\xi.$$

We have  $v^k \in L^p(-1, T; W_0^{1,p}(\Omega)) \cap L^\infty(Q_T)$ ,  $v_0^k \in L^\infty(\Omega)$  and, for  $l > 0$ ,

$$r^k \rightarrow 1 \text{ a.e.}, \quad T_l(v^k) \rightarrow T_l(u) \text{ in } L^p(-1, T; W_0^{1,p}(\Omega)), \quad v_0^k \rightarrow u_0 \text{ a.e.},$$

when  $k$  tends to  $+\infty$ . Test Eq. (19) against  $\varphi(t, x) r_k(\xi)$  to obtain

$$-\int_0^T \int_\Omega (v^k - v_0^k) \varphi_t + \int_0^T \int_\Omega \sigma \cdot \nabla \varphi r^k - \int_0^T \int_\Omega f \varphi r^k = \int_0^T \int_\Omega \int_{\mathbb{R}} \varphi \alpha_k d\mu. \quad (40)$$

This is the approximate form of (39). Now we want to use a kind of chain-rule formula to obtain an approximation of (38). To this purpose, we first infer from (40) the inequality

$$\left| \int_{Q_T} \varphi_t (v^k - v_0^k) - \int_0^T \langle G^k, \varphi \rangle dt \right| \leq \|\varphi\|_{L^\infty} \varepsilon_k, \quad (41)$$

where  $G^k := -(\operatorname{div}(\sigma r^k(u)) + f r^k(u)) \in L^{p'}(0, T; W^{-1,p'}(\Omega)) + L^1(Q)$  and  $\varepsilon_k := \mu_*((k-1, k+2)) + \mu_*((-k-2, -k+1)) \rightarrow 0$  when  $k \rightarrow +\infty$ . We then consider the following lemma.

**Lemma 2** *Let  $\varepsilon > 0$ ,  $v \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^1(Q_T)$ ,  $v_0 \in L^\infty(\Omega)$  and*

$$G \in L^{p'}(0, T; W^{-1,p'}(\Omega)) + L^1(Q_T)$$

*satisfy*

$$\left| \int_0^T \int_\Omega \varphi_t (v - v_0) - \int_0^T \langle G, \varphi \rangle dt \right| \leq \|\varphi\|_{L^\infty} \varepsilon, \quad (42)$$

for all  $\varphi \in \mathcal{D}(Q_T)$ . Then, for all  $\varphi \in \mathcal{D}(\mathbb{R}^N \times (-1, T))$ , for all  $h \in W^{1,\infty}(\mathbb{R})$  such that

$$(h(v)\varphi)(t) = 0 \text{ on } \partial\Omega, \text{ for a.e. } t \in (0, T), \quad (43)$$

we have

$$\left| \int_0^T \int_{\Omega} \varphi_t \int_{v_0}^v h(\xi) d\xi - \int_0^T \langle G, h(v)\varphi \rangle dt \right| \leq \|\varphi\|_{L^\infty} \|h\|_{L^\infty} \varepsilon. \quad (44)$$

The Dirichlet condition (43) makes sense since  $h(v)\varphi \in L^p(0, T; W^{1,p}(\Omega))$ . The proof of Lemma 2 is given in the following section. We apply Lemma 2 to (41), with  $\varphi \in \mathcal{D}(Q_T)$ ,  $h(v) = \theta(v)$  to deduce

$$\left| \int_0^T \int_{\Omega} \left( \int_{v_0^k}^{v^k} \theta(\xi) d\xi \right) \varphi_t - \int_0^T \int_{\Omega} \sigma r_k(u) \cdot \nabla (\varphi \theta(v^k)) + \int_0^T \int_{\Omega} f r_k(u) \varphi \theta(v^k) \right| \leq \|\varphi \otimes \theta\|_{L^\infty} \varepsilon_k.$$

By use of the Lebesgue dominated convergence Theorem, we obtain (38) at the limit  $k \rightarrow +\infty$ . Recall that  $\psi = \partial_\xi \theta$ , so that we actually proved that  $\partial_\xi(\mu - \nu) = 0$ . By a classical Lemma in the theory of distributions, this shows that  $\mu - \nu$  is constant with respect to  $\xi$ , or, more precisely, that for every  $\kappa \in \mathcal{D}(Q_T)$  the distribution on  $\mathbb{R}$  defined by  $\psi \mapsto \langle \mu - \nu, \kappa \otimes \psi \rangle$  is represented by a constant  $c_\kappa$ . There remains to show that  $c_\kappa = 0$ .

In the case of nonnegative solution, *i.e.* under Hypothesis (20), this is straightforward since both  $\mu$  and  $\nu$  vanish on  $Q_T \times (-\infty, 0)_\xi$ . In the general case, *i.e.* under Hypothesis (21), we show that  $\nu$  actually satisfies the condition at infinity

$$\lim_{k \rightarrow +\infty} \langle \nu, \varphi \otimes \mathbf{1}_{(k, k+1)} \rangle = 0, \quad \forall \varphi \in C_c(Q_T). \quad (45)$$

Since  $c_\kappa = \langle \mu - \nu, \kappa \otimes \mathbf{1}_{(k, k+1)} \rangle$  for all  $k$ , this will give the result by (18).

