Lévy Flights in Evolutionary Ecology

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Abstract

We are interested in modeling Darwinian evolution resulting from the interplay of phenotypic variation and natural selection through ecological interactions. The population is modeled as a stochastic point process whose generator captures the probabilistic dynamics over continuous time of birth, mutation, and death, as influenced by each individual's trait values, and interactions between individuals. An offspring usually inherits the trait values of her progenitor, except when a random mutation causes the offspring to take an instantaneous mutation step at birth to new trait values. In the case we are interested in, the probability distribution of mutations has a heavy tail and belongs to the domain of attraction of a stable law. We investigate the large-population limit with allometric demographies: larger populations made up of smaller individuals which reproduce and die faster, as is typical for micro-organisms. We show that depending on the allometry coefficient the limit behavior of the population process can be approximated by nonlinear Lévy flights of different nature: either deterministic, in the form of nonlocal fractional reaction-diffusion equations, or stochastic, as nonlinear super-processes with the underlying reaction and a fractional diffusion operator. These approximation results demonstrate the existence of such nontrivial fractional objects; their uniqueness is also proved.

Key-words: Darwinian evolution, mutation law with heavy tail, birth-death-mutation-competition point process, mutation-selection dynamics, nonlinear fractional reaction-diffusion equations, nonlinear superprocesses with fractional diffusion.

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1 Introduction

In this paper, we are interested in modeling the dynamics of populations as driven by the interplay of phenotypic variation and natural selection operating through ecological interactions, i.e., Darwinian evolution. The population is modeled as a stochastic Markov point process whose generator captures the probabilistic dynamics over continuous time of birth, mutation and death, as influenced by each individual's trait values and interactions between individuals. The adaptive nature of a trait implies that an offspring usually inherits the trait values of her progenitor, except when a mutation occurs. In this case, the offspring makes an instantaneous mutation step at birth to new trait values. This microscopic point of view has been heuristically introduced in Bolker-Pacala [5] and Dieckmann-Law [12]. It has been rigorously developed first in Fournier-Méléard [15] for spatial seed models and by Champagnat-Ferrière-Méléard [8], [9] for phenotypic trait structured populations when the mutation kernel behaves essentially as a Gaussian law. The aim in this work is to study the case where a mutant individual can be significantely different from his ancestor. More precisely, the mutation kernel will be assumed to have a heavy tail and to belong to the domain of attraction of a stable law. If the traits describe a spatial dispersion, for instance for seeds, we thus assume that seed offsprings instantaneously jump far from the mother seed because of availability of resources or wind. In the case of phenotypic traits, the heavy tail mutation assumption says that if a mutant offspring is too close to the mother trait, then it is so deleterious that it cannot be observed. A trait which quantifies the aggressivity level is an example of such a situation.

In the context of ecology several authors have considered the fractional reaction-diffusion model and we would like to acknowledge some of their contributions here. In particular, Baeumer, Kovacs, and Meerschaert [2] considered fractional reproduction-dispersal equations and heavy tail dispersal kernels. Their paper also contains an exhaustive review of the literature on the population spreads with extreme patterns of dispersal and reproduction. Another population dynamics model introduced by Gurney and Nisbet [16] led to strongly nonlinear partial differential equation of the porous media type; we have studied a probabilistic fractional framework for related stochastic and partial differential equations in [19].

Each individual is characterized by a real-valued trait x describing a phenotypic, or a spatial parameter. The birth and death rates of this individual depend on its trait x and also on the environment through the influence of the others individuals alive. In case the offspring produced by the individual with trait x carries a mutated trait, then the difference between this mutated trait and x is distributed according to $M(x, dh)$ such that

$$\int_{|h|\geq y} M(x, dh) \sim_{y\to\infty} \frac{C(x)}{y^{\alpha}}, \tag{1.1}$$

with $\alpha \in (0, 2)$ not depending on x. The inclusion of the dependence of the mutation rate on the trait itself, as expressed in the dependence of M, and C, on x, is an essential part of our model even though it considerably complicates the mathematical framework of this paper.

An example for $M(x, dh)$ could be the Pareto law with density $\frac{\alpha}{2}\mathbf{1}_{\{|h|\geq 1\}}/|h|^{1+\alpha}$, but a mutation law equal to the Pareto law (up to a constant) outside a given interval of the form $[-a, +a]$, and constant inside, is also covered by our model. The latter example corresponds to a distribution of mutant traits which is uniform in a small neighborhood around the mother's trait, but decreases, with heavy tail for more distant traits.

We investigate the large population limits, $K \to \infty$, with allometric demographies: larger populations made up of smaller individuals which reproduce and die faster. This leads to systems in which organisms have short lives and reproduce fast while their colonies or populations grow or decline on a slow time scale. Typically, these assumptions are appropriate for microorganisms such as bacteria or plankton. The allometric effect will be modeled by a dominant birth and death rate of order K^η, $\eta > 0$.

In [9], such asymptotic approximations have been studied in the case of small mutations with a Gaussian, thin tail distribution. In such a situation, for the allometric exponent $\eta < 1$, and large enough population, the individual population stochastic process is approximated by the solution of a classical deterministic nonlinear reaction-diffusion equation. In our present heavy tail setting we will prove that in a similar limit, the population process is approximated by the solution of a nonlocal nonlinear partial differential equation driven by a fractional Laplacian and involving a reaction term. Note that, as a byproduct, this result provides a proof of the existence of weak solutions of such nonclassical equations. Separately, employing purely analytical techniques, we will also prove the uniqueness of their solutions. The case of the allometric exponent $\eta = 1$ is significantly different as stochasticity remains present in the limit due to the demographic acceleration. The population process is then approximated by a nonlinear super-process with underlying reaction and fractional diffusion.

Our work provides a rigorous derivation of macroscopic models from microscopic dynamics (hydrodynamic limit) for a large class of nonlinear equations involving Lévy flights which naturally appear in evolutionary ecology and population dynamics. Our use of probabilistic tools of interacting particle systems is essential and provides the information about the scales (between mutation amplitude and population size) at which such models are justified.

It should be mentioned at this point that fractional reaction-diffusion equations have been suggested and studied as models for several physical phenomena. Thus, Del-

Castillo-Negrete, Carreras, and Lynch [11] investigated front dynamics in reaction-diffusion systems with Lévy flights, and Henry, Langlands and Wearne [17], studied Turing pattern formation in fractional activator-inhibitor systems. Also, Saxena, Mathai, and Haubold [25] found explicit solutions of the fractional reaction-diffusions equations in terms of the Mittag-Leffler functions which are suitable for numerical computations; they also considered the situation where the time derivative is also replaced by a fractional derivative of order less than one. Another numerical method for finding solutions of such equations can be found in Baeumer, Kovacs, and Meerschaert [3].

This paper starts (Section 2) with the description of the reproduction and death mechanisms for the individuals of the population we are interested in. The main convergence results based on a large population limit are stated. Thus Theorem 3.2 shows that an allometric effect of order K^η, with $\eta \in (0,1)$, leads to a deterministic, nonlinear integro-differential equation driven by a nonlocal fractional Laplacian operator independent of η, while in Theorem 3.3, we show that an allometric effect of order K ($\eta = 1$) yields, in the large population limit, a stochastic measure-valued process depending on the acceleration rate of the birth-and-death process. Thus (demographic) stochasticity appears as the allometric exponent takes on the value $\eta = 1$. Also, Proposition 3.1 establishes that the mutation kernel conveniently renormalized behaves approximately as a jump kernel with the heavy tail jump measure $\sigma(x)dz/|z|^{1+\alpha}$. Section 4 contains technical lemmas needed in the proof of the main results of Section 3. In particular we clarify a key technical point that was eluded in [15], [9], allowing to deduce the tightness of the measure valued population process for the weak topology from the tightness for the vague topology (see Lemma 4.3 and Remark 4.4 and Step 2 of the proof of Theorem 3.2). Section 5 contains proofs of the main theorems stated in Section 3 using the measure-valued martingale properties of the population process.

2 Population point process

As in [9], the evolving population is modelled by a stochastic system of interacting individuals, where each individual is characterized by a phenotypic trait. This trait is described quantitatively by a real number. We assume that the parameter K scales the initial number of individuals. To observe a nontrivial limit behavior of the system as K grows to infinity it is necessary to attach to each individual the weight $\frac{1}{K}$. Thus our system evolves in the subset $\mathcal{M}_K$ of the set M_F of finite non-negative measures on $\mathbb{R}$ consisting of all finite point measures with weight $\frac{1}{K}$:

$$\mathcal{M}_K = \left\{ \frac{1}{K} \sum_{i=1}^{n} \delta_{x_i},\ n \geq 0, x_1, ..., x_n \in \mathbb{R} \right\}.$$

Here and below, δ_x denotes the Dirac mass at x. For any $m \in M_F$, any measurable function f on $\mathbb{R}$, we set $\langle m, f \rangle = \int_{\mathbb{R}} f dm$. Following [8], we describe the population by

$$\nu_t^K = \frac{1}{K} \sum_{i=1}^{I_t^K} \delta_{X_t^i}, \tag{2.1}$$

with $I_t^K \in \mathbb{N}$ denoting the number of individuals alive at time t, and $X_t^1, ..., X_t^{I_t^K}$ describing the individuals' traits (in $\mathbb{R}$).

The population measure-valued process ν^K evolves as a birth and death process with mutation and selection. More precisely, an individual can give birth or die. The death can be natural, or can be due to the competition pressure exerted by other individuals (for instance, by sharing food). At birth, the offspring can inherit the trait of its parent, or can mutate to another trait with some positive probability. We assume that the population is of order K and that for each individual, birth and death occur at the rate of order K^η, for some $0 < \eta \leq 1$, while preserving the demographic balance. More precisely, the main assumptions on the birth and death dynamics are summarized below.

- **Scaling Assumptions:** For a population $\nu = \frac{1}{K}\sum_{i=1}^{I} \delta_{x^i}$, and a trait $x \in \mathbb{R}$, the birth and death rates are scaled with the system's size according to the following rules:

$$\begin{aligned} b_K(x,\nu) &= K^\eta r(x) + b(x, \frac{1}{K}\sum_{i=1}^{I} V(x-x^i)) = K^\eta r(x) + b(x, V * \nu(x)), \\ d_K(x,\nu) &= K^\eta r(x) + d(x, \frac{1}{K}\sum_{i=1}^{I} U(x-x^i)) = K^\eta r(x) + d(x, U * \nu(x)), \end{aligned} \tag{2.2}$$

  where b and d are continuous functions on $\mathbb{R}^2$, and $*$ denotes the convolution operation. The allometric effect (smaller individuals reproduce and die faster) is parametrized by the exponent η and a trait-dependent function r which is assumed to be positive and bounded on $\mathbb{R}$.

