PENALISATION OF THE SYMMETRIC RANDOM WALK
by several functions of the supremum

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Abstract

Call $(\Omega, \mathcal{F}_\infty, \mathbb{P}, X, \mathcal{F})$ the canonical space for the standard random walk on $\mathbb{Z}$. Thus, Ω denotes the set of paths $\phi : \mathbb{N} \to \mathbb{Z}$ such that $|\phi(n+1) - \phi(n)| = 1$, $X = (X_n, n \geq 0)$ is the canonical coordinate process on Ω; $\mathcal{F} = (\mathcal{F}_n, n \geq 0)$ is the natural filtration of X, $\mathcal{F}_\infty$ the σ-field $\bigvee_{n\geq 0} \mathcal{F}_n$, and $\mathbb{P}_0$ the probability on $(\Omega, \mathcal{F}_\infty)$ such that under $\mathbb{P}_0$, X is the standard random walk started from 0, i.e., $\mathbb{P}_0\left(X_{n+1} = j \mid X_n = i\right) = \frac{1}{2}$ when $|j - i| = 1$.

Let $G : \mathbb{N} \times \Omega \to \mathbb{R}^+$ be a positive, adapted functional. For several types of functionals G, we show the existence of a positive $\mathcal{F}$-martingale $(M_n,\ n \geq 0)$ such that, for all n and all $\Lambda_n \in \mathcal{F}_n$,

$$\frac{\mathbb{E}_0[\mathbb{1}_{\Lambda_n} G_p]}{\mathbb{E}_0[G_p]} \longrightarrow \mathbb{E}_0[\mathbb{1}_{\Lambda_n} M_n] \qquad \text{when} \quad p \to \infty\,.$$

Thus, there exists a probability Q on $(\Omega, \mathcal{F}_\infty)$ such that $Q(\Lambda_n) = \mathbb{E}_0[\mathbb{1}_{\Lambda_n} M_n]$ for all $\Lambda_n \in \mathcal{F}_n$. We describe the behavior of the process $(\Omega, X, \mathcal{F})$ under Q.

We study here four kinds of G:
.G_p is a function of S_p where S_p is the unilateral supremum of X.
.G_p is a function of S_{g_p} where g_p is the last 0 at the left of p.
.G_p is a function of S_{d_p} where d_p is the first 0 at the right of p.
.G_p is a function of $S^*_{g_p}$ where S^*_p is the bilateral supremum of X.
.G_p is a function of S^*_p .
A similar study has been realized for other kinds of G (cf [Deb09]).

1 Introduction

Let $\left\{\Omega, (X_t, \mathcal{F}_t)_{t\geq 0}, \mathcal{F}_\infty, \mathbb{P}_x\right\}$ be the canonical one-dimensional Brownian motion. For several types of positive functionals $\Gamma : \mathbb{R}^+ \times \Omega \to \mathbb{R}^+$, B. Roynette, P. Vallois and M. Yor show in [RVY06] that, for fixed s and for all $\Lambda_s \in \mathcal{F}_s$,

$$\lim_{t\to\infty} \frac{\mathbb{E}_x[\mathbb{1}_{\Lambda_s} \Gamma_t]}{\mathbb{E}_x[\Gamma_t]}$$

exists and has the form $\mathbb{E}_x[\mathbb{1}_{\Lambda_s} M^x_s]$, where $(M^x_s, s \geq 0)$ is a positive martingale. This enables them to define a probability Q_x on $(\Omega, \mathcal{F}_\infty)$ by:

$$\forall \Lambda_s \in \mathcal{F}_s \qquad Q_x(\Lambda_s) = \mathbb{E}_x[\mathbb{1}_{\Lambda_s} M^x_s]\ ;$$

moreover, they precisely describe the behavior of the canonical process X under Q_x. They do this for numerous functionals Γ, for instance a function of the one-sided maximum, or of the local time, or of the age of the current excursion (cf. [RVY06], [RVY]).
We have already studied a discrete analogue of their results in [Deb09]. More precisely, let Ω denote the set of all functions ϕ from $\mathbb{N}$ to $\mathbb{Z}$ such that $|\phi(n+1) - \phi(n)| = 1$, let $X = (X_n, n \geq 0)$ be the process of coordinates on that space, $\mathcal{F} = (\mathcal{F}_n, n \geq 0)$ the canonical filtration, $\mathcal{F}_\infty$ the

σ-field $\bigvee_{n\geq 0}\mathcal{F}_n$, and $\mathbb{P}_x$ ($x\in\mathbb{N}$) the family of probabilities on $(\Omega,\mathcal{F}_\infty)$ such that under $\mathbb{P}_x$, X is the standard random walk started at x. For notational simplicity, we often write $\mathbb{P}$ for $\mathbb{P}_0$. Our aim is to establish that for several types of positive, adapted functionals $G:\mathbb{N}\times\Omega\to\mathbb{N}$,
i) for each $n\geq 0$ and each $\Lambda_n\in\mathcal{F}_n$,

$$\frac{\mathbb{E}_0[\mathbb{1}_{\Lambda_n}G_p]}{\mathbb{E}_0[G_p]},$$

tends to a limit when p tends to infinity;
ii) this limit is equal to $\mathbb{E}_0[\mathbb{1}_{\Lambda_n}M_n]$, for some $\mathcal{F}$-martingale M such that $M_0=1$.

Call $Q(\Lambda_n)$ this limit. Like the continuous case, Q describes a probability on $(\Omega,\mathcal{F}_\infty)$ by :

$$\forall n\geq 0,\forall\Lambda_n\in\mathcal{F}_n,\ Q(\Lambda_n)=\mathbb{E}_0[\mathbb{1}_{\Lambda_n}M_n],$$

and we also study the process X under Q.
A better definition of the principle of penalisation, for instance proof of existence and unicity, can be found in the introduction and the first part of [Deb09].

In this paper, G essentially depends on two functions $\varphi:\mathbb{N}\to\mathbb{R}^+$ and $\phi:\mathbb{N}\to\mathbb{R}^+$ such that :

$$\sum_{k\geq 0}\varphi(k)=1,\ \phi(x):=\sum_{k=x}^{\infty}\varphi(k)\,. \tag{1.1}$$

The following result comes from [Deb09], and is not proved in the following paper. Here, G is a function of the one-sided maximum, i.e. $G_p=\varphi(S_p)$, where $S_p:=\sup\{X_k,k\leq p\}$. We establish :

Theorem 1.1. *1. (a) For each $n\geq 0$ and each $\Lambda_n\in\mathcal{F}_n$, one has*

$$\lim_{p\to\infty}\frac{\mathbb{E}[\mathbb{1}_{\Lambda_n}\varphi(S_p)]}{\mathbb{E}[\varphi(S_p)]}=\mathbb{E}[\mathbb{1}_{\Lambda_n}M_n^{\varphi}]\,,$$

where $M_n^{\varphi}:=\varphi(S_n)(S_n-X_n)+\phi(S_n)$.

(b) $(M_n^{\varphi},n\geq 0)$ is a positive martingale, with $M_0^{\varphi}=1$, non uniformly integrable; in fact, M_n^{φ} tends a.s. to 0 when $n\to\infty$.

2. Call Q^{φ} the probability on $(\Omega,\mathcal{F}_\infty)$ characterized by

$$\forall n\in\mathbb{N},\Lambda_n\in\mathcal{F}_n,\quad Q^{\varphi}(\Lambda_n)=\mathbb{E}[\mathbb{1}_{\Lambda_n}M_n^{\varphi}]\,.$$

Then

(a) S_∞ is finite Q^{φ}-a.s. and satisfies for every $k\in\mathbb{N}$:

$$Q^{\varphi}(S_\infty=k)=\varphi(k)\,.$$

(b) Under Q^{φ}, the r.v. $T_\infty:=\inf\{n\geq 0,\ X_n=S_\infty\}$ (which is not a stopping time in general) is a.s. finite and

i. $(X_{n\wedge T_\infty},\ n\geq 0)$ and $(S_\infty-X_{T_\infty+n},\ n\geq 0)$ are two independent processes;

ii. conditional on the r.v. S_∞, the process $(X_{n\wedge T_\infty},\ n\geq 0)$ is a standard random walk stopped when it first hits the level S_∞;

iii. $(S_\infty - X_{T_\infty + n},\ n \geq 0)$ *is a 3-Bessel walk started from* 0.

