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Conformal Toda theory with a boundary

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ABSTRACT: We investigate $s\ell_n$ conformal Toda theory with maximally symmetric boundaries. There are two types of maximally symmetric boundary conditions, due to the existence of an order two automorphism of the $W_{n\geq 3}$ algebra. In one of the two cases, we find that there exist D-branes of all possible dimensions $0 \leq d \leq n-1$, which correspond to partly degenerate representations of the W_n algebra. We perform classical and conformal bootstrap analyses of such D-branes, and relate these two approaches by using the semi-classical light asymptotic limit. In particular we determine the bulk one-point functions. We observe remarkably severe divergences in the annulus partition functions, and attribute their origin to the existence of infinite multiplicities in the fusion of representations of the $W_{n\geq 3}$ algebra. We also comment on the issue of the existence of a boundary action, using the calculus of constrained functional forms, and derive the generating function of the Bäcklund transformation for $s\ell_3$ Toda classical mechanics, using the minisuperspace limit of the bulk one-point function.

Contents

1. Introduction

There are good reasons for studying $s\ell_n$ conformal Toda theories, as in principle these non-rational two-dimensional conformal field theories have all the usual applications of two-dimensional CFTs, applications to quantum gravity, string theory and critical phenomena. (See [1] for more details and references.) In particular, the simplest and well-studied case of Liouville theory ($n = 2$) is an essential tool in the study of non-critical string theories and two-dimensional quantum gravity. And the study of Liouville theory with a boundary plays an important role in the understanding of non-critical open strings and of the corresponding D-branes, which account for the non-perturbative effects in non-critical string theory. The other cases ($n \geq 3$) are directly related to the so-called W-strings and W-gravity theories, whose names come from the W_n symmetry algebra of $s\ell_n$ conformal Toda theory. And the non-perturbative effects in W string theory are expected to be due to D-branes, which can be technically described using conformal Toda theory with a boundary. In addition to such applications, another motivation for investigating the higher Toda theories is their beautiful, intricate and challenging nature, which suggests that their study can reveal qualitatively new structures and phenomena in two-dimensional conformal field theory.

The W_n algebra, which is an extension of the Virasoro algebra, was discovered [2] [3] soon after the seminal work of Belavin, Polyakov and Zamolodchikov on two-dimensional CFTs [4], and rational CFTs with W_n symmetries were then constructed [5] [3]. The study of $s\ell_n$ conformal Toda theories, which are non-rational CFTs with W_n symmetries, is much more recent [1] [6]. The case of Liouville theory had to be studied first, and $s\ell_{n\geq 3}$ Toda theory is considerably more complicated than Liouville theory. The reasons for these extra complications can be found in the properties of the W_n algebras, as we will demonstrate.

Our most powerful tool in the study of non-rational CFTs is the conformal bootstrap method, which purposes to determine all correlation functions once the spectrum of the theory is given, and the W_n symmetry of the theory is assumed. So far this has been achieved only in the case of Liouville theory; however this is in principle doable also in $s\ell_{n\geq 3}$ Toda theories. The conformal bootstrap equations for say the three-point functions are vastly overdetermined, the problem is to find closed subsystems of manageable numbers of equations. In the present article we will achieve this in the case of the one-point function in the presence of a boundary. Introducing a boundary in the two-dimensional space on which our field theory lives of course makes the theory more complicated, but the advantage is that simple correlation functions like the one point function, which has to vanish in the absence of a boundary, now become interesting observables.

In the case of CFTs with boundaries, the fundamental relations between the properties of the symmetry algebra and the physical observables of the theory were discovered by Cardy [7], and we will refer to them as “Cardy’s ideas”. First of all, maximally symmetric D-branes are related to the representations of the symmetry algebra which appear in the bulk spectrum. Then, the spectrum of open strings with their ends on two D-branes is given by the fusion product of the two corresponding representations. We will find that these properties mostly hold in $s\ell_n$ conformal Toda theory, in the cases where we can determine the relevant objects. There will be restrictions, some of which were already observed in the case of Liouville theory: there exist not only continuous D-branes associated to the continuous representations which do appear in the bulk spectrum, but also discrete D-branes associated to degenerate representations which do not.

So we will begin with a study of W_n algebras and their representations (Section 2), where we will emphasize the features which will play an important role in $s\ell_n$ Toda theory: the properties of the characters, the existence of an order 2 automorphism, the existence of infinite fusion multiplicities, the existence of a hierarchy of partly degenerate representations. We hope that this review will be enough for understanding the rest of the article, but we also recommend the reviews [8][9] on W_n algebras and [10] on conformal field theory. Then, we will solve the classical Toda equations on the disc (Section 3). The resulting picture of the moduli spaces of D-branes will turn out to be qualitatively correct, as will be confirmed by the conformal bootstrap analysis (Section 4). There, the analysis of the differential equations obeyed by certain two-point functions will result in explicit expressions for the one-point functions, which characterize how D-branes couple to bulk operators. The calculation of annulus partition functions will also provide some information on the boundary sector. The relation between the classical and bootstrap analyses will be made precise thanks to the light asymptotic limit (Section 5), which will also allow us to predict some correlation functions which are at present out of reach of the bootstrap analysis. The conclusion (Section 6) will summarize the main results and remaining puzzles. Then come two Appendices, which are devoted to interesting but peripheral topics: Appendix A to the minisuperspace limit, which will turn out to lead to the determination of the generating function of the Bäcklund transformation which relates $s\ell_3$ Toda classical mechanics to a free system, and Appendix B to the existence of boundary actions, which we will be able to predict or rule out based on the properties of the boundary conditions.

2. W_n algebras and their representations

The symmetry algebra of the $s\ell_n$ conformal Toda theory is the so-called W_n algebra. The Virasoro algebra coincides with the W_2 algebra, and is a subalgebra of the $W_{n>2}$ algebra, so that $s\ell_n$ conformal Toda theory indeed has conformal symmetry. The spectrum of the theory decomposes into representations of the W_n algebra, which we will therefore study.

The infinite-dimensional W_n algebra is related to the finite-dimensional $s\ell_n$ algebra in a number of ways. For example, the Virasoro algebra can be obtained from the affine extension $\widehat{s\ell_2}$ of the $s\ell_2$ algebra by a quantum Hamiltonian reduction of the Drinfeld-Sokolov type. The W_n algebra can similarly be obtained from $\widehat{s\ell_n}$. Moreover, a fully degenerate representation of W_n can be associated to each pair of two highest-weight representations of $s\ell_n$. This is our motivation for reviewing the representations of $s\ell_n$ (more on this in [10]), as an introduction to the study of representations of W_n.

2.1 Representation theory of $s\ell_n$ Lie algebras

Representations of $s\ell_n$ are parametrized by vectors in an $n-1$-dimensional space spanned by the simple roots $e_1 \cdots e_{n-1}$ whose scalar products $K_{i,j} = (e_i, e_j)$ form the Cartan matrix, whose only nonzero entries are $K_{ii} = 2, K_{i,i-1} = K_{i,i+1} = -1$. The $\frac{1}{2}n(n-1)$ positive roots are the sums of any numbers of consecutive simple roots, in the $s\ell_3$ case they are $\{e > 0\} = \{e_1, e_2, e_1 + e_2\}$. The fundamental weights are the vectors ω_i such that $(\omega_i, e_j) = \delta_{ij}$. The Weyl vector is

$$\rho = \tfrac{1}{2} \sum_{e>0} e = \sum_i \omega_i \ , \tag{2.1}$$

and $\rho^2 \equiv (\rho, \rho) = \frac{1}{12}(n-1)n(n+1)$. In the $s\ell_3$ case we have

$$\begin{cases} \omega_1 = \frac{2}{3}e_1 + \frac{1}{3}e_2 \\ \omega_2 = \frac{1}{3}e_1 + \frac{2}{3}e_2 \end{cases}, \quad \begin{cases} e_1 = 2\omega_1 - \omega_2 \\ e_2 = 2\omega_2 - \omega_1 \end{cases}, \quad K = \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right), \; \{e > 0\} = \{e_1, e_2, \rho\} \,. \tag{2.2}$$

The Weyl group, a finite group, acts on the root space while preserving the scalar product. In the case of $s\ell_2$ it is a $\mathbb{Z}_2$ group whose nontrivial element is the reflection $r(v) = -v$. In the case of $s\ell_3$ the Weyl group has six elements $\{1, r, s, rs, sr, rsr = srs\}$ and can be identified with the group of permutations of the three elements $\{h_i\} \equiv \{\omega_1, \omega_2 - \omega_1, -\omega_2\}$ with the action

1	r	s	rs	sr	rsr
e_1	ρ	$-e_1$	$-\rho$	e_2	$-e_2$
e_2	$-e_2$	ρ	e_1	$-\rho$	$-e_1$
ρ	e_1	e_2	$-e_2$	$-e_1$	$-\rho$
ω_1	ω_1	$\omega_2 - \omega_1$	$-\omega_2$	$\omega_2 - \omega_1$	$-\omega_2$
$\omega_2 - \omega_1$	$-\omega_2$	ω_1	ω_1	$-\omega_2$	$\omega_2 - \omega_1$
$-\omega_2$	$\omega_2 - \omega_1$	$-\omega_2$	$\omega_2 - \omega_1$	ω_1	ω_1

(2.3)

In the general $s\ell_n$ case, the Weyl group is generated by the $n-1$ reflections s_i such that $s_i(e_j) = e_j - K_{ji}e_i$. The signature of an element of the group is the function ϵ such that $\epsilon(s_i) = -1$ and $\epsilon(ww') = \epsilon(w)\epsilon(w')$.

To an integral dominant weight, that is a vector $\Omega = \sum_i \lambda_i \omega_i \in \sum_i \mathbb{N}\omega_i$, we can associate a finite-dimensional irreducible representation R_Ω of $s\ell_n$. The vector Ω is then called its highest weight. A finite number of weights $h \in H_\Omega$ such that $\Omega - h \in \sum_i \mathbb{N}e_i$ are associated to the representation. The weights are the eigenvalues of the generators of the Cartan subalgebra when acting on a basis of the representation, so that the number of weights, taking into account their possible integer multiplicities, is the dimension of the representation. For example, the fundamental representation of $s\ell_n$ has dimension n and weights $H_{\omega_1} = \{h_k = \omega_1 - \sum_{i=1}^k e_i | k = 0 \cdots n-1\}$. The adjoint representation of $s\ell_3$ has dimension 8 and weights $H_\rho = \{\pm e_1, \pm e_2, \pm\rho, 2 \cdot 0\}$ where the weight 0 appears with multiplicity 2. Multiplicities higher than one appear only in the cases $s\ell_{n\geq 3}$.

The character $\chi_\Omega(p)$ of a representation is defined as a function of a vector p by

$$\chi_\Omega(p) \equiv \sum_{h \in H_\Omega} e^{(h,p)} \,. \tag{2.4}$$

Given the highest weight Ω of a representation, the other weights can be found thanks to the Weyl formula

$$\chi_\Omega(p) = \frac{\sum_{w\in W} \epsilon(w) e^{(\rho+\Omega, w(p))}}{\sum_{w\in W} \epsilon(w) e^{(\rho, w(p))}} \,, \tag{2.5}$$

whose denominator can be rewritten as

$$\sum_{w\in W} \epsilon(w) e^{(\rho, w(p))} = \prod_{e>0} (e^{\frac{1}{2}(e,p)} - e^{-\frac{1}{2}(e,p)}) \,. \tag{2.6}$$

Characters behave nicely under tensor products of representations, thanks to the property

$$R_\Omega \otimes R_{\Omega'} = \sum_{\Omega''} m_{\Omega,\Omega'}^{\Omega''} R_{\Omega''} \ \Rightarrow\ \chi_\Omega(p)\chi_{\Omega'}(p) = \sum_{\Omega''} m_{\Omega,\Omega'}^{\Omega''} \chi_{\Omega''}(p) \ . \tag{2.7}$$

The hyperplanes $\{(e,p)=0\}_{e>0}$ divide the p-space into $n!$ Weyl chambers, which are fundamental domains for the action of the Weyl group.

For $n \geq 3$, the algebra $s\ell_n$ has an order two automorphism, called the Dynkin diagram automorphism, which maps R_Ω to R_{Ω^*}, where the conjugation $\Omega \to \Omega^*$ is the linear map characterized by $e_i^* = e_{n-i}$. This map is trivial in the case of $s\ell_2$. We will see that this automorphism induces an automorphism of the $W_{n\geq 3}$ algebra.

2.2 Representation theory of W_n algebras

The algebra W_n is generated by $n-1$ operators $W^{(2)}, W^{(3)}, \cdots W^{(n)}$, where $W^{(2)} = T$ is the stress-energy tensor. (See the review [8].) Let us explicitly write Zamolodchikov's W_3 algebra, where for simplicity we denote $W^{(3)} = W$:

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \mathcal{O}(1) \ , \tag{2.8}$$

$$T(z)W(w) = \frac{3W(w)}{(z-w)^2} + \frac{\partial W(w)}{(z-w)} + \mathcal{O}(1) \ , \tag{2.9}$$

$$\begin{aligned} W(z)W(w) = {}& \frac{c/3}{(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^3} + \frac{1}{(z-w)^2}\left[2\beta\Lambda(w) + \frac{3}{10}\partial^2 T(w)\right] \\ & + \frac{1}{z-w}\left[\beta\partial\Lambda(w) + \frac{1}{15}\partial^3 T(w)\right] + \mathcal{O}(1) \ , \end{aligned} \tag{2.10}$$

where $\Lambda(w) = (TT)(w) - \frac{3}{10}\partial^2 T(w)$ and $\beta = \frac{16}{22+5c}$. The algebra depends on a central charge c, which we parametrize in terms of a real number b as $c = (n-1)(1+n(n+1)(b+b^{-1})^2)$. The generators of the algebra can be decomposed into modes $W_n^{(s)}$ as $W^{(s)}(z) = \sum_{n\in\mathbb{Z}} W_n^{(s)} z^{-n-s}$; there is a special notation L_n for the modes of $T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$.

A representation of the W_n algebra can be encoded in a vertex operator $V(z)$, and the action of the algebra is encoded in the operator product $W^{(s)}(z)V(w) = \sum_{n\in\mathbb{Z}} \frac{W_n^{(s)}V(w)}{(z-w)^{s-n}}$. A standard assumption in conformal field theory is that the spectrum is a sum of highest-weight representations, generated by primary operators such that $W_{n>0}^{(s)}V(w) = 0$ and $W_0^{(s)}V(w) = q^{(s)}V(w)$. The product of a generator $W^{(s)}$ with a primary operator therefore contains a finite number of singular terms. In the W_3 case a primary operator $V(w)$ obeys

$$T(z)V(w) = \frac{\Delta V(w)}{(z-w)^2} + \frac{\partial V(w)}{z-w} + \mathcal{O}(1) \ , \tag{2.11}$$

$$W(z)V(w) = \frac{qV(w)}{(z-w)^3} + \frac{W_{-1}V(w)}{(z-w)^2} + \frac{W_{-2}V(w)}{z-w} + \mathcal{O}(1) \ , \tag{2.12}$$

where we denote $\Delta = q^{(2)}$ the conformal dimension and $q = q^{(3)}$ the W-charge, and we use the identification of L_{-1} with the generator of translations ∂, which is another standard assumption in conformal field theory. All operators of interest are assumed to be linear combinations of operators of the type $DV(w) = (\prod_{i=1}^N W_{-n_i}^{(s_i)})V(w)$ where $V(w)$ is primary and $n_i > 0, N \geq 0$. A descendent operator of level $L > 0$ is a linear combination of such operators with $\sum_i n_i = L$. A descendent which is itself primary is called a null vector.

Representations are classified according to their numbers of algebraically independent null vectors. Each such null vector implies the existence of a relation between the $n-1$ charges $q^{(s)}$; so that the presence of $n-1$ null vectors would fully determine all charges as functions of the central charge c, and further null vectors would constrain c itself. We work for generic values of c and therefore consider only representations with a number $0 \leq k \leq n-1$ of algebraically independent null vectors. Such representations are called non-degenerate or continuous if $k = 0$, simply degenerate if $k = 1$, doubly degenerate if $k = 2$, up to fully degenerate if $k = n-1$. Now the existence of k algebraically independent null vectors implies the existence of $\frac{1}{2}k(k+1)$ linearly independent null vectors, and therefore $\frac{1}{2}k(k+1)$ null vector equations for correlation functions involving the corresponding primary operators. In particular, in the $s\ell_3$ case, each simply degenerate operator comes with one null-vector equation, whereas each fully degenerate operator comes with 3 null-vector equations.

A primary operator is in principle characterized by the corresponding $W_0^{(s)}$ eigenvalues $q^{(s)}$, but it is convenient to introduce a redundant parametrization of these eigenvalues and to label operators by an $(n-1)$-dimensional vector α called the momentum,

$$\alpha = \sum_i \alpha_i \omega_i \quad \text{so that} \quad \alpha_i = (e_i, \alpha) \ . \tag{2.13}$$

The corresponding conformal dimension is supposed to be

$$q_\alpha^{(2)} = \Delta_\alpha = \frac{1}{2}(\alpha, 2Q - \alpha) \ , \tag{2.14}$$

where we introduce the vector

$$Q = (b + b^{-1})\rho \ . \tag{2.15}$$

In the case of the algebra W_3 we also have

$$q_\alpha^{(3)} = q_\alpha = \frac{i}{27}(\alpha_1 - \alpha_2)(2\alpha_1 + \alpha_2 - 3b - 3b^{-1})(\alpha_1 + 2\alpha_2 - 3b - 3b^{-1}) \ . \tag{2.16}$$

In general, $q^{(s)}$ is Weyl-invariant and homogeneous of degree s as a function of $\alpha - Q$. The representations which appear in the spectrum of $s\ell_3$ conformal Toda theory have momenta

$$\alpha \in Q + i(\mathbb{R}\omega_1 + \mathbb{R}\omega_2) \ , \tag{2.17}$$

so that Δ_α and q_α are real numbers. Under a Weyl transformation of the momentum, an operator $V_\alpha(z)$ is supposed to behave as

$$V_\alpha(z) = R_w(\alpha)\, V_{Q+w(\alpha-Q)}(z) \ , \tag{2.18}$$

for some reflection coefficients $R_w(\alpha)$. On the other hand, the conjugation of α does not leave the charges invariant, but they transform according to $q_{\alpha^*}^{(s)} = (-1)^s q_\alpha^{(s)}$, because α^* is related to $2Q - \alpha$ by a Weyl transformation. The conjugation of α therefore corresponds to the automorphism

$W^{(s)} \to (-1)^s W^{(s)}$ of the algebra W_n. (This assumes that the charges $W^{(s\geq 3)}$ are defined so that they are primary operators of dimensions s with respect to $T(z)$; other definitions are in principle possible.)

The character of a representation R_α of the W_n algebra is defined by

$$\xi_\alpha(\tau) \equiv \mathrm{Tr}_{R_\alpha}\, e^{2i\pi\tau(L_0-\frac{c}{24})} . \tag{2.19}$$

This is easily computed in the case of a continuous representation, with the result

$$\xi_\alpha(\tau) = \frac{e^{-i\pi\tau(Q-\alpha)^2}}{\eta(e^{2i\pi\tau})^{n-1}} , \tag{2.20}$$

where η is the Dedekind eta function. Let us now consider the case of a fully degenerate representation of the W_n algebra. Such a representation $R_{-b\Omega^+-b^{-1}\Omega^-}$, which is generated by the primary operator $V_{-b\Omega^+-b^{-1}\Omega^-}$, can be associated to any pair (Ω^+, Ω^-) of integral dominant weights of $s\ell_n$. The character $R_{-b\Omega^+-b^{-1}\Omega^-}$ is a sum over the Weyl group [11],

$$\xi_{-b\Omega^+-b^{-1}\Omega^-}(\tau) = \frac{\sum_{w\in W}\epsilon(w)\, e^{-i\pi\tau(b(\rho+\Omega^+)+b^{-1}w(\rho+\Omega^-))^2}}{\eta(e^{2i\pi\tau})^{n-1}} . \tag{2.21}$$

Now we observe that this degenerate character can be expressed in terms of the characters $\chi_{\Omega^\pm}$ of the two representations of $s\ell_n$ of highest weights $\Omega^\pm$,

$$\xi_{-b\Omega^+-b^{-1}\Omega^-}(\tau) = \frac{\sqrt{n}}{n!}\int d^{(n-1)}p\, \frac{e^{-\frac{2i\pi}{\tau}\frac{1}{2}p^2}}{\eta\left(e^{-\frac{2i\pi}{\tau}}\right)^{n-1}} \times \prod_{\pm}\left[\chi_{\Omega^\pm}(2\pi b^{\pm 1}p)\prod_{e>0}(e^{\frac{1}{2}(e,2\pi b^{\pm1}p)} - e^{-\frac{1}{2}(e,2\pi b^{\pm1}p)})\right] , \tag{2.22}$$

where the integration measure is defined as $d^{(n-1)}p = \prod_{i=1}^{n-1} dp_i$ with $p = \sum_i p_i e_i$, and we used the Weyl formula (2.5).