- **Assumptions (H1):** The interaction kernels V, and U, affecting, respectively, reproduction and mortality rates, are continuous functions on $\mathbb{R}$. In addition, there exist constants $\bar{r}$, $\bar{b}$, $\bar{d}$, $\bar{U}$, and $\bar{V}$, such that, for $x, z \in \mathbb{R}$,

$$0 \leq r(x) \leq \bar{r} \; ; \; 0 \leq b(x,z) \leq \bar{b}, \tag{2.3}$$

$$0 \leq d(x,z) \leq \bar{d}\,(1 + |z|), \tag{2.4}$$

$$0 \leq U(x) \leq \bar{U} \; ; \; 0 \leq V(x) \leq \bar{V}. \tag{2.5}$$

  Note that the death rate d is not assumed to be bounded but its growth in variable z is at most linear. This dependence models a possible competition between individuals (e.g., for shared resources), increasing their death rate.

Assumptions (H1) ensure that there exists a constant $\bar{C} > 0$, such that the total event rate for a population counting measure $\nu = \frac{1}{K}\sum_{i=1}^{I}\delta_{x^i}$, obtained as the sum of all event rates, is bounded by $\bar{C}I(1+I)$, where I is the population size.

- **Assumption (H2):** When an individual with trait x gives birth, it can produce a mutant offspring with probability $p(x)$. Otherwise, with probability $1-p(x)$, the offspring carries the same trait x as its ancestor. If a mutation occurs, the mutated offspring instantly acquires a new trait $x+h$, where h is selected randomly according to the mutation step measure $M_K(x, dh)$. We only consider mutations which have heavy tail distributions. More precisely, we assume that the probability measure $M_K(x, dh)$ is the law of the random variable

$$\frac{X(x)}{K^{\frac{\eta}{\alpha}}},$$

where $X(x)$ is a symmetric random variable such that, for some $\alpha \in (0,2)$, and a bounded $\sigma : \mathbb{R} \to \mathbb{R}_+$,

$$\lim_{u\to+\infty}\sup_{x\in\mathbb{R}}\left|u^\alpha \mathbb{P}(|X(x)| \geq u) - \frac{2\sigma(x)}{\alpha}\right| = 0. \tag{2.6}$$

As an immediate corollary of Assumption (H2) we have

$$\int_{|h|\geq y} M_K(x, dh) \sim_{yK^{\frac{\eta}{\alpha}}\to\infty} \frac{2\,\sigma(x)}{\alpha\, K^\eta\, y^\alpha},$$

and, for $\alpha \in (1,2)$, also

$$\int_{\mathbb{R}} hM_K(x, dh) = 0.$$

Example 2.1. One can choose the random variable $X(x)$ with Pareto's law independent of x, that is, with the density

$$\frac{\alpha}{2}\mathbf{1}_{\{|h|\geq 1\}}\frac{1}{|h|^{1+\alpha}}.$$

Example 2.2. Another possibility is to take $X(x)$ with the Pareto law (up to multiplicative a constant) outside a given interval of the form $[-a, +a]$, and constant inside. This choice corresponds to a distribution of mutant traits which is uniform in a small neighborhood around the mother's trait, but decreases, with heavy tail. for more distant traits.

We refer to Fournier-Méléard [15] or Champagnat-Ferrière-Méléard [8] for a pathwise construction of a point measure-valued Markov process $(\nu_t^K)_{t\geq 0}$ satisfying Assumptions (H1). The infinitesimal generator of its Markovian dynamics is given, for each finite point measure ν, by the expression

$$\begin{aligned}L^K\phi(\nu) &= K\int_{\mathbb{R}}(1-p(x))\ (K^\eta r(x)+b(x,V*\nu(x)))(\phi(\nu+\frac{1}{K}\delta_x)-\phi(\nu))\nu(dx)\\ &+K\int_{\mathbb{R}}p(x)\ (K^\eta r(x)+b(x,V*\nu(x)))\int_{\mathbb{R}}(\phi(\nu+\frac{1}{K}\delta_{x+h})-\phi(\nu))M_K(x,dh)\nu(dx)\\ &+K\int_{\mathbb{R}}(K^\eta r(x)+d(x,U*\nu(x)))(\phi(\nu-\frac{1}{K}\delta_x)-\phi(\nu))\nu(dx). \qquad (2.7)\end{aligned}$$

The first term of (2.7) captures the effect of births without mutation, the second term that of births with mutation, and the last term that of deaths.

If $E(\langle\nu_0^K,\mathbf{1}\rangle^2)<+\infty$, then for any $T<\infty$, $E(\sup_{t\in[0,T]}\langle\nu_t^K,1\rangle^2)<\infty$ (see Lemma 4.1). Thus, for any measurable functions ϕ on M_F such that $|\phi(\nu)|+|L^K\phi(\nu)|\leq C(1+\langle\nu,1\rangle^2)$, the process

$$\phi(\nu_t^K)-\phi(\nu_0^K)-\int_0^t L^K\phi(\nu_s^K)ds \qquad (2.8)$$

is a martingale. In particular, in view of (2.7), for each measurable bounded function f,

$$\begin{aligned}M_t^{K,f} &= \langle\nu_t^K,f\rangle-\langle\nu_0^K,f\rangle\\ &-\int_0^t\int_{\mathbb{R}}(b(x,V*\nu_s^K(x))-d(x,U*\nu_s^K(x)))f(x)\nu_s^K(dx)ds \qquad (2.9)\\ &-\int_0^t\int_{\mathbb{R}}p(x)\ (K^\eta r(x)+b(x,V*\nu_s^K(x)))\Big(\int_{\mathbb{R}}f(x+h)M_K(x,dh)-f(x)\Big)\nu_s^K(dx)ds,\end{aligned}$$

is a square integrable martingale with quadratic variation

$$\begin{aligned}\langle M^{K,f}\rangle_t &= \frac{1}{K}\Big\{\int_0^t\int_{\mathbb{R}}(2K^\eta r(x)+b(x,V*\nu_s^K(x))+d(x,U*\nu_s^K(x)))f^2(x)\nu_s^K(dx)ds\\ &+\int_0^t\int_{\mathbb{R}}p(x)\ (K^\eta r(x)+b(x,V*\nu_s^K(x)))\Big(\int_{\mathbb{R}}f^2(x+h)M_K(x,dh)-f^2(x)\Big)\nu_s^K(dx)ds\Big\}. \qquad (2.10)\end{aligned}$$

3 Scaling limits of population point processes

Our aim is now to make the population size's scaling K tend to infinity, in a scale accelerating the births and deaths, and making the mutation steps smaller and smaller. We begin by explaining why a fractional Laplacian term appears in the limit, $K\to\infty$, of the last term of the r.h.s. of (2.9).

Proposition 3.1 *Under* **(H2)**,

(i) if $\alpha \in (1,2)$*, then for any* $f \in C_b^2(\mathbb{R})$*,* $K^\eta \int_{\mathbb{R}}(f(x+h)-f(x))M_K(x,dh)$ *converges to* $\sigma(x)\int_{\mathbb{R}}(f(x+z)-f(x)-f'(x)z\mathbf{1}_{\{|z|\leq 1\}})\frac{dz}{|z|^{1+\alpha}}$ *uniformly for* $x \in \mathbb{R}$ *as* $K \to \infty$.

(ii) if $\alpha \in (0,1]$*, then the same result holds for* $f \in C_b^2(\mathbb{R})$ *compactly supported.*

Proof . By Fubini's theorem and since $f(x+h)-f(x) = \int_0^h f'(x+z)dz$,

$$\int_{\mathbb{R}} \mathbf{1}_{\{h\geq 0\}}(f(x+h)-f(x))M_K(x,dh) = \int_0^\infty f'(x+z)\mathbb{P}(X(x) \geq K^{\frac{\eta}{\alpha}}z)dz.$$

Treating in the same way the integral for $h<0$ and using the symmetry of $X(x)$, one deduces that

$$\int_{\mathbb{R}}(f(x+h)-f(x))M_K(x,dh) = \int_0^\infty \frac{f'(x+z)-f'(x-z)}{2}\mathbb{P}(|X(x)| \geq K^{\frac{\eta}{\alpha}}z)dz.$$

By integration by parts,

$$\begin{aligned}\int_0^{+\infty}(f'(x+z)-f'(x-z))\frac{dz}{z^\alpha} &= \alpha\int_0^{+\infty}(f(x+z)+f(x-z)-2f(x))\frac{dz}{z^{1+\alpha}}\\ &= \alpha\int_{\mathbb{R}}(f(x+z)-f(x)-f'(x)z\mathbf{1}_{\{|z|\leq 1\}})\frac{dz}{|z|^{1+\alpha}}.\end{aligned}$$

Combining both equalities one deduces that

$$\begin{aligned}R(x) &\overset{\text{def}}{=} K^\eta\int_{\mathbb{R}}(f(x+h)-f(x))M_K(x,dh) - \sigma(x)\int_{\mathbb{R}}(f(x+z)-f(x)-f'(x)z\mathbf{1}_{\{|z|\leq 1\}})\frac{dz}{|z|^{1+\alpha}}\\ &= \int_0^{+\infty}\frac{f'(x+z)-f'(x-z)}{2}\left(K^\eta z^\alpha\mathbb{P}(|X(x)| \geq K^{\frac{\eta}{\alpha}}z) - \frac{2\sigma(x)}{\alpha}\right)\frac{dz}{z^\alpha}.\end{aligned}$$

Therefore

$$\begin{aligned}|R(x)| \leq{}& \|f''\|_\infty \sup_{u>0,x\in\mathbb{R}}\left|u^\alpha\,\mathbb{P}(|X(x)|\geq u) - \frac{2\sigma(x)}{\alpha}\right|\int_0^{K^{-\frac{\eta}{2\alpha}}} z^{1-\alpha}dz\\ &+ \|f''\|_\infty \sup_{u>K^{\frac{\eta}{2\alpha}},x\in\mathbb{R}}\left|u^\alpha\mathbb{P}(|X(x)|\geq u) - \frac{2\sigma(x)}{\alpha}\right|\int_{K^{-\frac{\eta}{2\alpha}}}^1 z^{1-\alpha}dz\\ &+ \sup_{u>K^{\frac{\eta}{\alpha}},x\in\mathbb{R}}\left|u^\alpha\mathbb{P}(|X(x)|\geq u) - \frac{2\sigma(x)}{\alpha}\right|\int_1^{+\infty}|f'(x+z)-f'(x-z)|\frac{dz}{2z^\alpha}.\end{aligned} \tag{3.1}$$

By **(H2)** $\sup_{u>K^{\frac{\eta}{2\alpha}},x\in\mathbb{R}}\left|u^\alpha\mathbb{P}(|X(x)|\geq u) - \frac{2\sigma(x)}{\alpha}\right|$ tends to 0 as $K\to\infty$. Moreover for $v>0$,

$$\sup_{u>0,x\in\mathbb{R}}\left|u^\alpha\mathbb{P}(|X(x)|\geq u) - \frac{2\sigma(x)}{\alpha}\right| \leq v^\alpha + \frac{2}{\alpha}\sup_{x\in\mathbb{R}}\sigma(x) + \sup_{u>v,x\in\mathbb{R}}\left|u^\alpha\mathbb{P}(|X(x)|\geq u) - \frac{2\sigma(x)}{\alpha}\right|$$

with the right-hand-side finite when v is large enough.
Last, $\int_1^{+\infty} |f'(x+z) - f'(x-z)| \frac{dz}{2z^\alpha}$ is smaller than $\|f'\|_\infty \int_1^{+\infty} \frac{dz}{z^\alpha}$ if $\alpha > 1$. If $\alpha \in (0,1]$, it is smaller than $\|f'\|_\infty \int_1^{1+2C} \frac{dz}{z^\alpha}$ when the compactly supported function $f(y)$ vanishes for $|y| \geq C$. Hence (3.1) implies the desired uniform convergence as $K \to \infty$.

