From Hermite polynomials to multifractional processes

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Abstract

We establish an invariance principle where the limit process is a Hermite-type process. We also prove that this limit process is multifractional. Our main result is a generalization of results from [6] and [11] to a multifractional setting. It also generalizes the main result of [3] to a non-Gaussian framework.

1 Introduction

Hermite processes have attracted a lot of attention for many years because they have nice properties as they generalize fractional Brownian motion [6, 11]. Let $m \in \mathbb{N}^*$ and $H \in (1/2, 1)$. The Hermite process $W_{m,H}$ of order m and Hurst index H can be defined for instance in terms of Dobrushin-Wiener-Itô integrals [5] as, for every $t \in [0, \infty)$,

$$W_{m,H}(t) = \int_{\mathbb{R}^m} f_{m,H}(x_1, \dots, x_m, t) d\widehat{B}_{x_1} \cdots d\widehat{B}_{x_m} \quad (1)$$

with

$$f_{m,H}(x_1, \dots, x_m, t) = C(m, H) \frac{\exp(it(x_1 + \dots + x_m)) - 1}{i(x_1 + \dots + x_m)|x_1 \cdots x_m|^{(2H-2+m)/2m}}$$

where $C(m, H)$ is a normalizing constant and $d\widehat{B}$ is the complex random measure corresponding to a standard Brownian motion B . Notice that for $m = 1$, the Hermite process $W_{1,H}$ is the fractional Brownian motion with Hurst index H .

An important property of Hermite processes is the invariance principle [4, 6, 10, 11], which can be stated as follows. Let $X = \{X_j\}_{j \in \mathbb{N}}$ be a Gaussian stationary sequence of centered random variables with $\mathbb{E}[X_0^2] = 1$ and satisfying the property

$$\mathbb{E}[X_0 X_j] \sim c j^{2(H-1)/m} \quad \text{as } j \rightarrow \infty \quad (2)$$

where c is a positive real number. Notice that (2) is a long range property. We consider a function $\phi \in L^2(e^{-x^2/2} dx)$ with Hermite rank equal to m , and define the partial sum $S_{\phi,H}^N(t)$ for every $N \in \mathbb{N}$ and $t \in [0, \infty)$ as

$$S_{\phi,H}^N(t) = \frac{1}{N^H} \sum_{j=1}^{\lfloor Nt \rfloor} \phi(X_j). \quad (3)$$

The invariance principle establishes that the finite-dimensional distributions of $S_{\phi,H}^N$ converge, as N goes to infinity, to the Hermite process $W_{m,H}$ with a suitable constant $C(m, H)$.

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As fractional Brownian motion, Hermite processes of index H are H -self-similar and H' -Hölder-continuous if and only if $H' < H$. As other fractional processes, a drawback of Hermite processes lies in the strong homogeneity of their properties, which are governed by the Hurst index H . In order to generalize fractional processes to less homogeneous processes, multifractional processes have been introduced, as for instance the class of multifractional Brownian motions [2, 8]. Multifractional processes have locally, but not globally, the same properties as fractional processes. These properties are governed by a function h that substitute for the constant H .

As for fractional Brownian motion and other Hermite processes, some nontrivial multifractional Gaussian processes satisfy invariance principle. Indeed, it is proven in [3] the following result. Let a Gaussian field $\{X_j(H)\}_{(j,H) \in \mathbb{N} \times (1/2,1)}$ satisfying some long-range assumptions and a continuous function h taking its values in $(1/2, 1)$. Then, the finite-dimensional distributions of the process

$$t \rightarrow S_h^N(t) = \sum_{j=1}^{\lfloor Nt \rfloor} \frac{X_j(h(j/N))}{N^{h(j/N)}} \quad (4)$$

converge to those of a centered Gaussian process S_h with covariance given for $t, s \geq 0$ by:

$$\mathbb{E}[S_h(t)S_h(s)] = \int_0^t d\theta \int_0^s d\sigma R(h(\theta), h(\sigma)) |\theta - \sigma|^{h(\theta)+h(\sigma)-2} \quad (5)$$

where R is a continuous function and derived from long-range assumptions of the field $\{X_j(H)\}_{(j,H) \in \mathbb{N} \times (1/2,1)}$. The process S_h is multifractional. If the function h is constant, then the process S_h is a fractional Brownian motion. The result above is then a generalization of classical invariance principle [9].

In this work, we generalize invariance principles presented above. We study the asymptotic behavior of a sequence generalizing both (3) and (4). In particular, this sequence is defined from a Gaussian field $\{X_j(H)\}_{j,H}$ satisfying long-range properties, a function $\phi \in L^2(e^{-x^2/2}dx)$ with Hermite rank equal to m and a Hurst function h . We get as a limit a multifractional process $S_{m,h}$ that depends on the integer m and the function h . If the function is a constant H , then the limit process is the Hermite process with Hurst index H and Hermite order m . If the integer m is equal to 1, then the limit process corresponds to a Gaussian multifractional process of the class obtained in [3].

The paper is organized as follows. In Section 2 we recall some definitions and preliminary results about Hermite polynomials and multiple stochastic integrals, which are used throughout the paper. In Section 3 we establish the main result of the paper. Section 4 is devoted to the proof of the main result.

2 Preliminaries

In this section we give some definitions and recall some results we use throughout this paper.

For each positive integer $m \in \mathbb{N}$, the m th Hermite polynomial P_m is defined as, for every $x \in \mathbb{R}$,

$$P_m(x) = (-1)^m e^{x^2/2} \frac{d^m}{dx^m} e^{-x^2/2}.$$

The family of the Hermite polynomials $\{P_m, m \in \mathbb{N}\}$ is an orthogonal basis of the space $L^2(e^{-x^2/2}dx)$ defined by

$$L^2(e^{-x^2/2}dx) = \left\{ \phi : \mathbb{R} \rightarrow \mathbb{C}, \phi \text{ measurable and } \int_{\mathbb{R}} |\phi(x)|^2 e^{-x^2/2} dx < \infty \right\}$$

with the inner product $\langle \cdot, \cdot \rangle$ defined as, for every ϕ_1 and ϕ_2 in $L^2(e^{-x^2/2}dx)$,

$$\langle \phi_1, \phi_2 \rangle = \int_{\mathbb{R}} \phi_1(x) \overline{\phi_2(x)} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}}$$

and whose the corresponding norm will be denoted as $\|\cdot\|$. For every function $\phi \in L^2(e^{-x^2/2}dx)$, there exists an integer m_ϕ such that $\langle \phi, P_{m_\phi} \rangle \neq 0$ and $\langle \phi, P_m \rangle = 0$ for every $m = 0, \dots, m_\phi - 1$. The integer m_ϕ is called the Hermite index of the function ϕ . Hence, for every $\phi \in L^2(e^{-x^2/2}dx)$,

$$\phi = \sum_{m=0}^{\infty} \frac{\langle \phi, P_m \rangle}{m!} P_m = \sum_{m=m_\phi}^{\infty} \frac{\langle \phi, P_m \rangle}{m!} P_m \quad (6)$$

where the convergence of the series holds for the norm $\|\cdot\|$. If X and Y are two Gaussian random variables $\mathcal{N}(0, 1)$, then, for every j and k in $\mathbb{N}^*$,

$$\mathbb{E}[P_j(X)P_k(Y)] = \begin{cases} k! \mathbb{E}[XY]^k & \text{if } k = j, \\ 0 & \text{if } k \neq j. \end{cases} \quad (7)$$