3. Put $R_n = 2S_n - X_n$. *Under* Q^φ, $(R_n, n \geq 0)$ *is a 3-Bessel walk independent of* S_∞.

The 3-Bessel walk is the Markov chain $(R_n, n \geq 0)$, with values in $\mathbb{N} = \{0, 1, 2, \ldots\}$, whose transition probabilities from $x \geq 0$ are given by

$$\pi(x, x+1) = \frac{x+2}{2x+2}\ ; \qquad \pi(x, x-1) = \frac{x}{2x+2}\ . \tag{1.2}$$

The 3-Bessel* walk is the Markov chain $(R^*_n, n \geq 0)$, valued in $\mathbb{N}^* = \{1, 2, \ldots\}$, such that $R^* - 1$ is a 3-Bessel walk. So its transition probabilities from $x \geq 1$ are

$$\pi^*(x, x+1) = \frac{x+1}{2x}\ ; \qquad \pi^*(x, x-1) = \frac{x-1}{2x}\ .$$

the 3-Bessel walk and the 3-Bessel* walk, will play a role in this work; they are identical up to a one-step space shift.
This result and those of [Deb09] can let think that the process of penalization gives rather intuitive results. Nevertheless, the following Theorems show that this intuition can be false and it is necessary to lead the calculations to their terms.

1) In the first section, G is a function of the one-sided maximum till the last zero before p, i.e. :

$$G_p = \varphi(S_{g_p})$$

where $g_p := \sup\{k \leq p, X_k = 0\}$ and where φ satisfies (1.1) and :

$$\sum_{k=0}^{\infty} k\varphi(k) < \infty. \tag{1.3}$$

To study this penalisation we have to introduce $(\gamma_n, n \geq 0)$ the number of 0 before n and we also recall that for all real a, $a^+ := \sup(a, 0)$. The result of this first section is summarized in the following statement :

Theorem 1.2. *1. (a) For all* $n \geq 0$ *et all* $\Lambda_n \in \mathcal{F}_n$*:*

$$\lim_{p\to\infty} \frac{E\left[\mathbb{1}_{\Lambda_n} \varphi\left(S_{g_p}\right)\right]}{E\left[\varphi\left(S_{g_p}\right)\right]} = E\left[\mathbb{1}_{\Lambda_n} M_n\right], \tag{1.4}$$

where $M_n = \frac{1}{2}\varphi\left(S_{g_n}\right)|X_n| + \varphi\left(S_n\right)\left(S_n - X_n^+\right) + \phi\left(S_n\right)$.

(b) Moreover, $(M_n, n \geq 0)$ *is a positive martingale, not uniformly integrable.*

2. Let Q *be the probability on* $(\Omega, \mathcal{F}_\infty)$*, induces by:*

$$\forall n \geq 0, \Lambda_n \in \mathcal{F}_n, Q\left(\Lambda_n\right) := E\left[\mathbb{1}_{\Lambda_n} M_n\right].$$

Then under the probability Q*:*

(a) let $g := \sup\{k \geq 0, X_k = 0\}$*. Then* $Q\left(0 \leq g < \infty\right) = 1$.

(b) $Q\left(S_\infty = \infty\right) = \frac{1}{2}$ *and, conditionally on* $S_\infty < \infty$, φ *is the density of* S_∞.

(c) (S_g, γ_g) *admits as density:*

$$f_{\gamma_g, S_g}(a,k) := \begin{cases} \left(\frac{1}{2}\right)^a \varphi(0) & , \text{ for } k = 0 \\ \frac{1}{2}\left\{\left(1 - \frac{1}{2(k+1)}\right)^{a-1} - \left(1 - \frac{1}{2k}\right)^{a-1}\right\}\varphi(k) & , \text{ otherwise.} \end{cases}$$

In particular, S_g *admits* φ *as density.*

3. *Under Q:*

   (a) $(X_n, n \leq g)$ *and* $(X_n, n > g)$ *are two independent processes.*

   (b) *With probability* $\frac{1}{2}$, $(X_{g+n}, n \geq 0)$ *(respectively to* $(-X_{g+n}, n \geq 0)$*) is a 3-Bessel* walk.*

   (c) *Conditionally on* $\gamma_g = a$ *and* $S_g = b$*, the process* $(X_n, n \leq g)$ *is a symmetric random walk stopped in* τ_a *and conditionally on* $S_{\tau_a} = b$*.*

2) In the second section, $G_p = \varphi\left(S_{d_p}\right)$ where $d_p := \inf\{k \geq p, X_k = 0\}$ the first zero after p and φ satisfies (1.1) and (1.3). Let $f : \mathbb{N} \times \mathbb{Z} \to \mathbb{R}^+$ such that:

$$f(b,a) := 1\!\!1_{b=0}\varphi(0) + 1\!\!1_{b\neq 0}\left[\varphi(b)\left(1 - \frac{a^+}{b}\right) + a^+\sum_{k=b}^{\infty}\frac{\varphi(k)}{k(k+1)}\right].$$

The main result of this section is :

Theorem 1.3. *For all* $n \geq 0$ *and all* $\Lambda_n \in \mathcal{F}_n$*:*

$$\lim_{p\to\infty}\frac{E\left[1\!\!1_{\Lambda_n}\varphi\left(S_{d_p}\right)\right]}{E\left[\varphi\left(S_{d_p}\right)\right]} = \lim_{p\to\infty}\frac{E\left[1\!\!1_{\Lambda_n}f\left(S_p, X_p\right)\right]}{E\left[f\left(S_p, X_p\right)\right]} = \lim_{p\to\infty}\frac{E\left[1\!\!1_{\Lambda_n}\varphi\left(S_p\right)\right]}{E\left[\varphi\left(S_p\right)\right]} = E\left[1\!\!1_{\Lambda_n}M_n^{\varphi}\right],$$

where $M_n^{\varphi} := \varphi\left(S_n\right)\left(S_n - X_n\right) + 1 - \phi\left(S_n\right)$*.*
We remark that we obtain the same martingale as the one obtained for the penalisation by a function of the maximum(cf [Deb09]), i.e. that the penalisation by $\varphi\left(S_{d_p}\right)$ *is the same as the penalisation by* $\varphi\left(S_p\right)$*.*

3) In the third section, φ has to satisfy a stronger integrability condition :

$$\sum_{k\geq 0}k^2\varphi(k) < \infty \tag{1.5}$$

and $G_p = \varphi\left(S_p^*\right)$ where $S_p^* = \sup_{n\leq p}|X_n|$, the bilateral supremum of $(X_n)_{n\geq 0}$. The result of this alinea is :

Theorem 1.4. 1. (a) *Let* $n \in \mathbb{N}$, $\Lambda_n \in \mathcal{F}_n$*:*

$$\lim_{p\to\infty}\frac{E\left[1\!\!1_{\Lambda_n}\varphi\left(S_{g_p}^*\right)\right]}{E\left[\varphi\left(S_{g_p}^*\right)\right]} = E\left[1\!\!1_{\Lambda_n}M_n^*\right],$$

where $M_n^* = \varphi\left(S_{g_n}^*\right)|X_n| + \varphi\left(S_n^*\right)\left(S_n^* - |X_n|\right) + \phi\left(S_n^*\right)$*.*

(b) *Moreover,* $(M_n^*, n \geq 0)$ *is a non uniformly, integrable positive* $\mathcal{F}_n$ *martingale .*

2. *Let* Q^* *be the probability on* $(\Omega, \mathcal{F}_\infty)$ *induced by:*

$$\forall n \in \mathbb{N}, \Lambda_n \in \mathcal{F}_n : \quad Q^*\left(\Lambda_n\right) := E\left[1\!\!1_{\Lambda_n}M_n^*\right].$$