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In what follows, we will denote by $\tilde{\sigma}$ the product function defined by

$$\tilde{\sigma}(x) = p(x)r(x)\sigma(x)$$

and by $\hat{\sigma}$ the function

$$\hat{\sigma}(x) = \tilde{\sigma}^{1/\alpha}(x) = (p(x)r(x)\sigma(x))^{1/\alpha}.$$

The nature of the scaling limit of the population point processes ν^K strongly depends on the value of the allometric exponent η. For $0 < \eta < 1$, the limit is deterministic and our convergence result is described in the following theorem:

Theorem 3.2 *(i) Suppose that Assumptions (H1) and (H2) are satisfied,* $0 < \eta < 1$ *and the product function* $\tilde{\sigma}$ *is continuous. Additionally, assume that, as* $K \to \infty$*, the initial conditions* ν_0^K *converge in law, and for the weak topology on* M_F*, to a finite deterministic measure* ξ_0*, and that*

$$\sup_K E(\langle \nu_0^K, 1\rangle^3) < +\infty. \tag{3.2}$$

Then, for each $T > 0$*, the laws* (Q^K) *of the processes* (ν^K) *in* $\mathbb{D}([0,T], M_F)$ *(with* M_F *endowed with the weak convergence topology) are tight. Moreover, the weak limit of each of their convergent subsequences gives full weight to the process* $(\xi_t)_{t\geq 0} \in C([0,T], M_F)$ *satisfying the following condition: for each function* $f \in C_b^2(\mathbb{R})$*,*

$$\begin{aligned}\langle \xi_t, f\rangle = \langle \xi_0, f\rangle &+ \int_0^t \int_{\mathbb{R}} (b(x, V * \xi_s(x)) - d(x, U * \xi_s(x))) f(x) \xi_s(dx) ds \\ &+ \int_0^t \int_{\mathbb{R}} \tilde{\sigma}(x) \Big(\int_{\mathbb{R}} (f(x+h) - f(x) - f'(x) h \mathbf{1}_{\{|h|\leq 1\}}) \frac{dh}{|h|^{1+\alpha}} \Big) \xi_s(dx) ds. \end{aligned} \tag{3.3}$$

(ii) If we assume additionally that $b(x,z)$*, and* $d(x,z)$*, are Lipschitz continuous in* z*, uniformly for* $x \in \mathbb{R}$*, and that* $\hat{\sigma}$ *is Lipschitz continuous, then* (3.3) *has at most one solution such that* $\sup_{t\in[0,T]} \langle \xi_t, 1\rangle < +\infty$*, and, as* $K \to \infty$*, the processes* (ν^K) *converge weakly to this unique solution.*

(iii) Finally, if $\hat{\sigma} \in C_b^3$*, i.e., it is bounded together with its derivatives of order* ≤ 3*, and the product function* $\tilde{\sigma}(x) > 0$*, for all* $x \in \mathbb{R}$*, then, for each* $t > 0$*, the measure* ξ_t *has a density with respect to the Lebesgue measure.*

The following remarks are immediate corollaries of the above Theorem.

Remark 3.1. The limit (3.3) does not depend on $\eta \in (0,1)$. As will appear in the proof, this is implied by the fact that the growth rate $b_K - d_K$ does not depend on η, and that the mutation kernel $M_K(x,z)$ compensates exactly the dispersion in the trait space induced by the acceleration of the births with mutations.

Remark 3.2. In the case considered in Theorem 3.2 *(iii)*, Eq. (3.3) may be written in the form

$$\partial_t \xi_t(x) = \Big(b(x, V * \xi_t(x)) - d(x, U * \xi_t(x))\Big)\xi_t(x) + D^\alpha(\tilde{\sigma}\,\xi_t)(x), \tag{3.4}$$

where

$$D^\alpha f(x) = \int_{\mathbb{R}} (f(x+h) - f(x) - f'(x)h\mathbf{1}_{\{|h|\le 1\}})\frac{dh}{|h|^{1+\alpha}},$$

denotes the fractional Laplacian of order α. Note that, by the change of variable $z = h/\hat{\sigma}(x)$,

$$\tilde{\sigma}(x)D^\alpha f(x) = \int_{\mathbb{R}} (f(x+\hat{\sigma}(x)z) - f(x) - f'(x)\hat{\sigma}(x)z\mathbf{1}_{\{|z|\le 1\}})\frac{dz}{|z|^{1+\alpha}}. \tag{3.5}$$

Remark 3.3. Theorem 3.2 also proves the existence of a weak solution for (3.4). Equation (3.4) generalizes the Fisher reaction-diffusion equation known from classical population genetics (see e.g. [6]) with a fractional Laplacian term replacing the classical Laplacian. It justifies the Lévy flight modeling in ecology as approximations for models in which mutations have heavy tail. Such models were judged justified by real-life data in many areas of physics and economics, and in some popular literature devoted to the 2008 global financial crisis.

In the case of the allometric exponent $\eta = 1$ the scaling limit of the population point processes ν^K has a richer structure of a nonlinear stochastic superprocess which is described below.

Theorem 3.3 *(i) Suppose that Assumptions (H1) and (H2) are satisfied, $\eta = 1$ and that $\tilde{\sigma}$ is continuous. Additionally, assume that, as $K \to \infty$, the initial conditions ν_0^K converge in law, and for the weak topology on M_F, to a finite (possibly random) measure X_0, and that*

$$\sup_K E(\langle \nu_0^K, 1\rangle^4) < +\infty. \tag{3.6}$$

Then, for each $T > 0$, the laws of the processes $\nu^K \in \mathbb{D}([0,T], M_F)$ are tight and the limiting values are superprocesses $X \in C([0,T], M_F)$ satisfying the following two conditions:

$$\sup_{t\in[0,T]} E\left(\langle X_t, 1\rangle^4\right) < \infty, \tag{3.7}$$

and, for any $f \in C_b^2(\mathbb{R})$,

$$
\begin{aligned}
\bar{M}_t^f &= \langle X_t, f\rangle - \langle X_0, f\rangle - \int_0^t \int_{\mathbb{R}} \tilde{\sigma}(x)\, D^\alpha f(x) X_s(dx)ds \\
&\quad - \int_0^t \int_{\mathbb{R}} f(x)\left(b(x, V * X_s(x)) - d(x, U * X_s(x))\right) X_s(dx)ds \qquad (3.8)
\end{aligned}
$$

is a continuous martingale with the quadratic variation

$$
\langle \bar{M}^f\rangle_t = 2\int_0^t \int_{\mathbb{R}} r(x) f^2(x) X_s(dx)ds. \qquad (3.9)
$$

(ii) Assume moreover that $\hat{\sigma}$ is Lipschitz continuous and that r is bounded from below by a positive constant. Then there is a unique such limiting superprocess.

Remark 3.4. Here again, as in the case of Theorem 3.2, Theorem 3.3 yields the existence of a measure-valued process X which is a weak solution of the stochastic partial differential equation with fractional diffusion operator

$$
\partial_t X_t(x) = \Big(b(x, V * X_t(x)) - d(x, U * X_t(x))\Big) X_t(x) + D^\alpha(\tilde{\sigma}\, X_t)(x) + \dot{M}_t,
$$

where $\dot{M}_t$ is a random fluctuation term reflecting the demographic stochasticity of this fast birth-and-death process; the process is faster than the accelerated birth-and-death process which led to the deterministic reaction-diffusion approximation (3.4).

4 Auxiliary Lemmas

In this section we provide auxiliary lemmas needed in the proofs of Theorems 3.2, and 3.3. The latter will be completed in Section 5. We begin with a lemma which gives uniform estimates for the moments of the population process. Its proof can be easily adapted from the the proof of an analogous result in [15] and is thus omitted.

Lemma 4.1 *Assume that $p \geq 2$, and $\sup_K E(\langle \nu_0^K, 1\rangle^p) < +\infty$. Then*

$$
\sup_K \left(E\Big(\sup_{t\in[0,T]} \langle \nu_t^K, 1\rangle^p \Big) \right) < \infty. \qquad (4.1)
$$

To provide mass control for ν^K, we first need to study the action of the fractional Laplacian on functions

$$
f_n(x) = \psi(0 \vee (|x| - (n-1)) \wedge 1), \qquad n \in \mathbb{N},
$$

where the function $\psi(x) = 6x^5 - 15x^4 + 10x^3 \in C^2$. Observe that ψ is non-decreasing on $[0, 1]$, and such that

$$
\psi(0) = \psi'(0) = \psi''(0) = 1 - \psi(1) = \psi'(1) = \psi''(1) = 0.
$$

Lemma 4.2 *For each $n \in \mathbb{N}^*$, the function f_n is in C^2, even, non-decreasing on $\mathbb{R}_+$, equal to 0 on $[-(n-1), n-1]$, and to 1 on $(-n,n)^c$. In particular $f_0 \equiv 1$. Moreover,*

$$\sup_{n\in\mathbb{N}^*, x\in\mathbb{R}} |D^\alpha f_n(x)| < +\infty,$$

and, for each $n \in \mathbb{N}^*$, and $x \in (-n+1, n-1)$,

$$|D^\alpha f_n(x)| \leq \frac{2}{\alpha(n-1-|x|)^\alpha}.$$

Finally, under Assumption (H2),

$$\lim_{K\to\infty} K^\eta \int_{\mathbb{R}} (f_n(x+h) - f_n(x)) M_K(x, dh) = \sigma(x) D^\alpha f_n(x),$$

uniformly, for $n \in \mathbb{N}^*$, and $x \in \mathbb{R}$.