Notice that the W_n characters $\xi_\alpha(\tau)$ keep track of the conformal dimensions (L_0 eigenvalues) of states, and not of their charges $q^{(s>2)}$. So if $n > 2$ they contain much less information than the $s\ell_n$ characters $\chi_\Omega(p)$, which depend on a vector p and not on a single number τ. This will make the modular bootstrap analysis less powerful in theories with $W_{n>2}$ symmetries than in theories with just the Virasoro symmetry.

2.3 Fusion multiplicity

We will now comment on the fusion product of W_n representations. The fusion product is a generalisation to vertex operator algebras of the tensor product of representations of Lie algebras. So we first comment on the tensor product of $s\ell_n$ representations. We consider generic representations, which are not necessarily finite-dimensional, and even do not necessarily have a highest weight.

The algebra $s\ell_n$ can be represented in terms of differential operators acting on functions of $\frac{1}{2}n(n-1)$ "isospin" variables. (This is also the number of the creation operators, the operators which generate the highest-weight representations from their highest-weight states.) For example,

$s\ell_2$ is represented by the operators $D^- = \frac{\partial}{\partial x}$, $D^3 = x\frac{\partial}{\partial x} - j$, $D^+ = x^2\frac{\partial}{\partial x} - 2jx$ acting on functions of one isospin variable x, where the number j is the spin of the representation. States in a representation of $s\ell_n$ with spin j can be represented as functions $\Psi^j(x)$, where the spin j is a vector with $n-1$ components, and the isospin x is a vector with $\frac{1}{2}n(n-1)$ components. We wish to analyse the possible appearances of a representation R_{j_3} in the tensor product $R_{j_1} \otimes R_{j_2}$. Such an appearance implies the existence of a nonzero invariant vector in $R_{j_1} \otimes R_{j_2} \otimes R^*_{j_3}$, where $R^*_{j_3}$ is the contragredient representation. In the representation of $s\ell_n$ in terms of differential operators, an invariant vector in $R_{j_1} \otimes R_{j_2} \otimes R^*_{j_3}$ is represented as a function $\Phi(x_1, x_2, x_3)$ of three isospin vectors, subject to $(\dim s\ell_n) = n^2 - 1$ equations. If the representations are generic and no more assumptions are made, solutions $\Phi(x_1, x_2, x_3)$ come with the number of parameters

$$d_n = 3\frac{n(n-1)}{2} - (n^2-1) = \frac{1}{2}(n-1)(n-2) \ . \tag{2.23}$$

If $n > 2$ then $d_n > 0$ which implies that R_{j_3} can appear an infinite number of times in $R_{j_1} \otimes R_{j_2}$. If however one of the three representations R_{j_1}, R_{j_2} or R_{j_3} is not generic, then extra equations on $\Phi(x_1, x_2, x_3)$ can follow, and the number of parameters may become lower. If the number of parameters is zero, as happens if $n = 2$ or one of the involved representations has a highest weight state, then multiplicities must be finite.

A similar counting of variables, and similar conclusions on fusion multiplicities, hold in the case of the fusion product of W_n representations. This is a consequence of the conformal Ward identities for the three-point correlation functions $\left\langle \prod_{i=1}^3 V_{\alpha_i}(z_i) \right\rangle$ where the momenta α_i label W_n representations; such correlation functions are analogous to the invariants $\Phi(x_1, x_2, x_3)$ of our $s\ell_n$ reasoning (although the positions z_i are not analogous to the isospins x_i). The fusion multiplicity is the minimum number of correlation functions of descendent operators $\left\langle \prod_{i=1}^3 D_i V_{\alpha_i}(z_i) \right\rangle$ in terms of which all other such correlation functions can be linearly expressed using the Ward identities. Such identities are obtained by inserting the identity $\oint_\infty \varphi_s(z) W^{(s)}(z) = 0$ in a correlation function, where $\oint_\infty$ denotes the integration along a contour which encloses all the positions z_i of the operators, $\varphi_s(z)$ is meromorphic with possible poles at $z = z_i$, and at infinity $|\varphi_s(z)| \leq |z|^{2s-2}$. (We assume $W^{(s)}(z) \underset{z\to\infty}{\sim} z^{-2s}$, which follows from the $W^{(s)}$-symmetry of the vacuum.) Local Ward identities are obtained for functions $\varphi_s(z)$ which do have poles; the case $\varphi_s(z) = (z - z_i)^{-k}$ (with $k \in \mathbb{N}$) yields the expression of a correlation function involving $W^{(s)}_{1-s-k} V_{\alpha_i}(z_i)$ in terms of correlation functions with descendents of the type $W^{(s)}_{-p} V_{\alpha_i}(z_i)$ with $1 \leq p \leq s-1$, as can be seen from the operator product $W^{(s)}(z) V_\alpha(w)$. (See eq. (2.12) for the case $s = 3$.) In the theory with W_n symmetry there are $\frac{1}{2}n(n-1)$ modes of the type $W^{(s)}_{-p}$ with $2 \leq s \leq n$, $1 \leq p \leq s-1$, for instance the three modes L_{-1}, W_{-1}, W_{-2} in the case $n = 3$, and these modes are analogous to the isospin variables x of the $s\ell_n$ algebra. Global Ward identities are obtained for holomorphic functions $\varphi_s(z)$, that is polynomials of degrees at most $2s-2$. The number of global Ward identities is therefore $\sum_{s=2}^n (2s-1) = n^2 - 1$.

Thus, the number of modes of the $W^{(s)}$ symmetry generators which cannot be eliminated from the correlation functions of the type $\left\langle \prod_{i=1}^3 D_i V_{\alpha_i}(z_i) \right\rangle$ using the Ward identities is d_n (2.23). For example, $d_3 = 1$ means that in a theory with W_3 symmetry all three-point functions can be expressed in terms of the correlation functions $\langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) (W_{-1})^k V_{\alpha_3}(z_3) \rangle$ where V_{α_i} are primary operators and $k \in \mathbb{N}$. (Instead of W_{-1} we may have written W_{-2}, but not L_{-1}, because the three global Ward identities from $T(z)$ close among themselves and can be solved.) Notice that a similar reasoning can be used to predict the number E of independent differential equations obeyed by an N-point function of primary operators, some of which may be degenerate and involve a total number of V linearly independent null vectors. We find $E = n^2 - 4 - N(\frac{n(n-1)}{2} - 1) + V$, where we subtract the three equations from global conformal symmetry, as well as the contribution of the L_{-1} generator as it is identified with a derivative. Such a counting of differential equations has previously been used in [12].

To conclude, infinite fusion multiplicities must appear in all theories with a $W_{n\geq 3}$ symmetry as soon as continuous representations are involved, which will be the case in conformal Toda theories.

2.4 Lagrangian formulation

Conformal $s\ell_n$ Toda theory on a Riemann surface without boundary has a Lagrangian formulation. The dynamical fields of the theory form a vector with $n-1$ components

$$\phi = \sum_i \phi_i e_i \qquad \text{so that} \qquad \phi_i = (\omega_i, \phi) \ , \tag{2.24}$$

and the Lagrangian is

$$L_n = \frac{1}{2\pi}(\partial\phi, \bar{\partial}\phi) + \mu \sum_{i=1}^{n-1} e^{b(e_i,\phi)} \ , \tag{2.25}$$

where μ is the bulk cosmological constant, and the derivatives with respect to the complex coordinates $z, \bar{z}$ are related to derivatives with respect to the real coordinates $x = \Re z$, $y = \Im z$ by $\partial = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$, $\bar{\partial} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$. After the rescaling $\phi \to b^{-1}\phi$, the classical equations of motion are

$$\partial\bar{\partial}\phi_i = \pi b^2 \mu e^{(e_i,\phi)} \ . \tag{2.26}$$

The Lagrangian formulation permits the calculation of certain particular correlation functions, and of general correlation functions in certain limits, but not of general correlation function [13, 1]. For our purposes, we will only make use of the classical equations of motion, and not of functional integrals involving the action $S = \int L_n$. We will actually solve the equations of motion in Section 3.

The W_n symmetry of $s\ell_n$ Toda theory manifests itself by the existence of charges $W^{(2)} = T, W^{(3)} \cdots W^{(n)}$ which are classically conserved in the sense that $\bar{\partial}W^{(s)} = 0$. In the case of Liouville theory, this is

$$T = -(\partial\phi)^2 + \partial^2\phi \ . \tag{2.27}$$

In the case of $s\ell_3$ Toda theory, $W^{(3)} = W$ has the ambiguity $W \to W - \xi\partial T$ for ξ an arbitrary number, which we lift by assuming $W(\phi^*) = -W(\phi)$, where $\phi \to \phi^*$ is the Dynkin diagram automorphism $\phi_1 \leftrightarrow \phi_2$. We then have

$$T = -\tfrac{1}{2}(\partial\phi, \partial\phi) + (\rho, \partial^2\phi) = -\partial\phi_1^2 - \partial\phi_2^2 + \partial\phi_1\partial\phi_2 + \partial^2\phi_1 + \partial^2\phi_2 \ , \tag{2.28}$$

$$W = \left(\partial^3\phi_2 - 2\partial^2\phi_2\partial\phi_2 - \partial^2\phi_2\partial\phi_1 + 2\partial\phi_2^2\partial\phi_1\right) - (\phi_1 \leftrightarrow \phi_2) \ , \tag{2.29}$$

where we neglect a possible normalization factor in the definition of W. Such classically conserved charges can alternatively be found as classical limits of the corresponding quantum symmetry generators of the W_3 algebra [5]. We use the same notation for the classical charge and the quantum generator; the context should clarify which one we are dealing with.

The momentum α, which we used as a label for W_n representations (see (2.14,2.16)), has a simple interpretation in the Lagrangian formulation. Namely, the classical counterpart of the quantum operator $V_\alpha(z)$ is $e^{(\alpha,\phi(z))}$.

3. Solutions of the Toda equations on a disc

The classical $s\ell_n$ Toda equations on the Riemann sphere have been solved in [14], see also [15]. We will now look for solutions on the disc, which are solutions on the sphere respecting certain boundary conditions. We will only consider maximally symmetric boundary conditions, that is conditions of the type $\bar{W}^{(s)} = f(\{W^{(s')}\})$ where f is an automorphism of the W_n algebra. The known automorphisms are the identity, and in the $W_{n\geq 3}$ algebra the automorphism $W^{(s)} \to (-1)^s W^{(s)}$. There will therefore be two possible types of boundary conditions if $n \geq 3$, and only one if $n = 2$. We will study the cases of the W_2 and W_3 algebras. The sphere will be identified with the complex plane and parametrized by the coordinates $z, \bar{z}$, and the disc will be identified with the upper half-plane $\{\Im z > 0\}$.

In this Section, we will study the solutions of the Toda equations and in particular their invariants, which we call the boundary parameters. The question whether our boundary conditions follow from boundary actions is postponed to Appendix B. We will consider the classical Toda equations (2.26) with the value

$$\mu = -\frac{1}{\pi b^2} \tag{3.1}$$

for the cosmological constant. The choice of a negative value for μ will allow real, globally defined, regular solutions to exist.

3.1 Case of Liouville theory

In order to solve the Liouville equation $\partial\bar{\partial}\phi = -e^{2\phi}$ together with the boundary condition $T = \bar{T}$ where $T = -(\partial\phi)^2 + \partial^2\phi$, we introduce the variable $X = e^{-\phi}$ which is such that $T = -\frac{\partial^2 X}{X}$ and the Liouville equation amounts to $\Delta_2(X) \equiv X\partial\bar{\partial}X - \partial X\bar{\partial}X = 1$. The solutions of this equation are of the form

$$X = \sum_{i=1}^{2} a_i(z)b_i(\bar{z}) \quad , \quad \mathrm{Wr}[a_1, a_2] = \mathrm{Wr}[b_1, b_2] = 1 \; , \tag{3.2}$$

where $\mathrm{Wr}[a_1, a_2] = a_1a_2' - a_2a_1'$ is the Wronskian. (By setting $\mathrm{Wr}[a_1, a_2] = \mathrm{Wr}[b_1, b_2] = 1$ we have eliminated the ambiguity $a_i \to \xi a_i,\ b_i \to \xi^{-1}b_i$.) The stress-energy tensor T associated with such a solution is $T = -\frac{a_1''}{a_1} = -\frac{a_2''}{a_2}$. The condition $T = \bar{T}$ is solved by assuming

$$a_i(z) = \sum_{j=1}^{2} N_{ij}b_j(z) \quad , \quad \det N = 1 \; . \tag{3.3}$$

The condition that X be real and positive will now be solved by assuming that $b(z) = \bar{b}(z)$ is real, and that the constant matrix N is Hermitian and positive. To summarize, our solutions are

$$X = \sum_{i,j=1}^{2} \overline{b_i(z)} N_{ij} b_j(z) \quad , \quad \mathrm{Wr}[b_1, b_2] = 1 \ , \ \det N = 1, \ N > 0 \ . \tag{3.4}$$

There remain some ambiguities in the solutions, because different choices for $N_{ij}, b_i(z)$ can lead to the same X. In particular, the following action of $SL(2,\mathbb{R})$ leaves X invariant:

$$\begin{cases} b \to \Lambda^{-1} b \\ N \to \Lambda^T N \Lambda \end{cases} \quad , \quad \Lambda \in SL(2,\mathbb{R}) \ , \tag{3.5}$$

where Λ^T denotes the transpose of the matrix Λ.

Let us define a boundary parameter λ_L associated to a given solution. We assume this parameter to be a z-independent function of the solution X. Independence from z implies being a function of the matrix N. Being a function of X implies being invariant under the action (3.5) of $SL(2,\mathbb{R})$. The only such invariant function of N is

$$\lambda_L \equiv \frac{1}{2i} \mathrm{Tr}\, NP \quad , \quad P \equiv \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \ . \tag{3.6}$$

Notice that the matrix P obeys $\forall \Lambda \in SL(2,\mathbb{R}),\ \Lambda P \Lambda^T = P$. The role of λ_L as a boundary parameter can be demonstrated by rewriting the boundary condition in terms of the field $X = e^{-\phi}$. At the boundary $z = \bar{z}$ we find:

$$(\partial - \bar{\partial}) X = 2i\lambda_L \quad , \quad (\partial - \bar{\partial})\phi = -2i\lambda_L e^{\phi} \ . \tag{3.7}$$

This implies that the boundary conditions could be derived by adding a boundary term $\int L_2^{bdy}$ to the action, with

$$L_2^{bdy} = \lambda_L e^{\phi} \ , \tag{3.8}$$

and λ_L would be the boundary cosmological constant.

We conclude this Subsection with a remark. Given the solution of the Liouville equation, it is easy to write the Bäcklund transformation from Liouville theory to a free field theory. The free field can be defined as $\psi = -\log \frac{\sum_i u_i a_i(z)}{\sum_i v_i b_i(\bar{z})}$ where (u_1, u_2) and (v_1, v_2) are constant vectors, and the stress-energy tensors are $T = -(\partial\psi)^2 + \partial^2\psi$, $\bar{T} = -(\bar{\partial}\psi)^2 - \bar{\partial}^2\psi$. The field ψ obeys the free equations of motion $\partial\bar{\partial}\psi = 0$, as well as Dirichlet boundary conditions $(\partial + \bar{\partial})\psi = 0$, and the value of ψ at the boundary is related to λ_L.

3.2 Case of $s\ell_3$ Toda theory

Let us solve the $s\ell_3$ Toda equations $\begin{cases} \partial\bar{\partial}\phi_1 = -e^{2\phi_1 - \phi_2} \\ \partial\bar{\partial}\phi_2 = -e^{2\phi_2 - \phi_1} \end{cases}$. In terms of $X_i = e^{-\phi_i}$, the $s\ell_3$ Toda equations amount to $\begin{cases} \Delta_2(X_1) = X_2 \\ \Delta_2(X_2) = X_1 \end{cases}$, where $\Delta_2(X)$ was defined in the previous Subsection, and is such that $\Delta_2(\Delta_2(X)) = X \det \begin{pmatrix} X & \partial X & \partial\partial X \\ \bar\partial X & \partial\bar\partial X & \partial\partial\bar\partial X \\ \bar\partial\bar\partial X & \partial\bar\partial\bar\partial X & \partial\partial\bar\partial\bar\partial X \end{pmatrix}$. The solutions of the $s\ell_3$ Toda equations are

$$\left\{\begin{array}{l} X_1 = \sum_{i=1}^3 a_i(z) b_i(\bar z) \\ X_2 = \sum_{i<j} \mathrm{Wr}[a_i, a_j](z) \mathrm{Wr}[b_i, b_j](\bar z) \end{array}\right. \quad , \quad \mathrm{Wr}[a_1, a_2, a_3] = \mathrm{Wr}[b_1, b_2, b_3] = 1 \ , \tag{3.9}$$

where $\mathrm{Wr}[a_1, a_2, a_3] = \epsilon_{ijk} a_i a_j' a_k''$ is the cubic Wronskian. In this solution, the Dynkin diagram automorphism $\phi_1 \leftrightarrow \phi_2$ manifests itself as $\left\{\begin{array}{l} a_i \leftrightarrow \frac12 \epsilon_{ijk} \mathrm{Wr}[a_j, a_k] \\ b_i \leftrightarrow \frac12 \epsilon_{ijk} \mathrm{Wr}[b_j, b_k] \end{array}\right.$. In these formulas we used the fully antisymmetric tensor ϵ_{ijk} such that $\epsilon_{123} = 1$.

Let us rewrite the symmetry charges T (2.28) and W (2.29) in terms of the variables X_1, X_2 or a_i, b_j:

$$T = -\frac{\partial^2 X_1}{X_1} - \frac{\partial^2 X_2}{X_2} + \frac{\partial X_1}{X_1}\frac{\partial X_2}{X_2} = \frac{\partial^3 X_1 \bar\partial X_1 - X_1 \partial^3 \bar\partial X_1}{X_2} = \frac{a_3 a_1''' - a_1 a_3'''}{\mathrm{Wr}[a_1, a_3]} \ , \tag{3.10}$$

$$W = T\left(\frac{\partial X_2}{X_2} - \frac{\partial X_1}{X_1}\right) - \frac{\partial^3 X_1}{X_1} + \frac{\partial^3 X_2}{X_2} = -T' + 2\frac{a_1' a_3''' - a_3' a_1'''}{\mathrm{Wr}[a_1, a_3]} \ , \tag{3.11}$$

where we could use any pair of functions a_i instead of (a_1, a_3), and the result would not change due to the identity $\mathrm{Wr}[a_1, a_2, a_3]' = 0$. The antiholomorphic charges $\bar T, \bar W$ are similarly written in terms of b_i.

We now consider $s\ell_3$ Toda theory on the half-plane, and the possible boundary conditions on the real line. We must impose $T = \bar T$ for conformal symmetry to be preserved. For the spin 3 current W we have the two choices $W = \pm \bar W$, where the minus sign corresponds to using the nontrivial automorphism of the W_3 algebra.