To prove (45), we first observe that it is sufficient to obtain (45) for regular test-functions  $\varphi$  in the multiplicative form

$$\varphi(x, t) = \varphi_1(t) \varphi_2(x), \quad \varphi_1 \in C_c^1(-1, T), \quad \varphi_2 \in C_c^1(\Omega), \quad \varphi_i \geq 0.$$

We then apply Lemma 2 to Eq. (40) with  $\varphi(x, t) = \varphi_1(t) \|\varphi_2\|_\infty$ ,  $h(v) = (T_{l+1} - T_l)(v)$ ,  $l > 0$  (observe that  $h \in W^{1,\infty}(\mathbb{R})$ , and  $h(0) = 0$  so that (43) is satisfied) and let  $k \rightarrow +\infty$  to obtain Eq. (38) as above with  $\varphi = \varphi_1(t) \|\varphi_2\|_\infty$ , *i.e.*

$$\begin{aligned} \int_0^T \int_{\Omega} (\sigma \cdot \nabla u) \varphi_1(t) \|\varphi_2\|_\infty \mathbf{1}_{(l, l+1)}(u) &= \int_0^T \int_{\Omega} \left( \int_{u_0}^u (T_{l+1} - T_l)(\xi) d\xi \right) \varphi_1'(t) \|\varphi_2\|_\infty \\ &\quad + \int_0^T \int_{\Omega} f \varphi_1(t) \|\varphi_2\|_\infty (T_{l+1} - T_l)(u) d\xi. \end{aligned}$$

Relabel  $l$  by  $k$  and take the limit  $[k \rightarrow +\infty]$  to obtain

$$\lim_{k \rightarrow +\infty} \langle \nu, \varphi_1 \|\varphi_2\|_\infty \otimes \mathbf{1}_{(k, k+1)} \rangle = 0. \quad (46)$$

Since  $-\varphi + \varphi_1 \|\varphi_2\|_\infty \in C(\overline{Q_T})$  is nonnegative, we also have, by (21):

$$\limsup_{k \rightarrow +\infty} \langle \nu, (-\varphi + \varphi_1 \|\varphi_2\|_\infty) \otimes \mathbf{1}_{(k, k+1)} \rangle \leq 0.$$

This, combined with (46), shows that  $\liminf_{k \rightarrow +\infty} \langle \nu, \varphi \otimes \mathbf{1}_{(k, k+1)} \rangle \geq 0$ . Using (21) again, we obtain (45). ■

### 3.3 Proof of Lemma 2

It is a variation on the proof of Lemma 4.3 in [CW99] (Lemma 4.3 of [CW99] corresponds to the case  $\varepsilon = 0$ ).

**Step 1.** Suppose that  $v_0$  additionally satisfies  $v_0 \in W_0^{1,p}(\Omega)$ . For  $t < 0$ , set  $v(t) = v_0$ .

Also first suppose  $h$  is non-increasing and  $\varphi$  nonnegative or  $h$  is non-decreasing and  $\varphi$  non-positive. We have

$$-\|\varphi\|_{L^\infty} \varepsilon \leq \int_0^T \int_\Omega \varphi_t (v - v_0) - \int_0^T \langle G, \varphi \rangle dt \leq \|\varphi\|_{L^\infty} \varepsilon \quad (47)$$

for all  $\varphi \in \mathcal{D}(Q_T)$  and thus, by regularity of  $v, G$ , for all  $\varphi \in L^p(-1, T; W_0^{1,p}(\Omega)) \cap L^\infty(Q_T)$  with  $\varphi_t \in L^p(Q_T)$ . To use the function  $h(v)$  as a test-function in (47), we have first to regularize its dependence on  $t$ : for fixed  $\varphi \in \mathcal{D}^+(Q_T)$  and for  $\eta > 0$  small enough (such that  $\text{supp}(\varphi) \subset \Omega \times (-1, T - 2\eta)$ ), we set  $\zeta := \varphi h(v)$ ,

$$\zeta_\eta: (x, t) \rightarrow \frac{1}{\eta} \int_{t-\eta}^t \zeta(x, s) ds.$$

In (47), this gives

$$\begin{aligned} \int_0^T \langle G, \zeta_\eta \rangle dt &\leq \|\varphi\|_{L^\infty} \|h\|_{L^\infty} \varepsilon + \int_0^T \int_\Omega (\varphi_\eta)_t (v - v_0) \\ &= \|\varphi\|_{L^\infty} \|h\|_{L^\infty} \varepsilon + \int_0^T \int_\Omega \frac{1}{\eta} (\zeta(x, t) - \zeta(x, t - \eta)) (v - v_0)(x, t) dx dt \\ &= \|\varphi\|_{L^\infty} \|h\|_{L^\infty} \varepsilon + \int_{\mathbb{R}} \int_\Omega \frac{1}{\eta} (v(x, t) - v(x, t + \eta)) \zeta(x, t) dx dt \\ &= \|\varphi\|_{L^\infty} \|h\|_{L^\infty} \varepsilon + \int_{\mathbb{R}} \int_\Omega \frac{1}{\eta} (v(t) - v(t + \eta)) h(v(t)) \varphi(t) dx dt. \end{aligned}$$