Proof . For $n \in \mathbb{N}^*$, let us check the three last statements for f_n, the others being obvious. Using the Taylor expansion for $|y| \leq 1$ and remarking that, for each $x, z \in \mathbb{R}$, $|f_n(z) - f_n(x)| \leq 1$, one has

$$\begin{aligned}
|D^\alpha f_n(x)| &\leq \int_{|y|\leq 1} \frac{\sup_{z\in[0,1]} |\psi''(z)| y^2}{2} \times \frac{dy}{|y|^{1+\alpha}} + \int_{|y|\geq 1} \frac{dy}{|y|^{1+\alpha}} \\
&\leq \frac{\sup_{z\in[0,1]} |\psi''(z)|}{2-\alpha} + \frac{2}{\alpha}.
\end{aligned}$$

For $x \in (-n+1, n-1)$, since $f_n(x) = f_n'(x) = 0$, and $f_n(x+y) = 0$ if $|y| \leq n-1-|x|$,

$$|D^\alpha f_n(x)| = \left| \int_{\mathbb{R}} \frac{f_n(x+y)dy}{|y|^{1+\alpha}} \right| \leq \int_{|y|>n-1-|x|} \frac{dy}{|y|^{1+\alpha}} = \frac{2}{\alpha(n-1-|x|)^\alpha}.$$

Since $\sup_{n\in\mathbb{N}^*, x\in\mathbb{R}} |f_n''(x)| = \sup_{x\in[0,1]} |\psi''(x)| < +\infty$, writing (3.1) for the function f_n we see that to establish the uniform convergence we only need to check that $\sup_{n\in\mathbb{N}^*, x\in\mathbb{R}} \int_1^{+\infty} |f_n'(x+z) - f_n'(x-z)| \frac{dz}{2z^\alpha} < +\infty$. Since $\sup_{n\in\mathbb{N}^*, x\in\mathbb{R}} |f_n'(x)| = \sup_{x\in[0,1]} |\psi'(x)| < +\infty$, and f_n' vanishes outside $[-n, -n+1] \cup [n-1, n]$,

$$\sup_{n\in\mathbb{N}^*, x\in\mathbb{R}} \int_1^{+\infty} |f_n'(x+z) - f_n'(x-z)| \frac{dz}{2z^\alpha} \leq \sup_{x\in[0,1]} |\psi'(x)| \int_1^2 \frac{dz}{z^\alpha},$$

which concludes the proof. ∎

The next lemma provides mass control for the sequence (ν^K).

Lemma 4.3 *Under the assumptions of Theorem 3.2 or Theorem 3.3,*

$$\lim_{n\to\infty} \limsup_{K\to\infty} E\left(\sup_{t\leq T} \langle \nu_t^K, f_n \rangle \right) = 0.$$

Proof . Let $M_t^{K,n}$ denote the square integrable martingale defined by (2.9) with f_n replacing f. The boundedness of r and σ together with the last assertion in Lemma 4.2 ensures the existence of a sequence $(\varepsilon_K)_K$ converging to 0 such that

$$\langle \nu_t^K, f_n\rangle \leq \langle \nu_0^K, f_n\rangle + M_t^{K,n} + \bar{b}\int_0^t \langle \nu_s^K, f_n\rangle ds + \varepsilon_K \int_0^t \langle \nu_s^K, 1\rangle ds \\ + \int_0^t \int_{\mathbb{R}} r(x)p(x)\sigma(x)D^\alpha f_n(x)\nu_s^K(dx)ds. \quad (4.2)$$

For $n \geq m > 1$, splitting $\mathbb{R}$ into $(-n+m, n-m)$ and its complement, and using Lemma 4.2, we get

$$\int_{\mathbb{R}} r(x)p(x)\sigma(x)D^\alpha f_n(x)\nu_s^K(dx) \\ \leq \bar{r}\sup_{x\in\mathbb{R}}\sigma(x)\left(\sup_{l,x}|D^\alpha f_l(x)|\langle \nu_s^K, f_{n-m}\rangle + \frac{2}{\alpha(m-1)^\alpha}\langle \nu_s^K, 1\rangle\right). \quad (4.3)$$

Since the sequence $(f_n)_n$ is non-increasing, $\langle \nu_s^K, f_n\rangle \leq \langle \nu_s^K, f_{n-m}\rangle$, and there is a constant C not depending on $n \geq 2$ and $m \in \{2, \ldots, n\}$, and a sequence $(\eta_m)_{m\geq 2}$ of positive numbers converging to 0 such that

$$\langle \nu_t^K, f_n\rangle \leq \langle \nu_0^K, f_n\rangle + M_t^{K,n} + C\int_0^t \langle \nu_s^K, f_{n-m}\rangle ds + (\varepsilon_K + \eta_m)\int_0^t \langle \nu_s^K, 1\rangle ds. \quad (4.4)$$

Let $\mu_t^{K,n} = E\left(\sup_{s\leq t}\langle \nu_s^K, f_n\rangle\right)$ and $\mu_t^K = E\left(\sup_{s\leq t}\langle \nu_s^K, 1\rangle\right)$ which is bounded uniformly in K and $t \in [0,T]$ since, using Lemma 4.1,

$$\sup_K E\big(\sup_{t\in[0,T]}\langle \nu_t^K, 1\rangle^3\big) < \infty. \quad (4.5)$$

Assume, first, that $0 < \eta < 1$. Observing that, in view of (2.10), $\langle M^{K,n}\rangle_t \leq CK^{\eta-1}\int_0^t \langle \nu_s^K, 1\rangle + \langle \nu_s^K, 1\rangle^2 ds$, and using Doob's inequality and (4.5), one deduces that

$$\mu_t^{K,n} \leq \mu_0^{K,n} + C\int_0^t \mu_s^{K,n-m}ds + \varepsilon_K + \eta_m \quad (4.6)$$

for the modified sequences $(\varepsilon_K)_K$ and $(\eta_m)_m$ that still converge to 0, as K and m, respectively, grow to ∞. Iterating this inequality yields, for $j \in \mathbb{N}^*$, and $m > 1$,

$$\mu_T^{K,jm} \leq \sum_{l=0}^{j-1}\mu_0^{K,(j-l)m}\frac{(CT)^l}{l!} + \frac{C^jT^{j-1}\int_0^T \mu_s^K ds}{(j-1)!} + (\varepsilon_K + \eta_m)\sum_{l=0}^{j-1}\frac{(CT)^l}{l!} \\ \leq \mu_0^{K,\lfloor j/2\rfloor m}e^{CT} + E\left(\langle \nu_0^K, 1\rangle\right)\sum_{l=\lfloor (j+3)/2\rfloor}^{+\infty}\frac{(CT)^l}{l!} + \frac{C'(CT)^j}{(j-1)!} + (\varepsilon_K + \eta_m)e^{CT},$$

where we used the monotonicity of $\mu_0^{K,n}$ w.r.t. n, and (4.5) to justify the second inequality.

The random variables $(\langle \nu_0^K, f_n\rangle)_K$ converge in law to $\langle \xi_0, f_n\rangle$, as $K\to\infty$, and are uniformly integrable according to (3.2). Therefore, for a fixed n, $\mu_0^{K,n}$ converges to $\langle \xi_0, f_n\rangle$, as $K\to\infty$. Hence,

$$\limsup_{K\to\infty} \mu_T^{K,jm} \leq \langle \xi_0, f_{\lfloor j/2\rfloor m}\rangle e^{CT} + \sup_K E\left(\langle \nu_0^K, 1\rangle\right) \sum_{l=\lfloor (j+3)/2\rfloor}^{+\infty} \frac{(CT)^l}{l!} + \frac{C'(CT)^j}{(j-1)!} + \eta_m e^{CT}.$$

For m, j large enough, the right-hand-side is arbitrarily small. Since $n \mapsto \mu_T^{K,n}$ is non-increasing, the proof is complete in the case $0 < \eta < 1$.

When $\eta = 1$, $\sup_n E(\sup_{t\leq T} \langle M^{K,n}\rangle_t)$ does not vanish anymore as $K\to\infty$, and the proof cannot be concluded in the same way as above. But, taking expectations in (4.4) we obtain that (4.6) holds with $\sup_{s\leq t} E(\langle \nu_s^K, f_n\rangle)$, $E(\langle \nu_0^K, f_n\rangle)$ and $\sup_{r\leq s} E(\langle \nu_r^K, f_{n-m}\rangle)$, replacing, respectively, $\mu_t^{K,n}$, $\mu_0^{K,n}$, and $\mu_s^{K,n-m}$. Following the previous line of reasoning we obtain that

$$\lim_{n\to\infty} \limsup_{K\to\infty} \sup_{t\leq T} E(\langle \nu_t^K, f_n\rangle) = 0.$$

Now, in view of (2.10), and because $f_n^2 \leq f_n$, one gets

$$\langle M^{K,n}\rangle_t \leq \bar{r} \int_0^t \left(2\langle \nu_s^K, f_n\rangle + \int_{\mathbb{R}^2} f_n(x+h) M_K(x, dh) \nu_s^K(dx)\right) ds + \varepsilon_K,$$

with $(\varepsilon_K)_K$ tending to 0 as K converges to infinity. Since by (2.6), for K large enough,

$$\begin{aligned}\int_{\mathbb{R}} f_n(x+h) M_K(x, dh) &= E\left(f_n\Big(x + \frac{X(x)}{K^{1/\alpha}}\Big)\right) \leq f_{n-1}(x) + P\Big(\frac{X(x)}{K^{1/\alpha}} > 1\Big) \\ &\leq f_{n-1}(x) + \frac{1}{K}\Big(\frac{2\sup_{y\in\mathbb{R}} \sigma(y)}{\alpha} + 1\Big),\end{aligned}$$

one deduces that $\lim_{n\to\infty} \limsup_{K\to\infty} E(\langle M^{K,n}\rangle_T) = 0$. Taking advantage of (4.4) and Doob's inequality one checks that (4.6) holds with ε_K replaced by $\varepsilon_{K,n}$ such that $\lim_{n\to\infty}\limsup_{K\to\infty} \varepsilon_{K,n} = 0$ on the right-hand-side. Then one easily adapts the end of the proof written for $\eta < 1$ to obtain the desired conclusion. ∎

Remark 4.4 *In the case of a mutation kernel with finite second order moment as in [9], the true Laplacian being a local operator,* (4.3) *may be replaced by*

$$\int_{\mathbb{R}} r(x)p(x)\sigma(x) f_n''(x) \nu_s^K(dx) \leq \bar{r} \sup_{x\in\mathbb{R}} \sigma(x) \sup_{l,x} |f_l''(x)|\, \langle \nu_s^K, f_{n-1}\rangle.$$

The conclusion of Lemma 4.3 may be derived by a similar argument.