So, under Q^**:*

   (a) *Let* $g := \sup\{k \geq 0, X_n = 0\}$*. then* g *is finite and* $S_\infty = \infty$ *a.s.*

   (b) *The law of the couple* (S_g, γ_g) *is:*

$$f_{\gamma_g,S_g}(a,k) = \begin{cases} \varphi(0) & , \text{ if } a = k = 0 \\ \left\{\left(1 - \frac{1}{k+1}\right)^a - \left(1 - \frac{1}{k}\right)^a\right\}\varphi(k) & , \text{ for } a \geq 0,\ k > 0. \end{cases}$$

We deduced that φ *is the density of* S_g*.*

3. *Under* Q^**:*

(a) $(X_n, n \leq g)$ *and* $(X_n, n > g)$ *are two independent processes.*

(b) *With probability* $\frac{1}{2}$, $(X_{g+n}, n \geq 0)$ *(respectively* $(-X_{g+n}, n \geq 0)$*) is a three dimensional Bessel* walk.*

(c) *Conditionally on* $\gamma_g = a$ *and* $S_g^* = b$, $(X_n, n \leq g)$ *is a symmetric random walk stopped in* τ_a *and conditionally on* $S_{\tau_a}^* = b$.

4) Finally, in order to be comprehensive, we fix an integer $a > 0$ and consider the penalisation functional :

$$G_p = \mathbb{1}_{\{S_p^* < a\}}.$$

We obtain :

Theorem 1.5. *1. For each* $n \geq 0$ *and each* $\Lambda_n \in \mathcal{F}_n$ *:*

$$\lim_{p\to\infty} \frac{\mathbb{E}\left[\mathbb{1}_{\{\Lambda_n,\, S_p^* < a\}}\right]}{\mathbb{E}\left[\mathbb{1}_{\{S_p^* < a\}}\right]} := \mathbb{E}\left[\mathbb{1}_{\{\Lambda_n, S_n^* < a\}} M_n\right],$$

where $M_n := \mathbb{1}_{\{\Lambda_n, S_n^* < a\}} \left(\cos\left(\frac{\pi}{2a}\right)\right)^{-n} \sin\left(\frac{\pi(a - X_n)}{2a}\right)$ *is a positive martingale non uniformly integrable.*

2. Let us define a new probability Q *on* $(\Omega, \mathcal{F}_\infty)$ *characterized by :*

$$\forall n \in \mathbb{N},\, \forall \Lambda_n \in \mathcal{F}_n,\, Q(\Lambda_n) := \mathbb{E}\left[\Lambda_n M_n\right].$$

Under this new probability Q, $(X_n, n \geq 0)$ *has the following transition probabilities for* $-b+1 \leq k \leq a-1$*:*

$$Q(X_{n+1} = k+1 | X_n = k) = \frac{\sin\left(\frac{a-k-1}{2a}\pi\right)}{2\cos\left(\frac{\pi}{2a}\right)\sin\left(\frac{a-k}{2a}\pi\right)},$$

$$Q(X_{n+1} = k-1 | X_n = k) = \frac{\sin\left(\frac{a-k+1}{2a}\pi\right)}{2\cos\left(\frac{\pi}{2a}\right)\sin\left(\frac{a-k}{2a}\pi\right)}.$$

2 Penalisation by a function of $\mathbf{S_{g_p}}$, proof of Theorem 1.2

1) To establish the first point of the Theorem (formula 1.4), we need the following lemma :

Lemma 2.1. $\forall x \in \mathbb{N}$, $\forall a \in \mathbb{Z}\backslash]x, +\infty[$ *:*

$$\frac{\mathbb{P}_a(S_{g_n} = x, T_0 < n)}{2\mathbb{P}_a(S_n = 0)}$$

is bounded above by 1 for all $n \geq 0$ *and tends to 1 when* $n \to \infty$.

Proof. We have to see that :

$$\mathbb{P}_a(S_{g_n} < x, T_0 < n) = \mathbb{P}_a(T_0 < n, g_n < T_x) = \mathbb{P}_a(T_0 < n < T_x + T_0 \circ \theta_{T_x}),$$

where $\{\theta_n\}_n$ denote the family of shifts operators. We split in two cases according to the sign of a.
First, for $a \leq 0$, according to the Desiré André's principle :

$$\mathbb{P}_a(S_{g_n} < x, T_0 < n) = \mathbb{P}_a(0 \leq S_n < 2x) = \mathbb{P}(|a| \leq S_n < 2x + |a|).$$

Which implies :

$$\begin{aligned}\mathbb{P}_a\left(S_{g_n}=x,T_0<n\right) &= \mathbb{P}\left(|a|\leq S_n<2x+2+|a|\right)-\mathbb{P}\left(|a|\leq S_n<2x+|a|\right)\\ &= \mathbb{P}\left(S_n=2x+|a|\right)+\mathbb{P}\left(S_n=2x+1+|a|\right).\end{aligned}$$

And for $a>0$:

$$\begin{aligned}\mathbb{P}_a\left(S_{g_n}<x,T_0<n\right) &= \mathbb{P}_a\left(n<T_x+T_0\circ\theta_{T_x},T_0<T_x\right)-\mathbb{P}_a\left(n<T_x+T_0\circ\theta_{T_x},n\leq T_0<T_x\right)\\ &= \mathbb{P}_a\left(n<T_x+T_0\circ\theta_{T_x},T_0<T_x\right)-\mathbb{P}_a\left(n\leq T_0<T_x\right)\\ &= \mathbb{P}_a\left(n<T_x+T_0\circ\theta_{T_x}\right)-\mathbb{P}_a\left(n<T_x+T_0\circ\theta_{T_x},T_x<T_0\right)-\mathbb{P}_a\left(n\leq T_0<T_x\right)\\ &= \mathbb{P}_a\left(n<T_x+T_0\circ\theta_{T_x}\right)-\mathbb{P}_a\left(n<T_0,T_x<T_0\right)-\mathbb{P}_a\left(n\leq T_0<T_x\right)\\ &= \mathbb{P}_a\left(n<T_x+T_0\circ\theta_{T_x}\right)-\mathbb{P}_a\left(n<T_0\right)+\mathbb{P}_a\left(n<T_0<T_x\right)-\mathbb{P}_a\left(n\leq T_0<T_x\right)\\ &= \mathbb{P}_a\left(n<T_{2x}\right)-\mathbb{P}_a\left(n<T_0\right)-\mathbb{P}_a\left(n=T_0<T_x\right).\end{aligned}$$

And consequently :

$$\begin{aligned}\mathbb{P}_a\left(S_{g_n}=x,T_0<n\right) &= \mathbb{P}_a\left(2x\leq S_n<2x+2\right)-\mathbb{P}_a\left(n=T_0<T_{x+1}\right)+\mathbb{P}_a\left(n=T_0<T_x\right)\\ &= \mathbb{P}_a\left(2x\leq S_n<2x+2\right)-\mathbb{P}_a\left(n=T_0,S_n=x\right)\\ &\leq \mathbb{P}\left(2x-a\leq S_n<2x+2-a\right).\end{aligned}$$

In the ratio $\frac{\mathbb{P}(S_n=x)}{\mathbb{P}(S_n=0)}$, the denominator is bounded below by $\mathbb{P}(X_1=\ldots=X_n=-1)=2^{-n}$; so it does not vanish. Observe that, for even n and even $k\geq 2$,

$$\frac{\mathbb{P}(S_n=k-1)}{\mathbb{P}(S_n=0)}=\frac{\mathbb{P}(S_n=k)}{\mathbb{P}(S_n=0)}=\frac{p_{n,k}}{p_{n,0}}=\left(\frac{n-k+2}{n+2}\right)\left(\frac{n-k+4}{n+4}\right)\cdots\left(\frac{n}{n+k}\right);$$

and for odd n and odd $k\geq 1$,

$$\frac{\mathbb{P}(S_n=k-1)}{\mathbb{P}(S_n=0)}=\frac{\mathbb{P}(S_n=k)}{\mathbb{P}(S_n=0)}=\frac{p_{n,k}}{p_{n,1}}=\left(\frac{n-k+2}{n+1}\right)\left(\frac{n-k+4}{n+3}\right)\cdots\left(\frac{n+1}{n+k}\right).$$