Since  $h$  is non-increasing and  $\varphi$  nonnegative or  $h$  is non-decreasing and  $\varphi$  non-positive, we have the inequality

$$(v(t) - v(t + \eta)) h(v(t)) \varphi(t) \leq \int_{v(t)}^{v(t+\eta)} h(r) dr \varphi(t), \quad t < T,$$

hence

$$\begin{aligned} \int_0^T \langle G, \zeta_\eta \rangle dt &\leq \|\varphi\|_{L^\infty} \|h\|_{L^\infty} \varepsilon + \int_{\mathbb{R}} \int_\Omega \varphi(t) \frac{1}{\eta} \int_{v(t)}^{v(t+\eta)} h(r) dr \\ &= \|\varphi\|_{L^\infty} \|h\|_{L^\infty} \varepsilon + \int_{\mathbb{R}} \int_\Omega \frac{1}{\eta} (\varphi(t) - \varphi(t - \eta)) \int_{v_0}^{v(t)} h(r) dr. \end{aligned}$$

At the limit  $\eta \rightarrow 0$ , a first inequality is obtained

$$\int_0^T \langle G, h(v)\varphi \rangle dt \leq \|\varphi\|_{L^\infty} \|h\|_{L^\infty} \varepsilon + \int_0^T \int_\Omega \varphi_t \int_{v_0}^{v(t)} h(r) dr.$$

By use of  $\zeta_\eta: (x, t) \rightarrow \frac{1}{\eta} \int_t^{t+\eta} \zeta(x, s) ds$  as a test-function, we derive in a similar way the second inequality

$$\int_0^T \langle G, h(v)\varphi \rangle dt \geq -\|\varphi\|_{L^\infty} \|h\|_{L^\infty} \varepsilon + \int_0^T \int_\Omega \varphi_t \int_{v_0}^{v(t)} h(r) dr,$$

which gives (44). In case  $h$  is non-decreasing and  $\varphi$  nonnegative or  $h$  is non-increasing and  $\varphi$  non-positive, proceed similarly (just exchanging the order of the different time-regularizations) to prove (44),

then decompose  $h$  as the sum of two monotone functions and  $\varphi$  as the sum of two signed functions to deduce the result in the general case.

**Step 2.** In the general case where  $v_0 \in L^\infty(Q)$ , regularize  $v_0$  by  $v_0^n$ ,  $v_0^n \in W_0^{1,p}(\Omega)$ ,  $\|v_0 - v_0^n\|_{L^1(\Omega)} \leq 1/n$ . Observe that, from (42), we deduce

$$\left| \int_0^T \int_\Omega \varphi_t (v - v_0^n) - \int_0^T \langle G, \varphi \rangle dt \right| \leq \|\varphi\|_{L^\infty} (\varepsilon + 1/n). \quad (48)$$

Apply Step 1. to get

$$\left| \int_0^T \int_\Omega \varphi_t \int_{v_0^n}^v h(\xi) d\xi - \int_0^T \langle G, h(v)\varphi \rangle dt \right| \leq \|\varphi\|_{L^\infty} \|h\|_{L^\infty} (\varepsilon + 1/n),$$

then pass to the limit [ $n \rightarrow +\infty$ ] to achieve the proof of Lemma 2. ■

### 3.4 Proof of the strong convergence of the gradient

We start from (34) and prove the strong convergence of the gradient by the arguments of Minty, Browder and Leray, Lions [Bro63, Min63, LL65]. Let  $\varphi \in C_c(\Omega \times (0, T))$ ,  $\varphi \geq 0$  be given. Consider the sum

$$\int_{Q_T} (a(\nabla T_k(u^n)) - a(\nabla T_k(u))) \cdot (\nabla T_k(u^n) - \nabla T_k(u)) \varphi dx dt. \quad (49)$$

We develop the product in this last term. The result (34) yields precisely the convergence of the term

$$\int_{Q_T} a(\nabla T_k(u^n)) \cdot \nabla T_k(u^n) dx dt.$$

The other terms, which are linear with respect to  $\nabla T_k(u^n)$  or  $a(\nabla T_k(u^n))$ , converge by weak convergence. At the limit  $n \rightarrow +\infty$  in (49), we obtain

$$\lim_{n \rightarrow +\infty} \int_{Q_T} (a(\nabla T_k(u^n)) - a(\nabla T_k(u))) \cdot (\nabla T_k(u^n) - \nabla T_k(u)) \varphi dx dt = 0.$$