Now, let $N(dt,dh)$ be a Poisson random measure on $\mathbb{R}_+ \times (-1,1)$ with intensity $dt\frac{dh}{|h|^{1+\alpha}}$. The process $Z_t = \int_{(0,t]\times(-1,1)} h\Big(N(dt,dh) - \frac{dh}{|h|^{1+\alpha}}\Big)$ is a Lévy process such that, for all $p \geq 1$, $E(|Z_1|^p) < +\infty$. Let X_t^x denote the solution of the stochastic differential equation

$$dX_t^x = \hat{\sigma}(X_{t^-}^x)dZ_t,\ X_0^x = x \tag{4.7}$$

which admits a unique solution when $\hat{\sigma}$ is Lipschitz continuous according to Theorem 7, p.259, in [22]. Let us denote by P_t the associated semigroup defined for all measurable and bounded $f : \mathbb{R} \to \mathbb{R}$ by the formula $P_t f(x) = E(f(X_t^x))$. The infinitesimal generator of the process X^x is

$$Lf(x) = \int_{(-1,1)} \Big(f(x+\hat{\sigma}(x)h) - f(x) - f'(x)\hat{\sigma}(x)h\Big)\frac{dh}{|h|^{1+\alpha}}. \tag{4.8}$$

Lemma 4.5 *Assume that $\hat{\sigma}$ is C^2 with a bounded first-order derivative, and a bounded, and locally Lipschitz second-order derivative. For $f \in C_b^2(\mathbb{R})$, $P_t f(x)$ belongs to $C_b^{1,2}([0,T]\times\mathbb{R})$, and solves the initial-value problem*

$$\begin{cases} \partial_t P_t f(x) = LP_t f(x),\ (t,x) \in [0,T]\times\mathbb{R} \\ P_0 f(x) = f(x),\ x \in \mathbb{R} \end{cases}.$$

Proof . By Theorem 40, p.317. in [22], the mapping $x \mapsto X_t^x$ is twice continuously differentiable with first, and second order derivatives solving, respectively, the equations

$$d\partial_x X_t^x = \hat{\sigma}'(X_{t^-}^x)\partial_x X_{t^-}^x dZ_t,\ \partial_x X_0^x = 1,$$

and

$$d\partial_{xx} X_t^x = \hat{\sigma}'(X_{t^-}^x)\partial_{xx} X_{t^-}^x dZ_t + \hat{\sigma}''(X_{t^-}^x)(\partial_x X_{t^-}^x)^2 dZ_t,\ \partial_{xx} X_0^x = 0.$$

For $q \geq 1$, since $E(|Z_T|^q) < +\infty$, by Theorem 66, p.346, in [22], there is a finite constant K such that, for any predictable process $(H_t)_{t\leq T}$, and any $t \leq T$,

$$E\left(\sup_{s\leq t}\left|\int_0^s H_r dZ_r\right|^q\right) \leq K\int_0^t E(|H_s|^q)ds. \tag{4.9}$$

This, combined with the regularity assumptions made on $\hat{\sigma}$, and Gronwall's Lemma, immediately implies that, for any $q \geq 1$,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} E(|\partial_x X_t^x|^q + |\partial_{xx} X_t^x|^q) < +\infty. \tag{4.10}$$

For $f \in C_b^2(\mathbb{R})$, one deduces that the mapping $x \mapsto P_t f(x) \in C_b^2(\mathbb{R})$, with $\partial_x P_t f(x) = E(f'(X_t^x)\partial_x X_t^x)$ and $\partial_{xx} P_t f(x) = E(f''(X_t^x)(\partial_x X_t^x)^2 + f'(X_t^x)\partial_{xx} X_t^x)$. Indeed, for instance, to differentiate for the second time under the expectation, we observe that, as y tends to x, the random variables $(f'(X_t^y)\partial_x X_t^y - f'(X_t^x)\partial_x X_t^x)/(y-x)$ converge a.s. to $f''(X_t^x)(\partial_x X_t^x)^2 + f'(X_t^x)\partial_{xx}X_t^x$, and by (4.10) are uniformly integrable since

$$\left|\frac{f'(X_t^y)\partial_x X_t^y - f'(X_t^x)\partial_x X_t^x}{y-x}\right| \leq \left|f'(X_t^y)\frac{\partial_x X_t^y - \partial_x X_t^x}{y-x}\right| + \left|\frac{f'(X_t^y) - f'(X_t^x)}{y-x}\right| |\partial_x X_t^x|$$
$$\leq \frac{\|f'\|_\infty}{y-x}\int_x^y |\partial_{xx}X_t^z|dz + |\partial_x X_t^x|\frac{\|f''\|_\infty}{y-x}\int_x^y |\partial_x X_t^z|dz.$$

Moreover, since $f(X_t^x)$, $f'(X_t^x)\partial_x X_t^x$, and $f''(X_t^x)(\partial_x X_t^x)^2 + f'(X_t^x)\partial_{xx}X_t^x$, are continuous w.r.t. x, and right-continuous and quasi left-continuous w.r.t. t, one deduces that the mapping $(t,x) \mapsto (P_t f(x), \partial_x P_t f(x), \partial_{xx}P_t f(x))$ is continuous and bounded on $[0,T]\times\mathbb{R}$. With the upper-bound,

$$|P_t f(x+\hat\sigma(x)h) - P_t f(x) - \partial_x P_t f(x)\hat\sigma(x)h| \leq \frac{1}{2}\|\partial_{xx}P_t f\|_\infty \hat\sigma^2(x)h^2,$$

one concludes that the mapping $(t,x)\mapsto LP_t f(x)$ is continuous and bounded on $[0,T]\times\mathbb{R}$. For $u>0$, by the Markov property stated in Theorem 32, p.300, of [22], $P_{t+u}f(x) = E(P_t f(X_u^x))$. By Itô's formula,

$$P_t f(X_u^x) = P_t f(x) + \int_0^u LP_t f(X_s^x)ds$$
$$+ \int_{(0,u]\times(-1,1)} (P_t f(X_{s^-}^x + \hat\sigma(X_{s^-}^x)h) - P_t f(X_{s^-}^x))\,(N(ds,dh) - \frac{dh}{|h|^{1+\alpha}}).$$

Since the last integral is a martingale, it is centered and one obtains that

$$\frac{P_{t+u}f(x) - P_t f(x)}{u} = E\left(\frac{1}{u}\int_0^u LP_t f(X_s^x)ds\right).$$

By Lebesgue's Theorem, one deduces that $\lim_{u\to 0^+}\frac{P_{t+u}f(x)-P_t f(x)}{u} = LP_t f(x)$. Hence $(t,x)\mapsto P_t f(x)$ belongs to $C^{1,2}([0,T]\times\mathbb{R})$ and solves the initial-value problem

$$\begin{cases}\partial_t P_t f(x) = LP_t f(x),\ (t,x)\in[0,T]\times\mathbb{R}\\ P_0 f(x) = f(x),\ x\in\mathbb{R}\end{cases}.$$

∎

Unfortunately, in the general case needed in the proofs of our theorems, $\hat\sigma$ is merely Lipschitz, and $P_{t-s}f(x)$ is not smooth enough in the spatial variable x. That is why, for $\varepsilon>0$, we set $\hat\sigma^\varepsilon(x) = \int_{\mathbb{R}}\hat\sigma(x-y)e^{-\frac{y^2}{2\varepsilon}}dy/\sqrt{2\pi\varepsilon}$, and define $X_t^{\varepsilon,x}$ as the solution of the SDE similar to (4.7), but with $\hat\sigma$ replaced by $\hat\sigma^\varepsilon$. The generator of $X_t^{\varepsilon,x}$ is the operator L^ϵ defined like L, but with $\hat\sigma^\varepsilon$ replacing $\hat\sigma$. Finally, we set $P_t^\varepsilon f(x) = E(f(X_t^{\varepsilon,x}))$. Now, we want to know what happens when ε tends to 0. The next lemma gives the Hölder's continuity of $\partial_x P_{t-s}^\varepsilon f$ and the order of convergence of $P_t^\varepsilon f(x)$ to $P_t f(x)$.

Lemma 4.6 *Assume that $\hat\sigma$ is Lipschitz, and $f \in C_b^2(\mathbb{R})$. Then there exists a constant $C > 0$, such that for all $t \in [0,T]$, and $x, y \in \mathbb{R}$ such that $|x-y| \le \|\hat\sigma\|_\infty$,*

$$|\partial_x P^\varepsilon_{t-s} f(x) - \partial_x P^\varepsilon_{t-s} f(y)| \le \frac{C|x-y|^{\alpha/2}}{\varepsilon^{\alpha/4}}.$$

Moreover,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} |P_t f(x) - P_t^\varepsilon f(x)| \le C\sqrt{\varepsilon}. \tag{4.11}$$

Proof . One has

$$\begin{aligned}\partial_x P_t^\varepsilon f(x) - \partial_x P_t^\varepsilon f(y) =& E\left(f'(X_t^{x,\varepsilon})(\partial_x X_t^{x,\varepsilon} - \partial_x X_t^{y,\varepsilon})\right)\\ &+ \int_y^x E\left(f''(X_t^{z,\varepsilon})\partial_x X_t^{z,\varepsilon}\partial_x X_t^{y,\varepsilon}\right) dz.\end{aligned}$$

The absolute value of the second term of the r.h.s. is smaller than $C|y-x|$ since, for each $q \ge 1$,

$$\sup_{\varepsilon>0}\ \sup_{(t,x)\in[0,T]\times\mathbb{R}} E(|\partial_x X_t^{x,\varepsilon}|^q) < +\infty, \tag{4.12}$$

because $\sup_{\varepsilon>0} \|\hat\sigma^{\varepsilon\,\prime}\|_\infty$ is not greater than the Lipschitz constant of $\hat\sigma$. To deal with the first term, we remark that

$$\begin{aligned}\partial_x X_t^{x,\varepsilon} - \partial_x X_t^{y,\varepsilon} =& \int_0^t \hat\sigma^{\varepsilon\,\prime}(X_{s^-}^{x,\varepsilon})(\partial_x X_{s^-}^{x,\varepsilon} - \partial_x X_{s^-}^{y,\varepsilon}) dZ_s\\ &+ \int_0^t (\hat\sigma^{\varepsilon\,\prime}(X_{s^-}^{x,\varepsilon}) - \hat\sigma^{\varepsilon\,\prime}(X_{s^-}^{y,\varepsilon}))\partial_x X_{s^-}^{y,\varepsilon} dZ_s\end{aligned}$$

Using (4.9), and the inequality $\|\hat\sigma^{\varepsilon\,\prime\prime}\|_\infty \le \frac{C}{\sqrt{\varepsilon}}$, one deduces that

$$\begin{aligned}E\left(\sup_{s\le t}(\partial_x X_s^{x,\varepsilon} - \partial_x X_s^{y,\varepsilon})^2\right) \le& C\int_0^t E((\partial_x X_s^{x,\varepsilon} - \partial_x X_s^{y,\varepsilon})^2) ds\\ &+ \frac{C(x-y)}{\varepsilon}\int_0^t\int_y^x E\left((\partial_x X_s^{z,\varepsilon})^2(\partial_x X_s^{y,\varepsilon})^2\right) dzds.\end{aligned}$$

In view of (4.12), and Gronwall's Lemma, one concludes that

$$E\left(\sup_{s\le T}(\partial_x X_s^{x,\varepsilon} - \partial_x X_s^{y,\varepsilon})^2\right) \le \frac{C(x-y)^2}{\varepsilon}. \tag{4.13}$$

Combining this bound with the inequality

$$|E\left(f'(X_t^{x,\varepsilon})(\partial_x X_t^{x,\varepsilon} - \partial_x X_t^{y,\varepsilon})\right)| \le CE\left(|\partial_x X_t^{x,\varepsilon} - \partial_x X_t^{y,\varepsilon}|^{\frac{\alpha}{2}}(|\partial_x X_t^{x,\varepsilon}| + |\partial_x X_t^{y,\varepsilon}|)^{1-\frac{\alpha}{2}}\right),$$

and Hölder's inequality, one easily deduces the first statement of the Lemma. Since $\|\hat\sigma - \hat\sigma^\varepsilon\|_\infty \le C\sqrt{\varepsilon}$, by a reasoning similar to the one employed to prove (4.13), one easily checks that $E\left(\sup_{s\le T}(X_s^x - X_s^{x,\varepsilon})^2\right) \le C\varepsilon$ and the second statement follows. ∎

5 Proofs of the Theorems

Proof of Theorem 3.2 The proof of the theorem will be carried out in five steps. Let us fix $T > 0$, and $\eta < 1$.