Clearly, these products are smaller than 1 and tend to 1 when n goes to infinity. This proves the first point of the lemma. Obviously, when $a\leq 0$, the ratio tends to 1 when n goes to infinity. In the other case, it appears clearly that $\mathbb{P}_a\left(n=T_0,S_n=x\right)\leq\mathbb{P}_a\left(I_n=0,S_n=x\right)$ tends to zero faster then the quantity $\mathbb{P}(S_n=0)$ (we have explicitely the expression of $\mathbb{P}_a\left(I_n=0,S_n=x\right)$ a little bit further in this paper). □

Remark 2.2. *Remark that we have also proved that for each $k\geq 0$ the ratio :*

$$\frac{\mathbb{P}(S_p=k)}{\mathbb{P}(S_p=0)}$$

is bounded above by 1 and tends to 1 when $p\to+\infty$.

Lemma 2.3. *For all $x\in\mathbb{Z}$ and $a\in\mathbb{Z}\backslash]x,+\infty[$:*

$$\frac{\mathbb{E}_a\left[\varphi\left(x\vee S_{g_n}\right)\mathbb{1}_{T_0<n}\right]}{2\mathbb{P}(S_n=0)}$$

is bounded above by $(x-a^+)\varphi(x)+\phi(x)$ and tends to $(x-a^+)\varphi(x)+\phi(x)$ when $n\to\infty$.

Proof. Write :

$$\begin{aligned}\frac{\mathbb{E}_a\left[\varphi\left(x\vee S_{g_n}\right)\mathbb{1}_{T_0<n}\right]}{2\mathbb{P}(S_n=0)} &= \frac{\mathbb{P}_a\left(S_{g_n}<x,T_0<n\right)}{2\mathbb{P}(S_n=0)}\varphi(x)\\ &+ \sum_{k=x}^{\infty}\frac{\mathbb{P}_a\left(S_{g_n}=k,T_0<n\right)}{2\mathbb{P}(S_n=0)}\varphi(k).\end{aligned}$$

By lemma 2.1, this sum is bounded above by $(x-a^+)\varphi(x)+\phi(x)$ and tends to this value by dominated convergence. □

To prove point 1.a, we split:

$$\mathbb{E}\left[\varphi\left(S_{g_p}\right) \mid \mathcal{F}_n\right] = \mathbb{E}\left[\varphi\left(S_{g_p}\right) 1\!\!1_{g_p<n} \mid \mathcal{F}_n\right] + \mathbb{E}\left[\varphi\left(S_{g_p}\right) 1\!\!1_{g_p\geq n} \mid \mathcal{F}_n\right] := (1) + (2).$$

a) As $0 \leq n \leq p$, we can write $(\tilde{X}_k, k \geq 0) := (X_{n+k} - X_n, k \geq 0)$, a standard random walk independent of $\mathcal{F}_n$. We denote by $\tilde{T}_a$ and $\tilde{S}$, the hitting time of the level a and the supremum associated to $\tilde{X}$. Obviously on $\{g_p < n\}$, $\{g_p = g_n\}$. Hence :

$$(1) = \varphi(S_{g_n})\tilde{\mathbb{P}}_{X_n}(\tilde{T}_0 > p - n) = \varphi(S_{g_n})\tilde{\mathbb{P}}\left(\tilde{S}_{p-n} \leq |X_n|\right),$$

where $\tilde{\mathbb{P}}$ only integrates over $\tilde{S}$, X_n being kept fixed. Eventually, according to remark 2.2 :

$$\frac{\mathbb{E}\left[1\!\!1_{\Lambda_n, g_p<n}\varphi\left(S_{g_p}\right)\right]}{\mathbb{P}(S_{p-n} = 0)} = \frac{\mathbb{E}\left[1\!\!1_{\Lambda_n}\varphi\left(S_{g_n}\right)\tilde{\mathbb{P}}\left(\tilde{S}_{p-n} \leq |X_n|\right)\right]}{\mathbb{P}(S_{p-n} = 0)}$$

is bounded above by $\mathbb{E}\left[1\!\!1_{\Lambda_n}\varphi(S_{g_n})|X_n|\right]$ which is integrable and tends to $\left[1\!\!1_{\Lambda_n}\varphi(S_{g_n})|X_n|\right]$ when p goes to ∞.

b) We now study the behaviour of (2). We use the same notations as before, adding for all $p \geq 0$, $\tilde{g}_p$ the last zero before p associated to $\tilde{X}$. Hence :

$$(2) = \frac{\mathbb{E}\left[\mathbb{E}\left[\varphi\left(S_n \vee S_{[n,g_p]}\right) 1\!\!1_{g_p\geq n} \mid \mathcal{F}_n\right]\right]}{2\mathbb{P}(S_{p-n} = 0)} = \mathbb{E}\left[\frac{\tilde{\mathbb{E}}_{X_n}\left[\varphi\left(S_n \vee \tilde{S}_{\tilde{g}_{p-n}}\right) 1\!\!1_{\tilde{T}_0\leq p-n}\right]}{2\mathbb{P}(S_{p-n} = 0)}\right],$$

where $\tilde{\mathbb{E}}$ integrates on $\tilde{S}$, $\tilde{g}$ and $\tilde{T}_0$, the variables S_n and X_n being kept fixed. When p tends to infinity, Lemma 2.1 says that the ratio in the right hand side tends to $(S_n - X_n^+)\varphi(S_n) + \phi(S_n)$ and is dominated by the same quantity, which is integrable. As a result :

$$\frac{\mathbb{E}\left[1\!\!1_{\Lambda_n}\varphi(S_{g_p})\right]}{\mathbb{P}(S_{p-n} = 0)} \begin{cases} \text{is bounded above by } \mathbb{E}\left[1\!\!1_{\Lambda_n}M_n\right] \text{ forall } p \geq n. \\ \text{and tends to } \mathbb{E}\left[1\!\!1_{\Lambda_n}M_n\right] \text{ when } p \to \infty. \end{cases}$$

Taking in particular $\Lambda_n = \Omega$, one also has

$$\frac{\mathbb{E}\left[\varphi\left(S_{g_p}\right)\right]}{2\mathbb{P}(S_{p-n} = 0)} \underset{p\to\infty}{\to} \mathbb{E}\left[M_n\right] = 1,$$

and to establish point 1 of Theorem 1.2, it suffices to take the ratio of these two limits.

ii) Let us prove now that $(M_n, n \geq 0)$ is a $(\mathcal{F}_n)$-martingale under $\mathbb{P}$. For typographical simplicity, we write :

$$M_n = \mathcal{A}_n + \mathcal{B}_n$$

where $\mathcal{A}_n := \varphi(S_n)(S_n - X_n^+) + \sum_{k=S_n}^{\infty} \varphi(k)$ and $\mathcal{B}_n := \frac{1}{2}\varphi(S_{g_n})|X_n|$.

We suppose that $n > 0$, the case $n = 0$ being trivial.

On $\{X_n \geq 1\}$, $\mathcal{A}_{n+1}$ is in fact the martingale found in the Theorem 1.1, then conditional on $\mathcal{F}_n$, this quantity is equal to $\mathcal{A}_n$.

On $\{X_n \leq -1\}$, $S_{n+1} = S_n = S_{g_n}$ and $X_{n+1}^+ = X_n^+ = 0$, obviously on this event $\mathcal{A}_{n+1} = \mathcal{A}_n$.