3.3 Boundary condition $W - \bar W = 0$

Now that we wrote the solutions of the bulk equations of motion in terms of the functions a_i, b_i, let us write boundary conditions for these functions. The conditions $T = \bar T, W = \bar W$ are obeyed provided we assume

$$a_i = \sum_{j=1}^3 N_{ij} b_j \quad , \quad \det N = 1 \ . \tag{3.12}$$

This implies $\epsilon_{ijk} \mathrm{Wr}[a_j, a_k] = (N^{-1T})_{ii'} \epsilon_{i'j'k'} \mathrm{Wr}[b_{j'}, b_{k'}]$, where N^{-1T} is the inverse of the transpose of N. Furthermore, in order for X_1, X_2 to be positive, we assume $b_i = \bar b_i$ and that N is a positive Hermitian matrix. To summarize,

$$\left\{\begin{array}{l} X_1 = \sum_{i,j=1}^3 \overline{b_i(z)} N_{ij} b_j(z) \\ X_2 = \sum_{i,j=1}^3 \overline{w_i(z)} (N^{-1T})_{ij} w_j(z) \end{array}\right. \quad , \quad \mathrm{Wr}[b_1, b_2, b_3] = 1 \ , \ \det N = 1, \ N > 0 \ , \tag{3.13}$$

where we introduced the notation $w_i = \epsilon_{ijk} b_j b_k'$ for the quadratic Wronskians, whose quadratic Wronskians are themselves $\mathrm{Wr}[w_i, w_j] = \epsilon_{ijk} b_k$.

As in the case of Liouville theory, our parametrization of the solutions is ambiguous, because X_1, X_2 are invariant under the action of an $SL(3,\mathbb{R})$ symmetry group, namely

$$\begin{cases} b \to \Lambda^{-1} b \\ w \to \Lambda^T w \\ N \to \Lambda^T N \Lambda \end{cases} \quad , \quad \Lambda \in SL(3,\mathbb{R}) \ . \tag{3.14}$$

The boundary parameters are the $SL(3,\mathbb{R})$-invariant functions of N. Such invariants can be constructed as Tr $(N^T N^{-1})^m$, $m = 1, 2 \cdots$. Given $\det N = 1$, all such invariants are functions of

$$\lambda_0 \equiv \det \tfrac{1}{2}(N + N^T) \ . \tag{3.15}$$

In the $s\ell_n$ case the number of invariants is the integer part of $\frac{n}{2}$.

We may wish to express the boundary conditions in terms of the fields $X_i = e^{-\phi_i}$. To this end, we may compute at the boundary

$$\begin{cases} (\partial - \bar{\partial}) X_1 = \frac{1}{2} \epsilon_{ijk} N_{ij} w_k \\ (\partial - \bar{\partial}) X_2 = \frac{1}{2} \epsilon_{ijk} N^{-1T}_{ij} b_k \end{cases} . \tag{3.16}$$

The right hand sides of these expressions are in general not functions of X_1, X_2. There is an exception in the special case when $N^T = N$, which corresponds to the free boundary conditions $(\partial - \bar{\partial}) X_i = 0$, in which case the boundary parameter is $\lambda_0 = 1$. Another exception occurs when $N_{ij} = U_i U_j + \epsilon_{ijk} A_k$, with $A_i U_i = 1$ so that $\det N = 1$. (We drop the assumption that N be hermitian.) In this case the boundary parameter is $\lambda_0 = -1$. Noticing $N^{-1T} = A_i A_j + \epsilon_{ijk} U_k$, we find $\begin{cases} (\partial - \bar{\partial}) X_1 = A_k w_k = \sqrt{X_2} \\ (\partial - \bar{\partial}) X_2 = U_k b_k = \sqrt{X_1} \end{cases}$. Such boundary conditions derive from the boundary Lagrangian

$$L_3^{bdy} = \frac{1}{2i} \left(e^{\phi_1 - \frac{1}{2}\phi_2} + e^{\phi_2 - \frac{1}{2}\phi_1} \right) \ . \tag{3.17}$$

One may be tempted to generalize this Lagrangian into $L_3^{bdy} = \nu_1 e^{\phi_1 - \frac{1}{2}\phi_2} + \nu_2 e^{\phi_2 - \frac{1}{2}\phi_1}$, which would depend on two boundary parameters ν_1, ν_2. However, it turns out that for general values of ν_1, ν_2 the W_3 symmetry would then be broken, in the sense that the boundary condition $W = \bar{W}$ would not be obeyed. Only the values of ν_1, ν_2 which we wrote in eq. (3.17) are therefore permitted.

3.4 Boundary condition $W + \bar{W} = 0$

Boundary conditions for the functions a_i, b_i which imply $W + \bar{W} = 0$ are now

$$a_i = \sum_{j=1}^{3} N_{ij} w_j \quad , \quad \det N = 1 \ . \tag{3.18}$$

This can be deduced from the case $W - \bar{W} = 0$ by using the Dynkin diagram automorphism, which exchanges the functions b_i with their Wronskians $w_i = \epsilon_{ijk} b_j b'_k$. It is however not clear how to guarantee the positivity of X_1, X_2. We refrain from making further assumptions on the matrix N, and we write the solutions of the classical Toda equations as

$$\begin{cases} X_1 = \sum_{i,j=1}^3 b_i(\bar{z}) N_{ij} w_j(z) \\ X_2 = \sum_{i,j=1}^3 w_i(\bar{z}) N^{-1T}_{ij} b_j(z) \end{cases} , \quad \mathrm{Wr}[b_1, b_2, b_3] = 1 \ , \ \det N = 1 \ . \tag{3.19}$$

As we do not impose reality conditions on b_i and N we find that X_1, X_2 are invariant under the action of an $SL(3, \mathbb{C})$ group of symmetries, instead of an $SL(3, \mathbb{R})$ group in the case $W - \bar{W} = 0$:

$$\begin{cases} b \to \Lambda^{-1} b \\ w \to \Lambda^T w \\ N \to \Lambda^T N \Lambda^{-1T} \end{cases} , \quad \Lambda \in SL(3, \mathbb{C}) \ . \tag{3.20}$$

The group acts by conjugation on the matrix N, and there are two invariants, which we interpret as boundary parameters:

$$\lambda_1 = \mathrm{Tr}\, N \quad , \quad \lambda_2 = \mathrm{Tr}\, N^{-1} \ . \tag{3.21}$$

In the $s\ell_n$ case the number of invariants is of course $n - 1$. This corresponds to the number of conserved charges $W^{(2)} \cdots W^{(n)}$. This already suggests that the boundary condition $W + \bar{W} = 0$ realizes Cardy's ideas on the correspondence between representations of the symmetry algebra and boundary parameters. We will demonstrate this further in our conformal bootstrap analysis in Section 4.

An interesting case happens when the matrix N obeys a second-order polynomial equation, that is when two of its eigenvalues coincide. Then we have $N^{-1} = uN + vI$ where I is the identity matrix and u, v are two complex numbers, and it follows from eq. (3.19) that $\phi_1 - \phi_2$ obeys Dirichlet boundary conditions and $\phi_1 + \phi_2$ obeys Neumann boundary conditions,

$$\phi_1 - \phi_2 = c \quad , \quad (\partial - \bar{\partial})(\phi_1 + \phi_2) = 0 \ , \tag{3.22}$$

where c is an arbitrary constant. These conditions can be derived from the $s\ell_3$ Toda action with no boundary terms. Due to the Dirichlet condition, the spacetime interpretation of this case is a one-dimensional D-brane, whereas the other cases describe two-dimensional D-branes. The one-dimensional D-branes extend along the direction of the Weyl vector ρ, which is consistent with the existence of a linear dilaton in that direction. (This linear dilaton can be seen in the expression for the stress-energy tensor T (2.28).)

We conclude this Subsection with a remark. Given the solution of the Toda equations, it is easy to write the Bäcklund transformation from Toda field theory to a free field theory. The free fields can be defined as $\psi_i = -\log \frac{\sum_j u_{ij} a_j(z)}{\sum_j v_{ij} w_j(\bar{z})}$ where u_{ij} and v_{ij} are constant matrices. The free fields obey free equations of motion $\partial\bar{\partial}\psi_i = 0$, and Dirichlet boundary conditions for ψ_i imply relations of the type $a_i = \sum_j N_{ij} w_j$ (3.18) at the boundary. Thus, we can interpret the boundary parameters λ_1, λ_2 encoded in the matrix N as the boundary values of the free fields. (See Appendix A for more details on the Bäcklund transformation.)

4. Conformal bootstrap study of $s\ell_3$ Toda theory with $W + \bar{W} = 0$

The conformal bootstrap method is the systematic exploitation of symmetry and consistency constraints on correlation functions in two-dimensional conformal field theories [4]. We will apply this method to the correlation functions of W_3 primary operators $V_\alpha(z)$. Such operators are defined up to normalizations by their operator products (2.11) and (2.12) with the symmetry generators $T(z), W(z), \bar{T}(\bar{z}), \bar{W}(\bar{z})$. In order to define correlation functions on the upper half-plane, the properties of the boundary must be characterized. This involves first of all imposing boundary conditions for the symmetry generators $T(z), W(z)$. Moreover, we saw in Section 3 that for each of the two boundary conditions $\begin{cases} T = \bar{T} \\ W = \pm\bar{W} \end{cases}$ there exist families of possible D-branes, parametrized by λ_0 or λ_1, λ_2. These parameters appeared in the classical analysis of the boundary conditions for the basic Toda fields ϕ_i, but such fields are not present in the conformal bootstrap formalism. Nevertheless, the equivalents of λ_0 or λ_1, λ_2 will appear when we will parametrize the solutions of the conformal bootstrap equations; a given solution will be called a D-brane or boundary state.

In the case of Liouville theory there is only one possible boundary condition $T = \bar{T}$, and there exist two types of D-branes: the continuous D-branes [16, 17], with a continuous parameter, and the discrete D-branes [18], which are parametrized by two integers. These two types of D-branes are associated to the two types of representations of the Virasoro algebra: the continuous and discrete representations. By analogy, we expect that in conformal $s\ell_n$ Toda theory there exists a hierarchy of D-branes, which would correspond to the hierarchy of representations of the W_n algebra which we discussed in Subsection 2.2. The dimension of a D-brane would be $n - 1 - k$, where k is the number of algebraically independent null vectors in the corresponding representation. In the case of $s\ell_3$ Toda theory, we would have three types of D-branes: two-dimensional continuous D-branes, one-dimensional "semi-degenerate" D-branes, and zero-dimensional discrete or fully-degenerate D-branes. We will see that these expectations are fulfilled when the boundary condition is $W + \bar{W} = 0$.

An important difference between the two boundary conditions $W = \pm\bar{W}$ manifests itself when analyzing the consequences of the W_3 symmetry on the correlation functions. We introduced the Ward identities which follow from the W_3 symmetry in Subsection 2.3, let us now sketch how such identities constrain the correlation function of N operators $V_{\alpha_i}(z_i)$ (with $\Im z_i > 0$) in the presence of a boundary at $z = \bar{z}$. It turns out that the Ward identities for such an N-point function are identical to the Ward identities for a $2N$-point function in the absence of a boundary, where the extra N operators are "reflected" operators located at $\bar{z}_i$. The reflected operators are $V_{\alpha_i}(\bar{z}_i)$ if the boundary condition is $W - \bar{W} = 0$, and $V_{\alpha_i^*}(\bar{z}_i)$ if the boundary condition is $W + \bar{W} = 0$. As far as the Ward identities are concerned, we thus have the relations

$$\langle V_\alpha(z) \rangle_{W-\bar{W}=0} \sim \langle V_\alpha(z) V_\alpha(\bar{z}) \rangle \ , \tag{4.1}$$

$$\langle V_\alpha(z) \rangle_{W+\bar{W}=0} \sim \langle V_\alpha(z) V_{\alpha^*}(\bar{z}) \rangle \ . \tag{4.2}$$

The Ward identities for a bulk two-point function $\langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) \rangle$ are known to imply that it vanishes unless $\Delta_{\alpha_1} = \Delta_{\alpha_2}$ and $q_{\alpha_1} + q_{\alpha_2} = 0$. Now eqs. (2.14) and (2.16) imply $\Delta_\alpha = \Delta_{\alpha^*}$ and $q_\alpha = -q_{\alpha^*}$. Therefore, while $\langle V_\alpha(z) \rangle_{W+\bar{W}=0}$ may be nonzero for all values of α, $\langle V_\alpha(z) \rangle_{W-\bar{W}=0}$ must vanish unless $q_\alpha = 0$. This restricts the momentum α to a one-dimensional space, which may be related to the fact that there is only one boundary parameter λ_0 in the case $W - \bar{W} = 0$. We will however not analyze this case further, and instead concentrate on the case $W + \bar{W} = 0$ from now on.

4.1 Continuous D-branes

Due to conformal symmetry, a one-point function on the upper half-plane must take the form

$$\langle V_\alpha(z)\rangle = \frac{U(\alpha)}{|z-\bar{z}|^{2\Delta_\alpha}} , \tag{4.3}$$

where $U(\alpha)$ is the bulk one-point structure constant, which we now want to determine. We will find constraints on $U(\alpha)$ by considering the two-point function $\langle V_{-b\omega_1}(y)V_\alpha(z)\rangle$, which can be factorized in two possible ways:

$$\langle V_{-b\omega_1}(y)V_\alpha(z)\rangle = \sum_{h\in H_{\omega_1}} C_h(\alpha)U(\alpha-bh)\mathcal{G}_h(\alpha|y,z) , \tag{4.4}$$

$$= \sum_j R_j S_j(\alpha)\mathcal{F}_j(\alpha|y,z) . \tag{4.5}$$

Let us explain these formulas. The first formula follows from the operator product expansion

$$V_{-b\omega_1}V_\alpha \to \sum_{h\in H_{\omega_1}} C_h(\alpha)V_{\alpha-bh} . \tag{4.6}$$

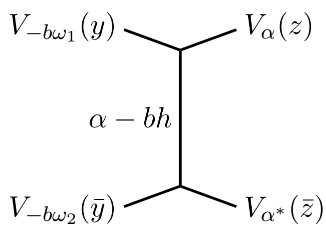
This OPE is a sum of three terms labelled by the set $H_{\omega_1} = \{\omega_1, \omega_2-\omega_1, -\omega_2\}$ of the weights of the fundamental representation of $s\ell_3$; this is analogous to the tensor product of $s\ell_3$ representations $R_{\omega_1}\otimes R_\Omega = \sum_{h\in H_{\omega_1}} R_{\Omega+h}$. We choose to study the correlation function $\langle V_{-b\omega_1}(y)V_\alpha(z)\rangle$ precisely because the fully degenerate operator $V_{-b\omega_1}$ has such simple OPEs; we could in principle use arbitrary operators instead, but the resulting constraints on $U(\alpha)$ could not necessarily be written explicitly. The OPE coefficients $C_h(\alpha)$ are [1]

$$C_{\omega_1}(\alpha) = 1 , \tag{4.7}$$

$$C_{\omega_2-\omega_1}(\alpha) = -\frac{\pi\mu}{\gamma(-b^2)}\frac{\gamma(b(e_1,\alpha-Q))}{\gamma(b(e_1,\alpha))} , \tag{4.8}$$

$$C_{-\omega_2}(\alpha) = \left(\frac{\pi\mu}{\gamma(-b^2)}\right)^2\frac{\gamma(b(e_2,\alpha-Q))}{\gamma(b(e_2,\alpha))}\frac{\gamma(b(\rho,\alpha-Q))}{\gamma(b(\rho,\alpha))} , \tag{4.9}$$

where we recall that μ is the bulk cosmological constant, b parametrizes the central charge, and $Q = (b+b^{-1})\rho$ (see Subsection 2.2). We also introduce the function $\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}$ where $\Gamma(x)$ is Euler's Gamma function. The last factor in eq. (4.4) is the conformal block $\mathcal{G}_h(\alpha|y,z)$. From our remark that N-point functions on the upper half-plane are equivalent to $2N$-point functions on the plane as far as Ward identities are concerned, it follows that $\mathcal{G}_h(\alpha|y,z)$ coincides with a bulk four-point t-channel conformal block,

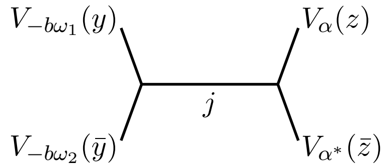
$\mathcal{G}_h(\alpha|y,z) =$ 

(4.10)

Explicit expressions for such conformal blocks can be deduced from [1], where more general conformal blocks were computed. Up to simple common prefactors, the three conformal blocks are

$$\mathcal{G}_h(\alpha|y,z) \propto \left|\tfrac{y-z}{y-\bar{z}}\right|^{2b(h,\alpha-Q)} {}_3F_2\left(\begin{matrix} -b^2, & b(e_h,\alpha-Q)-b^2, & b(e'_h,\alpha-Q)-b^2 \\ & b(e_h,\alpha-Q)+1, & b(e'_h,\alpha-Q)+1 \end{matrix}\middle| \left|\tfrac{y-z}{y-\bar{z}}\right|^2\right) \tag{4.11}$$

where for a given weight $h \in H_{\omega_1}$ we call e_h, e'_h the two roots such that $(h,e_h) = (h,e'_h) = 1$, for instance $e_{-\omega_2} = -e_2,\ e'_{-\omega_2} = -\rho$. Similarly, the quantity $\mathcal{F}_j(\alpha|y,z)$ in eq. (4.5) is an s-channel conformal block,

$\mathcal{F}_j(\alpha|y,z) =$ 

(4.12)

where j labels the operator which propagates in the s-channel. There is a subtlety here: the t-channel analysis predicts the existence of three independent blocks, but only two primary operators can appear in the OPE $V_{-b\omega_1}V_{-b\omega_2}$, namely V_0 and $V_{-b\rho}$. The point is that a descendent of $V_{-b\rho}$ can appear independently of $V_{-b\rho}$ itself; we will label as $j = \rho'$ the corresponding s-channel operator. In the presence of a boundary, we thus have the bulk-boundary OPE

$$V_{-b\omega_1} \to \sum_{j\in\{0,\rho,\rho'\}} R_j B_{-bj} \ , \tag{4.13}$$

where the coefficients R_j are unknown functions of b, and B_{-bj} are boundary operators, with the convention that $B_{-b\rho'}$ is some descendent of $B_{-b\rho}$. We will not dwell longer on this subtlety, as we are presently only interested in the s-channel operator $j = 0$. In this case, the bulk-boundary structure constant $S_j(\alpha)$ reduces to

$$S_0(\alpha) = U(\alpha) \ . \tag{4.14}$$

We can now obtain an equation for $U(\alpha)$ from the equality of (4.4) with (4.5) by using the fusion tranformation

$$\mathcal{G}_h(\alpha|y,z) = \sum_j F_{h,j}(\alpha)\mathcal{F}_j(\alpha|y,z) \ , \tag{4.15}$$

and extracting the $\mathcal{F}_0$ term in eq. (4.4). The result is

$$\sum_{h\in H_{\omega_1}} C_h(\alpha)F_{h,0}(\alpha)U(\alpha-bh) = R_0 U(\alpha) \ . \tag{4.16}$$

In order to make this equation explicit, let us compute the fusing matrix elements $F_{h,0}(\alpha)$ defined in eq. (4.15). Determining $F_{h,j}(\alpha)$ can be done by taking the limit $\Im z \to 0$ in that equation. By definition the blocks $\mathcal{F}_j(\alpha|y,z)$ are power-like functions of $\Im z$ in the limit $\Im z \to 0$, and we have

$$\mathcal{G}_h(\alpha|y,z) \underset{\Im z\to 0}{\sim} F_{h,\rho}(\alpha) + \frac{\Im y \Im z}{|y-\bar{z}|^2}F_{h,\rho'}(\alpha) + \left(\frac{\Im y\Im z}{|y-\bar{z}|^2}\right)^{2+3b^2} F_{h,0}(\alpha) \ . \tag{4.17}$$

As the block $\mathcal{G}_h(\alpha|y,z)$ is a ${}_3F_2$ hypergeometric function (4.11), let us study such functions, starting with their integral representation

$$ {}_3F_2\left(\begin{matrix} A_1 & A_2 & A_3 \\ & B_1 & B_2 \end{matrix}\middle| x\right) = \frac{\Gamma(B_2)}{\Gamma(A_3)\Gamma(B_2-A_3)} \int_0^1 dt\, t^{A_3-1}(1-t)^{B_2-A_3-1} \times {}_2F_1(A_1,A_2,B_1,tx)\ . \quad (4.18) $$