Step 1 We first endow M_F with the vague topology. To show the tightness of the sequence of laws $Q^K = \mathcal{L}(\nu^K)$ in $\mathcal{P}(\mathbb{D}([0,T],(M_F,v)))$, where M_F is endowed with the vague convergence topology, it suffices, following Roelly [23], to show that for any continuous bounded function f on $\mathbb{R}$ the sequence of laws of the processes $\langle \nu^K, f\rangle$ is tight in $\mathbb{D}([0,T],\mathbb{R})$. To this end we use the Aldous criterion [1] and the Rebolledo criterion (see [18]) which require us to show that

$$\sup_K E\big(\sup_{t\in[0,T]} |\langle \nu_t^K, f\rangle|\big) < \infty, \tag{5.1}$$

and the laws, respectively, of the predictable quadratic variation of the martingale part, and of the drift part of the semimartingales $\langle \nu^K, f\rangle$, are tight.

Since f is bounded, (5.1) is a consequence of (4.5): let us thus consider a pair (S, S') of stopping times satisfying a.s. the inequality $0 \leq S \leq S' \leq S+\delta \leq T$. Using (2.10) and (4.5), for some positive real numbers C and C', we get

$$E\left(\langle M^{K,f}\rangle_{S'} - \langle M^{K,f}\rangle_S\right) \leq CE\left(\int_S^{S+\delta} \left(\langle \nu_s^K, 1\rangle + \langle \nu_s^K, 1\rangle^2\right) ds\right) \leq C'\delta.$$

In a similar way, we show that the expectation of the finite variation part of $\langle \nu_{S'}^K, f\rangle - \langle \nu_S^K, f\rangle$ is bounded by $C'\delta$. Hence, the sequence $Q^K = \mathcal{L}(\nu^K)$ is tight in $\mathcal{P}(\mathbb{D}([0,T],(M_F,v)))$.

Step 2 Let us now denote by Q the weak limit in $\mathcal{P}(\mathbb{D}([0,T],(M_F,v)))$ of a subsequence of (Q^K) which we also denote (Q^K). We remark that by construction,

$$\sup_{t\in[0,T]} \sup_{f\in L^\infty(\mathbb{R}), ||f||_\infty \leq 1} |\langle \nu_t^K, f\rangle - \langle \nu_{t^-}^K, f\rangle| \leq 1/K.$$

Since, for each f in a countable measure-determining set of continuous and compactly supported functions on $\mathbb{R}$, the mapping $\nu \mapsto \sup_{t\leq T} |\langle \nu_t, f\rangle - \langle \nu_{t-}, f\rangle|$ is continuous on $\mathbb{D}([0,T],(M_F,v))$, one deduces that Q only charges the continuous processes from $[0,T]$ into (M_F,v). Let us now endow M_F with the weak convergence topology and check that Q only charges the continuous processes from $[0,T]$ into (M_F,w), and that the sequence (Q^K) in $\mathcal{P}(\mathbb{D}([0,T],(M_F,w))$ converges weakly to Q. For this purpose, we need to control the behavior of the total mass of the measures. We will employ the sequence (f_n) of smooth functions introduced in Lemma 4.3 which approximate the functions $\mathbf{1}_{\{|x|\geq n\}}$. For each $n \in \mathbb{N}$, the continuous and compactly supported functions $(f_{n,l} \overset{\text{def}}{=} f_n(1-f_l))_{l\in\mathbb{N}}$ increase to f_n, as $l \to \infty$. Continuity of the mapping $\nu \mapsto \sup_{t\leq T}\langle \nu_t, f_{n,l}\rangle$ on $\mathbb{D}([0,T],(M_F,v))$, and its uniform integrability deduced from (4.5), imply the bound

$$E^Q\left(\sup_{t\le T}\langle\nu_t,f_{n,l}\rangle\right)=\lim_{K\to\infty}E\left(\sup_{t\le T}\langle\nu_t^K,f_{n,l}\rangle\right)\le\liminf_{K\to\infty}E\left(\sup_{t\le T}\langle\nu_t^K,f_n\rangle\right).$$

Taking the limit, $l\to\infty$, in the left-hand-side, in view of the monotone convergence theorem and respectively, (4.5) and Lemma 4.3, one concludes that for $n=0$,

$$E^Q\left(\sup_{t\le T}\langle\nu_t,1\rangle\right)=E^Q\left(\sup_{t\le T}\langle\nu_t,f_0\rangle\right)<+\infty \tag{5.2}$$

and for general n,

$$\lim_{n\to\infty}E^Q\left(\sup_{t\le T}\langle\nu_t,f_n\rangle\right)=0. \tag{5.3}$$

As a consequence one may extract a subsequence of the sequence $(\sup_{t\le T}\langle\nu_t,f_n\rangle)_n$ that converges a.s. to 0 under Q, and the set $(\nu_t)_{t\le T}$ is tight Q-a.s. Since Q only charges the continuous processes from $[0,T]$ into (M_F,v), one deduces that Q also only charges the continuous processes from $[0,T]$ into (M_F,w).

Let X denote a process with law Q. According to Méléard and Roelly [21], to prove that the sequence (Q^K) converges weakly to Q in $\mathcal{P}(\mathbb{D}([0,T],(M_F,w))$, it is sufficient to check that the processes $(\langle\nu^K,1\rangle=(\langle\nu_t^K,1\rangle)_{t\le T})_K$ converge in law to $\langle X,1\rangle\overset{\text{def}}{=}(\langle X_t,1\rangle)_{t\le T}$ in $\mathbb{D}([0,T],\mathbb{R})$. For a Lipschitz continuous and bounded function F from $\mathbb{D}([0,T],\mathbb{R})$ to $\mathbb{R}$, we have

$$\begin{aligned}\limsup_{K\to\infty}|E(F(\langle\nu^K,1\rangle)-F(\langle X,1\rangle)|&\le\limsup_{n\to\infty}\limsup_{K\to\infty}|E(F(\langle\nu^K,1\rangle)-F(\langle\nu^K,1-f_n\rangle))|\\&+\limsup_{n\to\infty}\limsup_{K\to\infty}|E(F(\langle\nu^K,1-f_n\rangle)-F(\langle X,1-f_n\rangle))|\\&+\limsup_{n\to\infty}|E(F(\langle X,1-f_n\rangle)-F(\langle X,1\rangle))|.\end{aligned}$$

Since $|F(\langle\nu,1-f_n\rangle)-F(\nu,1\rangle)|\le C\sup_{t\le T}\langle\nu_t,f_n\rangle$, Lemma 4.3 and (5.3) respectively imply that the first and the third terms in the r.h.s. are equal to 0. The second term is 0 in view of the continuity of the mapping $\nu\mapsto\langle\nu,1-f_n\rangle$ in $\mathbb{D}([0,T],(M_F,w))$.

Step 3 Recall that the time $T>0$ is fixed, and $0<\eta<1$. Let us check that, almost surely, the process X solves (3.3). By (5.2), for each T, $\sup_{t\in[0,T]}\langle X_t,1\rangle$ is finite a.s. Now, we fix a function $f\in C_b^2(\mathbb{R})$, compactly supported if $\alpha\le1$ (in this case the extension of (3.3) to any function f in C_b^2 follows by Lebesgue's theorem), and a $t\le T$.
For $\nu\in\mathbb{D}([0,T],(M_F,w))$, denote

$$\begin{aligned}\Psi_t^1(\nu)&=\langle\nu_t,f\rangle-\langle\nu_0,f\rangle-\int_0^t\int_{\mathbb{R}}(b(x,V*\nu_s(x))-d(x,U*\nu_s(x)))f(x)\nu_s(dx)ds,\\\Psi_t^2(\nu)&=-\int_0^t\int_{\mathbb{R}}\tilde\sigma(x)D^\alpha f(x)\nu_s(dx)ds.\end{aligned} \tag{5.4}$$

We must show that

$$E^Q\left(|\Psi_t^1(X)+\Psi_t^2(X)|\right)=0. \tag{5.5}$$

By (2.9), we know that, for each K,

$$M_t^{K,f}=\Psi_t^1(\nu^K)+\Psi_t^{2,K}(\nu^K),$$

where

$$\Psi_t^{2,K}(\nu^K)=-\int_0^t\int_{\mathbb{R}}p(x)(K^\eta r(x)+b(x,V*\nu_s^K(x)))\\ \times\left(\int_{\mathbb{R}}f(x+h)M_K(x,h)dh-f(x)\right)\nu_s^K(dx)ds.$$

Moreover, (4.5) implies that for each K,

$$E\left(|M_t^{K,f}|^2\right)=E\left(\langle M^{K,f}\rangle_t\right)\leq\frac{C_fK^\eta}{K}E\left(\int_0^t\left\{\langle\nu_s^K,1\rangle+\langle\nu_s^K,1\rangle^2\right\}ds\right)\leq\frac{C_{f,T}K^\eta}{K}, \tag{5.6}$$

which goes to 0 as K tends to infinity, since $0<\eta<1$. Since, by Proposition 3.1, there exists a deterministic sequence (ε_K), converging to 0 as $K\to\infty$, such that

$$|\Psi_t^{2,K}(\nu^K)-\Psi_t^2(\nu^K)|\leq\varepsilon_K\int_0^t\langle\nu_s^K,1\rangle ds,$$

one deduces that

$$\lim_K E(|\Psi_t^1(\nu^K)+\Psi_t^2(\nu^K)|)=0.$$

Since $X\in C([0,T],(M_F,w))$ and $f\in C_b^2(\mathbb{R})$, due to the continuity of the parameters, the functions Ψ_t^1, and Ψ_t^2 are a.s. continuous at X. Furthermore, for any $\nu\in\mathbb{D}([0,T],M_F)$,

$$|\Psi_t^1(\nu)+\Psi_t^2(\nu)|\leq C_f\left(\langle\nu_t,1\rangle+\langle\nu_0,1\rangle+\int_0^t\langle\nu_s,1\rangle+\langle\nu_s,1\rangle^2ds\right). \tag{5.7}$$

Hence, in view of (4.5), the sequence $(\Psi_t^1(\nu^K)+\Psi_t^2(\nu^K))_K$ is uniformly integrable, and thus

$$\lim_K E\left(|\Psi_t^1(\nu^K)+\Psi_t^2(\nu^K)|\right)=E\left(|\Psi_t^1(X)+\Psi_t^2(X)|\right)=0, \tag{5.8}$$

which concludes the proof of the first part of Theorem 3.2.