Eventually, on $\{X_n = 0\}$, as $|X_{n+1}| = 1$, we have $S_{n+1} = S_n$. So, summing on the possible values of X_{n+1}, 1 and -1, it is easy to check that :

$$\mathbb{E}[1\!\!1_{X_n=0}\mathcal{A}_{n+1}|\mathcal{F}_n] = 1\!\!1_{X_n=0}\left(\mathcal{A}_n - \frac{1}{2}\varphi(S_{g_n})\right). \tag{2.6}$$

It just remains the quantity $\mathcal{B}_n := \frac{1}{2}\varphi(S_{g_n})|X_n|$.

On $\{|X_n| \geq 2\}$, $S_{g_{n+1}} = S_{g_n}$ and as the function $x \to |x|$ is harmonic for the symmetric random walk, except in 0, consequently $\mathbb{E}[1\!\!1_{|X_n|\geq 2}\mathcal{B}_{n+1}|\mathcal{F}_n] = 1\!\!1_{|X_n|\geq 2}\mathcal{B}_n$.

On $\{|X_n|=1\}$, either $|X_{n+1}|=2$ and in this case $S_{g_{n+1}}=S_{g_n}$ implies $\mathcal{B}_{n+1}=\varphi(S_{g_n})$, either $|X_{n+1}|=0$ and in this case $\mathcal{B}_{n+1}=0$. Then, immediately we have $\mathbb{E}[\mathbb{1}_{|X_n|=1}\mathcal{B}_{n+1}|\mathcal{F}_n]=\mathbb{1}_{|X_n|=1}\frac{1}{2}\varphi(S_{g_n})=\mathbb{1}_{|X_n|=1}\mathcal{B}_n$.
At last, on $\{X_n=0\}$, $S_{g_{n+1}}=S_{g_n}$ and consequently $\mathcal{B}_{n+1}=\frac{1}{2}\varphi(S_{g_n})$. So according to (2.6) :

$$\mathbb{E}[\mathbb{1}_{X_n=0}M_{n+1}|\mathcal{F}_n]=\mathbb{E}[\mathbb{1}_{X_n=0}(\mathcal{A}_{n+1}+\mathcal{B}_{n+1})|\mathcal{F}_n]=\mathbb{1}_{X_n=0}\mathcal{A}_n=\mathbb{1}_{X_n=0}M_n.$$

2) For p and n in $\mathbb{N}$, the event $\{S_n>p\}$ is equal to $\{T_p<n\}$. Using the definition of Q and the Doob's stopping Theorem :

$$Q\left(S_n>p\right)=Q\left(T_p<n\right)=\mathbb{E}[\mathbb{1}_{T_p<n}M_{T_p}]=\mathbb{E}\left[\mathbb{1}_{T_p<n}\left\{\frac{1}{2}\varphi\left(S_{g_{T_p}}\right)p+\phi\left(p\right)\right\}\right].$$

Moreover according to [LeG85] p.457-458, under $\mathbb{P}$, $S_{g_{T_p}}$ is a uniformly distributed random variable on $\{0,...,p-1\}$. As $n\to\infty$, the Lebesgue Theorem permits us to write:

$$\begin{aligned}Q\left(S_\infty>p\right) &= \lim_{n\to\infty}Q\left(S_n>p\right)=\mathbb{E}\left[\left\{\frac{1}{2}\varphi\left(S_{g_{T_p}}\right)p+\phi\left(p\right)\right\}\right]\\ &= \frac{p}{2}\mathbb{E}\left[\varphi\left(S_{g_{T_p}}\right)\right]+\phi\left(p\right)=\frac{p}{2}\sum_{k=0}^{p-1}\frac{1}{p}\varphi(k)+\phi(p)=\frac{1}{2}\sum_{k=0}^{p-1}\varphi(k)+\phi(p).\end{aligned}$$

Consequently $Q\left(S_\infty=\infty\right)=\lim_{p\to\infty}Q\left(S_\infty>p\right)=\frac{1}{2}$ and the half of the point 2.b is proved.
In order to prove point 2.a, we need, for $a>0$, the law under $\mathbb{P}$ of S_{d_a} conditionally on $\mathcal{F}_p$.

Lemma 2.4. *Let $k\geq a>0$, then:*

$$P_a\left(S_{T_0}=k\right)=\frac{a}{k(k+1)}.$$

Proof. A direct use of the stopping Theorem to the martingale $(X_n,n\geq 0)$ and the stopping time $T_0\wedge T_k$ gives us :

$$\mathbb{P}_a\left(T_0>T_k\right)=\frac{a}{k}.$$

We just have to remark that $\mathbb{P}_a\left(S_{T_0}=k\right)=\mathbb{P}_a\left(T_k<T_0\right)-\mathbb{P}_a\left(T_{k+1}<T_0\right)$ to achieve the proof. □

Lemma 2.5. *Let $\psi:\mathbb{N}\to\mathbb{R}^+$ be an integrable function. Then :*

$$\mathbb{E}\left[\psi\left(S_{d_p}\right)\mid\mathcal{F}_p\right]=\mathbb{1}_{S_p=0}\psi(0)+\mathbb{1}_{S_p\neq 0}\left\{\psi\left(S_p\right)\left(1-\frac{X_p^+}{S_p}\right)+X_p^+\sum_{k\geq S_p}\frac{\psi(k)}{k(k+1)}\right\}.$$

Proof. We easily obtain :

$$\mathbb{E}\left[\psi\left(S_{d_p}\right)\mid\mathcal{F}_p\right]=\mathbb{1}_{S_p=0}\psi\left(0\right)+\mathbb{E}\left[\mathbb{1}_{S_p\neq 0}\psi\left(S_p\vee S_{[p,d_p]}\right)\mid\mathcal{F}_p\right].$$

If $X_p\leq 0$, then $X_k\leq 0$ for all $p\leq k\leq d_p$ and consequently $X_p^+=S_{[p,d_p]}=0$. As a result, on $\{X_p\leq 0,S_p\neq 0\}$:

$$\psi\left(S_p\vee S_{[p,d_p]}\right)=\psi(S_p)=\psi(S_p)\left(1-\frac{X_p^+}{S_p}\right)+X_p^+\sum_{k\geq S_p}\frac{\psi(k)}{k(k+1)}.$$

Let $(\tilde{X}_q:=X_{q+p}\geq 0)$, a random walk starting from X_p and independent of $\mathcal{F}_p$ and we denote by $\tilde{S}$ and $\tilde{T}_0$ respectively the supremum and the hitting time of 0 associated to $\tilde{X}$. Then $S_{[p,d_p]}=\tilde{S}_{\tilde{T}_0}$.

In the following calculus, $\tilde{\mathbb{E}}$ only integrates $\tilde{S}_{\tilde{T}_0}$, X_p and S_p being kept fixed. Consequently, on $\{X_p > 0, S_p \neq 0\}$, according to lemma 2.4, $\mathbb{E}\left[\psi\left(S_{d_p}\right) \mid \mathcal{F}_p\right]$ equals to:

$$\begin{aligned} \tilde{\mathbb{E}}_{X_p}\left[\psi\left(S_p \vee \tilde{S}_{\tilde{T}_0}\right)\right] &= \sum_{k=X_p}^{S_p-1} \tilde{\mathbb{P}}_{X_p}\left(\tilde{S}_{\tilde{T}_0} = k\right)\psi\left(S_p\right) + \sum_{k \geq S_p}^{\infty} \tilde{\mathbb{P}}_{X_p}\left(\tilde{S}_{\tilde{T}_0} = k\right)\psi\left(k\right) \\ &= \sum_{k=X_p}^{S_p-1} \frac{X_p}{k(k+1)}\psi\left(S_p\right) + \sum_{k \geq S_p}^{\infty} \frac{X_p}{k(k+1)}\psi\left(k\right) \\ &= \left(1 - \frac{X_p}{S_p}\right)\psi\left(S_p\right) + \sum_{k \geq S_p}^{\infty} \frac{X_p}{k(k+1)}\psi\left(k\right). \end{aligned}$$