Since  $F_n := (a(\nabla T_k(u^n)) - a(\nabla T_k(u))) \cdot (\nabla T_k(u^n) - \nabla T_k(u)) \varphi$  is nonnegative (by monotonicity of  $a$ ), this shows that  $F_n \rightarrow 0$  in  $L^1(Q_T)$ . A subsequence of  $(F_n)$  (still denoted  $(F_n)$ ) therefore converges to 0 on a set  $A$  of full measure in  $Q_T$ . Let  $(x, t) \in A$  and let  $q$  be an adherence value of  $(\nabla T_k(u^n))$  in  $\overline{\mathbb{R}^N}$ . Without loss of generality, we can suppose that  $\varphi(x, t) > 0$ . The vector  $q$  has finite-valued components as a consequence of the growth of  $a(\nabla T_k(u^n)) \cdot \nabla T_k(u^n)$ , which gives

$$(\alpha |\nabla T_k(u^n)(x, t)|^p - C |\nabla T_k(u^n)(x, t)|) \varphi(x, t) \leq F_n(x, t) \rightarrow 0.$$

At the limit [ $n \rightarrow +\infty$ ] in  $F_n(x, t) \rightarrow 0$ , we thus obtain

$$(a(q) - a(\nabla T_k(u)(x, t))) \cdot (q - \nabla T_k(u)(x, t)) \varphi(x, t) = 0.$$

By strict monotonicity of the flux  $a$ ,  $q = \nabla T_k(u)(x, t)$ . The sequence  $(\nabla T_k(u^n)(x, t))$  has only one possible adherence value and is therefore convergent:  $\nabla T_k(u^n) \rightarrow \nabla T_k(u)$  a.e. on  $Q_T$ . Together with the uniform bound on  $\nabla T_k(u^n)$  in  $L^p(Q_T)$ , this shows the strong convergence of  $\nabla T_k(u^n)$  to  $\nabla T_k(u)$  in any  $L^r(Q_T)$ ,  $r < p$ . Similarly,  $a(\nabla T_k(u^n))$  converges to  $a(\nabla T_k(u))$  a.e. and in  $L^r(Q_T)$ ,  $r < p'$ . In particular,  $\sigma = a(\nabla u)$  a.e.

To conclude, notice that we can recover the strong convergence  $\nabla T_k(u^n) \rightarrow \nabla T_k(u)$  in  $L^p_{\text{loc}}(Q_T)$ . Let  $K$  be compact subset of  $Q_T$ . By the weak convergence of  $(a(\nabla T_k(u^n)))$  and  $(\nabla T_k(u^n))$  to  $a(\nabla T_k(u))$  and  $\nabla T_k(u)$  respectively, and by the convergence

$$(a(\nabla T_k(u^n)) - a(\nabla T_k(u))) \cdot (\nabla T_k(u^n) - \nabla T_k(u)) \rightarrow 0$$

in  $L^1(K)$ ,  $(a(\nabla T_k(u^n)) \cdot \nabla T_k(u^n))$  converges to  $a(\nabla T_k(u)) \cdot \nabla T_k(u)$  in  $L^1(K)$ -weak. By Dunford-Pettis' Theorem, the family  $\{a(\nabla T_k(u^n)) \cdot \nabla T_k(u^n)\}$  is equi-integrable on  $K$ . By hypothesis (2a),  $|\nabla T_k(u^n)|^p$  is dominated by  $\alpha^{-1}a(\nabla T_k(u^n)) \cdot \nabla T_k(u^n)$  and, therefore,  $\{|\nabla T_k(u^n)|^p\}$  is also equi-integrable on  $K$ . Since  $\nabla T_k(u^n) \rightarrow \nabla T_k(u)$  a.e., and by Vitali's Theorem, this implies  $|\nabla T_k(u^n)|^p \rightarrow |\nabla T_k(u)|^p$  in  $L^1(K)$ . Besides, the weak convergence  $\nabla T_k(u^n) \rightarrow \nabla T_k(u)$  in  $L^p(K)$ -weak and the convergence  $\nabla T_k(u^n) \rightarrow \nabla T_k(u)$  a.e. implies that

$$\lim_{n \rightarrow +\infty} (\|\nabla T_k(u^n)\|_{L^p(K)} - \|\nabla T_k(u^n) - \nabla T_k(u)\|_{L^p(K)}) = \|\nabla T_k(u)\|_{L^p(K)}$$

(this is a refinement of Fatou's Lemma by Brezis and Lieb [BL83]). By convergence of the  $L^p(K)$ -norms, we conclude that

$$\lim_{n \rightarrow +\infty} \|\nabla T_k(u^n) - \nabla T_k(u)\|_{L^p(K)} = 0.$$

## 4 Parabolic equation with a term with natural growth

In this section, we briefly indicate how to adapt the arguments and proofs given above to solve the question of the strong convergence of the gradient (and, therefore, prove the existence of a renormalized solution) in the approximation by regularization and truncation of the following problem:

$$u_t - \operatorname{div}(a(\nabla u)) + \gamma(u)|\nabla u|^p = f \text{ in } \Omega \times (0, T), \quad (50a)$$

$$u = u_0 \text{ on } \Omega \times \{0\}, \quad (50b)$$

$$u = 0 \text{ on } \Sigma, \quad (50c)$$

We keep the same assumptions on  $a$  and on the data: assumptions 1 and 2. The function  $\gamma \in C(\mathbb{R})$  is supposed to satisfies the sign condition

$$u\gamma(u) \geq 0, \quad \forall u \in \mathbb{R}. \quad (51)$$

This sign condition ensures *a priori* estimates for the additional term  $\gamma(u)|\nabla u|^p$ , with a bound in  $L^1(Q_T)$ . More generally, we may consider a term  $\gamma(u)|\nabla u|^r$  with a power  $r \in [1, p]$ , instead of the term  $\gamma(u)|\nabla u|^p$ .