Step 4 In this step we will prove part *(ii)* of Theorem 3.2 which asserts uniqueness of the solution of (3.3) under the additional assumption that $\hat\sigma$ is Lipschitz. According to (3.5), one has

$$\tilde\sigma(x)D^\alpha f(x)=Lf(x)+\int_{\mathbb{R}\setminus(-1,1)}\left(f(x+\hat\sigma(x)h)-f(x)\right)\frac{dh}{|h|^{1+\alpha}},$$

where L has been defined in (4.8). It is easy to prove that if ξ is a solution of (3.3) satisfying $\sup_{t\in[0,T]}\langle \xi_t, 1\rangle < \infty$, then, for each test function $\psi_t(x) = \psi(t,x) \in C_b^{1,2}(\mathbb{R}_+\times\mathbb{R})$, one gets

$$\begin{aligned}
\langle \xi_t, \psi_t\rangle = \langle \xi_0, \psi_0\rangle &+ \int_0^t \int_{\mathbb{R}} (\partial_s\psi(s,x) + \tilde{\sigma}(x) D^\alpha \psi_s(x))\xi_s(dx)ds \\
&+ \int_0^t \int_{\mathbb{R}} (b(x, V*\xi_s(x)) - d(x, U*\xi_s(x)))\psi(s,x)\xi_s(dx)ds \\
= \langle \xi_0, \psi_0\rangle &+ \int_0^t \int_{\mathbb{R}} (\partial_s\psi(s,x) + L\psi_s(x))\xi_s(dx)ds \\
+ \quad & \int_0^t \int_{\mathbb{R}} \Big((b(x, V*\xi_s(x)) - d(x, U*\xi_s(x)))\psi(s,x) \\
& \qquad + \int_{\mathbb{R}\setminus(-1,1)} (\psi(s, x+\hat{\sigma}(x)h) - \psi(s,x)) \frac{dh}{|h|^{1+\alpha}} \Big)\xi_s(dx)ds. \quad (5.9)
\end{aligned}$$

Let $t\in[0,T]$ and $f\in C_b^2(\mathbb{R})$. We would like to choose $\psi(s,x) = P_{t-s}f(x)$, where $P_tf(x) = E(f(X_t^x))$, and X_t^x is the unique solution of the stochastic differential equation (4.7), so that the second term in the right-hand-side above vanishes. In view of Lemma 4.5, this is immediately possible if $\hat{\sigma}$ is assumed to be C^2, with a bounded first order derivative, and a bounded and locally Lipschitz second order derivative.

Unfortunately, when $\hat{\sigma}$ is merely Lipschitz continuous, $P_{t-s}f(x)$ is not smooth enough in the spatial variable x. That is why, for $\varepsilon > 0$, we set $\hat{\sigma}^\varepsilon(x) = \int_{\mathbb{R}} \hat{\sigma}(x-y)e^{-\frac{y^2}{2\varepsilon}}\frac{dy}{\sqrt{2\pi\varepsilon}}$ and define $X_t^{\varepsilon,x}$ as the solution to the SDE similar to (4.7) but with $\hat{\sigma}$ replaced by $\hat{\sigma}^\varepsilon$. The generator of $X_t^{\varepsilon,x}$ is the operator L^ϵ defined like L but with $\hat{\sigma}^\varepsilon$ replacing $\hat{\sigma}$ and we set $P_t^\varepsilon f(x) = E(f(X_t^{\varepsilon,x}))$. According to Lemma 4.5, for the choice $\psi(s,x) = P_{t-s}^\varepsilon f(x)$, equation (5.9) takes the form

$$\begin{aligned}
\langle \xi_t, f\rangle = & \langle \xi_0, P_t^\varepsilon f\rangle + \int_0^t \int_{\mathbb{R}} (L - L^\varepsilon) P_{t-s}^\varepsilon f(x)\xi_s(dx)ds \\
& + \int_0^t \int_{\mathbb{R}} \Big((b(x, V*\xi_s(x)) - d(x, U*\xi_s(x)))P_{t-s}^\varepsilon f(x) \\
& + \int_{\mathbb{R}\setminus(-1,1)} \big(P_{t-s}^\varepsilon f(x+\hat{\sigma}(x)h) - P_{t-s}^\varepsilon f(x)\big) \frac{dh}{|h|^{1+\alpha}} \Big)\xi_s(dx)ds. \qquad (5.10)
\end{aligned}$$

One now whishes to make ε tend to 0. Since,

$$(L - L^\varepsilon)P_{t-s}^\varepsilon f(x) = \int_{(-1,1)} \int_{\hat{\sigma}^\varepsilon(x)h}^{\hat{\sigma}(x)h} \big(\partial_x P_{t-s}^\varepsilon f(x+y) - \partial_x P_{t-s}^\varepsilon f(x)\big)\, dy \frac{dh}{|h|^{1+\alpha}},$$

Lemma 4.6, and the estimation $\|\hat{\sigma} - \hat{\sigma}^\varepsilon\|_\infty \le C\sqrt{\varepsilon}$, imply that

$$|(L - L^\varepsilon)P_{t-s}^\varepsilon f(x)| \le C\frac{|\hat{\sigma}(x) - \hat{\sigma}^\varepsilon(x)|}{\varepsilon^{\alpha/4}} \int_{(-1,1)} \frac{dh}{|h|^{\alpha/2}} \le C\varepsilon^{(2-\alpha)/4}.$$

Letting $\varepsilon \to 0$ in (5.10), one concludes using (4.11) that

$$\langle \xi_t, f\rangle = \langle \xi_0, P_t f\rangle + \int_0^t \int_{\mathbb{R}} \Big((b(x, V * \xi_s(x)) - d(x, U * \xi_s(x)))P_{t-s}f(x) + \int_{\mathbb{R}\setminus(-1,1)} (P_{t-s}f(x+\hat{\sigma}(x)h) - P_{t-s}f(x)) \frac{dh}{|h|^{1+\alpha}} \Big) \xi_s(dx)ds. \tag{5.11}$$

Next, we consider the variation norm defined for $\mu_1, \mu_2 \in M_F$ as follows:

$$||\mu_1 - \mu_2|| = \sup_{f \in L^\infty(\mathbb{R}),\, ||f||_\infty \le 1} |\langle \mu_1 - \mu_2, f\rangle|. \tag{5.12}$$

By the Jordan-Hahn decomposition of the measure $\mu_1 - \mu_2$, there exists a Borel subset A of $\mathbb{R}$ such that $||\mu_1 - \mu_2|| = \langle \mu_1 - \mu_2, 1_A - 1_{\mathbb{R}\setminus A}\rangle$. In view of the inner regularity of the measure $\mu_1 + \mu_2$ there exists a closed set $B \subset A$ such that $\langle \mu_1 + \mu_2, A \setminus B\rangle$ is arbitrarily small. The function $f_k(x) = (1 - kd(x, B)) \vee (-1)$ is Lipschitz continuous and tends to $1_B - 1_{\mathbb{R}\setminus B}$, as k tends to ∞. By Lebesgue's theorem, $\langle \mu_1 - \mu_2, f_k(x)\rangle$ tends to $\langle \mu_1 - \mu_2, 1_B - 1_{\mathbb{R}\setminus B}\rangle$, as $k \to \infty$. Since

$$\big|||\mu_1 - \mu_2|| - \langle \mu_1 - \mu_2, 1_B - 1_{\mathbb{R}\setminus B}\rangle\big| = 2|\langle \mu_1 - \mu_2, A \setminus B\rangle| \le 2\langle \mu_1 + \mu_2, A \setminus B\rangle,$$

for B and k well chosen, $\langle \mu_1 - \mu_2, f_k\rangle$ is arbitrarily close to $||\mu_1 - \mu_2||$. Now, utilizing the convolution, f_k may be approximated by a sequence of C_b^2 functions globally bounded by 1 which converge uniformly on compact sets. Thus one deduces that $||\mu_1 - \mu_2|| = \sup_{f \in C_b^2(\mathbb{R}),\, ||f||_\infty \le 1} |\langle \mu_1 - \mu_2, f\rangle|$.

Now, we are ready to prove the uniqueness of a solution of (5.11). For two solutions $(\xi_t)_{t\ge0}$, and $(\bar{\xi}_t)_{t\ge0}$, of (5.11), such that $\sup_{t\in[0,T]} \langle \xi_t + \bar{\xi}_t, 1\rangle = A_T < +\infty$, and $f \in C_b^2(\mathbb{R})$ satisfying the condition $||f||_\infty \le 1$, one has

$$\begin{aligned} |\langle \xi_t - \bar{\xi}_t, f\rangle| \le \int_0^t \Big| \int_{\mathbb{R}} [\xi_s(dx) - \bar{\xi}_s(dx)] \Big(& (b(x, V * \xi_s(x)) - d(x, U * \xi_s(x)))P_{t-s}f(x) \\ & + \int_{|h|\ge1} (P_{t-s}f(x+\hat{\sigma}(x)h) - P_{t-s}f(x)) \frac{dh}{|h|^{1+\alpha}} \Big) \Big| ds \\ + \int_0^t \Big| \int_{\mathbb{R}} \bar{\xi}_s(dx)(b(x, V * \xi_s(x)) - b(x, V * \bar{\xi}_s(x))) \Big| ds \\ + \int_0^t \Big| \int_{\mathbb{R}} \bar{\xi}_s(dx)(d(x, U * \xi_s(x)) - d(x, U * \bar{\xi}_s(x)))P_{t-s}f(x) \Big| ds. \end{aligned} \tag{5.13}$$

Since $||f||_\infty \le 1$, then $||P_{t-s}f||_\infty \le 1$ and, for all $x \in \mathbb{R}$,

$$|(b(x, V * \xi_s(x)) - d(x, U * \xi_s(x)))P_{t-s}f(x)| \le \bar{b} + \bar{d}(1 + \bar{U}A_T),$$

and

$$\left| \int_{|h|\geq 1} (P_{t-s}f(x+\hat{\sigma}(x)h) - P_{t-s}f(x)) \frac{dh}{|h|^{1+\alpha}} \right| \leq \frac{4}{\alpha}.$$

Moreover, b and d are Lipschitz continuous in their second variable with respective constants K_b, and K_d. Thus we obtain from (5.13) that

$$| \langle \xi_t - \bar{\xi}_t, f \rangle | \leq \left[\bar{b} + \bar{d}(1+\bar{U}A_T) + \frac{4}{\alpha} + K_b A_T \bar{V} + K_d A_T \bar{U} \right] \int_0^t ||\xi_s - \bar{\xi}_s|| ds. \quad (5.14)$$

Taking the supremum over C_b^2 functions f bounded by 1, one obtains

$$||\xi_t - \bar{\xi}_t|| \leq \left[\bar{b} + \bar{d}(1+\bar{U}A_T) + \frac{4}{\alpha} + K_b A_T \bar{V} + K_d A_T \bar{U} \right] \int_0^t ||\xi_s - \bar{\xi}_s|| ds,$$

and uniqueness follows by an application of Gronwall's Lemma.