□

Fixing $a > 0$, according to Doob's stopping Theorem:

$$Q\left(g_p > a\right) = Q\left(d_a < p\right) = \mathbb{E}\left[\mathbb{1}_{d_a<p} M_{d_a}\right] = \mathbb{E}\left[\mathbb{1}_{d_a<p}\left\{\varphi\left(S_{d_a}\right) S_{d_a} + \phi\left(S_{d_a}\right)\right\}\right].$$

The events $\{g_p > a\}$ form an increasing sequence with limit $\{g > a\}$. Hence :

$$Q\left(g > a\right) = \lim_{p\to\infty} Q\left(g_p > a\right) = \mathbb{E}\left[\varphi\left(S_{d_a}\right) S_{d_a} + \phi\left(S_{d_a}\right)\right] = \mathbb{E}\left[\varphi\left(S_{d_a}\right) S_{d_a}\right] + \mathbb{E}\left[\phi\left(S_{d_a}\right)\right].$$

To achieve the proof of the point 2.a, we have to prove that each term tends to zero as $a \to \infty$. According to the Lebesgue Theorem $\mathbb{E}\left[\phi\left(S_{d_a}\right)\right] \underset{a\to\infty}{\to} 0$. We use lemma 2.5 with $\psi(x) := x\varphi(x)$:

$$\begin{aligned} \mathbb{E}\left[S_{d_a}\varphi\left(S_{d_a}\right)\right] &= \mathbb{E}\left[\mathbb{1}_{S_a\neq 0}\varphi\left(S_a\right)\left(S_a - X_a^+\right) + X_a^+ \sum_{k\geq S_a} \frac{\varphi(k)}{k+1}\right] \\ &\leq \mathbb{E}\left[\mathbb{1}_{S_a\neq 0}\varphi\left(S_a\right) S_a + \frac{X_a^+}{S_a+1}\sum_{k\geq S_a}\varphi(k)\right] \\ &\leq \mathbb{E}\left[\varphi\left(S_a\right) S_a + \phi\left(S_a\right)\right] = \mathbb{E}\left[\varphi\left(S_a\right) S_a\right] + \mathbb{E}\left[\phi\left(S_a\right)\right]. \end{aligned}$$

On the one hand, according to the Lebesgue Theorem $\mathbb{E}\left[\phi\left(S_a\right)\right] \underset{a\to\infty}{\to} 0$.
On the other hand $\mathbb{E}\left[\varphi\left(S_a\right) S_a\right] = \sum_{k=0}^{\infty} \mathbb{P}\left(S_a = k\right)\varphi(k)k$ is bounded above by $\sum_{k=0}^{\infty}\varphi(k)k < \infty$ and $\mathbb{P}(S_a = k)$ tends to 0 when $a \to \infty$. Again, according to the Lebesgue Theorem, $\mathbb{E}\left[\varphi\left(S_a\right) S_a\right]$ tends to 0 when $a \to \infty$.
3) First of all, let us establish preliminary results and remind that $\gamma_n := \sum_{k=0}^{n} \mathbb{1}_{X_k=0}$ is the number of visits to zero up to time n and denote $\tau_a := \inf\{p \geq 0, \gamma_n = a\}$.

Lemma 2.6. *For all $c > 0$ and $a \geq 1$:*

$$\mathbb{P}\left(S_{\tau_a} = c\right) = \begin{cases} \left(\frac{1}{2}\right)^{a-1} & \text{, if } c = 0 \\ \left(1 - \frac{1}{2(c+1)}\right)^{a-1} - \left(1 - \frac{1}{2c}\right)^{a-1} & \text{, otherwise.} \end{cases}$$

Lemma 2.7. *For all $n \geq 0$:*

$$\sum_{k=n}^{\infty} \frac{1}{k(k+1)}\left[k\varphi(k) + \phi(k)\right] = \frac{1}{n}\left(1 - \phi(n)\right).$$

Proof of lemma 2.6. This is obvious for $a = 1$ so let us suppose that $a \geq 2$. If $c = 0$, we have, with an obvious recurrence :

$$\mathbb{P}\left(S_{\tau_a} = 0\right) = \mathbb{P}\left(X_1 = -1\right)\mathbb{P}\left(S_{\tau_a} = 0 \mid X_1 = -1\right) = \frac{1}{2}\mathbb{P}\left(S_{\tau_{a-1}} = 0\right) = \left(\frac{1}{2}\right)^{a-1}.$$

Now suppose that $c > 0$. With those notations, using the strong Markov property and an obvious recurrence:

$$\mathbb{P}\left(S_{\tau_a} < c\right) = \mathbb{P}\left(\tau_a < T_c\right) = \mathbb{P}(\tau_a < T_c | \tau_2 < T_c)\mathbb{P}(\tau_2 < T_c) = \mathbb{P}(\tau_{a-1} < T_c)\mathbb{P}(\tau_2 < T_c)$$
$$= \mathbb{P}(\tau_2 < T_c)^{a-1} = \left[\frac{1}{2}(\mathbb{P}_1(\tau_1 < T_c) + \mathbb{P}_{-1}(\tau_1 < T_c))\right]^{a-1} = \left[\frac{1}{2}(\mathbb{P}_1(\tau_1 < T_c) + 1)\right]^{a-1}.$$

We have already seen that $\mathbb{P}_1\left(T_c < T_0\right) = \frac{1}{c}$, then $\mathbb{P}\left(S_{\tau_a} < c\right) = \left(1 - \frac{1}{2c}\right)^{a-1}$.
We can note that the law of γ_{T_c} is a geometric law of parameter $\frac{1}{2c}$. Finally:

$$\mathbb{P}\left(S_{\tau_a} = c\right) = \mathbb{P}\left(S_{\tau_a} < c\right) - \mathbb{P}\left(S_{\tau_a} < c+1\right) = \left(1 - \frac{1}{2(c+1)}\right)^a - \left(1 - \frac{1}{2c}\right)^a.$$

□

Proof of lemma 2.7. We have :

$$\sum_{k=n}^{\infty} \frac{\phi(k)}{k(k+1)} = \sum_{k=n}^{\infty}\sum_{l}^{\infty} \frac{\varphi(l)}{k(k+1)} = \sum_{l=n}^{\infty}\sum_{k=n}^{l} \frac{\varphi(l)}{k(k+1)} = \sum_{l=n}^{\infty} \varphi(l)\left(\frac{1}{n} - \frac{1}{k+1}\right),$$

hence :

$$\sum_{k=n}^{\infty} \frac{1}{k(k+1)}\left[k\varphi(k) + \phi(k)\right] = \frac{1}{n}\sum_{k=n}^{\infty} \varphi(k).$$

□

Let F be a functional, f_1 and f_2 be two functions from $\mathbb{N}$ to $\mathbb{R}^+$.

$$\begin{aligned}
\mathcal{A} &:= \mathbb{E}^Q\left[F\left(X_u, u \le g\right) f_1\left(\gamma_g\right) f_2\left(S_g\right)\right] \\
&= \sum_{a\ge1} \mathbb{E}^Q\left[F\left(X_u, u \le \tau_a\right) f_1\left(\gamma_{\tau_a}\right) f_2\left(S_{\tau_a}\right) \mathbb{1}_{\tau_a<\infty,\, \tau_{a+1}=\infty}\right] \\
&= \sum_{a\ge1} \mathbb{E}^Q\left[F\left(X_u, u \le \tau_a\right) f_1\left(\gamma_{\tau_a}\right) f_2\left(S_{\tau_a}\right) \left(\mathbb{1}_{\tau_a<\infty} - \mathbb{1}_{\tau_{a+1}<\infty}\right)\right] \\
&= \sum_{a\ge1} \mathbb{E}\left[F\left(X_u, u \le \tau_a\right) f_1\left(\gamma_{\tau_a}\right) f_2\left(S_{\tau_a}\right) \mathbb{1}_{\tau_a<\infty} M_{\tau_a}\right] \\
&- \mathbb{E}\left[F\left(X_u, u \le \tau_a\right) f_1\left(\gamma_{\tau_a}\right) f_2\left(S_{\tau_a}\right) \mathbb{1}_{\tau_{a+1}<\infty} M_{\tau_{a+1}}\right] \\
&= \sum_{a\ge1} \mathbb{E}\left[F\left(X_u, u \le \tau_a\right) f_1\left(\gamma_{\tau_a}\right) f_2\left(S_{\tau_a}\right) \left(M_{\tau_a} - M_{\tau_{a+1}}\right)\right].
\end{aligned}$$