We wish to study the $x \to 1$ limit, where the critical exponents are 0, 1 and $B_1+B_2-A_1-A_2-A_3$, and to focus on the last one of those three exponents. Consider the region $t \to 1,\ x \to 1$, where we can use the approximation ${}_2F_1(A_1,A_2,B_1,tx) \underset{tx\to 1}{\sim} (1-tx)^{B_1-A_1-A_2}\frac{\Gamma(B_1)\Gamma(A_1+A_2-B_1)}{\Gamma(A_1)\Gamma(A_2)}$. Then we obtain the term with critical exponent $B_1+B_2-A_1-A_2-A_3$,

$$ {}_3F_2\left(\begin{matrix} A_1 & A_2 & A_3 \\ & B_1 & B_2 \end{matrix}\middle| z\right) \underset{z\to 1}{\supset} \frac{\Gamma(B_1)\Gamma(B_2)\Gamma(A_1+A_2+A_3-B_1-B_2)}{\Gamma(A_1)\Gamma(A_2)\Gamma(A_3)} \times (1-z)^{B_1+B_2-A_1-A_2-A_3}\ . \quad (4.19) $$

This term may or may not be the leading term of ${}_3F_2\left(\begin{matrix} A_1 & A_2 & A_3 \\ & B_1 & B_2 \end{matrix}\middle| z\right)$ in the limit $x \to 1$, depending on the values of A_i, B_j. In the case of $\mathcal{G}_h(\alpha|y,z)$ (4.11), it is actually subleading, as we assume $b>0$ and therefore $B_1+B_2-A_1-A_2-A_3 = 2+3b^2 > 2$. What we are interested in is the coefficient of $(1-z)^{2+3b^2}$, which is

$$ F_{h,0}(\alpha) = \frac{\Gamma(-2-3b^2)}{\Gamma(-b^2)} \frac{\Gamma(b(e_h,\alpha-Q)+1)}{\Gamma(b(e_h,\alpha-Q)-b^2)} \frac{\Gamma(b(e'_h,\alpha-Q)+1)}{\Gamma(b(e'_h,\alpha-Q)-b^2)}\ . \quad (4.20) $$

Combining this formula with the formulas for $C_h(\alpha)$, eq. (4.7)-(4.9), we obtain

$$ F_{h,0}(\alpha)C_h(\alpha) = \frac{\Gamma(-2-3b^2)}{\Gamma(-b^2)} \left[-\frac{\pi\mu}{\gamma(-b^2)}\right] \frac{A(\alpha-bh)}{A(\alpha)}\ , \quad (4.21) $$

where we introduced the function

$$ A(\alpha) \equiv \left[\pi\mu\gamma(b^2)\right]^{\frac{(\rho,\alpha-Q)}{b}} \prod_{e>0} \Gamma(b(e,\alpha-Q))^{-1}\Gamma(1+b^{-1}(e,\alpha-Q))^{-1}\ . \quad (4.22) $$

This function already appeared in conformal Toda theory [19], as a building block for the reflection coefficient $R_w(\alpha) = \frac{A(Q+w(\alpha-Q))}{A(\alpha)}$ defined in eq. (2.18). From that equation and the definition (4.3) of $U(\alpha)$, it follows that $A(\alpha)U(\alpha)$ must be invariant under the reflections $\alpha \to Q+w(\alpha-Q)$. The equation for $U(\alpha)$ (4.16) can now be rewritten as

$$ R_0 A(\alpha)U(\alpha) = \sum_{h\in H_{\omega_1}} A(\alpha-bh)U(\alpha-bh)\ , \quad (4.23) $$

where R_0 is still an unknown function of b, in which we actually absorbed the α-independent prefactors of $F_{h,0}(\alpha)C_h(\alpha)$ in eq. (4.21). Three more equations for $A(\alpha)U(\alpha)$ can similarly be obtained, by replacing the fully degenerate operator $V_{-b\omega_1}$ in eqs. (4.4) and (4.5) with one of the similar operators $V_{-b\omega_2}, V_{-b^{-1}\omega_1}$ or $V_{-b^{-1}\omega_2}$. The resulting equations for $A(\alpha)U(\alpha)$ are obtained from eq. (4.23) by replacing H_{ω_1} with H_{ω_2} and/or b by b^{-1}. The coefficient R_0 can also change, and we rename it $\lambda_{i,\pm}$ depending on the case. So we obtain the four equations

$$\lambda_{i,\pm} A(\alpha)U(\alpha) = \sum_{h\in H_{\omega_i}} A(\alpha - b^{\pm 1}h)U(\alpha - b^{\pm 1}h) \ . \tag{4.24}$$

The smooth, reflection-invariant solutions of these equations are $A(\alpha)U(\alpha) = \sum_{w\in W} e^{(w(s),\alpha-Q)}$ where W is the Weyl group and the arbitary vector s is the boundary parameter. The coefficients $\lambda_{i,\pm}$ might be called the boundary cosmological constants, and their values are

$$\lambda_{i,\pm} = \chi_{\omega_i}(-b^{\pm 1}s) \ , \tag{4.25}$$

where we recall that the fundamental character is $\chi_{\omega_1}(p) = e^{(\omega_1,p)} + e^{(\omega_2-\omega_1,p)} + e^{-(\omega_2,p)}$. The full formula for the solution $U_s(\alpha)$ is

$$\boxed{U_s(\alpha) = \left[\pi\mu\gamma(b^2)\right]^{\frac{(\rho,Q-\alpha)}{b}} \prod_{e>0} \Gamma(b(e,\alpha-Q))\Gamma(1+b^{-1}(e,\alpha-Q)) \sum_{w\in W} e^{(w(s),\alpha-Q)}} \tag{4.26}$$

Our equations (4.24) being linear, this formula holds up to an α-independent factor. We will say that $U_s(\alpha)$ defines a continuous D-brane when s is such that $U_s(\alpha)$ does not diverge exponentially in the limit of large momentum $|\alpha - Q| \to \infty$. As the operators in the spectrum of $s\ell_n$ Toda theory have purely imaginary values of $\alpha - Q$ (see eq. (2.17)), the continuous D-branes must have real values of s.

Notice that we wrote the one-point structure constant $U_s(\alpha)$ in a form which makes sense in $s\ell_n$ Toda theory for arbitrary n, and even in conformal Toda theories based on arbitrary simply-laced Lie algebras. We conjecture that this result, and most of the results in the rest of this Section, are valid in the general case and not only in $s\ell_3$ Toda theory.

4.2 Degenerate D-branes

In the previous Subsection we found continuous D-branes, whose parameter space has the same dimension as the space of continuous representations. According to the classical analysis of Subsection 3.4, such D-branes should be interpreted as covering the two-dimensional Toda space whose coordinates are ϕ_1, ϕ_2, because the boundary conditions for the fields ϕ_1, ϕ_2 are of the Neumann type. We will now investigate degenerate D-branes, whose dimensions and parameter spaces should be smaller. We will argue in Subsection 4.3 that the dimension of a D-brane is related to the divergence of its one-point function in the limit $\alpha - Q \to 0$: the higher the dimension, the more severe the divergence. So let us look for solutions $U(\alpha)$ to the equations (4.24) whose divergences at $\alpha - Q \to 0$ would be less severe than the divergence of $U_s(\alpha)$ (4.26). Cancelling some of the divergences from the poles of the $\Gamma(b(e, \alpha - Q))$ factor can be achieved by taking a linear combination of several solutions $U_s(\alpha)$ with different values of s. However, the resulting combination will still be a solution of eq. (4.24) only provided the four parameters $\lambda_{i,\pm}$ are the same for all the involved values of s.

In order to find two different values s, s' which have the same parameters $\lambda_{i,\pm}$, we make two observations: first, the parameters $\lambda_{i,\pm} = \chi_{\omega_i}(-bs)$ (4.25) are Weyl-invariant, second, they are invariant under shifts $s \to s + 2\pi i b^{\mp 1}(\mathbb{Z}e_1 + \mathbb{Z}e_2)$. Thus, there must exist two elements $w_\pm$ of the Weyl group such that $s - w_\pm(s') \in 2\pi i b^{\mp 1}(\mathbb{Z}e_1 + \mathbb{Z}e_2)$. Assuming the value of b^2 to be non-rational, this restricts s to a one-dimensional space. For example, the pair

$$s = \kappa\omega_1 + \pi i(\ell b + m b^{-1})e_2 \quad , \quad s' = \kappa\omega_1 + \pi i(\ell b - m b^{-1})e_2 \quad , \quad (\kappa, \ell, m) \in \mathbb{R} \times \mathbb{N}^2 \tag{4.27}$$

is such that $s - s' = 2\pi i m b^{-1} e_2$ and $s - r(s') = 2\pi i \ell b e_2$. Therefore,

$$U_{\kappa|\ell,m}(\alpha) \equiv U_{\kappa\omega_1 + \pi i(\ell b + m b^{-1})e_2}(\alpha) - U_{\kappa\omega_1 + \pi i(\ell b - m b^{-1})e_2}(\alpha) \tag{4.28}$$

is a solution of eq. (4.24), and an explicit calculation yields

$$\boxed{U_{\kappa|\ell,m}(\alpha) = \frac{4}{A(\alpha)} \sum_{e>0} e^{\kappa(h_e, \alpha - Q)} \sin 2\pi \ell b(e, \alpha - Q) \sin 2\pi m b^{-1}(e, \alpha - Q)} \, , \tag{4.29}$$

where to a positive root e we associate the weight $h_e \in H_{\omega_1}$ such that $(e, h_e) = 0$. In this expression the sine factors compensate some of the $\alpha - Q \to 0$ divergences of the prefactor $A(\alpha)$ (4.22). We hold that $U_{\kappa|\ell,m}(\alpha)$ with $\kappa \in \mathbb{R}$ and $\ell, m \in \mathbb{N}$ define the family of semi-degenerate D-branes. The boundary parameters of such D-branes are

$$\lambda_{1,+} = e^{-\frac{2}{3}b\kappa} + 2e^{\frac{1}{3}b\kappa}(-1)^m \cos \pi \ell b^2 \quad , \quad \lambda_{1,-} = e^{-\frac{2}{3}b^{-1}\kappa} + 2e^{\frac{1}{3}b^{-1}\kappa}(-1)^\ell \cos \pi m b^{-2} \, , \tag{4.30}$$

and $\lambda_{2,\pm}$ are obtained by $\kappa \to -\kappa$.

We expect that in the more general $s\ell_n$ case this construction generalizes to a hierarchy of partly degenerate D-branes. The difference of two terms in eq. (4.28) should be interpreted as a sum over the $\mathbb{Z}_2$ subgroup of the Weyl group which leaves ω_1 invariant, weighted by the signatures of the elements of that subgroup. In $s\ell_n$, there is a hierarchy of subgroups of the Weyl group which leave certain hyperplanes invariant, and summing over such subgroups should yield the partly degenerate D-branes. The case of fully degenerate D-branes corresponds to a sum over the full Weyl group, which we now study in the $s\ell_3$ case.

Given two integral dominant weights $\Omega, \Omega' \in \mathbb{N}\omega_1 + \mathbb{N}\omega_2$ we consider the combination

$$U_{\Omega|\Omega'}(\alpha) \equiv \sum_{w \in W} \epsilon(w)\, U_{2\pi i(b(\Omega+\rho) + b^{-1}w(\Omega'+\rho))}(\alpha) \, . \tag{4.31}$$

This can be rewritten as

$$\boxed{U_{\Omega|\Omega'}(\alpha) = \frac{1}{A(\alpha)} \sum_{w \in W} \epsilon(w)\, e^{2\pi i b(w(\Omega+\rho), \alpha - Q)} \sum_{w' \in W} \epsilon(w')\, e^{2\pi i b^{-1}(w'(\Omega'+\rho), \alpha - Q)}} \, . \tag{4.32}$$

This can be shown to have a finite limit as $\alpha - Q \to 0$, by first using the Weyl formula (2.5) in order to reduce the problem to the case of $U_{0|0}(\alpha)$,

$$\frac{U_{\Omega|\Omega'}(\alpha)}{U_{0|0}(\alpha)} = \chi_\Omega(2\pi i b(\alpha - Q))\, \chi_{\Omega'}(2\pi i b^{-1}(\alpha - Q)) \, , \tag{4.33}$$

and then the formula (2.6) in order to prove the regularity of $U_{0|0}(\alpha)$. We interpret $U_{\Omega|\Omega'}(\alpha)$ as defining localized (zero-dimensional) D-branes, which we will call discrete D-branes or fully degenerate D-branes. But first we should check that the six values of s which are involved in the sum (4.31) do have the same boundary parameters $\lambda_{i,\pm}$ (4.25). This is actually true, as a consequence of the fact that the weights $h \in H_{\omega_1}$ differ from one another by elements of $\mathbb{Z}e_1+\mathbb{Z}e_2$. And we find

$$\lambda_{1,+} = e^{-2\pi i(\omega_1,\Omega')}\chi_{\omega_1}(-2\pi i b^2\Omega) \quad , \quad \lambda_{1,-} = e^{-2\pi i(\omega_1,\Omega)}\chi_{\omega_1}(-2\pi i b^{-2}\Omega') \,. \tag{4.34}$$

In the case of discrete D-branes, these boundary cosmological constants are expected to be directly related to the values $U_s(-b^{\pm 1}\omega_1)$ of the one-point structure constant, as was argued in the case of Liouville theory in [18]. For example, $\lambda_{1,+}$ originally appeared as the bulk-boundary structure constant R_0 in the bulk-boundary OPE of $V_{-b\omega_1}$ (4.13). If the one-point structure constant was normalized so that $U_{\Omega|\Omega'}(0) = 1$, then $\lambda_{1,+}$ would coincide with $U_{\Omega|\Omega'}(-b^{\pm 1}\omega_1)$. Therefore we expect

$$\lambda_{i,\pm} = \frac{U_{\Omega|\Omega'}(-b^{\pm 1}\omega_i)}{U_{\Omega|\Omega'}(0)} \,. \tag{4.35}$$

And indeed the expressions (4.32) and (4.34) obey such relations, as can be shown with the help of the Weyl formula (2.5). (For this to be completely true we would have to reinstate the simple factor $\frac{\Gamma(-2-3b^2)}{\Gamma(-b^2)}\left[-\frac{\pi\mu}{\gamma(-b^2)}\right]$ in $\lambda_{i,\pm}$; such a factor was present in eq. (4.21) but we neglected it in what followed.) It is amusing to note that we did already apply the Weyl formula to $U_{\Omega|\Omega'}(\alpha)$ before, but in a different, "dual" way, in order to prove eq. (4.33).

Our expression for the bulk one-point function can be used to show that the equations of motion derived from the Lagrangian L_n (2.25) of Toda theory are obeyed in the presence of discrete D-branes. If we identify $e^{(\alpha,\phi)}$ with the operator V_α, and therefore ϕ_i with $\left.\frac{\partial}{\partial\alpha_i}\right|_{\alpha=0} V_\alpha$, then the quantum version of the equations of motion is

$$\left.\frac{\partial}{\partial\alpha_i}\right|_{\alpha=0} \partial\bar{\partial}\left\langle V_\alpha(z)\right\rangle = \pi b\mu \left\langle V_{be_i}(z)\right\rangle \,. \tag{4.36}$$

Using the form $\langle V_\alpha(z)\rangle = \frac{U(\alpha)}{|z-\bar{z}|^{2\Delta_\alpha}}$ (4.3) of the one-point function, this reduces to the following identity for the structure constant $U(\alpha)$:

$$\frac{U(be_i)}{U(0)} = \frac{2(b+b^{-1})}{\pi b\mu} \,. \tag{4.37}$$

It can be checked that this identity is obeyed by the one-point structure constant $U_{\Omega|\Omega'}(\alpha)$ (4.32) of a discrete D-brane.

4.3 Modular bootstrap analysis

The factorization constraint (4.4)-(4.5) whose solutions were our one-point structure constants is only one of the many equations of the conformal bootstrap formalism. Another one of these equations is relatively tractable: the modular bootstrap constraint, which relates two different decompositions of the annulus partition function

$$Z_{s_1;s_2}(\tau) = \frac{1}{6}\int d^2p\ U_{s_1}(Q+ip)U_{s_2}(Q-ip)\,\xi_{Q+ip}(\tau) \,, \tag{4.38}$$

$$= \mathrm{Tr}_{H_{s_1;s_2}}\, e^{-\frac{2i\pi}{\tau}\left(L_0-\frac{c}{24}\right)} \,. \tag{4.39}$$

Let us explain these formulas. We consider an annulus, the simplest Riemann surface with two boundaries. The two boundaries are characterized by their boundary parameters s_1, s_2, which may each correspond to any type of D-brane: continuous, semi-degenerate, or discrete; and the geometry of the annulus is characterized by the modular parameter τ. The annulus partition function (or zero-point correlation function) $Z_{s_1;s_2}(\tau)$ first has a "bulk channel" decomposition, which describes the exchange of bulk operators between the two boundaries. The resulting formula (4.38) for $Z_{s_1;s_2}(\tau)$ therefore involves a sum over the bulk spectrum. This sum decomposes into an integral over the physical values (2.17) of the momenta $\alpha = Q + ip$ which characterize the highest-weight representations of the algebra W_3 (with the factor $\frac{1}{6}$ due to the Weyl symmetry) , and sums over the descendent states in each representation, which are encoded in the characters $\xi_\alpha(\tau)$. The one-point structure constants $U_{s_1}(Q + ip)$ and $U_{s_2}(Q - ip)$ involve ingoing and outgoing momenta respectively, and no normalization factors appear due to the normalization assumption of [1], $\langle V_{Q+ip_1}(z_1)V_{Q-ip_2}(z_2)\rangle = \frac{\delta(p_1-p_2)}{|z_{12}|^{4\Delta_{Q+ip_1}}}$ (where we write only one of the six terms of a sum over the Weyl group).

The annulus partition function also has a "boundary channel" decomposition, which describes a one-loop partition function of open strings. The resulting formula (4.39) for $Z_{s_1;s_2}(\tau)$ is the trace over the boundary spectrum $H_{s_1;s_2}$ of the propagator $e^{-\frac{2i\pi}{\tau}\left(L_0-\frac{c}{24}\right)}$. This propagator is the operator which appears in the definition (2.19) of the characters $\xi_\alpha(\tau)$; in the boundary channel it appears with the dual value $-\frac{1}{\tau}$ of the modular parameter. Although we do not know the boundary spectrum $H_{s_1;s_2}$, the modular bootstrap method will produce tests of the one-point structure constants $U_s(\alpha)$. This is because for some choices of s_1, s_2 the spectrum is discrete, and the requirement that each representation should appear with a positive integer multiplicity is a nontrivial constraint.

Let us compute $Z_{s_1;s_2}(\tau)$ by using the bulk channel decomposition (4.38) with the bulk structure constants $U_s(\alpha)$ which we found in the previous two Subsections. We start with the case when both D-branes are discrete, and use eq. (4.33):

$$Z_{\Omega_1|\Omega'_1;\Omega_2|\Omega'_2}(\tau) = \frac{1}{6}\int d^2p\; \xi_{Q+ip}(\tau)\; U_{0|0}(Q+ip)U_{0|0}(Q-ip) \\ \times \chi_{\Omega_1^*}(2\pi bp)\chi_{\Omega_2}(2\pi bp)\chi_{\Omega_1'^*}(2\pi b^{-1}p)\chi_{\Omega'_2}(2\pi b^{-1}p)\;, \quad (4.40)$$

where we used the property $\chi_\Omega(-p) = \chi_{\Omega^*}(p)$. Now let us decompose the products of characters using eq. (2.7), while computing the product $U_{0|0}(Q + ip)U_{0|0}(Q - ip)$ using eq. (2.6),

$$Z_{\Omega_1|\Omega'_1;\Omega_2|\Omega'_2}(\tau) = \frac{(2\pi b^{-1})^6}{6}\sum_{\Omega} m^{\Omega}_{\Omega_1^*,\Omega_2}\sum_{\Omega'} m^{\Omega'}_{\Omega_1'^*,\Omega'_2}\int d^2p\; \xi_{Q+ip}(\tau) \\ \times \chi_\Omega(2\pi bp)\chi_{\Omega'}(2\pi b^{-1}p)\prod_{e>0}\prod_{\pm}\left(e^{\frac{1}{2}(e,2\pi b^{\pm1}p)} - e^{-\frac{1}{2}(e,2\pi b^{\pm1}p)}\right)\;. \quad (4.41)$$

The value of the (Gaussian) integral is given by eq. (2.22),

$$Z_{\Omega_1|\Omega'_1;\Omega_2|\Omega'_2}(\tau) = \frac{(2\pi b^{-1})^6}{\sqrt{3}}\sum_{\Omega,\Omega'} m^{\Omega}_{\Omega_1^*,\Omega_2} m^{\Omega'}_{\Omega_1'^*,\Omega'_2}\; \xi_{-b\Omega-b^{-1}\Omega'}(-\tfrac{1}{\tau})\;. \quad (4.42)$$

This is a sum of characters with positive integer coefficients $m^{\Omega}_{\Omega_1^*,\Omega_2} m^{\Omega'}_{\Omega_1'^*,\Omega'_2}$, up to a factor which could be absorbed in a renormalization of the one-point structure constant $U_{\Omega|\Omega'}(\alpha)$. The characters are those of fully degenerate representations. As we pointed out in Subsection 2.2, characters

$\chi_\alpha(\tau)$ do not fully characterize representations of the W_3 algebra, nevertheless we conjecture that the boundary spectrum is $H_{\Omega_1|\Omega'_1;\Omega_2|\Omega'_2} = \oplus_{\Omega,\Omega'} m^{\Omega}_{\Omega_1^*,\Omega_2} m^{\Omega'}_{\Omega_1'^*,\Omega'_2}\ R_{-b\Omega-b^{-1}\Omega'}$. If we associate the representation $R_{-b\Omega-b^{-1}\Omega'}$ to the discrete D-brane of parameter $s = \Omega|\Omega'$, fusing the representations associated to the two involved D-branes (after conjugating one of them) produces the boundary spectrum, which agrees with Cardy's ideas.