Numerous works have been devoted to the study of Problem (50) (or to its elliptic version). Let us cite in particular [BMP83, BMP89, BG92b, BGM93, Por00, SdL03] and references therein.

In case  $p = 2$ ,  $a = \operatorname{Id}$ , there is a change of variables that transforms the equation in a classical Heat Equation:

$$v_t - \Delta v = g, \quad v = \int_0^u e^{-\int_0^\xi \gamma d\xi}, \quad g = f e^{-\int_0^u \gamma}.$$

It is this change of variables that we will adapt to the nonlinear case by use of the kinetic formulation (or level-set PDE).

A renormalized solution to (50) is defined as follows.

**Definition 2** *A function  $u \in L^\infty(0, T; L^1(\Omega))$  is a renormalized solution to (50) if*

$$T_k(u) \in L^p(0, T; W_0^{1,p}(\Omega)), \quad \forall k > 0,$$

and, for every function  $S \in W^{2,\infty}(\mathbb{R})$  such that  $S'$  has compact support and  $S(0) = 0$ ,

$$S(u)_t - \operatorname{div}(S'(u)a(\nabla u)) + S'(u)\gamma(u)|\nabla u|^p = S(u_0) \otimes \delta_{t=0} + S'(u)f - S''(u)a(\nabla u) \cdot \nabla u$$

and

$$\lim_{k \rightarrow +\infty} \int_{Q_T \cap \{k < u < k+1\}} a(\nabla u) \cdot \nabla u dx dt = 0.$$

We can also use directly the level-set PDE and define a renormalized solution to (50) as a function  $u \in L^\infty(0, T; L^1(\Omega))$  having the regularity of the truncates  $T_k(u) \in L^p(0, T; W_0^{1,p}(\Omega))$ ,  $\forall k > 0$ , which satisfies the equation:

$$\partial_t \chi_u - \operatorname{div}(a(\nabla u)\delta_{u=\xi}) + \gamma(\xi)|\nabla u|^p \delta_{u=\xi} = \chi_{u_0} \otimes \delta_{t=0} + f\delta_{u=\xi} + \partial_\xi \mu,$$

where  $\mu := a(\nabla u) \cdot \nabla u \delta_{u=\xi}$ , and satisfies the condition at infinity  $\lim_{k \rightarrow \pm\infty} \mu_*(k, k+1) \rightarrow 0$ .

We now explain how to prove the existence of a renormalized solution to Problem (50). For the sake of simplicity, we will suppose that the solution has a sign: we assume

$$u_0 \geq 0 \text{ a.e. } f \geq 0 \text{ a.e.} \quad (52)$$

**Step 1.** Approximation. Let  $(u_0^n)$  and  $(f^n)$  be some nonnegative approximating sequences of, respectively,  $u_0$  and  $f$  in, respectively,  $L^1(\Omega)$  and  $L^1(\Omega \times (0, T))$  such that  $u_0^n \in L^p(\Omega)$ ,  $f^n \in L^{p'}(\Omega \times (0, T))$ . For each  $n$ , the problem

$$u_t^n - \operatorname{div}(a(\nabla u^n)) + \gamma(u^n)|\nabla u^n|^p = f^n \text{ in } \Omega \times (0, T), \quad (53a)$$

$$u^n = u_0^n \text{ on } \Omega \times \{0\}, \quad (53b)$$

$$u^n = 0 \text{ on } \Sigma, \quad (53c)$$

has a unique solution  $u^n$  in the space of functions  $v \in L^p(0, T; W_0^{1,p}(\Omega))$  with  $v_t \in L^{p'}(0, T; W^{-1,p}(\Omega))$ . The function  $u^n$  is a weak solution to (24), hence a renormalized solution and therefore satisfies the equation

$$\partial_t \chi_{u^n} - \operatorname{div}(a(\nabla u^n)\delta_{u^n=\xi}) + \gamma(\xi)|\nabla u^n|^p \delta_{u^n=\xi} = \chi_{u_0^n} \otimes \delta_{t=0} + f^n \delta_{u^n=\xi} + \partial_\xi \mu^n, \quad (54)$$

where  $\mu^n$  is defined by

$$\mu^n := a(\nabla u^n) \cdot \nabla u^n \delta_{u^n=\xi}.$$

**Step 2.** Estimates. As in Section 2.3.2, we show that, up to a subsequence,  $u_n \rightarrow u \in L^\infty(0, T; L^1(\Omega))$  in  $L^1(Q_T)$ ,  $a(\nabla T_k(u^n)) \rightarrow \sigma \mathbf{1}_{|u|<k}$  and  $\mu^n \rightarrow \mu$  weakly. We also prove, by the same technique as in Section 2.3.2, the conditions at infinity

$$\lim_{k \rightarrow \pm\infty} \mu_*((k, k+1)) = 0. \quad (55)$$

Since  $u^n \geq 0$  a.e., we also have  $u \geq 0$  a.e. and  $\mu$  is supported in  $\overline{Q_T} \times [0, +\infty)$ .