As a consequence, for every $\phi \in L^2(e^{-x^2/2}dx)$ and $X \sim \mathcal{N}(0, 1)$,

$$\mathbb{E}[|\phi(X)|^2] = \sum_{m=0}^{\infty} \frac{\langle \phi, P_m \rangle^2}{m!} < \infty \quad (8)$$

Others objects we strongly use in this paper are multiple Wiener-Itô integrals [5, 7]. Many notions of multiple Wiener-Itô integrals with respect to Brownian motion exist and are used to define processes as Hermite processes [6, 11]. Here we have chosen to use the so-called Dobrushin-Wiener-Itô integrals introduced in [5]. Let $d \in \mathbb{N}^*$, $f : \mathbb{R}^d \rightarrow \mathbb{C}$ be a square-integrable function, and $B = \{B_x\}_{x \in \mathbb{R}}$ be a standard Brownian motion in $\mathbb{R}$. In this paper the Dobrushin-Wiener-Itô integral of f is denoted $\int_{\mathbb{R}^d} f d\widehat{B}^{\otimes d}$ or

$$\int_{\mathbb{R}^d} f(x_1, \dots, x_d) d\widehat{B}_{x_1} \cdots d\widehat{B}_{x_d}.$$

It is well-defined if f is even and symmetric, that is, if f satisfies, for every $(x_1, \dots, x_d) \in \mathbb{R}^d$,

$$f(x_1, \dots, x_d) = \overline{f(-x_1, \dots, -x_d)},$$

and for every permutation ς on $\{1, \dots, d\}$,

$$f(x_1, \dots, x_d) = f(x_{\varsigma(1)}, \dots, x_{\varsigma(d)}).$$

We refer the reader to [5] for the precise definition of $\int_{\mathbb{R}^d} f d\widehat{B}^{\otimes d}$. Here we only recall some properties that we use in the proof of the main result. The integral $\int_{\mathbb{R}^d} f d\widehat{B}^{\otimes d}$ is Gaussian if and only if $d = 1$. In any case, it is a centered random variable and we can express its variance as

$$\mathbb{E} \left[\left(\int_{\mathbb{R}^d} f d\widehat{B}^{\otimes d} \right)^2 \right] = d! \int_{\mathbb{R}^d} |f(x_1, \dots, x_d)|^2 dx_1 \cdots dx_d.$$

We have a substitution formula for multiple integrals, using the self-similarity of the Brownian motion. For every $a > 0$, we have the equality in distribution

$$\int_{\mathbb{R}^d} f(x_1, \dots, x_d) d\widehat{B}_{x_1} \cdots d\widehat{B}_{x_d} \stackrel{\text{dist.}}{=} a^{d/2} \int_{\mathbb{R}^d} f(ax_1, \dots, ax_d) d\widehat{B}_{x_1} \cdots d\widehat{B}_{x_d}. \quad (9)$$

Another formula for change of variables is applied in this paper and is a consequence of Proposition 4.2 of [5]. Let $z : \mathbb{R} \rightarrow \mathbb{C}$ be a bounded and measurable function satisfying $z(x) = \overline{z(-x)}$ and $|z(x)| = 1$ for every $x \in \mathbb{R}$. Then, we have the equality in distribution

$$\int_{\mathbb{R}^d} f(x_1, \dots, x_d) d\widehat{B}_{x_1} \cdots d\widehat{B}_{x_d} \stackrel{\text{dist.}}{=} \int_{\mathbb{R}^d} f(x_1, \dots, x_d) z(x_1) \cdots z(x_d) d\widehat{B}_{x_1} \cdots d\widehat{B}_{x_d}. \quad (10)$$

By linearity of the integral and the bounded convergence theorem, we can prove the following convergence lemma.

Lemma 1. *Let $\{f_N\}_N$ be a sequence of even and symmetric functions in $L^2(\mathbb{R}^d, \mathbb{C})$. We assume that there exist two even and symmetric functions f and f^* in $L^2(\mathbb{R}^d, \mathbb{C})$ such that, for a. e. $x \in \mathbb{R}^d$, $\lim_{N \rightarrow \infty} f_N(x) = f(x)$ and $\sup_N |f_N(x)| \leq f^*(x)$. Then,*

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[\left| \int_{\mathbb{R}^d} f_N d\widehat{B}^{\otimes d} - \int_{\mathbb{R}^d} f d\widehat{B}^{\otimes d} \right|^2 \right] = 0.$$

To conclude this section, we mention a result that relate Hermite polynomials to multiple integrals. Let ψ be an even and symmetric function in $L^2(\mathbb{R}, \mathbb{C})$, m be a positive integer and P_m be the m th Hermite polynomial defined as previously. The random integral $\int_{\mathbb{R}} \psi(\xi) d\widehat{B}_\xi$ defines a centered Gaussian variable. If $\mathbb{E} [\int_{\mathbb{R}} |\psi(\xi)|^2 d\xi] = 1$ then we have almost surely

$$P_m \left(\int_{\mathbb{R}} \psi(x) d\widehat{B}_x \right) = \int_{\mathbb{R}^m} \psi(x_1) \cdots \psi(x_m) d\widehat{B}_{x_1} \cdots d\widehat{B}_{x_m}. \quad (11)$$

3 Main result

We let $m \in \mathbb{N}^*$ and define

$$b_m = 1 - \frac{1}{2m}.$$

We consider the Gaussian field $X = \{X_n(H), n \in \mathbb{N}, H \in (b_m, 1)\}$ defined as, for every $n \geq 0$ and $H \in (b_m, 1)$,

$$X_n(H) = \int_{-\pi}^{\pi} \exp(inx) g(H, x) |x|^{1/2-H} d\widehat{B}_x \quad (12)$$

where B is a standard Brownian motion and $g : (b_m, 1) \times (-\pi, \pi) \rightarrow \mathbb{C}$ is a measurable function satisfying the following properties.

- For every $(H, x) \in (b_m, 1) \times [-\pi, \pi]$, $g(H, x) = \overline{g(H, -x)}$. This property ensures that the field X is real.
- For every $H \in (b_m, 1)$,

$$\int_{-\pi}^{\pi} |g(H, x)|^2 |x|^{1-2H} dx = 1 \quad (13)$$

so that $\mathbb{E}[X_n(H)^2] = 1$.