In particular, the $0|0$ D-brane corresponds to the identity representation, and for any D-brane of parameter s, the spectrum $H_{0|0;s}$ should be the single representation which is associated to that D-brane. In the case of a continuous D-brane,

$$Z_{0|0;s}(\tau) = \frac{(2\pi b^{-1})^6}{6}\int d^2p\ \xi_{Q+ip}(\tau)\ \sum_{w\in W} e^{i(w(s),p)} = \frac{(2\pi b^{-1})^6}{\sqrt{3}}\xi_{Q-\frac{s}{2\pi i}}(-\tfrac{1}{\tau})\ , \qquad (4.43)$$

so that the continuous representation of momentum $Q-\frac{s}{2\pi i}$ is associated to the continuous D-brane of parameter s. This immediately generalizes to semi-degenerate D-branes, using the formula (4.28) for their one-point structure constants. Another generalization is to replace the identity D-brane with an arbitrary discrete D-brane,

$$Z_{\Omega|\Omega';s}(\tau) = \frac{(2\pi b^{-1})^6}{6}\int d^2p\ \xi_{Q+ip}(\tau)\ \chi_{\Omega^*}(2\pi bp)\chi_{\Omega'^*}(2\pi b^{-1}p)\sum_{w\in W} e^{i(w(s),p)} \qquad (4.44)$$

$$= \frac{(2\pi b^{-1})^6}{\sqrt{3}}\sum_{h\in H_\Omega}\sum_{h'\in H_{\Omega'}} \xi_{Q-\frac{s}{2\pi i}-bh-b^{-1}h'}(-\tfrac{1}{\tau})\ , \qquad (4.45)$$

which lends support to the conjecture that the boundary spectrum is obtained by fusing the representations which correspond to the two D-branes. (Notice that a weight h may appear several times in H_Ω, as happened in the definition of the character $\chi_\Omega(p)$ (2.4).)

4.4 Continuous boundary spectra

Let us investigate the boundary spectrum of $s\ell_n$ Toda theory in the presence of two continuous D-branes using the modular bootstrap approach. Up to numerical factors, the annulus partition function is formally written as

$$Z_{s_1;s_2}(\tau) = \int d^{n-1}p\ \frac{\sum_{w_1,w_2\in W} e^{i(w_1(s_1)-w_2(s_2),p)}}{\prod_\pm \prod_{e>0}\sinh \pi b^{\pm 1}(e,p)}\ \xi_{Q+ip}(\tau)\ . \qquad (4.46)$$

This expression suffers from infrared divergences near $p = 0$. In the case of Liouville theory ($n = 2$), the divergence is of the type $\int \frac{dp}{p^2}$ and therefore linear in a long distance cutoff L. This is attributed to the geometry of the continuous D-branes, which are supposed to extend up to infinity in the Liouville space of coordinate ϕ. One may therefore naively expect that for general n the divergence should be L^{n-1}. However, it is actually $L^{(n-1)^2}$, so there is an extra divergence $L^{(n-1)(n-2)}$ beyond what is expected on geometrical grounds.

We observe that this extra divergence is governed by the number $2d_n = (n-1)(n-2)$ of parameters which are necessary to account for the infinite fusion multiplicities of continuous representations, where the factor 2 in $2d_n$ is meant to take into account the antiholomorphic multiplicities. (See Subsection 2.3.) Combining this observation with Cardy's ideas suggests a heuristic explanation of the divergence. Indeed, if the boundary spectrum is obtained by fusing the representations associated to the two D-branes, and fusion multplicity is infinite, then the boundary spectrum is a sum of representations with infinite multiplicities. This must then lead to divergences in the annulus partition function, in addition to the ordinary "geometrical" divergences. Therefore, we conjecture that the boundary spectrum in the presence of two continuous D-branes is the sum of all continuous representations of the W_n algebra, each one appearing with an infinite multiplicity.

It is not obvious to us how these considerations generalize to annulus partition functions involving arbitrary D-branes. For example, in the case of $s\ell_3$ Toda theory with one continuous and one semi-degenerate D-brane, the annulus partition function diverges as L^2. The fusion multiplicity is finite in this case, and we conjecture that the spectrum is the sum of all continuous representations, each one appearing a finite number of times.

5. Light asymptotic limits of some correlation functions

The main purpose of this Section is to establish a link between the classical analysis of Section 3, and the conformal bootstrap analysis of Section 4, in the case of $s\ell_3$ Toda theory with the boundary condition $W + \bar{W} = 0$. We will use the classical solutions of the Toda equations for predicting the bulk one-point function in a certain limit, and we will find that the classical predictions agree with the bootstrap results up to unimportant details. In addition we will also predict the light asymptotic limits of the boundary two-point function in the case $W + \bar{W} = 0$, and of bulk one-point and boundary two-point functions in the case $W - \bar{W} = 0$, for which we did not perform the conformal bootstrap analysis. For pedagogical purposes, we will begin with the computation of the analogous correlation functions in the much simpler case of Liouville theory.

The light semi-classical asymptotic limit, or light asymptotic limit, of a correlation function $\left\langle \prod_i V_{\alpha_i}(z_i) \right\rangle_s$ in the presence of a boundary with parameter s, where $V_{\alpha_i}(z_i)$ may be a bulk or a boundary operator, is defined by

$$b \to 0 \quad , \quad \eta_i \equiv b^{-1}\alpha_i \text{ and } \sigma \equiv bs \text{ fixed} . \tag{5.1}$$

If the correlation function is formally represented as a functional integral over the Toda field ϕ, with the weight $e^{-S[\phi]}$ where $S[\phi]$ is the action, then the light asymptotic limit reduces that functional integral to a finite-dimensional integral over field configurations such that $W^{(s)} = \bar{W}^{(s)} = 0$ for all spins s [1]. In $s\ell_n$ Toda theory, such field configurations are those where $X_i = e^{-\phi_i}$ are polynomials of degree $n-1$ as functions of the coordinates $z, \bar{z}$. These polynomials must obey further constraints like the boundary conditions and the reality of ϕ_i. The functional integral reasoning therefore predicts that the quantity

$$\left\langle \prod_i e^{(\eta_i, \phi(z_i))} \right\rangle_\sigma^{light} \equiv \int_{M_\sigma} d\phi_i \, \prod e^{(\eta_i, \phi(z_i))} , \tag{5.2}$$

where M_σ is a finite-dimensional space of field configurations, should be related to the limit (5.1) of $\left\langle \prod_i V_{\alpha_i}(z_i) \right\rangle_s$,

$$\lim_{b\to 0} \frac{\left\langle \prod_i V_{b\eta_i}(z_i) \right\rangle_{b^{-1}\sigma}}{\left\langle V_0(0) \right\rangle_{b^{-1}\sigma}} = \frac{\left\langle \prod_i e^{(\eta_i, \phi(z_i))} \right\rangle_\sigma^{light}}{\left\langle 1 \right\rangle_\sigma^{light}} . \tag{5.3}$$

Here we normalize the correlation functions by dividing them by the partition function. This eliminates the dependences on the overall undetermined factor in the integration measure on M_σ, and on the value $S[\phi]$ of the action for the polynomial field configurations, which is difficult to compute. As a result, nothing in eq. (5.3) depends on the action, and we conjecture that that equation holds whether a boundary action exists or not. (See Appendix B for a discussion of that point.)

5.1 Case of Liouville theory

Let us consider the solutions (3.4) of the Liouville equation which obey the "light asymptotic condition" $\partial^2 X = \bar\partial^2 X = 0$, where $X = e^{-\phi}$. Such solutions are built from two functions b_1, b_2 which are polynomials of degrees at most one. Given the freedom to choose the matrix N, we can fix these functions without loss of generality, and we choose $b_1(z) = 1,\ b_2(z) = z$. So we have $X(z) = (1, \bar z)N\left(\begin{smallmatrix}1\\ z\end{smallmatrix}\right)$ where N is a positive Hermitian matrix of determinant one. According to equation (5.2), the one-point function of a bulk operator in the presence of a boundary with parameter λ_L (3.6) is of the type

$$\left\langle e^{\eta\phi(z)}\right\rangle_{\lambda_L}^{light} = \int dN\ \delta\left(\lambda_L - \tfrac{1}{2i}\mathrm{Tr}\,(NP)\right)\ X(z)^{-\eta}\ , \tag{5.4}$$

where dN is an integration measure, and we expect the boundary parameters λ_L and σ to be related.

Assuming that the integration measure dN is invariant under the $SL(2,\mathbb{R})$ symmetry $N \to \Lambda^T N \Lambda$ (3.5), let us show that this symmetry determines the z-dependence of the one-point function. The subgroup of $SL(2,\mathbb{R})$ which survives our fixing of b_1, b_2 is the set of matrices of the type $\Lambda(x,y) \equiv \begin{pmatrix}\sqrt{y} & 0 \\ -\frac{x}{\sqrt{y}} & \frac{1}{\sqrt{y}}\end{pmatrix}$, which are such that $\Lambda(x,y)\left(\begin{smallmatrix}1\\ x+iy\end{smallmatrix}\right) = \sqrt{y}\left(\begin{smallmatrix}1\\ i\end{smallmatrix}\right)$. By using this residual subgroup in eq. (5.4) we obtain $\left\langle e^{\eta\phi(z)}\right\rangle_{\lambda_L}^{light} = (\Im z)^{-\eta}\left\langle e^{\eta\phi(i)}\right\rangle_{\lambda_L}^{light}$, which agrees with what we would expect from conformal symmetry.

There is a simple method for computing the integral in eq. (5.4), which unfortunately does not easily generalize to $s\ell_{n>2}$ Toda theory. It uses the parametrization $N = \left(\begin{smallmatrix} Y_0 - Y_1 & Y_2 + iY_3 \\ Y_2 - iY_3 & Y_0 + Y_1\end{smallmatrix}\right)$, where the constraint $\det N = 1$ still has to be imposed. We have

$$\left\langle e^{\eta\phi(i)}\right\rangle_{\lambda_L}^{light} = \int\left(\prod_{i=0}^{3} dY_i\right)\ \delta(\lambda_L + Y_3)\ \delta(Y_0^2 - Y_1^2 - Y_2^2 - Y_3^2 - 1)\ (2Y_0 - 2Y_3)^{-\eta}\ . \tag{5.5}$$

Integrating over Y_3 then Y_1 and Y_2, we obtain $\left\langle e^{\eta\phi(i)}\right\rangle_{\lambda_L}^{light} = \frac{\pi}{2^\eta}\int_{\sqrt{\lambda_L^2+1}}^\infty dY_0 (Y_0 + \lambda_L)^{-\eta}$ from which we deduce

$$\left\langle e^{\eta\phi(z)}\right\rangle_{\lambda_L}^{light} = (\Im z)^{-\eta}\frac{\pi}{2^\eta}\frac{\left(\lambda_L + \sqrt{\lambda_L^2+1}\right)^{1-\eta}}{\eta - 1}\ . \tag{5.6}$$

Let us now present another calculation of the integral (5.4), which can more easily be generalized to the case of $s\ell_n$ Toda theory with $n \geq 3$. We adopt the parametrization $N = \bar M^T M = \left(\begin{smallmatrix}\rho^{-2}+|a|^2 & \rho\bar a \\ \rho a & \rho^2\end{smallmatrix}\right)$ where $M = \left(\begin{smallmatrix}\rho^{-1} & 0 \\ a & \rho\end{smallmatrix}\right)$ is a function of a real parameter ρ and a complex parameter a. The integral (5.4) becomes

$$\left\langle e^{\eta\phi(z)}\right\rangle_{\lambda_L}^{light} = \int \rho d\rho\ d^2a\ \delta(\lambda_L + \rho\Im a)\ \left(\rho^{-2} + |a + \rho z|^2\right)^{-\eta}\ . \tag{5.7}$$

It is possible, but tedious, to compute this integral directly. Instead, let us introduce the notation

$$\lambda_L = i \cosh \sigma \ . \tag{5.8}$$

For the moment this is a rather awkward notation, as we have to assume that ie^σ is real (and we further take it to be positive). In addition we perform the change of variables $a \to a - \rho z$, and we obtain

$$\left\langle e^{\eta\phi(z)} \right\rangle_\sigma^{light} = \int \rho d\rho \, d^2 a \, \delta\left(\tfrac{i}{2}e^\sigma + \tfrac{i}{2}e^{-\sigma} + \rho\Im a - \rho^2 \Im z\right) \left(\rho^{-2} + |a|^2\right)^{-\eta} \ . \tag{5.9}$$

In the limit

$$\begin{cases} ie^\sigma \to \xi i e^\sigma \\ z \to \xi z \end{cases} \quad , \quad \xi \to \infty \ , \tag{5.10}$$

the integral greatly simplifies,

$$\left\langle e^{\eta\phi(z)} \right\rangle_\sigma^{light} \sim \frac{1}{\xi \Im z} \int d^2 a \, \left(\tfrac{2\Im z}{ie^\sigma} + |a|^2\right)^{-\eta} = (\xi\Im z)^{-\eta} \frac{\pi}{2^\eta} \frac{(i\xi e^\sigma)^{\eta-1}}{\eta - 1} \ . \tag{5.11}$$

This agrees with the result (5.6), in spite of the limit (5.10) which we have taken. This is because the result (5.6) has a very simple behaviour under the rescalings involved in the definition of the limit. The behaviour under the rescaling of z is of course a consequence of conformal symmetry, but we have no a priori reason for the behaviour under the rescaling of ie^σ to be simple. We will assume that a similar behaviour persists in $s\ell_n$ Toda theory with arbitrary n, and this will allow us to take limits analogous to (5.10) before performing integrals which would otherwise seem intractable.

Now let us investigate whether the relation (5.3) between the classical and quantum calculations of the one-point function holds. According to eq. (5.11), the normalized classical one-point function is

$$\frac{\left\langle e^{\eta\phi(z)} \right\rangle_\sigma^{light}}{\langle 1 \rangle_\sigma^{light}} = (2\Im z)^{-\eta} \frac{(ie^\sigma)^\eta}{1 - \eta} \ . \tag{5.12}$$

The behaviour of the exact Liouville one-point function (4.26) in the light asymptotic limit is

$$\langle V_{b\eta}(z) \rangle_{b^{-1}\sigma} \underset{b\to 0}{\sim} (2\Im z)^{-\eta} (\pi b^{-2}\mu)^{-\frac{\eta}{2}} \Gamma(\eta - b^{-2}) \, e^{|\sigma| b^{-2}} \, \frac{e^{|\sigma|(1-\eta)}}{\eta - 1} \ , \tag{5.13}$$

$$\frac{\langle V_{b\eta}(z) \rangle_{b^{-1}\sigma}}{\langle 1 \rangle_{b^{-1}\sigma}} \underset{b\to 0}{\sim} (2\Im z)^{-\eta} (-1)^\eta (\pi b^2 \mu)^{-\frac{\eta}{2}} \frac{e^{-\eta|\sigma|}}{1 - \eta} \ , \tag{5.14}$$

where we assume σ to be real, and we used $\frac{\Gamma(\eta - b^{-2})}{\Gamma(-b^{-2})} \underset{b\to 0}{\sim} (-b^{-2})^\eta$. The classical and quantum results agree, provided the cosmological constant takes the value $\mu = -\frac{1}{\pi b^2}$ (3.1) which was assumed in the classical analysis, and the classical boundary parameter σ defined in eq. (5.8) is identified with the quantum boundary parameter σ which appears in eq. (5.13). However, ie^σ is assumed to be real in the classical analysis, whereas σ is assumed to be real in the calculation of the limit of the exact one-point function. Thus, the comparison between the two methods must involve an analytic continuation of the results. This problem ultimately comes from the fact that the bulk cosmological constant μ is assumed to be negative in the classical analysis, and positive in the conformal bootstrap analysis. And the known relation [16] between the boundary cosmological constant λ_L and σ is in our notations $\left(\frac{\lambda_L}{\pi b^2}\right)^2 = \frac{\mu}{\sin \pi b^2}\cosh^2\sigma$, which agrees with eq. (5.8) in the $b \to 0$ limit if $\mu = -\frac{1}{\pi b^2}$. This confirms our earlier identification of λ_L as the boundary cosmological constant, see eq. (3.8).

Finally, let us compute the light asymptotic limit of the boundary two-point function:

$$\left\langle e^{\eta_1\phi(z_1)}e^{\eta_2\phi(z_2)}\right\rangle_{\lambda_L}^{light} = \int \rho d\rho\, d^2a\, \delta(\lambda_L + \rho\Im a) \prod_{i=1}^{2}\left(\rho^{-2} + |a+\rho x_i|^2\right)^{-\eta_i} . \tag{5.15}$$

Let us use conformal invariance, and fix $x_1 = 0, x_2 = \infty$. This makes the computation elementary, and the result is

$$\left\langle e^{\eta_1\phi(0)}e^{\eta_2\phi(\infty)}\right\rangle_{\lambda_L}^{light} = \delta(\eta_1-\eta_2)\,\frac{\Gamma(\eta_1-\frac12)}{\Gamma(\eta_1)}(\lambda_L^2+1)^{\frac12-\eta_1} . \tag{5.16}$$

This agrees with the expectations from the conformal bootstrap analysis [16], provided our relation (5.8) between the classical and quantum boundary parameters is assumed.