**Step 3.** Limit of the equation. To pass to the limit of Eq. (54), there is a difficulty in the fact that the term  $\gamma(\xi)|\nabla u^n|^p \delta_{u^n=\xi}$  is uniformly bounded in  $L^1$  and that no stronger *a priori* bound is available. We define

$$\Gamma_+(\xi) = \begin{cases} \frac{1}{\alpha} \int_0^\xi \gamma & \text{if } \xi > 0, \\ -\frac{1}{\beta} \int_\xi^0 \gamma & \text{if } \xi < 0. \end{cases}$$

The function  $\Gamma_+$  is continuous, not  $C^1$ , on  $\mathbb{R}$ , but a step of regularization shows that we have

$$\partial_t e^{-\Gamma_+(\xi)} \chi_{u^n} - \operatorname{div}(e^{-\Gamma_+(\xi)} a(\nabla u^n) \delta_{u^n=\xi}) = e^{-\Gamma_+(\xi)} (\chi_{u_0^n} \otimes \delta_{t=0} + f^n \delta_{u^n=\xi}) + \partial_\xi (e^{-\Gamma_+(\xi)} \mu^n) + R$$

where

$$R := \gamma(\xi)e^{-\Gamma+(\xi)}\{(\alpha^{-1}\mathbf{1}_{\xi>0} + \beta^{-1}\mathbf{1}_{\xi<0})\mu_n - |\nabla u^n|^p \delta_{u^n=\xi}\}$$

(observe that the function  $\xi \mapsto \gamma(\xi)(\alpha^{-1}\mathbf{1}_{\xi>0} + \beta^{-1}\mathbf{1}_{\xi<0})$  is continuous since  $\gamma(0) = 0$ ). Since  $\mu_n = a(\nabla u^n) \cdot \nabla u^n \delta_{u^n=\xi}$ , the hypotheses (2a) and (2b) on the flux  $a$  ensure that  $R \geq 0$  and, therefore, that

$$\partial_t e^{-\Gamma+(\xi)} \chi_{u^n} - \operatorname{div}(e^{-\Gamma+(\xi)} a(\nabla u^n) \delta_{u^n=\xi}) \geq e^{-\Gamma+(\xi)} (\chi_{u_0^n} \otimes \delta_{t=0} + f^n \delta_{u^n=\xi}) + \partial_\xi (e^{-\Gamma+(\xi)} \mu^n). \quad (56)$$

Similarly, we define

$$\Gamma_-(\xi) = \begin{cases} \frac{1}{\beta} \int_0^\xi \gamma & \text{if } \xi > 0, \\ -\frac{1}{\alpha} \int_\xi^0 \gamma & \text{if } \xi < 0 \end{cases}$$

and show the inequality

$$\partial_t e^{-\Gamma_-(\xi)} \chi_{u^n} - \operatorname{div}(e^{-\Gamma_-(\xi)} a(\nabla u^n) \delta_{u^n=\xi}) \leq e^{-\Gamma_-(\xi)} (\chi_{u_0^n} \otimes \delta_{t=0} + f^n \delta_{u^n=\xi}) + \partial_\xi (e^{-\Gamma_-(\xi)} \mu^n). \quad (57)$$

It is then possible to pass to the limit  $[n \rightarrow +\infty]$  in (56) and (57) to obtain

$$\partial_t e^{-\Gamma+(\xi)} \chi_u - \operatorname{div}(e^{-\Gamma+(\xi)} \sigma \delta_{u=\xi}) \geq e^{-\Gamma+(\xi)} (\chi_{u_0} \otimes \delta_{t=0} + f \delta_{u=\xi}) + \partial_\xi (e^{-\Gamma+(\xi)} \mu), \quad (58)$$

$$\partial_t e^{-\Gamma_-(\xi)} \chi_u - \operatorname{div}(e^{-\Gamma_-(\xi)} \sigma \delta_{u=\xi}) \leq e^{-\Gamma_-(\xi)} (\chi_{u_0} \otimes \delta_{t=0} + f \delta_{u=\xi}) + \partial_\xi (e^{-\Gamma_-(\xi)} \mu). \quad (59)$$

What information do we extract from (58) and (59)? At a *formal* level, we can do the following computations: sum each inequality with respect to  $\xi \in \mathbb{R}$  and use the condition at infinity (55) to obtain the (formal) weak equations

$$\partial_t \int_{\mathbb{R}} e^{-\Gamma+(\xi)} \chi_u d\xi - \operatorname{div}(e^{-\Gamma+(u)} \sigma) \geq \int_{\mathbb{R}} e^{-\Gamma+(\xi)} \chi_{u_0} d\xi \otimes \delta_{t=0} + e^{-\Gamma+(u)} f, \quad (60)$$

$$\partial_t \int_{\mathbb{R}} e^{-\Gamma_-(\xi)} \chi_u d\xi - \operatorname{div}(e^{-\Gamma_-(u)} \sigma) \leq \int_{\mathbb{R}} e^{-\Gamma_-(\xi)} \chi_{u_0} d\xi \otimes \delta_{t=0} + e^{-\Gamma_-(u)} f. \quad (61)$$