- The function g is twice continuously differentiable on $(b_m, 1) \times (-\pi, \pi)$. We then define, for every $(H, x) \in (b_m, 1) \times (-\pi, \pi)$, $g_0(H) = g(H, 0)$ and $g_1(H, x) = \int_0^x (\partial g / \partial \xi)(H, \xi) d\xi$ so that $g = g_0 + g_1$ and, for every compact set K of $(b_m, 1)$,

$$\lim_{x \rightarrow 0} \sup_{H \in K} \left(|g_1(H, x)| + \left| \frac{\partial g_1}{\partial H}(H, x) \right| \right) = 0.$$

The assumptions above ensure that the covariance function satisfies the uniform long-range property of [3]. In particular, for every compact set $K \subset (b_m, 1)$, we have

$$\lim_{j-k \rightarrow \infty} \sup_{(H_1, H_2) \in K^2} |(j-k)^{2-H_1-H_2} \mathbb{E}[X_j(H_1)X_k(H_2)] - R(H_1, H_2)| = 0 \quad (14)$$

where

$$R(H_1, H_2) = g_0(H_1)g_0(H_2) \int_{\mathbb{R}} \exp(ix) |x|^{1-H_1-H_2} dx \quad (15)$$

for every $(H_1, H_2) \in (b_m, 1)$.

We consider a continuously differentiable function $h : [0, \infty) \rightarrow (1/2, 1)$ and a function $\phi \in L^2(e^{-x^2/2}dx)$ with Hermite rank equal to $m \in \mathbb{N}^*$. We let

$$\tilde{h} := 1 + \frac{h-1}{m} : [0, \infty) \rightarrow (b_m, 1).$$

We define for every $t \geq 0$ and $N > 0$

$$S_{\phi,h}^N(t) := \sum_{j=1}^{\lfloor Nt \rfloor} \frac{\phi(X_j(h_j^N))}{N^{h(j/N)}}, \quad (16)$$

with

$$h_j^N := 1 + \frac{h(j/N) - 1}{m} = \tilde{h}(j/N).$$

Now we can state the main result of this paper.

Theorem 1. *As $N \rightarrow \infty$, the finite-dimensional distributions of $S_{\phi,h}^N$ converge to those of $S_{m,h}$ defined for every $t \geq 0$ as*

$$S_{m,h}(t) := \int_{\mathbb{R}^m} f_{m,h}(x_1, \dots, x_m, t) d\widehat{B}_{x_1} \cdots d\widehat{B}_{x_m} \quad (17)$$

with, for every $(x_1, \dots, x_m, t) \in \mathbb{R}^m \times [0, \infty)$,

$$f_{m,h}(x_1, \dots, x_m, t) = \int_0^t \exp\left(i\theta \sum_{l=1}^m x_l\right) \tilde{g}(\theta) |x_1 \cdots x_m|^{1/2 - \tilde{h}(\theta)} d\theta$$

where

$$\tilde{g} = \frac{\langle \phi, P_m \rangle}{m!} (g_0 \circ \tilde{h})^m.$$

The process $S_{m,h}$ is continuous (up to a modification) and locally self-similar: for every $t \geq 0$,

$$\lim_{\varepsilon \rightarrow 0^+}^{\text{dist.}} \left\{ \frac{S_{m,h}(t + \varepsilon u) - S_{m,h}(t)}{\varepsilon^{h(t)}} \right\}_{u \geq 0} = \{T_{m,h,t}(u)\}_{u \geq 0}$$

where $\lim_{\varepsilon \rightarrow 0^+}^{\text{dist.}}$ stands for the limit in distribution in the space of continuous functions endowed with the uniform norm on every compact set and, for every $u \geq 0$,

$$T_{m,h,t}(u) = \tilde{g}(t) \int_{\mathbb{R}^m} \frac{\exp(iu \sum_{l=1}^m x_l) - 1}{i(\sum_{l=1}^m x_l) |x_1 \cdots x_m|^{\tilde{h}(t)-1/2}} d\widehat{B}_{x_1} \cdots d\widehat{B}_{x_m}.$$

Theorem 1 establishes that sequences of processes defined as (16), in particular from a Hurst function h , converge to multifractional processes with Hurst function h . This has been observed in [3] in the particular case $\phi \equiv 1$ where the limit process is $S_{1,h}$, which is a centered Gaussian process of covariance

$$(t, s) \mapsto \mathbb{E}[S_{1,h}(t)S_{1,h}(s)] = \int_0^t d\theta \int_0^s d\sigma R(h(\theta), h(\sigma)) \quad (18)$$

with R defined by (15). Theorem 1 is then an extension of the main result of [3], which assumes $\phi \equiv 1$, to any case where $\phi \in L^2(e^{-x^2/2}dx)$.

If we assume that $h \equiv H \in (b_m, 1)$, then Theorem 1 is the main result of [6, 11]. In particular, the limit process $S_{m,H}$ can be written as $W_{m,H}$ in (1) with the constant

$$C(m, H) = \frac{\langle \phi, P_m \rangle}{m!} (g_0(\tilde{H}))^m = \frac{\langle \phi, P_m \rangle}{m!} \left(\frac{R(\tilde{H}, \tilde{H})}{\int_{\mathbb{R}} e^{i\xi} |\xi|^{1-2\tilde{H}} d\xi} \right)^{m/2}$$

where

$$\tilde{H} := 1 + \frac{H-1}{m} \in (b_m, 1).$$

Hence, Theorem 1 generalizes results from [6] and [11] to a multifractional setting.

4 Proof of Theorem 1

The proof of Theorem 1 is organized as follows. In Subsection 4.1 we establish a technical lemma we then use throughout the proof of Theorem 1. We prove the convergence of $S_{\phi,h}^N$ in Subsection 4.2 and the regularity properties of $S_{m,h}$ in Subsection 4.3.

4.1 Technical lemma

In the following lemma, we prove for every $T > 0$ the existence of a function $\tilde{f}_T$ that is useful in the sequel of the proof to establish uniform bounds.