Step 5 In the final step we shall prove the existence of a density claimed in Part *(iii)*. The image of the Poisson random measure N on $\mathbb{R}_+ \times [-1,1]$ with intensity $dt\frac{dh}{|h|^{1+\alpha}}$, by the mapping $(t,h) \mapsto (t, \frac{\mathrm{sgn}(h)}{\alpha|h|^\alpha})$, is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R} \setminus [-\frac{1}{\alpha}, \frac{1}{\alpha}]$ with intensity $dtdz$ since, for $z = \frac{\mathrm{sgn}(h)}{\alpha|h|^\alpha}$, one has $dz = \frac{dh}{|h|^{1+\alpha}}$. Let us denote by $\tilde{\mu}(dt,dz)$ the associated compensated measure. The stochastic differential equation (4.7) can now be written in the form,

$$X_t^x = x + \int_{(0,t]\times\mathbb{R}\setminus[-\frac{1}{\alpha},\frac{1}{\alpha}]} c(X_{s^-}^x, z)\tilde{\mu}(ds,dz),$$

for $c(x,z) = \hat{\sigma}(x) \times \frac{\mathrm{sgn}(z)}{(\alpha|z|)^{\frac{1}{\alpha}}}$. When the strictly positive function $\hat{\sigma} \in C_b^3$, i.e., it is bounded together with its derivatives up to order 3, one may apply Theorem 2.14, p.11, [4], to deduce that, for $t \in (0,T]$, X_t^x admits a density $p_t(x,y)$ with respect to the Lebesgue measure on the real line. With (5.11), one deduces that, for $t > 0$, ξ_t admits a density equal to

$$y \mapsto \int_{\mathbb{R}} p_t(x,y)\xi_0(dx) + \int_0^t \int_{\mathbb{R}} (b(x, V*\xi_s(x)) - d(x, U*\xi_s(x)))p_{t-s}(x,y)\xi_s(dx)ds$$
$$+ \int_0^t \int_{\mathbb{R}} \int_{|h|\geq 1} (p_{t-s}(x+\hat{\sigma}(x)h, y) - p_{t-s}(x,y)) \frac{dh}{|h|^{1+\alpha}} \xi_s(dx)ds.$$

This completes the proof of Theorem 3.2. $\square$

Let us now turn to the proof of Theorem 3.3. In this case the allometric exponent $\eta = 1$.

Proof of Theorem 3.3 We will use a method similar to the one employed in the proof of Theorem 3.2. Actually, Steps 1, 2 and 3 are completely analogous and we omit them. Thus, we only have to prove the uniqueness (in law) of the solution of the martingale problem (3.7)–(3.9), and the fact that any accumulation point of the sequence of laws of ν^K is a solution of (3.7)–(3.9).

Uniqueness. The uniqueness in the general case can be deduced from the special case when $b = d = 0$ by using the Dawson-Girsanov transform for measure-valued processes (cf. Evans and Perkins [13] (Theorem 2.3)). Indeed,

$$E\left(\int_0^t \int_{\mathbb{R}^d} [b(x, V * X_s(x)) - d(x, U * X_s(x))]^2 X_s(dx)ds\right) < +\infty,$$

which allows us to use this transform.

In the case $b = d = 0$ the proof of uniqueness can be adapted from Fitzsimmons [14], Corollary 2.23. This proof is based on the identification of the Laplace transform of the process, using the extension of the martingale problem (3.8) to functions $\psi(s, x) = P_{t-s}f(x)$ with bounded functions f (Fitzsimmons [14] Proposition 2.13). For $\psi(s, x)$ being C^1 in time, and C^2 in x, (3.8) extends with an additional term $\partial_s\psi$ appearing in the drift part. As we have seen above, the function $P_{t-s}f(x)$ is not smooth enough. Thus we firstly apply (3.8) to $\psi(s, x) = P^{\varepsilon}_{t-s}f(x)$, as already done in the proof of Theorem 3.2, thanks to Lemma 4.5, and we make ε tend to 0 as in Lemma 4.6.

Identification of the limit. Fnally, let us identify the limit of the sequence of laws of ν^K as a solution of (3.7)–(3.9). Write $Q^K = \mathcal{L}(\nu^K)$, and denote by Q a limiting value in $\mathcal{P}(\mathbb{D}([0, T], (M_F, w))$ of a subsequence (denoted also Q^K), and by $X = (X_t)_{t\geq 0}$ a process with law Q. Because of Step 4, X belongs a.s. to $C([0, T], (M_F, w))$. We have to show that X satisfies the conditions (3.7), (3.8) and (3.9). First, note that (3.7) is straightforward from (3.6) and Lemma 4.1.

Next we show that for any function f in $C^2_b(\mathbb{R})$, compactly supported when $\alpha \leq 1$, the process $\bar{M}^f_t$ defined by (3.8) is a martingale (the extension to every function in C^2_b is not hard). Indeed, consider $0 \leq s_1 \leq ... \leq s_n < s < t$, and continuous bounded maps $\phi_1, ...\phi_n$ on M_F. Our goal is to prove that, if the function Ψ from $\mathbb{D}([0, T], M_F)$ into $\mathbb{R}$ is defined by the expression

$$\Psi(\nu) = \prod_{k=1}^n \phi_k(\nu_{s_k})\Big\{\langle \nu_t, f\rangle - \langle \nu_s, f\rangle - \int_s^t \int_{\mathbb{R}^d}\Big(\tilde{\sigma}(x)D^\alpha f(x) + f(x)\,[b(x, V * \nu_u(x)) - d(x, U * \nu_u(x))]\Big)\nu_u(dx)du\Big\}, \tag{5.15}$$

then

$$E\left(\Psi(X)\right) = 0. \tag{5.16}$$

It follows from (2.9) that

$$0 = E\left(\prod_{k=1}^{n}\phi_k(\nu^K_{s_k})\left\{M^{K,f}_t - M^{K,f}_s\right\}\right) = E\left(\Psi(\nu^K)\right) - A_K, \tag{5.17}$$

where

$$\begin{aligned} A_K = E\Big(\prod_{k=1}^{n}\phi_k(\nu^K_{s_k})\int_s^t\int_{\mathbb{R}^d}\Big\{p(x)b(x,V*\nu^K_u(x))\Big[\int_{\mathbb{R}^d}(f(x+h)-f(x))M_K(x,dh)\Big] \\ + p(x)r(x)K\int_{\mathbb{R}}(f(x+h)-f(x))M_K(x,dh) - \tilde{\sigma}(x)D^\alpha f(x)\Big\}\nu^K_u(dx)du\Big). \end{aligned}$$

In view of Proposition 3.1, A_K tends to zero, as K grows to infinity. Applying Lemma 4.1, for $p = 3$, we see that the sequence $(|\Psi(\nu^K)|)_K$ is uniformly integrable, so that

$$\lim_K E\left(|\Psi(\nu^K)|\right) = E_Q\left(|\Psi(X)|\right), \tag{5.18}$$

since the function ψ is continuous a.s. at X. Collecting the previous results allows us to conclude that (5.16) holds true, and thus $\bar{M}^f$ is a martingale.

Finally, we have to show that the bracket of $\bar{M}^f$ is of the form

$$\langle\bar{M}^f\rangle_t = 2\int_0^t\int_{\mathbb{R}} r(x)f^2(x)X_s(dx)ds.$$

To this end, we first check that

$$\begin{aligned} \bar{N}^f_t &= \langle X_t, f\rangle^2 - \langle X_0, f\rangle^2 - \int_0^t\int_{\mathbb{R}} 2r(x)f^2(x)X_s(dx)ds \\ &- 2\int_0^t\langle X_s, f\rangle\int_{\mathbb{R}} f(x)\left[b(x,V*X_s(x)) - d(x,U*X_s(x))\right]X_s(dx)ds \\ &- 2\int_0^t\langle X_s, f\rangle\int_{\mathbb{R}}\tilde{\sigma}(x)D^\alpha f(x)X_s(dx)ds \end{aligned} \tag{5.19}$$

is a martingale. This can be done exactly as for $\bar{M}^f_t$, using the semimartingale decomposition of $\langle\nu^K_t, f\rangle^2$, given by (2.8) with $\phi(\nu) = \langle\nu, f\rangle^2$, and applying Lemma 4.1 with $p = 4$. On the other hand, Itô's formula implies that

$$\begin{aligned} \langle X_t, f\rangle^2 - \langle X_0, f\rangle^2 - \langle\bar{M}^f\rangle_t - 2\int_0^t\langle X_s, f\rangle\int_{\mathbb{R}}\tilde{\sigma}(x)D^\alpha f(x)X_s(dx)ds \\ - 2\int_0^t\langle X_s, f\rangle\int_{\mathbb{R}} f(x)\big[b(x,V*X_s(x)) - d(x,U*X_s(x))\big]X_s(dx)ds \end{aligned}$$

is a martingale. Comparing this formula with (5.19), we obtain (3.9). $\square$

6 Concluding remarks

We have developed models for population dynamics in the context of evolutionary ecology permitting heavy tailed distribution of mutations. Depending on the value of the allometric exponent η, the continuum (macro) limits of the individual (micro) dynamics turned out to be described by deterministic solutions of fractional nonlocal reaction-diffusion equations driven by fractional Laplacians (the case $0 < \eta < 1$), or measure-valued nonlinear stochastic superprocesses driven by Lévy- stable processes. These limiting models can now be used as approximate objects for numerical simulation of evolutionary Darwinian dynamics in presence of non-negligible large mutations. Of course, estimators of the relevant parameters of the phenomena under study have to be obtained first.

It the future it may also be worthwhile, from the perspective of practical applications, to elucidate the situation where the mutations have distributions intermediate between the heavy tailed distributions studied in this paper and the Gaussian distributions considered in [8]. Such distributions, which can display a multiscaling behavior, α-stable type for small mutations, and exponential for large mutations, have been recently suggested in the physical and economics literature and studied under different names such as truncated Lévy, and tempered Lévy distributions, see, e.g., [20], [7], [24], [26], and [10].

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