Multiply the first inequality by  $e^{\Gamma+(\xi)-\Gamma_-(\xi)} \delta_{u=\xi}$  and the second inequality by  $e^{-\Gamma+(\xi)+\Gamma_-(\xi)} \delta_{u=\xi}$  to obtain (still after formal computations)

$$\partial_t e^{-\Gamma_-(\xi)} \chi_u - \operatorname{div}(e^{-\Gamma_-(\xi)} a(\nabla u) \delta_{u=\xi}) \geq e^{-\Gamma_-(\xi)} (\chi_{u_0} \otimes \delta_{t=0} + f \delta_{u=\xi}) - e^{-\Gamma_-(\xi)} \sigma \cdot \nabla \delta_{u=\xi},$$

$$\partial_t e^{-\Gamma+(\xi)} \chi_u - \operatorname{div}(e^{-\Gamma+(\xi)} \sigma \delta_{u=\xi}) \leq e^{-\Gamma+(\xi)} (\chi_{u_0} \otimes \delta_{t=0} + f \delta_{u=\xi}) - e^{-\Gamma+(\xi)} \sigma \cdot \nabla \delta_{u=\xi}.$$

At last, use the identity  $e^{-\Gamma\pm(\xi)} \sigma \cdot \nabla \delta_{u=\xi} = -\partial_\xi (e^{-\Gamma\pm(\xi)} \nu)$ , where

$$\nu := \sigma \cdot \nabla u \delta_{u=\xi},$$

(this is also a very *formal* identity) to obtain

$$\partial_t e^{-\Gamma_-(\xi)} \chi_u - \operatorname{div}(e^{-\Gamma_-(\xi)} a(\nabla u) \delta_{u=\xi}) \geq e^{-\Gamma_-(\xi)} (\chi_{u_0} \otimes \delta_{t=0} + f \delta_{u=\xi}) + \partial_\xi (e^{-\Gamma_-(\xi)} \nu), \quad (62)$$

$$\partial_t e^{-\Gamma+(\xi)} \chi_u - \operatorname{div}(e^{-\Gamma+(\xi)} \sigma \delta_{u=\xi}) \leq e^{-\Gamma+(\xi)} (\chi_{u_0} \otimes \delta_{t=0} + f \delta_{u=\xi}) + \partial_\xi (e^{-\Gamma+(\xi)} \nu). \quad (63)$$

Come back to the starting point (58)-(59) to deduce the inequalities

$$\partial_\xi (e^{-\Gamma+(\xi)} \mu) \leq \partial_\xi (e^{-\Gamma+(\xi)} \nu), \quad \partial_\xi (e^{-\Gamma_-(\xi)} \nu) \leq \partial_\xi (e^{-\Gamma_-(\xi)} \mu). \quad (64)$$

Assume for the moment that (64) is satisfied in  $\mathcal{D}'(U_T)$ . A test-function  $\varphi \in \mathcal{D}^+(Q_T)$  being fixed, we consider the distributions on  $\mathbb{R}$  defined by

$$\mu_\varphi : \psi \mapsto \langle \mu, \varphi \otimes \psi \rangle, \quad \nu_\varphi : \psi \mapsto \langle \nu, \varphi \otimes \psi \rangle.$$

They satisfy the inequalities

$$\partial_\xi(e^{-\Gamma+(\xi)}\mu_\varphi) \leq \partial_\xi(e^{-\Gamma+(\xi)}\nu_\varphi), \quad \partial_\xi(e^{-\Gamma-(\xi)}\nu_\varphi) \leq \partial_\xi(e^{-\Gamma-(\xi)}\mu_\varphi)$$

in  $\mathcal{D}'(\mathbb{R})$ . Consider the first of these inequalities. Since  $\mu_\varphi = \nu_\varphi = 0$  on  $(-\infty, 0)$ , we have  $e^{-\Gamma+(\xi)}\mu_\varphi \leq e^{-\Gamma+(\xi)}\nu_\varphi$  in  $\mathcal{D}'(\mathbb{R})$ . Similarly, using the second inequality, we obtain  $e^{-\Gamma-(\xi)}\mu_\varphi \geq e^{-\Gamma-(\xi)}\nu_\varphi$  and conclude that  $\mu_\varphi = \nu_\varphi$ . This being true for every  $\varphi \in \mathcal{D}^+(Q_T)$ , we have the desired result  $\mu = \nu$ .