Lemma 2. *For every $T > 0$, there exists a function $\tilde{f}_T \in L^2(\mathbb{R}^m, \mathbb{R})$ so that, for almost every $x \in \mathbb{R}^m$ and for every $t \in [0, T]$ and $H \in [\min \tilde{h}, \max \tilde{h}]$,*

$$\left| \frac{e^{it \sum_{l=1}^m x_l} - 1}{|x_1 \cdots x_m|^{H-1/2} \sum_{l=1}^m x_l} \right| (1 + |\ln |x_1 \cdots x_m||) \leq \tilde{f}_T(x),$$

Proof. For every $(x_1, \dots, x_m) \in \mathbb{R}^m$ we define

$$L(x_1, \dots, x_m) = (1 + |\ln |x_1 \cdots x_m||)^2$$

and

$$\tilde{f}_T(x_1, \dots, x_m) = \sqrt{\sum_{H \in \{\min \tilde{h}, \max \tilde{h}\}} \frac{T^2 1_{|\sum_{l=1}^m x_l| \leq 1} + 4 |\sum_{l=1}^m x_l|^{-2} 1_{|\sum_{l=1}^m x_l| > 1}}{|x_1 \cdots x_m|^{2H-1}} L(x_1, \dots, x_m)}$$

We fix $T > 0$. For every $t \in [0, T]$, we can write

$$\left| \frac{e^{it \sum_{l=1}^m x_l} - 1}{\sum_{l=1}^m x_l} \right|^2 \max_{H \in [\min \tilde{h}, \max \tilde{h}]} \frac{L(x_1, \dots, x_m)}{|x_1 \cdots x_m|^{2H-1}} \leq \tilde{f}_T(x_1, \dots, x_m)^2.$$

It is then enough to prove that, for $H \in \{\min \tilde{h}, \max \tilde{h}\}$, the function

$$(x_1, \dots, x_m) \mapsto \frac{T^2 1_{|\sum_{l=1}^m x_l| \leq 1} + 4 |\sum_{l=1}^m x_l|^{-2} 1_{|\sum_{l=1}^m x_l| > 1}}{|x_1 \cdots x_m|^{2H-1}} L(x_1, \dots, x_m)$$

is integrable. We successively make the substitutions $y_j = x_1 + \cdots + x_j$ for every $j \in \{1, \dots, m\}$, $z_k = y_k / y_{k+1}$ for every $k \in \{1, \dots, m-1\}$ and $z_m = y_m$ to get

$$\begin{aligned} & \int_{\mathbb{R}^m} \frac{T^2 1_{|\sum_{l=1}^m x_l| \leq 1} + 4 |\sum_{l=1}^m x_l|^{-2} 1_{|\sum_{l=1}^m x_l| > 1}}{|x_1 \cdots x_m|^{2H-1}} L(x_1, \dots, x_m) dx_1 \cdots dx_m \\ &= \int_{\mathbb{R}^m} \frac{T^2 1_{|y_m| \leq 1} + 4 |y_m|^{-2} 1_{|y_m| > 1}}{|y_1(y_2 - y_1) \cdots (y_m - y_{m-1})|^{2H-1}} L(y_1, y_2 - y_1, \dots, y_m - y_{m-1}) dy_1 \cdots dy_m \\ &= \int_{\mathbb{R}} \frac{t^2 1_{|z_m| \leq 1} + 4 |z_m|^{-2} 1_{|z_m| > 1}}{|z_m|^{2m(H-1)+1}} dz_m \int_{\mathbb{R}} \frac{dz_{m-1}}{|1 - z_{m-1}|^{2H-1} |z_{m-1}|^{2(m-1)(H-1)+1}} \times \cdots \\ & \quad \cdots \times \int_{\mathbb{R}} \frac{dy_1}{|1 - z_1|^{2H-1} |z_1|^{2H-1}} L\left(\prod_{k=1}^m z_k, (1 - z_1) \prod_{k=2}^m z_k, \dots, (1 - z_{m-1}) z_m\right). \end{aligned}$$

The right-hand side above can be bounded by a finite sum of terms of the form

$$\begin{aligned} & \int_{\mathbb{R}} \frac{t^2 1_{|z_m| \leq 1} + 4 |z_m|^{-2} 1_{|z_m| > 1}}{|z_m|^{2m(H-1)+1}} dz_m \int_{\mathbb{R}} \frac{dz_{m-1}}{|1 - z_{m-1}|^{2H-1} |z_{m-1}|^{2(m-1)(H-1)+1}} \times \cdots \\ & \quad \cdots \times \int_{\mathbb{R}} \frac{dy_1}{|1 - z_1|^{2H-1} |z_1|^{2H-1}} |\ln |z_k||^\mu |\ln |1 - z_j||^\nu \end{aligned} \quad (19)$$

where k, j, μ and ν are integer. The terms of the form (19) are finite since $H \in (1 - 1/(2m), 1)$ and Bertrand's test. This concludes the proof. $\square$

4.2 Convergence of $S_{\phi,h}^N$

We first deal with the study of $S_{P_m,h}^N$ defined for every $t \geq 0$ by

$$S_{P_m,h}^N(t) = \sum_{j=1}^{\lfloor Nt \rfloor} \frac{P_m(X_j(h_j^N))}{N^{h(j/N)}}.$$

From now on, we denote $\prod_{l=1}^m d\widehat{B}_{x_l}$ by $d\widehat{B}_x^{\otimes m}$ when $x = (x_1, \dots, x_d)$.

Lemma 3. *The process $S_{P_m,h}^N$ is equal in distribution to the process $\widetilde{S}_{m,h}^N$ defined for every $t \geq 0$ by*

$$\widetilde{S}_{m,h}^N(t) = \int_{(-N\pi, N\pi)^m} d\widehat{B}_x^{\otimes m} \frac{1}{N} \sum_{j=1}^{\lfloor Nt \rfloor} \prod_{l=1}^m \exp(ijx_l/N) g(h_j^N, x_l/N) |x_l|^{1/2-h_j^N}.$$

Proof. Using (11) we obtain, almost surely,

$$P_m(X_j(h_j^N)) = \int_{(-\pi, \pi)^m} \prod_{l=1}^m \exp(ijx_l) g(h_j^N, x_l) |x_l|^{1/2-h_j^N} d\widehat{B}_{x_l}.$$

We then have

$$S_{P_m,h}^N(t) = \sum_{j=1}^{\lfloor Nt \rfloor} \frac{1}{N^{1-m/2}} \int_{(-\pi, \pi)^m} d\widehat{B}_x^{\otimes m} \prod_{l=1}^m \exp(ijx_l) g(h_j^N, x_l) |Nx_l|^{1/2-h_j^N}.$$

Making the substitution $x \rightarrow x/N$ and using (9) we get

$$S_{P_m,h}^N \stackrel{\text{dist.}}{=} t \mapsto \sum_{j=1}^{\lfloor Nt \rfloor} \frac{1}{N} \int_{(-N\pi, N\pi)^m} d\widehat{B}_x^{\otimes m} \prod_{l=1}^m \exp(ijx_l/N) g(h_j^N, x_l/N) |x_l|^{1/2-h_j^N}.$$

This concludes the proof by linearity of the multiple integral. $\square$

Now we aim to prove the convergence of $\widetilde{S}_{m,h}^N(t)$ in $L^2(\Omega, \mathbb{R})$ for every t . To this goal, we introduce the functions

$$\begin{aligned} f^N : [0, \infty) \times \mathbb{R}^m &\rightarrow \mathbb{C} \\ (t, x) &\mapsto 1_{(-N\pi, N\pi)^m}(x) \frac{1}{N} \sum_{j=1}^{\lfloor Nt \rfloor} \prod_{l=1}^m \exp(ijx_l/N) \frac{g(h_j^N, x_l/N)}{|x_l|^{h_j^N-1/2}} \end{aligned}$$

and we state the following lemma.