5.2 Case of $s\ell_3$ Toda theory with $W - \bar{W} = 0$

Let us consider the solutions (3.13) of the $s\ell_3$ Toda equations which obey the "light asymptotic condition" $\partial^3 X_i = \bar\partial^3 X_i = 0$. Such solutions are built from three functions b_1, b_2, b_3 which are polynomials of degrees at most two. Given the freedom to choose the matrix N, we can fix these functions without loss of generality, and we choose $B(z) \equiv (b_1(z), b_2(z), b_3(z)) = (\frac12 z^2, z, 1)$. The corresponding Wronskians are $(w_1(z), w_2(z), w_3(z)) = (1, -z, \frac12 z^2)$ and they obey $w_i = P_{ij}b_j$ with $P \equiv \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right)$. This leads to

$$X_1 = B(\bar z)NB(z)^T \quad , \quad X_2 = B(\bar z)PN^{-1T}PB(z)^T . \tag{5.17}$$

According to equation (5.2), the one-point function of a bulk operator $e^{(\eta,\phi(z))} = e^{\eta_1\phi_1(z)+\eta_2\phi_2(z)}$ in the presence of a boundary with parameter λ_0 (3.15) is of the type

$$\left\langle e^{(\eta,\phi(z))}\right\rangle_{\lambda_0}^{light} = \int dN\, \delta\left(\lambda_0 - \det\tfrac12(N+N^T)\right)\, X_1^{-\eta_1}X_2^{-\eta_2} . \tag{5.18}$$

We may use the $SL(3,\mathbb{R})$ symmetry $N \to \Lambda N \Lambda^T$ (3.14) in order to determine the dependence of the one-point function on $z = x+iy$. Introducing the family of matrices $\Lambda(x,y) \equiv \left(\begin{smallmatrix} y^{-1} & -xy^{-1} & \frac12 x^2y^{-1} \\ 0 & 1 & -x \\ 0 & 0 & y \end{smallmatrix}\right)$, we have $\Lambda(x,y)^T P\Lambda(x,y) = P$ and $\Lambda(x,y)B(x+iy)^T = yB(i)^T$. Together with the assumption that the measure dN in eq. (5.18) is invariant under the symmetry, this implies $\left\langle e^{(\eta,\phi(z))}\right\rangle_{\lambda_0}^{light} = (\Im z)^{-2\eta_1-2\eta_2}\left\langle e^{(\eta,\phi(i))}\right\rangle_{\lambda_0}^{light}$. The power of $\Im z$ thus obtained is $-2(\rho,\eta) = \lim_{b\to 0}(-2\Delta_{b\eta})$ where Δ_α given by eq. (2.14), as expected from conformal symmetry. Now it turns out that the $SL(3,\mathbb{R})$ symmetry can yield further information on the one-point function. This is because after assuming $z = i$ there still is a residual subgroup of matrices $\Lambda(d,e,f) \equiv \left(\begin{smallmatrix} f & -2d & 2f-2e \\ d & e & -2d \\ \frac12 f - \frac12 e & d & f \end{smallmatrix}\right)$ where d, e, f are real parameters constrained by $\det\Lambda(d,e,f) = (e^2 +$

$4d^2)(2f-e)=1$, and such matrices obey the relations $\Lambda(d,e,f)^{-1T}B(i)^T = \frac{1}{e-2id}B(i)^T$ and $\Lambda(d,e,f)PB(i)^T = (e+2id)PB(i)^T$. Thus, under transformations $N \to \Lambda^{-1}N\Lambda^{-1T}$, we have $X_1(i) \to \frac{1}{e^2+4d^2}X_1(i)$ and $X_2(i) \to (e^2+4d^2)X_2(i)$. This shows that $\left\langle e^{(\eta,\phi(i))}\right\rangle_{\lambda_0}^{light}$ must vanish unless $\eta_1 = \eta_2$. The consequences of the $SL(3,\mathbb{R})$ symmetry may be summarized as

$$\left\langle e^{(\eta,\phi(z))}\right\rangle_{\lambda_0}^{light} = \delta(\eta_1-\eta_2)\,(\Im z)^{-4\eta_1}\left\langle e^{\eta_1(\rho,\phi(z))}\right\rangle_{\lambda_0}^{light} . \tag{5.19}$$

Let us now introduce a parametrization $N = \bar{M}^T M$ in terms of a triangular matrix $M = \left(\begin{smallmatrix}\rho & a & b\\ 0 & \nu & c\\ 0 & 0 & \tau\end{smallmatrix}\right)$ which depends on three real parameters ρ,ν,τ such that $\rho\nu\tau = 1$, and three complex parameters a,b,c. Then $N = \left(\begin{smallmatrix}\rho^2 & \rho a & \rho b\\ \rho\bar{a} & \nu^2+|a|^2 & \bar{a}b+\nu c\\ \rho\bar{b} & a\bar{b}+\nu\bar{c} & |b|^2+|c|^2+\tau^2\end{smallmatrix}\right)$ and

$$dN = \tfrac{1}{\pi^3}\rho^3 d\rho\, \nu d\nu\, d^2a\, d^2b\, d^2c\,, \tag{5.20}$$

$$X_1 = \tau^2 + |c+\nu z|^2 + |b+az+\tfrac{1}{2}\rho z^2|^2\,, \tag{5.21}$$

$$X_2 = \rho^{-2} + |\tau a + \nu^{-1}z|^2 + |ac-\nu b+\rho cz+\tfrac{1}{2}\tau^{-1}z^2|^2\,. \tag{5.22}$$

In terms of such variables, the boundary parameter is

$$\lambda_0 = \det\tfrac{1}{2}(N+N^T) = \rho^2\left[(\nu\Im b - \Re c\Im a)^2 + (\nu^2+\Im a^2)(\tau^2+\Im c^2)\right] . \tag{5.23}$$

We consider the one-point function (5.18) and perform the simultaneous shifts

$$a \to a-\rho z \quad , \quad b \to b-az+\tfrac{1}{2}\rho z^2 \quad , \quad c \to c-\nu z\,, \tag{5.24}$$

thereby obtaining

$$\begin{aligned}\left\langle e^{(\eta,\phi(z))}\right\rangle_{\lambda_0}^{light} &= \int\frac{dN}{\rho^2}\,(\tau^2+|b|^2+|c|^2)^{-\eta_1}(\rho^{-2}+\tau^2|a|^2+|ac-\nu b|^2)^{-\eta_2}\\ \delta\bigg(\frac{\lambda_0}{\rho^2} &- [\nu\Im b-\Re c\Im a+\Im z(\rho\Re c-\nu\Re a)]^2 - [\nu^2+(\Im a-\rho\Im z)^2][\tau^2+(\Im c-\nu\Im z)^2]\bigg) .\end{aligned} \tag{5.25}$$

In the limit

$$\begin{cases}\lambda_0 \to \xi^4\lambda_0\\ z \to \xi z\end{cases} \quad , \quad \xi\to\infty\,, \tag{5.26}$$

the integral simplifies and reduces to

$$\left\langle e^{(\eta,\phi(z))}\right\rangle_{\lambda_0}^{light} \sim \frac{1}{\xi^4}(\Im z)^{-2\eta_1-2\eta_2}\lambda_0^{\frac{\eta_1+\eta_2-2}{2}}\int\frac{d\nu}{\nu}\nu^{\eta_1-\eta_2}\times I_{\eta_1,\eta_2}\,, \tag{5.27}$$

$$I_{\eta_1,\eta_2} \equiv \tfrac{1}{\pi^3}\int d^2a\,d^2b\,d^2c\,(1+|b|^2+|c|^2)^{-\eta_1}(1+|a|^2+|ac-b|^2)^{-\eta_2}\,. \tag{5.28}$$

The integral I_{η_1,η_2} can be computed for arbitrary values of η_1,η_2 by making repeated use of the formula $\frac{1}{\pi}\int d^2z\,(|z|^2+A)^{-\eta} = \frac{A^{1-\eta}}{\eta-1}$, with the result

$$I_{\eta_1,\eta_2} = \frac{1}{(\eta_1-1)(\eta_2-1)(\eta_1+\eta_2-2)} = \prod_{e>0}\frac{1}{(e,\eta-\rho)}\,. \tag{5.29}$$

Using this result in the particular case $\eta_1 = \eta_2$, we obtain the expression for the one-point function in the limit (5.26),

$$\xi^4 \left\langle e^{(\eta,\phi(z))} \right\rangle_{\lambda_0}^{light} \sim \delta(\eta_1 - \eta_2)\, (\Im z)^{-4\eta_1} \frac{\lambda_0^{\eta_1 - 1}}{(\eta_1 - 1)^3} \, . \tag{5.30}$$

By analogy with the case of Liouville theory, we conjecture that this is the light asymptotic limit of the one-point function for all values of λ_0 and z (up to a possible redefinition of λ_0), and not just in the limit (5.26). What we have rigorously established is however only the presence of the $\delta(\eta_1 - \eta_2)\, (\Im z)^{-4\eta_1}$ factor, which follows from the $SL(3,\mathbb{R})$ symmetry. This is already significant evidence that our classical and conformal bootstrap analyses of the $W - \bar{W} = 0$ boundary condition actually describe the same D-branes.

We conclude with a few words on the boundary two-point function $\left\langle e^{(\eta,\phi(x))} e^{(\mu,\phi(y))} \right\rangle_{\lambda_0}^{light}$. The $SL(3,\mathbb{R})$ symmetry determines its dependence on the boundary coordinates x, y, and implies that it must vanish unless the momenta of the two boundary operators are conjugate to each other, $\eta = \mu^*$. This relation between the two momenta can be confirmed by a direct calculation using the parametrization (5.20)-(5.22), which yields

$$\begin{aligned} &\left\langle e^{(\eta,\phi(0))} e^{(\mu,\phi(\infty))} \right\rangle_{\lambda_0}^{light} = \delta^{(2)}(\eta - \mu^*) \int d^2a\, d^2b\, d^2c \\ &\delta\left(\lambda_0 - (\Im a^2 + 1)(\Im c^2 + 1) - (\Im b - \Re c \Im a)^2\right) \; (1 + |b|^2 + |c|^2)^{-\eta_1} (1 + |a|^2 + |ac - b|^2)^{-\eta_2} \, . \end{aligned} \tag{5.31}$$

This integral is the same as I_{η_1,η_2} (5.28) with an additional delta-function, and at this moment we do not know how to compute it. Also, we have no conformal bootstrap results to compare it with. In the special case $\lambda_0 = -1$ when the boundary action is local (3.17), the exact result is known [19], but does not have a light asymptotic limit. Correspondingly, our light asymptotic calculation is meaningful only if $\lambda_0 \geq 1$, because if $\lambda_0 < 1$ the argument of the δ function in eq. (5.31) cannot vanish.

5.3 Case of $s\ell_3$ Toda theory with $W + \bar{W} = 0$

Let us consider the solutions (3.19) of the $s\ell_3$ Toda equations which obey the "light asymptotic condition" $\partial^3 X_i = \bar{\partial}^3 X_i = 0$. We will write them in terms of the same functions $B(z) = (\frac{1}{2}z^2, z, 1)$ as in the $W - \bar{W} = 0$ case, together with the same matrix $P = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right)$ which relates them to their Wronskians,

$$X_1 = B(\bar{z}) N P B(z)^T \quad , \quad X_2 = B(\bar{z}) P N^{-1T} B(z)^T \, . \tag{5.32}$$

Now that we choose these particular functions $B(z)$, it is possible to ensure that $\phi_i = -\log X_i$ are real by imposing the simple condition that NP be a positive Hermitian matrix. This restricts the $SL(3,\mathbb{C})$ symmetry group (3.20) to its $SL(3,\mathbb{R})$ subgroup, which still acts as $N \to \Lambda^T N \Lambda^{-1T}$. (Beware that the matrix P transforms nontrivially under this symmetry.) In addition, this implies that the eigenvalues of N must be real, because there is a matrix M such that $NP = \bar{M}^T M$, and N is thus conjugate to the Hermitian matrix $MP\bar{M}^T$.

According to eq. (5.2), the one-point function of a bulk operator $e^{(\eta,\phi(z))} = e^{\eta_1\phi_1(z)+\eta_2\phi_2(z)}$ in the presence of a boundary with parameters λ_1, λ_2 (3.21) is of the type

$$\left\langle e^{(\eta,\phi(z))} \right\rangle^{light}_{\lambda_1,\lambda_2} = \int dN\, \delta\left(\lambda_1 - \text{Tr } N\right)\, \delta\left(\lambda_2 - \text{Tr } N^{-1}\right)\, X_1^{-\eta_1} X_2^{-\eta_2}\ , \tag{5.33}$$

The consequences of the $SL(3,\mathbb{R})$ symmetry on this integral can be evaluated using the same particular symmetry transformations as in the case $W - \bar{W} = 0$. Using $\Lambda(x,y)$ we similarly obtain the expected dependence on z. Using $\Lambda(d,e,f)$ however teaches us nothing new, because such transformations now leave X_1, X_2 invariant.

Let us now use the parametrization $N = \bar{M}^T M P$ in terms of an upper-triangular matrix $M = \left(\begin{smallmatrix} \rho & a & b \\ 0 & \nu & c \\ 0 & 0 & \tau \end{smallmatrix}\right)$ which depends on three real parameters ρ, ν, τ such that $\rho\nu\tau = 1$, and three complex parameters a, b, c. The expressions (5.20)-(5.22) for dN, X_1, X_2 still hold, and we find

$$\lambda_1 = \text{Tr } N = 2\rho\Re b - \nu^2 - |a|^2\ , \tag{5.34}$$

$$\lambda_2 = \text{Tr } N^{-1} = 2\rho\nu\Re(ac) - 2\rho\nu^2\Re b - \rho^2|c|^2 - \rho^2\tau^2\ . \tag{5.35}$$

We perform the shifts (5.24) in the expression (5.33) for the one-point function, and obtain

$$\begin{aligned} \left\langle e^{(\eta,\phi(z))} \right\rangle^{light}_{\lambda_1,\lambda_2} = \int dN\ & \left(\tau^2 + |b|^2 + |c|^2\right)^{-\eta_1} \left(\rho^{-2} + \tau^2|a|^2 + |\nu b - ca|^2\right)^{-\eta_2} \\ & \times \delta\left(\lambda_1 + \nu^2 + |a|^2 - 2\rho\Re b + 2\rho^2(\Im z)^2 - 4\Im a \Im z\right) \\ & \times \delta\left(\lambda_2 + \nu^2\lambda_1 + \nu^4 + \nu^{-2} + |\rho c - \nu\bar{a} - 2i\rho\nu\Im z|^2\right)\ . \end{aligned} \tag{5.36}$$

We do not know how to perform this integral, except by taking a particular limit of the variables z, λ_1, λ_2. To this end, we parametrize the eigenvalues of N as $\{e^{-(h,\sigma)}\} = \{e^{-\sigma_1}, e^{\sigma_1-\sigma_2}, e^{\sigma_2}\}$ for a vector $\sigma = \sigma_1 e_1 + \sigma_2 e_2$ in the Cartan subalgebra of $s\ell_3$. Then the parameters λ_i coincide with values of the fundamental and antifundamental characters of $s\ell_3$,

$$\lambda_1 = \chi_{\omega_2}(\sigma)\quad ,\quad \lambda_2 = \chi_{\omega_1}(\sigma)\ . \tag{5.37}$$

Notice however that the matrix N is not positive, and σ_1, σ_2 are not expected to be real numbers. Rather, we assume that e^{σ_1} and e^{σ_2} are negative real numbers. We now introduce the limit

$$\begin{cases} z \to \xi z \\ e^{\sigma_i} \to \xi^2 e^{\sigma_i} \end{cases}\quad ,\quad \xi \to \infty\ , \tag{5.38}$$

so that $\lambda_i \to \xi^2 e^{\sigma_i}$. In this limit, the delta-functions in the integral (5.36) fix the variables ρ, ν, τ to values proportional to the eigenvalues of N, namely

$$\rho^2 = -\tfrac{1}{2}\frac{1}{(\Im z)^2}e^{\sigma_2}\quad ,\quad \nu^2 = \tfrac{1}{2}e^{\sigma_1-\sigma_2}\quad ,\quad \tau^2 = -4(\Im z)^2 e^{-\sigma_1}\ . \tag{5.39}$$

And we find that the integral takes the value

$$\left\langle e^{(\eta,\phi(z))} \right\rangle^{light}_{\sigma} \sim \frac{1}{\xi^4}(\Im z)^{-2\eta_1-2\eta_2}(-e^{\sigma_1})^{\eta_1-1}(-e^{\sigma_2})^{\eta_2-1} I_{\eta_1,\eta_2}\ , \tag{5.40}$$

$$\frac{\left\langle e^{(\eta,\phi(z))} \right\rangle^{light}_{\sigma}}{\langle 1 \rangle^{light}_{\sigma}} = \frac{-2}{(\Im z)^{2(\rho,\eta)}}(-e^{\sigma_1})^{\eta_1}(-e^{\sigma_2})^{\eta_2} \prod_{e>0} \frac{1}{(e,\eta-\rho)}\ , \tag{5.41}$$

where we used the definition (5.28) and value (5.29) of the integral I_{η_1,η_2}. We derived this result in the case of $s\ell_3$ Toda theory, but it is not very difficult to generalize it to the case of $s\ell_n$ Toda theory with arbitrary n. By analogy with Liouville theory, we conjecture that this result holds for general values of z and σ, and not just in the limit (5.38).

Let us investigate the light asymptotic limit (5.1) of the conformal bootstrap result (4.26). The behaviour of the factor $\sum_{w\in W} e^{(w(s),\alpha-Q)}$ depends on which Weyl chamber $s=b^{-1}\sigma$ belongs to; there is a Weyl chamber such that

$$\langle V_{b\eta}(z)\rangle_{b^{-1}\sigma} \underset{b\to 0}{\sim} \frac{(\pi\mu b^{-2})^{-(\rho,\eta)}}{(2\Im z)^{2(\rho,\eta)}} e^{-b^{-2}(\rho,\sigma)} e^{(\eta-\rho,\sigma)} \prod_{e>0} \frac{\Gamma(1+(e,\eta-\rho)-b^{-2}(e,\rho))}{(e,\eta-\rho)} \tag{5.42}$$

$$\frac{\langle V_{b\eta}(z)\rangle_{b^{-1}\sigma}}{\langle V_0(i)\rangle_{b^{-1}\sigma}} \underset{b\to 0}{\sim} -2\frac{(\pi\mu b^2)^{-(\rho,\eta)}}{(\Im z)^{2(\rho,\eta)}} (-1)^{2(\rho,\eta)} e^{(\eta,\sigma)} \prod_{e>0} \frac{1}{(e,\eta-\rho)}\ , \tag{5.43}$$

which generalizes the Liouville result (5.13). Checking the agreement (5.3) between the classical calculation (5.41) and the conformal bootstrap result (5.43) involves assuming that the boundary parameters e^{σ_i} which we introduced in both analyses coincide. Then the boundary cosmological constants $\lambda_{i,+}$ (4.25) agree with the classical boundary parameters $\lambda_{1,2}$ (5.37). However, as in Liouville theory, we must analytically continue the one-point function, because the boundary parameters e^{σ_i} take positive values in the bootstrap analysis and negative values in the classical calculation.

This agreement between the classical and bootstrap analyses, and the identification of their respective boundary parameters, have interesting consequences in the case when σ belongs to the boundary of a Weyl chamber, that is $(e,\sigma)=0$ for e some positive root. This is the case when two of the eigenvalues $\{e^{-\sigma_1}, e^{\sigma_1-\sigma_2}, e^{\sigma_2}\}$ of the matrix N coincide; in the classical analysis of Subsection 3.4 this corresponded to the D-brane being one-dimensional. In the conformal bootstrap analysis, this case corresponds to the semi-degenerate D-branes, as is clear from eq. (4.30) for the boundary cosmological constants, where two of the three terms coincide. In the limit $b\to 0$ with $c=-\frac{1}{3}b\kappa$ fixed and m odd, the boundary cosmological constants become $\left\{\begin{array}{l}\lambda_{1,+}=e^{2c}-2e^{-c}\\ \lambda_{2,+}=e^{-2c}-2e^{c}\end{array}\right.$. This allows us to identify c with the position of the D-brane, as given by the Dirichlet conditions eq. (3.22) from the classical analysis.

We conclude with a few words on the boundary two-point function $\left\langle e^{(\eta,\phi(x))}e^{(\mu,\phi(y))}\right\rangle^{light}_{\lambda_1,\lambda_2}$. The $SL(3,\mathbb{R})$ symmetry determines its dependence on the boundary coordinates x,y, and implies that it must vanish unless the momenta obey $(\rho,\eta-\mu)=0$. This condition can be interpreted as the conformal invariance of the boundary theory; however, assuming the momenta to be related to the $W^{(3)}$ charges as in eq. (2.16), we would expect a stronger constraint from the full W_3 symmetry, namely $\eta=\mu^*$. And indeed this is the constraint we found in the $W-\bar{W}=0$ case. But we can confirm the absence of this constraint in the present case by the explicit calculation

$$\begin{aligned}\left\langle e^{(\eta,\phi(0))}e^{(\mu,\phi(\infty))}\right\rangle^{light}_{\lambda_1,\lambda_2} &= \delta((\rho,\eta-\mu))\int d^2a\, d^2b\, d^2c\,\frac{d\nu}{\nu}\,\nu^{2(\eta_1-\mu_2)}\\ &\times\delta\left(\lambda_1-2\nu^{-1}\Re b+\nu^2+\nu^2|a|^2\right)\,\delta\left(\lambda_2+2\nu\Re b+\nu^{-2}+\nu^{-2}|c|^2-2\nu\Re(ac)\right)\\ &\times(1+|b|^2+|c|^2)^{-\eta_1}(1+|a|^2+|ac-b|^2)^{-\eta_2}\ .\end{aligned} \tag{5.44}$$

Curiously, the limit in which we are able to compute this integral is different from the limit (5.38) which we used in the case of the one-point function, and in particular no longer forbids the coincidence of two eigenvalues of N. This limit is chosen so that a rescaling of ν can match the behaviours of λ_1 and λ_2 in the delta-functions:

$$\begin{cases} e^{\sigma_1} \to \xi^{-1} e^{\sigma_1} \\ e^{\sigma_2} \to \xi e^{\sigma_2} \end{cases} , \quad \xi \to \infty . \tag{5.45}$$

We obtain in this limit

$$\xi^{3+2\eta_2-2\mu_1} \left\langle e^{(\eta,\phi(0))} e^{(\mu,\phi(\infty))} \right\rangle_\sigma^{light} \sim \delta((\rho, \eta-\mu)) \frac{1}{(\eta_2-1)(\mu_2-1)} \frac{\Gamma(\eta_1+\eta_2-\frac{3}{2})}{\Gamma(\eta_1+\eta_2-1)} \\ \times \left|e^{\sigma_2-\sigma_1}\right|^{\mu_1-\eta_2-\frac{3}{2}} \left|\sinh \tfrac{1}{2}(\sigma_1+\sigma_2)\right|^{-2\eta_1-2\eta_2+3} . \tag{5.46}$$

This result is invariant under the exchange of the two operators $\eta \leftrightarrow \mu$. On the other hand there is no invariance under $\eta \to \eta^*, \mu \to \mu^*$. This is because our limit (5.45) treats the boundary parameters λ_1, λ_2 in an asymmetric way; in particular λ_1 and λ_2 do not go to infinity at the same rate.