**Step 4.** Strong convergence of the gradient. The identity  $\mu = \nu$  is the key point in the proof of the strong convergence of the gradient. Once this has been proved, we proceed as in Section 2.3.3. We prove in particular that  $\nabla T_k(u^n) \rightarrow \nabla T_k(u)$  in  $L^p_{\text{loc}}(Q_T)$ , and this allows to pass to the limit in Eq. (54) to obtain

$$\partial_t \chi_u - \text{div}(a(\nabla u)\delta_{u=\xi}) + \gamma(\xi)|\nabla u|^p \delta_{u=\xi} = \chi_{u_0} \otimes \delta_{t=0} + f\delta_{u=\xi} + \partial_\xi(a(\nabla u) \cdot \nabla u \delta_{u=\xi}),$$

i.e. the fact that  $u$  is a renormalized solution.

**Step 5.** Rigorous proof of (64). This is a variation on the proof of Lemma 1 given in Section 3.2. Let us explain the main arguments. Introduce  $\alpha_k := \rho_k * \mathbf{1}_{[k, k+1]}$ , and define

$$r^k = r^k(u) = \int_{|u|}^\infty \alpha_k, \quad v^k := \int_{\mathbb{R}} e^{-\Gamma+(\xi)} \chi_u(\xi) r^k(\xi) d\xi, \quad v_0^k := \int_{\mathbb{R}} e^{-\Gamma+(\xi)} \chi_{u_0}(\xi) r^k(\xi) d\xi.$$

Set also  $\tilde{r}^k = e^{-\Gamma+(u)} r^k$  and

$$v := \int_{\mathbb{R}} e^{-\Gamma+(\xi)} \chi_u(\xi) d\xi, \quad v_0 := \int_{\mathbb{R}} e^{-\Gamma+(\xi)} \chi_{u_0}(\xi) d\xi.$$

We have  $v^k \in L^p(-1, T; W_0^{1,p}(\Omega)) \cap L^\infty(Q_T)$ ,  $v_0^k \in L^\infty(\Omega)$  and  $T_l(v^k) \rightarrow T_l(v)$  in  $L^p(-1, T; W_0^{1,p}(\Omega))$  ( $l > 0$ ),  $v_0^k \rightarrow v_0$ ,  $r^k \rightarrow 1$  a.e. when  $k$  tends to  $+\infty$ . Test Eq. (58) against  $\varphi(t, x)r^k(\xi)$  (with  $\varphi \in \mathcal{D}^+(Q_T)$ ), to obtain

$$-\int_0^T \int_\Omega (v^k - v_0^k) \varphi_t + \int_0^T \int_\Omega \sigma \cdot \nabla \varphi \tilde{r}^k - \int_0^T \int_\Omega f \varphi \tilde{r}^k \geq \int_0^T \int_\Omega \int_{\mathbb{R}} \varphi e^{-\Gamma+(\xi)} \alpha_k d\mu.$$

We deduce the inequality

$$\int_{Q_T} \varphi_t (v^k - v_0^k) - \int_0^T \langle G^k, \varphi \rangle dt \leq \|\varphi\|_{L^\infty} \varepsilon_k,$$

where  $G^k := -(\text{div}(\sigma \tilde{r}^k(u)) + f \tilde{r}^k(u)) \in L^{p'}(0, T; W^{-1,p'}(\Omega)) + L^1(Q)$  and  $\varepsilon_k := \mu_*((k-1, k+2)) \rightarrow 0$  when  $k \rightarrow +\infty$ . The analogue of Lemma 2 then shows that, for every  $h \in W^{1,\infty}(\mathbb{R})$ ,  $v^k$  satisfies the following inequality:

$$\int_{Q_T} \varphi_t \int_{v_0^k}^{v^k} h(\zeta) d\zeta - \int_0^T \langle G^k, \varphi h(v^k) \rangle dt \leq \|\varphi\|_{L^\infty} \|h\|_{L^\infty} \varepsilon_k.$$

Taking  $h$  with compact support, we obtain at the limit  $k \rightarrow +\infty$  the inequality

$$\int_{Q_T} \varphi_t \int_{v_0}^v h(\zeta) d\zeta - \int_0^T \langle G, \varphi h(v) \rangle dt \leq 0,$$

i.e.

$$\int_{Q_T} \varphi_t \int_{v_0}^v h(\zeta) d\zeta - \int_{Q_T} e^{-\Gamma+(u)} \sigma \cdot \nabla(\varphi h(v)) dt + \int_{Q_T} e^{-\Gamma+(u)} f \varphi h(v) \leq 0. \quad (65)$$

We then fix  $\theta \in \mathcal{D}(\mathbb{R})$  and apply (65) with

$$h(\zeta) := e^{-(\Gamma_- - \Gamma_+)(\phi^{-1}(\zeta))} \theta(\phi^{-1}(\zeta)), \quad \phi(\xi) := \int_0^\xi e^{-\Gamma_+},$$

in such a way that

$$\int_{v_0}^v h(\zeta) d\zeta = \int_{u_0}^u e^{-\Gamma_-(\xi)} \theta(\xi) d\xi, \quad h(v) = e^{-(\Gamma_- - \Gamma_+)(u)} \theta(u),$$

to obtain the weak form of (62). Similarly, we prove (63). As explained in Step 3., these two inequalities combined with (58) and (59) imply (64).

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