Lemma 4. *For every $t \geq 0$, there exists a function $f_t^* \in L^2(\mathbb{R}^m, \mathbb{R})$ so that, for every $x \in \mathbb{R}^m$ and $N \in \mathbb{N}$,*

$$|f^N(t, x)| \leq f_t^*(x).$$

Proof. We have

$$f^N(t, x) = 1_{(-N\pi, N\pi)^m}(x) \frac{i \sum_{l=1}^m x_l/N}{1 - e^{-i \sum_{l=1}^m x_l/N}} \sum_{j=1}^{\lfloor Nt \rfloor} \frac{e^{ij \sum_{l=1}^m x_l/N} - e^{i(j-1) \sum_{l=1}^m x_l/N}}{i \sum_{l=1}^m x_l} G_j^N(x)$$

where

$$G_j^N(x) = \prod_{l=1}^m \frac{g(h_j^N, x_l/N)}{|x_l|^{h_j^N-1/2}}.$$

We write

$$f^N(t, x) = f^{N,1}(t, x) - f^{N,2}(t, x)$$

with

$$\begin{aligned} f^{N,1}(t, x) &= \mathbb{I}_{(-N\pi, N\pi)}(x) \frac{i \sum_{l=1}^m x_l/N}{1 - e^{-i \sum_{l=1}^m x_l/N}} \sum_{j=1}^{\lfloor Nt \rfloor} \frac{1}{i \sum_{l=1}^m x_l} \\ &\quad \times \left(G_j^N(x) \left(e^{ij \sum_{l=1}^m x_l/N} - 1 \right) - G_{j-1}^N(x) \left(e^{i(j-1) \sum_{l=1}^m x_l/N} - 1 \right) \right) \\ &= \mathbb{I}_{(-N\pi, N\pi)}(x) \frac{i \sum_{l=1}^m x_l/N}{1 - e^{-i \sum_{l=1}^m x_l/N}} G_{\lfloor Nt \rfloor}^N(x) \frac{e^{i \lfloor Nt \rfloor \sum_{l=1}^m x_l/N} - 1}{i \sum_{l=1}^m x_l} \end{aligned}$$

and

$$\begin{aligned} f^{N,2}(t, x) &= \mathbb{I}_{(-N\pi, N\pi)}(x) \frac{i \sum_{l=1}^m x_l/N}{1 - e^{-i \sum_{l=1}^m x_l/N}} \\ &\quad \times \sum_{j=1}^{\lfloor Nt \rfloor} \frac{e^{i(j-1) \sum_{l=1}^m x_l/N} - 1}{i \sum_{l=1}^m x_l} (G_j^N(x) - G_{j-1}^N(x)). \end{aligned}$$

We first deal with $f^{N,1}$. Because g is bounded, there exists $M_1 > 0$ such that for every x and N

$$|f^{N,1}(t, x)| \leq M_1 \left| \frac{e^{i \lfloor Nt \rfloor \sum_{l=1}^m x_l/N} - 1}{|x_1 \cdots x_m|^{h_{\lfloor Nt \rfloor}^N - 1/2} \sum_{l=1}^m x_l} \right|.$$

Then, by Lemma 2, there exists a function $\tilde{f}_{t,1} \in L^2(\mathbb{R}^m, \mathbb{R})$ so that for every x and N ,

$$\left| \frac{e^{i \lfloor Nt \rfloor \sum_{l=1}^m x_l/N} - 1}{|x_1 \cdots x_m|^{h_{\lfloor Nt \rfloor}^N - 1/2} \sum_{l=1}^m x_l} \right| \leq \tilde{f}_{t,1}(x),$$

so that we get

$$|f^{N,1}(t, x)| \leq M_1 \tilde{f}_{t,1}(x). \quad (20)$$

Now we deal with $f^{N,2}$. By using Taylor formula we obtain, for almost every x ,

$$\begin{aligned} &|G_j^N(x) - G_{j-1}^N(x)| \\ &\leq \frac{\max |\tilde{h}'|}{N} \max_{H \in [\min h, \max \tilde{h}]} \left| \frac{-\ln |x_1 \cdots x_m|}{|x_1 \cdots x_m|^{H-1/2}} \prod_{l=1}^m g\left(H, \frac{x_l}{N}\right) \right. \\ &\quad \left. + \frac{1}{|x_1 \cdots x_m|^{H-1/2}} \sum_{k=1}^m \left(\frac{\partial g}{\partial H} \left(H, \frac{x_k}{N} \right) \right) \prod_{l=1, l \neq k}^m g\left(H, \frac{x_l}{N}\right) \right|. \end{aligned}$$

Since g and $\partial g / \partial H$ are bounded, there exists a constant $M_2 > 0$, which depends only on h and g , such that for almost every x and every N

$$|f^{N,2}(t, x)| \leq \frac{M_2}{N} \sum_{j=1}^{\lfloor Nt \rfloor} \frac{|e^{i(j-1) \sum_{l=1}^m x_l/N} - 1|}{|\sum_{l=1}^m x_l|} \max_{H \in [\min h, \max \tilde{h}]} \frac{1 + |\ln |x_1 \cdots x_m||}{|x_1 \cdots x_m|^{H-1/2}}.$$

As for $f^{N,2}$, by Lemma 2, there exists a function $\tilde{f}_{t,2} \in L^2(\mathbb{R}^m, \mathbb{R})$ so that for almost every x and every N and j ,

$$\left| \frac{e^{i(j-1) \sum_{l=1}^m x_l/N} - 1}{\sum_{l=1}^m x_l} \right| \max_{H \in [\min h, \max \tilde{h}]} \frac{1 + |\ln |x_1 \cdots x_m||}{|x_1 \cdots x_m|^{H-1/2}} \leq \tilde{f}_{t,2}(x),$$

so that we get

$$|f^{N,2}(t, x)| \leq M_2 \tilde{f}_{t,2}(x). \quad (21)$$

Hence, taking $\tilde{f}_t = M_1 \tilde{f}_{t,1} + M_2 \tilde{f}_{t,2}$ and combining (20) and (21) we conclude the proof. $\square$

The convergence of $\tilde{S}_{m,h}^N$ can now be established.