6. Conclusion

Combining classical and conformal bootstrap analyses yields a consistent picture of the moduli space of maximally symmetric D-branes in $s\ell_n$ conformal Toda theory. Our results and conjectures on these moduli spaces and on the existence of a boundary action in the $s\ell_2$ and $s\ell_3$ cases can be summarized in the following table:

Theory	Type of Brane	d	Parameters	Classical parameters	Action
Liouville	Continuous	1	$s \in \mathbb{R}$	$\lambda_L = i \cosh bs$	$\lambda_L \int e^\phi$
	Discrete	0	$\ell \vert \ell' \in \mathbb{N}^2$	n. a.	n. a.
$s\ell_3$, $W = \bar{W}$	Continuous	2	?	λ_0	nonlocal
	Degenerate	?	?	?	?
$s\ell_3$, $W = -\bar{W}$	Continuous	2	$s \in \mathbb{R}^2$	$\lambda_i = \chi_{\omega_i}(bs)$	inexistent
	Semi-degenerate	1	$\kappa \vert \ell, m \in \mathbb{R} \times \mathbb{N}^2$	$c = -\frac{1}{3} b\kappa$	0
	Discrete	0	$\ell, m \vert \ell', m' \in \mathbb{N}^4$	n. a.	n. a.

In the case of the boundary conditions $W^{(s)} - \bar{W}^{(s)} = 0$, the dimension of the moduli space is the integer part of $\frac{n}{2}$ (Subsection 3.3). This coincides with the number of nonzero charges $\{q^{(s)}\}_{2 \leq s \leq n-1 | s \text{ even}}$ for bulk operators whose one-point functions do not vanish (Section 4), in accordance with a generalization of Cardy's idea. In the $s\ell_3$ case, we provide predictions for certain correlation functions in the light asymptotic limit, namely the bulk one-point function (5.30) and boundary two-point function (5.31).

We examined the case of the boundary conditions $W^{(s)} - (-1)^s \bar{W}^{(s)} = 0$ in more detail. We propose that there exists a hierarchy of D-branes of dimensions $d = 0 \cdots n-1$, which correspond to the partly degenerate representations of the W_n algebra. In particular, there are continuous

D-branes of dimension $n-1$, and discrete D-branes of dimension 0. The moduli space of d-dimensional D-branes is itself d-dimensional, although there are also discrete parameters. This was the result of classical (Subsection 3.4) and bootstrap (Section 4) analyses, which were shown to agree in detail (Subsection 5.3). In particular, we found explicit formulas for the bulk one-point functions of continuous (4.26) and discrete (4.32) D-branes. In the $s\ell_3$ case, we also computed the bulk one-point functions of the semi-degenerate D-branes (4.29). As our D-branes conform to Cardy's ideas by corresponding to representations of the W_n algebra, they also correspond to the topological defects of the very interesting article [20] (where such defects are related to certain operators in four-dimensional gauge theories). And a D-brane's one-point function is closely related to the corresponding defect operator's coefficients.

The calculation of annulus partition functions leads to natural conjectures for the spectra of open strings with one end on a discrete D-brane (Subsection 4.3). These spectra coincide with what can be obtained by fusing the two representations which correspond to the two involved D-branes. If this structural property persists in the case of all D-branes, then it can help explain the divergences of the annulus partition functions (Subsection 4.4). Infinite fusion multiplicities indeed appear in the fusion of two continuous representations (Subsection 2.3), so that we expect infinite multiplicities in the spectra of continuous D-branes. This might also explain the apparent violation of the W_3 symmetry in the minisuperspace prediction (5.46) for the boundary two-point function. A boundary spectrum with infinite multiplicities can certainly not be adequately parametrized by momenta $\alpha = b\eta$, and an operator with a given momentum might correspond to a combination of states belonging to different representations of the W_3 algebra.

Thus, the moduli space of D-branes may now be well-understood, but the boundary operators and their correlation functions remain problematic, and they certainly have new and complicated features.

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A. Minisuperspace limits of some correlation functions

In addition to the light asymptotic limit which we studied in Section 5, there is another semi-classical limit in which Toda correlation functions simplify and can in certain cases be independently predicted: the minisuperspace limit, where our two-dimensional field theory reduces to a one-dimensional system. In this limit, a bulk primary operator $V_{Q+ip}(z)$ corresponds to a wave-function $\Psi_p(\phi)$, which is a solution of the Schrödinger equation of Toda quantum mechanics [1],

$$\left[-\left(\frac{\partial}{\partial\phi}\right)^2 + 2\pi\mu b^{-2}\sum_{i=1}^{n-1} e^{(e_i,\phi)}\right]\Psi_p(\phi) = p^2\Psi_p(\phi)\ . \tag{A.1}$$

(Compare with the $s\ell_n$ Toda Lagrangian L_n (2.25), and notice the rescaling $\phi \to b^{-1}\phi$.) Here the variable ϕ can be interpreted as the z-independent zero-mode of the Toda field $\phi(z)$. The

Schrödinger equation is deduced from the Hamiltonian picture of the dynamics of ϕ, which is associated to radial quantization in the z-plane.

In the minisuperspace limit, a boundary with parameter σ corresponds to a boundary wavefunction $\Psi^{bdy}_\sigma(\phi)$, which can be interpreted as the density of the corresponding D-brane. If a boundary Lagrangian $L^{bdy}[\phi]$ is known, then the boundary wavefunction can be obtained by computing this Lagrangian for constant values of the field $\phi(z)$, namely

$$\Psi^{bdy}_\sigma(\phi) = e^{-L^{bdy}(\phi)} \; . \tag{A.2}$$

In any case, the minisuperspace one-point function is defined as

$$\langle \Psi_p \rangle^{mini}_\sigma \equiv \int d\phi \; \Psi_p(\phi) \; \Psi^{bdy}_\sigma(\phi) \; , \tag{A.3}$$

and we expect that it is related to a $b \to 0$ limit of the one-point function $\langle V_\alpha(z) \rangle_s$,

$$\lim_{b\to 0} (\Im z)^{2\Delta_{Q+ibp}} \, \langle V_{Q+ibp}(z) \rangle_{b^{-1}\sigma} = \langle \Psi_p \rangle^{mini}_\sigma \; . \tag{A.4}$$

(Compare with the light asymptotic limit (5.3).)

In the case of Liouville theory, a boundary Lagrangian is known. Then it is possible to compute the minisuperspace one-point function (A.3) and to compare it with the conformal bootstrap one-point function. It turns out that eq. (A.4) is obeyed, which provides a test of the conformal bootstrap one-point function [16]. In the case of $s\ell_3$ Toda theory with $W + \bar{W} = 0$, no boundary action exists, as we will see in Appendix B. We will reason in the opposite direction, and deduce the minisuperspace boundary wavefunction $\Psi^{bdy}_\sigma(\phi)$ from the conformal bootstrap one-point function. We will do this first in Liouville theory, as a preparation for the case of $s\ell_3$ Toda theory. The boundary wavefunction will turn out to have interesting properties; in particular it provides the generating function of the Bäcklund transformation which maps the Toda classical mechanics of ϕ to the free classical mechanics of σ. As we saw in Section 3, there is a good reason why the Toda boundary parameter σ can be interpreted as a free field: there exists a Bäcklund transformation from conformal Toda theory to a free field theory, such that the $W + \bar{W} = 0$ boundary conditions in Toda theory are mapped to Dirichlet boundary conditions in the free theory, and the Toda boundary parameter σ is mapped to the free field boundary parameter, which is the boundary value of the free field.

A.1 Case of Liouville theory

In this case the Schrödinger equation (A.1) becomes $\left[-\frac{1}{2}\frac{\partial^2}{\partial\phi^2} + 2\pi\mu b^{-2} e^{2\phi}\right] \Psi_p(\phi) = \frac{1}{2}p^2 \Psi_p(\phi)$. The solution is [16]

$$\Psi_p(\phi) = (\pi\mu b^{-2})^{-\frac{i}{2}p} \frac{2}{\Gamma(-ip)} K_{ip}\left(2\sqrt{\pi\mu b^{-2}} e^{\phi}\right) \; , \tag{A.5}$$

where $K_\nu(z)$ is a Bessel function, and Ψ_p is normalized so that

$$\int d\phi \; \Psi_{p_1}(\phi) \Psi_{p_2}(\phi) = 2\pi\delta(p_1 + p_2) \; . \tag{A.6}$$

The minisuperspace limit (A.4) of the Liouville one-point function (4.26) is

$$\langle \Psi_p \rangle_\sigma^{mini} = 2(\pi\mu b^{-2})^{-\frac{i}{2}p} \Gamma(ip) \cos(p\sigma) \; . \tag{A.7}$$

According to eq. (A.2) and eq. (A.3) we can deduce the boundary wavefunction from the knowledge of $\langle \Psi_p \rangle_\sigma^{mini}$,

$$\Psi_\sigma^{bdy}(\phi) = \frac{1}{2\pi} \int dp \ \langle \Psi_{-p} \rangle_\sigma^{mini} \, \Psi_p(\phi) = \frac{2}{\pi} \int dp \ \cos(p\sigma) \ K_{ip} \left(2\sqrt{\pi\mu b^{-2}} e^{\phi} \right) \; . \tag{A.8}$$

The calculation is performed using the formula $\int_0^\infty dp \ \cos(ap) \ K_{ip}(z) = \frac{\pi}{2} e^{-z\cosh a}$, with the result

$$\Psi_\sigma^{bdy}(\phi) = e^{-L_2^{bdy}} \quad , \quad L_2^{bdy} = \sqrt{4\pi\mu b^{-2}} \cosh(\sigma) \, e^{\phi} \; . \tag{A.9}$$

The function $L_2^{bdy}(\phi, \sigma)$ generates a canonical transformation between Toda and free classical mechanics, as follows from the identity

$$\left(\frac{\partial L_2^{bdy}}{\partial \phi} \right)^2 - \left(\frac{\partial L_2^{bdy}}{\partial \sigma} \right)^2 = 4\pi\mu b^{-2} \ e^{2\phi} \; , \tag{A.10}$$

whose right hand-side is the bulk Liouville potential, see the Lagrangian (2.25). Considering indeed ϕ, σ as time-dependent variables with associated momenta $\dot{\phi} = -\frac{\partial L_2^{bdy}}{\partial \phi}$ and $\dot{\sigma} = \frac{\partial L_2^{bdy}}{\partial \sigma}$, the Liouville equation of motion $\ddot{\phi} = 4\pi\mu b^{-2} \ e^{2\phi}$ amounts to the ϕ-derivative of eq. (A.10), and the free equation of motion $\ddot{\sigma} = 0$ amounts to the σ-derivative of eq. (A.10).

A.2 Case of $s\ell_3$ Toda theory with $W + \bar{W} = 0$

The solution of the Schrödinger equation (A.1) in the case of $s\ell_3$ Toda theory is [21] [1]

$$\begin{aligned} \Psi_p(\phi) = & \frac{8(\pi\mu b^{-2})^{-i(\rho,p)}}{\prod_{e>0} \Gamma(-i(e,p))} e^{\frac{i}{6}(e_1 - e_2, \phi)(e_1 - e_2, p)} \\ & \times \int_0^\infty \frac{dt}{t} t^{i(e_2 - e_1, p)} K_{i(\rho,p)} \left(2\sqrt{1+t^{-2}} \sqrt{\frac{\pi\mu}{b^2}} e^{\frac{1}{2}(e_1,x)} \right) K_{i(\rho,p)} \left(2\sqrt{1+t^2} \sqrt{\frac{\pi\mu}{b^2}} e^{\frac{1}{2}(e_2,x)} \right) \; , \end{aligned} \tag{A.11}$$

and it is normalized such that

$$\int d^2\phi \ \Psi_{p_1}(\phi) \Psi_{p_2}(\phi) = (2\pi)^2 \delta^{(2)}(p_1 + p_2) \; . \tag{A.12}$$

The minisuperspace limit (A.4) of the $s\ell_3$ Toda one-point function (4.26) is

$$\langle \Psi_p \rangle_\sigma^{mini} = (\pi\mu b^{-2})^{-i(\rho,p)} \prod_{e>0} \Gamma(i(e,p)) \sum_{w\in W} e^{i(w(\sigma),p)} \; . \tag{A.13}$$

According to eq. (A.2) and eq. (A.3) we can deduce the boundary wavefunction from the knowledge of $\langle \Psi_p \rangle_\sigma^{mini}$,

$$\Psi_\sigma^{bdy}(\phi) = \frac{1}{(2\pi)^2} \int d^2p \ \langle \Psi_{-p} \rangle_\sigma^{mini} \, \Psi_p(\phi) \; . \tag{A.14}$$

The calculation can be performed using the formula

$$\int_0^\infty dp\ \cos(p\sigma)\ K_{ip}(z_1)K_{ip}(z_2) = \frac{\pi}{2}K_0\left(\sqrt{z_1^2+z_2^2+2bc\cosh\sigma}\right)\ . \tag{A.15}$$

The result is

$$\Psi^{bdy}_\sigma(\phi) = K_0\left(L_3^{(0)}\right)\ , \tag{A.16}$$

where we define

$$L_3^{(0)} \equiv \sqrt{4\pi\mu b^{-2}}\sqrt{e^{(e_1,\phi)}+e^{(e_2,\phi)}+e^{(\omega_1,\phi)}\chi_{\omega_1}(\sigma^*)+e^{(\omega_2,\phi)}\chi_{\omega_2}(\sigma^*)}\ , \tag{A.17}$$

$$= \sqrt{4\pi\mu b^{-2}}\prod_{h\in H_{\omega_1}}\sqrt{e^{\frac{1}{3}(e_1,\phi)}+(-1)^{(\rho,h)}e^{\frac{1}{3}(e_2,\phi)}e^{(h,\sigma^*)}}\ . \tag{A.18}$$

In the strong coupling region where μ is large, we have $\Psi^{bdy}_\sigma(\phi) \sim e^{-L_3^{(0)}}$ as follows from $K_0(z) \underset{z\to\infty}{\sim} \sqrt{\frac{\pi}{2z}}e^{-z}$. And $L_3^{(0)}$ generates the canonical transformation from Toda classical mechanics to the free classical mechanics. (The transformation itself is written in [22].) As in the case of Liouville theory, this follows from the identity

$$\left(\frac{\partial L_3^{(0)}}{\partial\phi}\right)^2 - \left(\frac{\partial L_3^{(0)}}{\partial\sigma}\right)^2 = 4\pi\mu b^{-2}\left[e^{(e_1,\phi)}+e^{(e_2,\phi)}\right]\ , \tag{A.19}$$

which can be proved with the help of the formulas

$$\begin{cases}\left(\frac{\partial\chi_{\omega_1}(\sigma)}{\partial\sigma}\right)^2 = \frac{2}{3}\chi_{2\omega_1}(\sigma)-\frac{4}{3}\chi_{\omega_2}(\sigma) \\ \left(\frac{\partial\chi_{\omega_1}(\sigma)}{\partial\sigma},\frac{\partial\chi_{\omega_2}(\sigma)}{\partial\sigma}\right) = \frac{1}{3}\chi_\rho(\sigma)-\frac{8}{3}\end{cases}\ ,\quad \begin{cases}\chi_{\omega_1}^2 = \chi_{2\omega_1}+\chi_{\omega_2} \\ \chi_{\omega_1}\chi_{\omega_2} = \chi_\rho+1\end{cases}\ . \tag{A.20}$$

To conclude, let us come back to the interpretation of the boundary wavefunction $\Psi^{bdy}_\sigma(\phi)$ as the density of the continuous D-brane of parameter σ, as suggested by eq. (A.3). In the weak coupling region where $L_3^{(0)}$ is small, we have $\Psi^{bdy}_\sigma(\phi) \underset{(\rho,\phi)\to-\infty}{\sim} -(\rho,\phi)$ as follows from $K_0(z) \underset{z\to 0}{\sim} -\log\frac{z}{2}$. Thus, the density of the D-brane grows linearly with ϕ. The minisuperspace annulus partition function $Z^{mini}_{\sigma_1;\sigma_2} = \int d^2\phi\ \Psi_{\sigma_1}(\phi)\Psi_{\sigma_2}(\phi)$ therefore has an L^4 infrared divergence, where L is a large distance cutoff. This confirms the divergence which was found by modular bootstrap methods in Subsection 4.3. This contrasts with the case of Liouville theory, where the density of a continuous D-brane is constant in the weak coupling region, and correspondingly the annulus partition function diverges as L, which is the volume of the ϕ-space in that case.

B. On the existence of a boundary action in $s\ell_3$ Toda theory

The functional integral formalism is often useful in the study of conformal field theories, although in general it permits the calculation of only a subset of the correlation functions. In this formalism, correlation functions are expressed as functional integrals over the fields ϕ_i, where field configurations come with weights e^{-S}. Here S is the action, which may or may not be written as the integral of a certain Lagrangian L, namely $S = \int d^2z\ L(z)$. If the Lagrangian exists and is local, that is if $L(z)$ is a function of the fields ϕ_i and finitely many of their derivatives at the point z, then the action is also called local. If the space has a boundary $z = \bar{z}$, the boundary action or boundary terms of the action are the terms which depend only on the values of the fields at the boundary, and local boundary actions are those of the type $S = \int_{z=\bar{z}} dx\ L(x)$ where $L(x)$ is a local boundary Lagrangian.

The choice of an action S is constrained by the classical theory. Namely, the solutions of the classical equations of motion and boundary conditions should be functional critical points of the action. This constraint does not fully determine S; here we will however only be concerned with the question of the existence of at least one action which obeys this constraint.

In $s\ell_3$ conformal Toda theory on surfaces with no boundaries, the Lagrangian L_3 (2.25) is known [1]. In Liouville theory on surfaces with boundaries, we have the boundary Lagrangian L_2^{bdy} (3.8), see [16]. In the case of $s\ell_3$ Toda theory, we could so far derive our boundary conditions from boundary Lagrangians only in particular subcases of the two cases $W = \pm\bar{W}$. We will now investigate systematically for which boundary conditions ($W = \pm\bar{W}$) and boundary parameters (λ_0 or λ_1, λ_2) boundary actions can exist.

B.1 Boundary conditions as functional one-forms

Let us assume the existence of a boundary action $S^{bdy}[\phi_i]$, that is a functional of the values of the Toda fields ϕ_1, ϕ_2 at the boundary. We however do not assume that $S^{bdy}[\phi_i]$ is local. In particular we do not forbid introducing auxiliary boundary fields in addition to ϕ_i, so long as these auxiliary fields can be eliminated using their equations of motion. We only exclude the possibility for fields to obey Dirichlet boundary conditions, which excludes the particular case (3.22) from the analysis.