Since:

$$M_{\tau_a} - M_{\tau_{a+1}} = \varphi\left(S_{\tau_a}\right) S_{\tau_a} + \phi\left(S_{\tau_a}\right) - \varphi\left(S_{\tau_{a+1}}\right) S_{\tau_{a+1}} - \phi\left(S_{\tau_{a+1}}\right).$$

One can write $S_{\tau_a+1} = S_{\tau_a} \vee \tilde{S}_{\tilde{\tau}_2}$ where $\tilde{S}$ and $\tilde{\tau}_2$ are respectively the unilateral supremum and the time of the return in 0 of the standard random walk $(X_{n+\tau_a}, n \ge 0)$ which is independent of $\mathcal{F}_{\tau_a}$. Hence :

$$M_{\tau_a} - M_{\tau_{a+1}} = \mathbb{1}_{\tilde{S}_{\tilde{\tau}_2} > S_{\tau_a}} \left(\varphi(S_{\tau_a})S_{\tau_a} - \varphi(\tilde{S}_{\tilde{\tau}_2})\tilde{S}_{\tilde{\tau}_2} + \sum_{k=S_{\tau_a}}^{\tilde{S}_{\tilde{\tau}_2}-1} \varphi(k)\right).$$

Then, we condition this quantity by $\mathcal{F}_{\tau_a}$:

$$\begin{aligned}
\mathbb{E}\left[M_{\tau_a}-M_{\tau_{a+1}}|\mathcal{F}_{\tau_a}\right] &= \sum_{l>S_{\tau_a}}\mathbb{P}(\tilde{S}_{\tilde{\tau}_1}=l)(\varphi(S_{\tau_a})S_{\tau_a}-\varphi(l)l+\sum_{k=l}^{l-1}\varphi(k))\\
&= \sum_{l>S_{\tau_a}}\frac{1}{2l(l+1)}(\varphi(S_{\tau_a})S_{\tau_a}-\varphi(l)l+\sum_{k=l}^{l-1}\varphi(k))\\
&= \frac{\varphi(S_{\tau_a})S_{\tau_a}}{2(S_{\tau_a}+1)}-\sum_{l>S_{\tau_a}}\frac{\varphi(l)}{2(l+1)}+\sum_{k\geq S_{\tau_a}}\varphi(k)\sum_{l\geq k+1}\frac{1}{2l(l+1)}\\
&= \frac{1}{2}\left(\frac{\varphi(S_{\tau_a})S_{\tau_a}}{S_{\tau_a}+1}-\sum_{l>S_{\tau_a}}\frac{\varphi(l)}{l+1}+\sum_{k\geq S_{\tau_a}}\frac{\varphi(k)}{k+1}\right)=\frac{\varphi(S_{\tau_a})}{2}.
\end{aligned}$$

Consequently: $\mathcal{A}=\sum_{a\geq 1}\frac{1}{2}\mathbb{E}\left[F\left(X_u,u\leq\tau_a\right)f_1\left(a\right)f_2\left(S_{\tau_a}\right)\varphi\left(S_{\tau_a}\right)\right]$ and with $F\equiv 1$:

$$\begin{aligned}
\mathcal{A}=\frac{1}{2}\sum_{a\geq 1}\sum_{k\geq 0}\mathbb{P}\left(S_{\tau_a}=k\right)f_1(a)f_2(k)\varphi(k) &= \frac{1}{2}\sum_{a\geq 1}\left(\frac{1}{2}\right)^{a-1}f_1(a)f_2(0)\varphi(0)\\
&+\frac{1}{2}\sum_{a\geq 1}\sum_{k\geq 1}\left\{\left(1-\frac{1}{2(k+1)}\right)^{a-1}-\left(1-\frac{1}{2k}\right)^{a-1}\right\}f_1(a)f_2(k)\varphi(k)
\end{aligned}$$

which gives us the density of (γ_g,S_g).
Now, summing over a we easily find that φ is the density of S_g under Q.
For proving 3.iii, we write the formula $\mathcal{A}$ in two different ways:

$$\begin{aligned}
\mathcal{A} &= \sum_{a\geq 1}\sum_{k\geq 0}f_{\gamma_g,S_g}(a,k)\mathbb{E}_Q\left[F\left(X_u,u\leq g\right)\mid S_g=k,\gamma_g=a\right]f_1\left(a\right)f_2\left(k\right)\\
&= \frac{1}{2}\sum_{a\geq 1}\sum_{k\geq 0}f_1\left(a\right)f_2\left(k\right)\varphi\left(k\right)\mathbb{P}\left(S_{\tau_a}=k\right)\mathbb{E}\left[F\left(X_u,u\leq\tau_a\right)\mid S_{\tau_a}=k\right]
\end{aligned}$$

The formulas that we obtained for $Q\left(S_g=k,\gamma_g=a\right)$ and $\mathbb{P}\left(S_{\tau_a}=k\right)$ imply obviously that for all $k,a\geq 0$:

$$\mathbb{E}_Q\left[F\left(X_u,u\leq g\right)\mid S_g=k,\gamma_g=a\right]=\mathbb{E}\left[F\left(X_u,u\leq\tau_a\right)\mid S_{\tau_a}=k\right].$$

3.ii) The study of the process $(X_n,\ n\geq 0)$ under Q^{h^+,h^-} starts with the next three lemmas.

Lemma 2.8. *Under $\mathbb{P}_1$ and conditional on the event $\{T_p<T_0\}$, the process $(X_n,\ 0\leq n\leq T_p)$ is a 3-Bessel* walk started from 1 and stopped when it first hits the level p (cf. [LeG85]).*

For typographical simplicity, call $T_{p,n}:=\inf\{k>n,\ X_k=p\}$ the time of the first visit to p after n, and $\mathcal{H}_l:=\left\{T_{p,\tau_l}<\tau_{l+1,\,X_{\tau_l+1}=1}\right\}$, the event that the l-th excursion is positive and reaches level p.

Lemma 2.9. *Under the law Q and conditional on the event $\mathcal{H}_l$, the process $(X_{n+\tau_l},\ 1\leq n\leq T_{p,\tau_l}-\tau_l)$ is a 3-Bessel* walk started from 1 and stopped when it first hits the level p.*

Proof. Let G be a functional on $\mathbb{Z}^n$:

$$\begin{aligned}
\mathcal{D} &:= Q\left[G\left(X_{\tau_l+1},...,X_{\tau_l+n}\right)\mathbb{1}_{n+\tau_l<T_{p,\tau_l}}\mid\mathcal{H}_l\right]\\
&= \frac{Q\left[G\left(X_{\tau_l+1},...,X_{\tau_l+n}\right)\mathbb{1}_{n+\tau_l<T_{p,\tau_l}<\tau_{l+1},X_{\tau_l+1}=1}\right]}{Q\left(\mathcal{H}_l\right)}\\
&= \frac{\mathbb{E}\left[G\left(X_{\tau_l+1},...,X_{\tau_l+n}\right)\mathbb{1}_{n+\tau_l<T_{p,\tau_l}<\tau_{l+1},X_{\tau_l+1}=1}M_{\tau_{l+1}}\right]}{\mathbb{E}\left[\mathbb{1}_{\mathcal{H}_l}M_{\tau_{l+1}}\right]}.
\end{aligned}$$