Lemma 5. *For every $t \geq 0$, as $N \rightarrow \infty$, $\tilde{S}_{m,h}^N(t)$ converges in $L^2(\Omega, \mathbb{R})$ to $\tilde{S}_{m,h}^\infty(t)$ given by*

$$\tilde{S}_{m,h}^\infty(t) = \int_{\mathbb{R}^m} d\hat{B}_x^{\otimes m} \int_0^t \exp\left(i\theta \sum_{l=1}^m x_l\right) g_0(\tilde{h}(\theta))^m |x_1 \cdots x_m|^{1/2-\tilde{h}(\theta)} d\theta.$$

Proof. Because of Lemmas 1 and 4, it suffices to prove that the function $f^N(t, x)$ converges for almost every x to $f^\infty(t, x)$ defined by

$$f^\infty(t, x) := \int_0^t \exp\left(i\theta \sum_{l=1}^m x_l\right) g_0(\tilde{h}(\theta))^m |x_1 \cdots x_m|^{1/2-\tilde{h}(\theta)} d\theta.$$

We let

$$G_{j,0}^N(x) = g_0(h_j^N)^m |x_1 \cdots x_m|^{1/2-h_j^N} \quad \text{and} \quad G_{j,1}^N(x) = G_j^N(x) - G_{j,0}^N(x)$$

where $G_j^N(x)$ is defined as in the proof of Lemma 4. We also consider the same decomposition $f^N = f^{N,1} - f^{N,2}$ as in the proof of Lemma 4 and we let

$$f^{N,1} = f^{N,1,0} - f^{N,1,1} \quad \text{and} \quad f^{N,2} = f^{N,2,0} - f^{N,2,1}$$

where, for $\kappa \in \{0, 1\}$,

$$f^{N,1,\kappa}(t, x) = \mathbb{I}_{(-N\pi, N\pi)}(x) \frac{i \sum_{l=1}^m x_l / N}{1 - e^{-i \sum_{l=1}^m x_l / N}} G_{\lfloor Nt \rfloor, \kappa}^N(x) \frac{e^{i \lfloor Nt \rfloor \sum_{l=1}^m x_l / N} - 1}{i \sum_{l=1}^m x_l}$$

and

$$\begin{aligned} f^{N,2,\kappa}(t, x) &= \mathbb{I}_{(-N\pi, N\pi)}(x) \frac{i \sum_{l=1}^m x_l / N}{1 - e^{-i \sum_{l=1}^m x_l / N}} \\ &\quad \times \sum_{j=1}^{\lfloor Nt \rfloor} \frac{e^{i(j-1) \sum_{l=1}^m x_l / N} - 1}{i \sum_{l=1}^m x_l} (G_{j,\kappa}^N(x) - G_{j-1,\kappa}^N(x)). \end{aligned}$$

Because h and g_0 are continuously differentiable we get, for almost every x ,

$$\lim_{N \rightarrow \infty} f^{N,1,0}(t, x) = g_0(\tilde{h}(t))^m \frac{e^{it \sum_{l=1}^m x_l} - 1}{i |x_1 \cdots x_m|^{\tilde{h}(t)-1/2} \sum_{l=1}^m x_l}$$

and

$$\lim_{N \rightarrow \infty} f^{N,2,0}(t, x) = \int_0^t \frac{e^{i\theta \sum_{l=1}^m x_l} - 1}{i \sum_{l=1}^m x_l} \tilde{h}'(\theta) \frac{\partial}{\partial H} \left(\frac{g_0(H)^m}{|x_1 \cdots x_m|^{H-1/2}} \right) \Big|_{H=\tilde{h}(\theta)} d\theta,$$

so that

$$\lim_{N \rightarrow \infty} (f^{N,1,0}(t, x) - f^{N,2,0}(t, x)) = f^\infty(t, x).$$

Now we deal with $f^{N,1,1}$ and $f^{N,2,1}$. We remark that we can express $G_{j,1}^N(x)$ as

$$G_{j,1}^N(x) = |x_1 \cdots x_m|^{1/2-h_j^N} \sum_{k=1}^m g_1\left(h_j^N, \frac{x_k}{N}\right) g_0(h_j^N)^{k-1} \prod_{l=k+1}^m g\left(h_j^N, \frac{x_l}{N}\right).$$

Then, because of Lemma 2 and the boundedness of g_0 and g , there exist a constant $M_3 > 0$ and a function $\tilde{f}_{t,3} \in L^2(\mathbb{R}^m, \mathbb{R})$ such that for almost every x ,

$$|f^{N,1,1}(t, x)| \leq M_3 \tilde{f}_{t,3}(x) \sum_{k=1}^m \sup_{H \in [\min \tilde{h}, \max \tilde{h}]} \left| g_1 \left(H, \frac{x_k}{N} \right) \right|,$$

so that $\lim_{N \rightarrow \infty} f^{N,1,1}(t, x) = 0$. Similarly, using Lemma 2, there exist a constant $M_4 > 0$ and a function $\tilde{f}_{t,4} \in L^2(\mathbb{R}^m, \mathbb{R})$ such that for almost every x ,

$$|f^{N,2,1}(t, x)| \leq M_4 \tilde{f}_{t,4}(x) \sum_{k=1}^m \sup_{H \in [\min \tilde{h}, \max \tilde{h}]} \left(\left| g_1 \left(H, \frac{x_k}{N} \right) \right| + \left| \frac{\partial g_1}{\partial H} \left(H, \frac{x_k}{N} \right) \right| \right),$$

so that $\lim_{N \rightarrow \infty} f^{N,2,1}(t, x) = 0$ and then

$$\lim_{N \rightarrow \infty} (f^{N,2,1}(t, x) - f^{N,2,1}(t, x)) = 0,$$

which concludes the proof. $\square$

The following lemma establishes that the convergence of $S_{\phi,h}^N$ can be reduced to the one of $S_{P_m,h}^N$ and, as a consequence of Lemma 3, to the one of $\tilde{S}_{m,h}^N$.

Lemma 6. *For every $t \geq 0$, we have*

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[\left(S_{\phi,h}^N(t) - \frac{\langle \phi, P_m \rangle}{m!} S_{P_m,h}^N(t) \right)^2 \right] = 0.$$

Proof. Since (6) we have

$$\begin{aligned} & \mathbb{E} \left[\left(S_{\phi,h}^N(t) - \frac{\langle \phi, P_m \rangle}{m!} S_{P_m,h}^N(t) \right)^2 \right] \\ &= \mathbb{E} \left[\left(\sum_{j=1}^{\lfloor Nt \rfloor} \frac{1}{N^{h(j/N)}} \sum_{n=m+1}^{\infty} \frac{\langle \phi, P_n \rangle}{n!} P_n(X_j(h_j^N)) \right)^2 \right] \\ &= \sum_{j=1}^{\lfloor Nt \rfloor} \sum_{k=1}^{\lfloor Nt \rfloor} \frac{1}{N^{h(j/N)+h(k/N)}} \sum_{n=m+1}^{\infty} \frac{\langle \phi, P_n \rangle^2}{(n!)^2} \mathbb{E}[P_n(X_j(h_j^N)) P_n(X_k(h_k^N))]. \end{aligned}$$