Let us derive the Neumann-type boundary conditions from the action $S = \int d^2z \frac{1}{2}(\partial\phi, \bar{\partial}\phi) + S_3^{bdy}[\phi_i]$, where the interaction terms in the bulk action (2.25) can be omitted as they will not contribute. We find

$$\frac{1}{2i}(\partial - \bar{\partial})(2\phi_1 - \phi_2) = \frac{\delta S^{bdy}}{\delta\phi_1} \quad , \quad \frac{1}{2i}(\partial - \bar{\partial})(2\phi_2 - \phi_1) = \frac{\delta S^{bdy}}{\delta\phi_2}\ . \tag{B.1}$$

In terms of the $X_i = e^{-\phi_i}$, this becomes

$$2\frac{(\partial - \bar{\partial})X_1}{X_1^2} - \frac{(\partial - \bar{\partial})X_2}{X_1X_2} = 2i\frac{\delta S^{bdy}}{\delta X_1} \quad , \quad 2\frac{(\partial - \bar{\partial})X_2}{X_2^2} - \frac{(\partial - \bar{\partial})X_1}{X_1X_2} = 2i\frac{\delta S^{bdy}}{\delta X_2}\ . \tag{B.2}$$

The existence of the boundary action S^{bdy} can now be interpreted as the condition that the functional one-form

$$g = \left(2\frac{(\partial - \bar{\partial})X_1}{X_1^2} - \frac{(\partial - \bar{\partial})X_2}{X_1X_2}\right)\delta X_1 + \left(2\frac{(\partial - \bar{\partial})X_2}{X_2^2} - \frac{(\partial - \bar{\partial})X_1}{X_1X_2}\right)\delta X_2 \tag{B.3}$$

be exact, namely $g = \delta(2iS^{bdy})$. It follows that g must be closed, $\delta g = 0$. In order to be able to work with this condition, we will study functional calculus in the next Subsection.

Before that, let us point out that the natural variables to work with are not X_1, X_2 but the functions b_1, b_2, b_3 in terms of which we wrote the solutions of the Toda equations (3.13) and (3.19). These variables are subject to the constraint $\mathrm{Wr}[b_1, b_2, b_3] = 1$, so that we must include the possibility of such constraints in our study of functional calculus.

B.2 Technical interlude: functional calculus

We wish to study functional forms which depend on functions $b_i(x)$. A zero-form is a functional $S[b_i]$. A one-form is an object $g = \int dx \sum_i g_i(x)\delta b_i(x)$, where $g_i(x)$ are x-dependent functionals of b_i. An example of a one-form is the differential of a zero-form, namely $\delta S = \int dx \sum_i \frac{\delta S}{\delta b_i(x)}\delta b_i(x)$. A two-form is an object $k = \int dxdy \sum_{ij} k_{ij}(x,y)\delta b_i(x)\wedge\delta b_j(y)$, where $k_{ij}(x,y)$ are x,y-dependent functionals of b_i. The basic two-forms $\delta b_i(x)\wedge\delta b_j(y) = -\delta b_j(y)\wedge\delta b_i(x)$ are antisymmetric, which however does not imply the vanishing of $\delta b_i(x)\wedge\delta b_i(y) = -\delta b_i(y)\wedge\delta b_i(x)$. So the differential of a one-form is

$$\delta\left(\sum_i \int dx\, g_i(x)\delta b_i(x)\right) = \sum_{i<j}\int dxdy\left(\frac{\delta g_i(x)}{\delta b_j(y)} - \frac{\delta g_j(y)}{\delta b_i(x)}\right)\delta b_i(x)\wedge\delta b_j(y)$$
$$+\sum_i \int dxdy \frac{\delta g_i(x)}{\delta b_i(y)}\delta b_i(x)\wedge\delta b_i(y)\ . \quad \text{(B.4)}$$

As an exercise, we can compute the differential of an action functional $S = \int dx\, L(b(x), b'(x))$,

$$\delta S = \int dzdy\left[\frac{\partial L}{\partial b}(z)\delta(z-y) + \frac{\partial L}{\partial b'}(z)\delta'(z-y)\right]\delta b(y) = \int dz\left[\frac{\partial L}{\partial b} - \frac{\partial}{\partial z}\frac{\partial L}{\partial b'}\right](z)\delta b(z)\ , \quad \text{(B.5)}$$

and we can check that $\delta^2 S = 0$.

Now we will be interested in variables b_1, b_2, b_3 which are not independent, as they obey the constraint $\mathrm{Wr}[b_1,b_2,b_3] = 1$. If these were ordinary variables instead of functions, the condition for the form $g = \sum g_i db_i$ to be closed modulo a constraint $C(b_1,b_2,b_3) = 1$ would simply be $dg\wedge dC = 0$, and the integral S of the one-form g would be characterized by $(dS - g)\wedge dC = 0$. Let us generalize these notions to the case of functional forms. Let $g = \int dx \sum_{i=1}^3 g_i(x)\delta b_i(x)$ be a one-form, let us study the condition that it is closed modulo the constraint Wr. We denote $\delta g = \int dxdy \sum_{ij} k_{ij}(x,y)\delta b_i(x)\wedge\delta b_j(y)$ with $k_{ij}(x,y) = -k_{ji}(y,x)$.

We assume for a moment that the constraint can be inverted and rewritten as $b_3 = \phi[b_1,b_2]$. Then it is straightforward to rewrite $g = \int dx \sum_{i=1}^2 \tilde{g}_i(x)\delta b_i(x)$ and to compute δg in terms of ϕ and k_{ij}. We find that the vanishing of δg modulo the constraint $\mathrm{Wr}[b_1,b_2,b_3] = 1$ is equivalent to

$$K_{12} - K_{13} - K_{32} + K_{33} = K_{11} - K_{13} - K_{31} + K_{33} = K_{22} - K_{23} - K_{32} + K_{33} = 0\ , \quad \text{(B.6)}$$
$$\text{where}\quad K_{ij} \equiv \left(\frac{\delta\mathrm{Wr}}{\delta b_i}\right)^{-1t} k_{ij}\left(\frac{\delta\mathrm{Wr}}{\delta b_j}\right)^{-1}\ . \quad \text{(B.7)}$$

In the definition of K_{ij} we have used new notations for functions of two variables $f(x,y)$ such as $\frac{\delta\mathrm{Wr}(x)}{\delta b_i(y)}$ or $k_{ij}(x,y)$. Namely, the products and inverses of such functions are defined with respect to the product law $(f_1 f_2)(x,y) \equiv \int dz\, f_1(x,z)f_2(z,y)$, and the transposition is defined as the exchange of the two variables, $f^t(x,y) \equiv f(y,x)$. In the case when the functions b_i are x-independent, the product law becomes commutative, the objects k_{ii} and K_{ii} vanish, and the conditions (B.6) boil down to $K_{12} + K_{23} + K_{31} = 0$ which is equivalent to $dg\wedge d\mathrm{Wr} = 0$ as we found by the direct analysis of that case. Notice that the conditions (B.6) on the matrix K_{ij} are equivalent to $\sum_{ij} v_i K_{ij} v'_j = 0$ for any two vectors v, v' such that $\sum_i v_i = \sum_i v'_j = 0$.

Then the conditions for an "action" functional S to be the integral of the functional one-form g modulo the constraint Wr is:

$$\left(\frac{\delta S}{\delta b_1}-g_1\right)\left(\frac{\delta \mathrm{Wr}}{\delta b_1}\right)^{-1}=\left(\frac{\delta S}{\delta b_2}-g_2\right)\left(\frac{\delta \mathrm{Wr}}{\delta b_2}\right)^{-1}=\left(\frac{\delta S}{\delta b_3}-g_3\right)\left(\frac{\delta \mathrm{Wr}}{\delta b_3}\right)^{-1}. \tag{B.8}$$

Now the Wronskian constraint is not invertible, as b_3 cannot be fully determined in terms of b_1, b_2. So the quantities $\left(\frac{\delta \mathrm{Wr}}{\delta b_i}\right)^{-1}$ are ambiguous. We indeed find that $\frac{\delta \mathrm{Wr}}{\delta b_i}$ has several inverses, parametrized by numbers κ^i_{jk},

$$\left(\frac{\delta \mathrm{Wr}}{\delta b_i}\right)^{-1}(x,y)=\frac{1}{w_i^2}\left[\Theta(y-x)\epsilon_{ijk}b_j(x)b_k(y)+\sum_{j,k\neq i}\kappa^i_{jk}b_j(x)b_k(y)\right], \tag{B.9}$$

where $\Theta(x)$ is a step function such that $\Theta'(x)=\delta(x)$, and we recall the notations $w_i=\epsilon_{ijk}b_jb'_k$ and $\mathrm{Wr}=\epsilon_{ijk}b_ib'_jb''_k$. Then for δg to vanish modulo the constraint, the condition (B.6) must hold for all values of κ^i_{jk}. Similarly, integrating the functional one-form g modulo the constraint requires the equation (B.8) to be satisfied for all values of κ^i_{jk}.

B.3 Existence of the boundary action if $W-\bar{W}=0$

We have found that the boundary conditions $W-\bar{W}=0$ lead to the expressions (3.13) for the Toda fields X_1, X_2 in terms of functions b_i subject to the Wronskian constraint. The expressions (3.16) for $(\partial-\bar{\partial})X_i$ are also known. These expressions depend on a constant matrix N_{ij} of size 3 and determinant 1; it will be convenient to decompose both N and N^{-1T} into symmetric and antisymmetric parts, according to $N_{ij}=S_{ij}+\epsilon_{ijk}A_k$ and $N^{-1T}=\sigma_{ij}+\epsilon_{ijk}\alpha_k$.

So we can compute the one-form g (B.3) and its differential $k=\delta g$ in terms of the functions b_i. Taking the ambiguities κ^i_{jk} to vanish in the inversion (B.9) of the Wronskian constraint, the quantities K_{ij} (B.7) turn out to be of the form

$$K_{ij}(x,y)=\frac{2\epsilon_{i\ell m}\epsilon_{jpq}b_\ell(x)b_p(y)}{w_i^2(x)w_j^2(y)}\int dz\ \Theta(x-z)\Theta(y-z)\ \Lambda^{ij}_{mq}(z), \tag{B.10}$$

where we sum over repeated indices except i, j, and the tensor Λ^{ij}_{mq}, which is defined for $q\neq j$ and $m\neq i$ and obeys $\Lambda^{ij}_{mq}=-\Lambda^{ji}_{qm}$, is

$$\begin{aligned}\Lambda^{ij}_{mq}=&\ \frac{2\sigma_{qr}w_rw_jb_m}{X_1X_2^2}(\alpha_iS_{uv}-\alpha_uS_{iv})b_ub_v-\frac{2\sigma_{mr}w_rw_ib_q}{X_1X_2^2}(\alpha_jS_{uv}-\alpha_uS_{jv})b_ub_v\\&+\frac{2S_{ir}b_rw_jb_m}{X_1^2X_2}(A_u\sigma_{qv}-A_q\sigma_{uv})w_uw_v-\frac{2S_{jr}b_rw_ib_q}{X_1^2X_2}(A_u\sigma_{mv}-A_m\sigma_{uv})w_uw_v\\&+(\alpha_iS_{jr}-\alpha_jS_{ir})\frac{b_rb_mb_q}{X_1X_2}+(A_m\sigma_{qr}-A_q\sigma_{mr})\frac{w_iw_jw_r}{X_1X_2}.\end{aligned} \tag{B.11}$$

In the special case of the free boundary conditions we have $A_m=\alpha_m=0$ thus $\Lambda^{ij}_{mq}=0$. In the special case when the boundary Lagrangian given by eq. (3.17) we have $S_{ij}=U_iU_j$ thus $\alpha_iS_{uv}-\alpha_uS_{iv}=A_u\sigma_{qv}-A_q\sigma_{uv}=0$ thus again $\Lambda^{ij}_{mq}=0$.

For a quantity K_{ij} of the form (B.10), the condition (B.6) amounts to

$$\begin{cases} \forall m \neq i \\ \forall q \neq j \end{cases}, \int \Lambda^{ij}_{mq} = 0 \text{ and } \begin{cases} \forall j, k \\ \forall i \neq m \end{cases}, \epsilon_{jpq} \frac{b_p(y)}{w_j^2(y)} \int^y \Lambda^{ij}_{mq} = \epsilon_{kpq} \frac{b_p(y)}{w_k^2(y)} \int^y \Lambda^{ik}_{mq}, \tag{B.12}$$

where $\int^y \Lambda$ is the primitive of the function Λ. Curiously, taking into account the ambiguities parametrized by κ^i_{jk} does not yield extra equations.

We wish to find out whether the equation (B.12) holds for any triples (b_1, b_2, b_3) obeying the Wronskian constraint. We do not know how to do this except by testing the equation for a number of triples. Large families of solutions of the Wronskian constraint can be built from functions of the type z^ν or $e^{\nu z}$. This raises the questions of the admissible behaviour of $b_i(z)$ at $z = \infty$ and at generic points z, and of the appropriate contours of integration in our equation (B.12). We have no satisfactory answers to these questions. So we will test only the purely algebraic consequences of our equation.

Consider an equation of the type $u_0(y) = 0$ where $u_0(y) = \sum_{i=1}^n u_i(y) \int^y v_i$. Let us build the matrix of size $n+1$ formed by u_i and their first n derivatives, $M = [u_i^{(j)}]_{i,j=0\cdots n}$. Then $\det M = 0$ is a purely algebraic consequence of the original equation, in the sense that the terms involving primitives $\int^y v_i$ cancel. Applying this treatment to eq. (B.12) removes the need to deal with integrals and to worry about the regularity of $b_i(z)$. For all the numerous cases which we tested, we found that the condition $\det M = 0$ held. This is strong evidence that the form δg is closed. This is strong evidence that it is in fact exact, and we conjecture that there exists a boundary action from which the boundary condition (3.12) can be derived.

This action is expected to be a functional S^{bdy} of the values of the Toda fields ϕ_1, ϕ_2 at the boundary $z = \bar{z}$. In addition, S^{bdy} is expected to depend on the boundary parameter λ_0. Comparing its definition (B.2) with the formulas (3.13) for X_i and (3.16) for $(\partial - \bar{\partial})X_i$, we see that S^{bdy} cannot be local, that is of the type $\int L^{bdy}[\phi_1, \phi_2]$ where L^{bdy} is a function of ϕ_i and finitely many of their derivatives, except in the two special cases $\lambda_0 = \pm 1$ which we considered in Subsection 3.3. It is possible that the nonlocal boundary action has a simple expression as a local functional of b_i. Even so, it would not be very easy to use such an action in free-field computations of correlation functions.

B.4 No boundary action if $W + \bar{W} = 0$

We have found that the boundary condition $W + \bar{W} = 0$ led to the expressions (3.19) for the Toda fields X_1, X_2 in terms of functions b_i subject to the Wronskian constraint. Expressions for $(\partial - \bar{\partial})X_i$ can easily be derived. These expressions depend on a constant matrix N_{ij} of size 3 and determinant 1; it is not restrictive to assume that N is diagonal with eigenvalues ν_1, ν_2, ν_3.

The rest of the reasoning is similar to the case $W - \bar{W} = 0$, with a different formula for the object Λ^{ij}_{mq} which appears in eq. (B.10):

$$\begin{aligned} \Lambda^{ij}_{mq} = (\nu_i - \nu_m)(\nu_j - \nu_q) \Bigg[&\left(\frac{2}{X_1^2} - \frac{2\nu_\ell \nu_p}{X_2^2} + \frac{\nu_p - \nu_\ell}{X_1 X_2} \right) w_j b_q (w_i' b_m - w_i b_m') \\ &- \left(\frac{2}{X_1^2} - \frac{2\nu_\ell \nu_p}{X_2^2} - \frac{\nu_p - \nu_\ell}{X_1 X_2} \right) w_i b_m (w_j' b_q - w_j b_q') + \frac{\nu_\ell - \nu_p}{X_1 X_2} \left(\frac{\Delta_1}{X_1} - \frac{\Delta_2}{X_2} \right) w_i b_m w_j b_q \Bigg], \end{aligned} \tag{B.13}$$

where the indices p and ℓ are such that $\epsilon_{im\ell}$ and ϵ_{jqp} do not vanish. With such an expression for Λ^{ij}_{mq}, we find that eq. (B.12) no longer holds, by numerically testing it in various examples of values of b_i. This proves that there is no boundary action from which the boundary condition $W + \bar{W} = 0$ can be derived.

Remember however that this proof of the non-existence of the boundary action relies on our assumption that only Neumann-type boundary conditions are allowed, and Dirichlet-type boundary conditions do not occur. So there is no contradiction with the special case (3.22) when Dirichlet-type conditions could be derived by varying an action (whose boundary term was actually zero). But we saw in Section 4 that in the generic case the boundary condition $W + \bar{W} = 0$ corresponds to two-dimensional D-branes, and we do not expect Dirichlet conditions to apply.

References

[1] V. A. Fateev, A. V. Litvinov, *Correlation functions in conformal Toda field theory I*, JHEP 11 p. 002 (2007), `arXiv:0709.3806 [hep-th]`

[2] A. B. Zamolodchikov, *Infinite Additional Symmetries in Two-Dimensional Conformal Quantum Field Theory*, Theor. Math. Phys. 65 pp. 1205–1213 (1985)

[3] V. A. Fateev, S. L. Lukyanov, *The Models of Two-Dimensional Conformal Quantum Field Theory with Z(n) Symmetry*, Int. J. Mod. Phys. A3 p. 507 (1988)

[4] A. A. Belavin, A. M. Polyakov, A. B. Zamolodchikov, *Infinite conformal symmetry in two-dimensional quantum field theory*, Nucl. Phys. B241 pp. 333–380 (1984)

[5] V. A. Fateev, A. B. Zamolodchikov, *Conformal quantum field theory models in two dimensions having Z_3 symmetry*, Nucl. Phys. B280 pp. 644–660 (1987)

[6] V. A. Fateev, A. V. Litvinov, *Correlation functions in conformal Toda field theory II*, JHEP 01 p. 033 (2009), `0810.3020`

[7] J. L. Cardy, *Boundary conditions, fusion rules and the Verlinde formula*, Nucl. Phys. B324 p. 581 (1989)

[8] P. Bouwknegt, K. Schoutens, *W symmetry in conformal field theory*, Phys. Rept. 223 pp. 183–276 (1993), `hep-th/9210010`

[9] S. L. Lukyanov, V. A. Fateev, *Physics reviews: Additional symmetries and exactly soluble models in two-dimensional conformal field theory* Chur, Switzerland: Harwood (1990) 117 p. (Soviet Scientific Reviews A, Physics: 15.2)

[10] P. Di Francesco, P. Mathieu, D. Senechal, *Conformal field theory* New York, USA: Springer (1997) 890 p

[11] P. Bowcock, G. M. T. Watts, *Null vectors, three point and four point functions in conformal field theory*, Theor. Math. Phys. 98 pp. 350–356 (1994), `hep-th/9309146`

[12] S. Ribault, *On sl3 Knizhnik-Zamolodchikov equations and W3 null-vector equations*, JHEP 10 p. 002 (2009), `0811.4587`

[13] A. B. Zamolodchikov, A. B. Zamolodchikov, *Structure constants and conformal bootstrap in Liouville field theory*, Nucl. Phys. B477 pp. 577–605 (1996), `hep-th/9506136`

[14] A. N. Leznov, M. V. Saveliev, *Representation of zero curvature for the system of nonlinear partial differential equations $x_{\alpha,z\bar{z}} = \exp(kx)_\alpha$ and its integrability.*, Lett. Math. Phys. 3(6) p. 489 (1979)

[15] V. de Alfaro, A. T. Filippov, *Multi-exponential models of (1+1)-dimensional dilaton gravity and Toda-Liouville integrable models*, Theor. Math. Phys. 162 pp. 34–56 (2010), `0902.4445`

[16] V. Fateev, A. B. Zamolodchikov, A. B. Zamolodchikov, *Boundary Liouville field theory. I: Boundary state and boundary two-point function* (2000), `hep-th/0001012`

[17] J. Teschner, *Remarks on Liouville theory with boundary* (2000), `hep-th/0009138`

[18] A. B. Zamolodchikov, A. B. Zamolodchikov, *Liouville field theory on a pseudosphere* (2001), `hep-th/0101152`

[19] V. A. Fateev, *Normalization factors, reflection amplitudes and integrable systems* (2001), `hep-th/0103014`

[20] N. Drukker, D. Gaiotto, J. Gomis, *The Virtue of Defects in 4D Gauge Theories and 2D CFTs* (2010), `1003.1112`

[21] L. Takhtajan, A. Vinogradov, *Theory of the Eisenstein series for the group $SL(3,R)$ and its application to the binary problem I*, Notes of the LOMI seminars 76 p. 5 (1978)

[22] A. Anderson, B. E. W. Nilsson, C. N. Pope, K. S. Stelle, *The Multivalued free field maps of Liouville and Toda gravities*, Nucl. Phys. B430 pp. 107–152 (1994), `hep-th/9401007`