Because of (7) and (13) we get, for every $n \geq m+1$,

$$\begin{aligned} \mathbb{E}[P_n(X_j(h_j^N)) P_n(X_k(h_k^N))] &= n! \mathbb{E}[X_j(h_j^N) X_k(h_k^N)]^n \\ &\leq n! |\mathbb{E}[X_j(h_j^N) X_k(h_k^N)]|^{m+1}, \end{aligned}$$

so that

$$\begin{aligned} & \mathbb{E} \left[\left(S_{\phi,h}^N(t) - \frac{\langle \phi, P_m \rangle}{m!} S_{P_m,h}^N(t) \right)^2 \right] \\ &\leq \left(\sum_{n=m+1}^{\infty} \frac{\langle \phi, P_n \rangle^2}{n!} \right) \sum_{j=1}^{\lfloor Nt \rfloor} \sum_{k=1}^{\lfloor Nt \rfloor} \frac{|\mathbb{E}[X_j(h_j^N) X_k(h_k^N)]|^{m+1}}{N^{h(j/N)+h(k/N)}}. \end{aligned}$$

Let $\eta > 0$. Using the representation of the field X , there exists $N_\eta \in \mathbb{N}^*$ such that, for $|j-k| > N_\eta$ and $N \in \mathbb{N}^*$, $|\mathbb{E}[X_j(h_j^N) X_k(h_k^N)]| \leq \eta$, so that

$$\begin{aligned} & \sum_{j=1}^{\lfloor Nt \rfloor} \sum_{k=1}^{\lfloor Nt \rfloor} \frac{|\mathbb{E}[X_j(h_j^N) X_k(h_k^N)]|^{m+1}}{N^{h(j/N)+h(k/N)}} \\ &\leq \sum_{j=1}^{\lfloor Nt \rfloor} \sum_{k=1}^{\lfloor Nt \rfloor} \frac{1_{|j-k| \leq N_\eta}}{N^{h(j/N)+h(k/N)}} + \eta \sum_{j=1}^{\lfloor Nt \rfloor} \sum_{k=1}^{\lfloor Nt \rfloor} 1_{|j-k| \geq 1} \frac{|\mathbb{E}[X_j(h_j^N) X_k(h_k^N)]|^m}{N^{h(j/N)+h(k/N)}}. \end{aligned}$$

There exists $C_1(\eta) > 0$ such that

$$\sum_{j=1}^{\lfloor Nt \rfloor} \sum_{k=1}^{\lfloor Nt \rfloor} \frac{1_{|j-k| \leq N_\eta}}{N^{h(j/N)+h(k/N)}} \leq \frac{C_\eta}{N^{2 \min h-1}}.$$

Moreover, because of the assumptions on X , there exists a constant $C_2 > 0$, which is independent on η , such that, for every j, k and N ,

$$|\mathbb{E}[X_j(h_j^N)X_k(h_k^N)]| \leq C_2|j-k|^{h_j^N+h_k^N-2}.$$

We then obtain

$$\begin{aligned} & \sum_{j=1}^{\lfloor Nt \rfloor} \sum_{k=1}^{\lfloor Nt \rfloor} \frac{|\mathbb{E}[X_j(h_j^N)X_k(h_k^N)]|^{m+1}}{N^{h(j/N)+h(k/N)}} \\ & \leq \frac{C_\eta}{N^{2 \min h-1}} + \frac{\eta C_2}{N^2} \sum_{j=1}^{\lfloor Nt \rfloor} \sum_{k=1}^{\lfloor Nt \rfloor} 1_{|j-k| \geq 1} \left| \frac{j-k}{N} \right|^{m(h_j^N+h_k^N-2)}. \end{aligned}$$

Hence, for every $\eta > 0$

$$\begin{aligned} & \limsup_{N \rightarrow \infty} \mathbb{E} \left[\left(S_{\phi,h}^N(t) - \frac{\langle \phi, P_m \rangle}{m!} S_{P_m,h}^N(t) \right)^2 \right] \\ & \leq \eta C_2 \left(\sum_{n=m+1}^{\infty} \frac{\langle \phi, P_n \rangle^2}{n!} \right) \int_0^t \int_0^t |\theta - \sigma|^{h(\theta)+h(\sigma)-2} d\theta d\sigma. \end{aligned}$$

The constants $\sum_{n=m+1}^{\infty} \langle \phi, P_n \rangle^2 / n!$ and $\int_0^t \int_0^t |\theta - \sigma|^{h(\theta)+h(\sigma)-2} d\theta d\sigma$ are finite since (8) and $h > 1/2$ respectively. This concludes the proof. $\square$

Now we conclude this subsection by the following lemma.

Lemma 7. *As $N \rightarrow \infty$, the finite-dimensional distributions of $S_{\phi,h}^N$ converge to those of $S_{m,h}$, which can be defined for every $t \geq 0$ as:*

$$S_{m,h}(t) := \frac{\langle \phi, P_m \rangle}{m!} \int_{\mathbb{R}^m} f^\infty(t, x_1, \dots, x_m) d\hat{B}_{x_1} \cdots d\hat{B}_{x_m}. \quad (22)$$

Proof. We fix $n \in \mathbb{N}$, $(t_1, \dots, t_n) \in [0, \infty)^n$ and a Lipschitz bounded function $\Psi : \mathbb{R}^n \rightarrow \mathbb{R}$. We define $\phi_m = \langle \phi, P_m \rangle / m!$. We have

$$|\mathbb{E}[\Psi(S_{\phi,h}^N(t_1), \dots, S_{\phi,h}^N(t_n))] - \mathbb{E}[\Psi(S_{m,h}(t_1), \dots, S_{m,h}(t_n))]| \leq E_1^N + E_2^N \quad (23)$$

where

$$E_1^N = |\mathbb{E}[\Psi(S_{\phi,h}^N(t_1), \dots, S_{\phi,h}^N(t_n)) - \Psi(\phi_m S_{P_m,h}^N(t_1), \dots, \phi_m S_{P_m,h}^N(t_n))]|$$

and

$$E_2^N = |\mathbb{E}[\Psi(\phi_m S_{P_m,h}^N(t_1), \dots, \phi_m S_{P_m,h}^N(t_n))] - \mathbb{E}[\Psi(S_{m,h}(t_1), \dots, S_{m,h}(t_n))]|$$

Because Ψ is Lipschitz and using Cauchy-Schwartz inequality, there exists $C_1 > 0$ so that, for every N ,

$$E_1^N \leq C_1 \sum_{j=1}^n \sqrt{\mathbb{E} \left[\left(S_{\phi,h}^N(t_j) - \phi_m S_{P_m,h}^N(t_j) \right)^2 \right]}.$$

Then, since Lemma 6,

$$\lim_{N \rightarrow \infty} E_1^N = 0. \quad (24)$$

Because Lemma 3 we have

$$E_2^N = \left| \mathbb{E} \left[\Psi \left(\phi_m \tilde{S}_{m,h}^N(t_1), \dots, \phi_m \tilde{S}_{m,h}^N(t_n) \right) - \Psi(S_{m,h}(t_1), \dots, S_{m,h}(t_n)) \right] \right|.$$

Thus, as for E_1^N , because Ψ is Lipschitz and using Cauchy-Schwartz inequality, there exists $C_2 > 0$ so that, for every N ,

$$E_2^N \leq C_2 \sum_{j=1}^n \sqrt{\mathbb{E} \left[\left(\phi_m \tilde{S}_{m,h}^N(t_j) - S_{m,h}(t_j) \right)^2 \right]}.$$

As a consequence, from Lemma 5,

$$\lim_{N \rightarrow \infty} E_2^N = 0. \quad (25)$$

We conclude the proof by combining (23), (24) and (25). $\square$

4.3 Continuity and local self-similarity of $S_{m,h}$

We first prove the local self-similarity in the sense of the finite-dimensional distributions. Then we prove the continuity of $S_{m,h}$. Finally, we establish a tightness property for the family $\left\{ \left\{ \varepsilon^{-h(t)} (S_{\phi,h}(t + \varepsilon u) - S_{\phi,h}(t)) \right\}_{u \geq 0} \right\}_{\varepsilon > 0}$ using Kolmogorov lemma [1] to conclude with the proof of the local self-similarity property in the space of continuous functions.

By making the substitution $\theta \rightarrow \varepsilon\theta + t$, we get

$$\frac{S_{m,h}(t + \varepsilon u) - S_{m,h}(t)}{\varepsilon^{h(t)}} = \int_{\mathbb{R}^m} \exp \left(i t \sum_{l=1}^m x_l \right) \check{f}_1(t, u, x, \varepsilon) d\hat{B}_{x_1} \cdots d\hat{B}_{x_m}$$

where

$$\check{f}_1(t, u, x, \varepsilon) = \varepsilon^{1-h(t)} \int_0^u \exp \left(i \varepsilon \theta \sum_{l=1}^m x_l \right) \tilde{g}(\varepsilon\theta + t) |x_1 \cdots x_m|^{1/2 - \tilde{h}(\varepsilon\theta + t)} d\theta.$$

Since (10) and (9) we have

$$\left\{ \frac{S_{m,h}(t + \varepsilon u) - S_{m,h}(t)}{\varepsilon^{h(t)}} \right\}_{u \geq 0} \stackrel{\text{dist.}}{=} \left\{ \int_{\mathbb{R}^m} \check{f}_2(t, u, x, \varepsilon) d\hat{B}_{x_1} \cdots d\hat{B}_{x_m} \right\}_{u \geq 0}$$

with

$$\check{f}_2(t, u, x, \varepsilon) = \int_0^u \varepsilon^{m(\tilde{h}(\varepsilon\theta + t) - \tilde{h}(t))} \exp \left(i \theta \sum_{l=1}^m x_l \right) \tilde{g}(\varepsilon\theta + t) |x_1 \cdots x_m|^{1/2 - \tilde{h}(\varepsilon\theta + t)} d\theta.$$

For almost every $x \in \mathbb{R}^m$, every u and t ,

$$\lim_{\varepsilon \rightarrow 0} \check{f}_2(t, u, x, \varepsilon) = \tilde{g}(t) \frac{\exp(iu \sum_{l=1}^m x_l) - 1}{i(\sum_{l=1}^m x_l) |x_1 \cdots x_m|^{\tilde{h}(t) - 1/2}}$$

By an integration by part, we have

$$\begin{aligned} & \check{f}_2(t, u, x, \varepsilon) \\ &= \varepsilon^{m(\tilde{h}(\varepsilon u + t) - \tilde{h}(t))} \tilde{g}(\varepsilon u + t) \frac{\exp(iu \sum_{l=1}^m x_l) - 1}{i(\sum_{l=1}^m x_l) |x_1 \cdots x_m|^{\tilde{h}(\varepsilon u + t) - 1/2}} \\ & \quad + \varepsilon \int_0^u \varepsilon^{m(\tilde{h}(\varepsilon\theta + t) - \tilde{h}(t))} \frac{\exp(i\theta \sum_{l=1}^m x_l) - 1}{i(\sum_{l=1}^m x_l) |x_1 \cdots x_m|^{\tilde{h}(\varepsilon\theta + t) - 1/2}} \\ & \quad \times \left\{ \tilde{g}'(\varepsilon\theta + t) + \tilde{h}'(\varepsilon\theta + t) \tilde{g}(\varepsilon\theta + t) (\ln(\varepsilon^m) - \ln|x_1 \cdots x_m|) \right\} d\theta \end{aligned}$$

We fix $T > 0$ and $U > 0$. As a consequence of the identity above and because of the regularity of $\tilde{h}$ and $\tilde{g}$, there exists a constant $M_{T,U} > 0$ such that, for every $u \in [0, U]$, $t \in [0, T]$, $\varepsilon \in (0, 1]$ and $x \in \mathbb{R}^m$,

$$|\tilde{f}_2(t, u, x, \varepsilon)| \leq M_{T,U} \tilde{f}_{T+U,2}(x) \quad (26)$$

where $\tilde{f}_{T+U,2}(x)$ is defined in Lemma 2. Since $\tilde{f}_{T+U,2}$ is square integrable and because of Lemma 1, this proves the local self-similarity of $S_{m,h}$ in the sense of the finite-dimensional distributions.

To prove the continuity of the $S_{m,h}$ we use Kolmogorov lemma. By making the same calculations as above we have, for every $t > s > 0$,

$$\mathbb{E}[(S_{\phi,h}(t) - S_{\phi,h}(s))^2] = m!(t-s)^{2h(s)} \int_{\mathbb{R}^m} (\tilde{f}_2(s, 1, x, t-s))^2 dx_1 \cdots dx_m$$

If $t - s < 1$, because of (26) we then have

$$\mathbb{E}[(S_{\phi,h}(t) - S_{\phi,h}(s))^2] \leq m!M_{1,1}(t-s)^{2h(s)} \int_{\mathbb{R}^m} \tilde{f}_{2,2}(x)^2 dx_1 \cdots dx_m,$$

which concludes the proof of the continuity of $S_{\phi,h}$.

Finally, in a similar way as just previously there exists a constant $C > 0$ such that, for every u and v satisfying $|u - v| < 1$,

$$\mathbb{E}\left[\left(\frac{S_{\phi,h}(t + \varepsilon u) - S_{\phi,h}(t + \varepsilon v)}{\varepsilon^{h(t)}}\right)^2\right] \leq C|u - v|^{2h(t)},$$

This prove the tightness of the family $\left\{\left\{\varepsilon^{-h(t)}(S_{\phi,h}(t + \varepsilon u) - S_{\phi,h}(t))\right\}_{u \geq 0}\right\}_{\varepsilon > 0}$ thanks to Kolmogorov lemma [1], and then the local self-similarity property of $S_{\phi,h}$.

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