Obviously $M_{\tau_{l+1}} = \varphi(S_{\tau_{l+1}})S_{\tau_{l+1}}+\phi(S_{\tau_{l+1}})$ and conditionning by $\mathcal{F}_{T_{p,\tau_l}}$, we obtain $\mathbb{E}\left[M_{\tau_{l+1}} \mid \mathcal{F}_{T_{p,\tau_l}}\right] = \mathbb{E}_p\left[\varphi(S_{T_0})S_{T_0}+\phi(S_{T_0})\right]$, a constant. Conditioning by $\mathcal{F}_{T_{p,\tau_l}}$ the denominator and numerator of $\mathcal{D}$:

$$\begin{aligned}\mathcal{D} &= \frac{\mathbb{E}\left[G\left(X_{\tau_l+1},...,X_{\tau_l+n}\right)\mathbb{1}_{n+\tau_l<T_{p,\tau_l}<\tau_{l+1},X_{\tau_l+1}=1}\mathbb{E}\left[M_{\tau_{l+1}} \mid \mathcal{F}_{T_{p,\tau_l}}\right]\right]}{\mathbb{E}\left[\mathbb{1}_{\mathcal{H}_l}\mathbb{E}\left[M_{\tau_{l+1}} \mid \mathcal{F}_{T_{p,\tau_l}}\right]\right]} \\ &= \frac{\mathbb{E}\left[G\left(X_{\tau_l+1},...,X_{\tau_l+n}\right)\mathbb{1}_{n+\tau_l<T_{p,\tau_l}<\tau_{l+1},X_{\tau_l+1}=1}\right]}{\mathbb{E}\left[\mathbb{1}_{\mathcal{H}_l}\right]}\end{aligned}$$

Using the conditionning by $\mathcal{F}_{\tau_l+1}$ and the Markov property :

$$\mathcal{D} = \frac{\mathbb{E}_1\left[G\left(X_0,...,X_{n-1}\right)\mathbb{1}_{n-1<T_p<T_0}\right]}{\mathbb{P}_1\left(T_p<T_0\right)} = \mathbb{E}_1\left[G\left(X_0,...,X_{n-1}\right)\mathbb{1}_{n-1<T_p} \mid T_p<T_0\right].$$

□

On the other part, according to [LeG85] conditionally on $\{T_p<T_0\}$, the law of $(X_n, n<T_p)$ under $\mathbb{P}_1$ is the law of the 3-dimensional Bessel* walk. We deduce that, conditionally on $\{T_p<T_0\}$ under Q_1, $(X_n, n<T_p)$ is a 3-dimensional Bessel* walk. Making p go to infinity, we obtain that under Q_1 conditionally on $\{T_0=\infty\}$, $(X_n, n\geq 0)$ is a 3-dimensional Bessel* walk.
Obviously, by symmetry, under Q_{-1}, conditionally on $\{T_0=\infty\}$, $(-X_n, n\geq 0)$ is a three dimensional Bessel* walk. We deduce that $(X_n, n\geq g)$, is either a three dimensional Bessel* walk, either a reversed three dimensional Bessel* walk. It remains to know with what probability we have one or the other.
We have seen in 2.ii that S_∞ under Q was finished with probability $\frac{1}{2}$. As a three dimensional Bessel* walk goes to infinity in infinity, we deduce that $(X_n, n\geq g)$ is one or the other walk with probability $\frac{1}{2}$.

3 Penalisation by a function of $\mathbf{S_{d_p}}$

We've already seen according to lemma 2.5 that $\mathbb{E}\left[\varphi\left(S_{d_p}\right) \mid \mathcal{F}_p\right] = f\left(S_p, X_p\right)$. Moreover :

$$\begin{aligned}\mathbb{E}\left[f\left(S_p,X_p\right)\mathbb{1}_{\Lambda_n}\right] &= \mathbb{E}\left[\varphi\left(S_p\right)\mathbb{1}_{\Lambda_n}\right] - \mathbb{E}\left[\mathbb{1}_{S_p\neq 0}\varphi\left(S_p\right)\frac{X_p^+}{S_p}\mathbb{1}_{\Lambda_n}\right] \\ &+ \mathbb{E}\left[\mathbb{1}_{S_p\neq 0}X_p^+\sum_{k\geq S_p}\frac{\varphi(k)}{k(k+1)}\mathbb{1}_{\Lambda_n}\right] \\ &= (1)-(2)+(3).\end{aligned}$$

We already know (cf. [Deb09]) that $\forall n\geq 0$ and $\Lambda_n\in\mathcal{F}_n$:

$$\frac{\mathbb{E}[\mathbb{1}_{\Lambda_n}\varphi(S_p)]}{\mathbb{P}(S_{p-n}=0)}\begin{cases}\text{is bounded above by } \mathbb{E}[\mathbb{1}_{\Lambda_n}M_n^\varphi] \text{ for all } p\geq n \\ \text{and tends to } \quad \mathbb{E}[\mathbb{1}_{\Lambda_n}M_n^\varphi] \quad \text{when } p\to\infty\,.\end{cases} \tag{3.1}$$

Then we just have to prove that :

$$\frac{\mathbb{E}\left[(f(S_p,X_p)-\varphi(S_p))\mathbb{1}_{\Lambda_n}\right]}{\mathbb{P}(S_{p-n}=0)}$$

goes to 0 when $p\to\infty$.
In particular, if we take $\Lambda_n=\Omega$, we have :

$$\frac{\mathbb{E}\left[\varphi(S_{d_p})\mathbb{1}_{\Lambda_n}\right]}{\mathbb{P}(S_{p-n}=0)} \underset{p\to\infty}{\rightarrow} \mathbb{E}\left[M_n^\varphi\right]=1$$

and to establish Theorem 1.3, we take the ratio of the two limits.

ii) To study the behaviour of the last two terms, we need the following lemma:

Lemma 3.1. *For $b \geq 0$ and $a \leq b$:*

$$\frac{\mathbb{P}\left(S_p = b, X_p = a\right)}{\mathbb{P}(S_p = 0)}$$

is bounded above by 1 and tends to 0 when $p \to \infty$.

Proof. Remark that a and p must have the same parity, otherwise $\mathbb{P}\left(S_p = b, X_p = a\right)$ is equal to zero and the lemma is obvious. According to remark 2.2 :

$$\frac{\mathbb{P}(S_p = b, X_p = a)}{\mathbb{P}(S_p = 0)} \leq \frac{\mathbb{P}(S_p = b)}{\mathbb{P}(S_p = 0)} \leq 1.$$

With these hypothesis, according to the Desiré André's reflexion principle:

$$\begin{aligned}
\mathbb{P}\left(S_p = b, X_p = a\right) &= \mathbb{P}\left(S_p \geq b, X_p = a\right) - \mathbb{P}\left(S_p \geq b+1, X_p = a\right) \\
&= \mathbb{P}\left(X_p = 2b - a\right) - \mathbb{P}\left(X_p = 2b + 2 - a\right) = \left(\frac{1}{2}\right)^p \left[C_p^{\frac{p+2b-a}{2}} - C_p^{\frac{p+2b-a}{2}+1}\right] \\
&= \left(\frac{1}{2}\right)^p C_p^{\frac{p+2b-a}{2}} \left[1 - \frac{p-2b+a}{p+2b-a+2}\right] = \mathbb{P}(X_p = 2b-a)\frac{4b-2a+2}{p+2b-a+2}.
\end{aligned}$$

As $\mathbb{P}(X_p = 2b-a) = \mathbb{P}(S_p = 2b-a)$ (see for instance [Fel50] p.75) and using the remark 2.2:

$$\frac{\mathbb{P}(S_p = b, X_p = a)}{\mathbb{P}(S_p = 0)} = \frac{\mathbb{P}(S_p = 2b-a)}{\mathbb{P}(S_p = 0)} \frac{4b-2a+2}{p+2b-a+2} \underset{p\to\infty}{\to} 0.$$

□

Lemma 3.2. *Let $y \in \mathbb{N}$ and $x \in \mathbb{Z}$. Then :*

$$\frac{\mathbb{E}\left[\mathbb{1}_{y \vee (x+S_p) \neq 0} \varphi\left(y \vee (x+S_p)\right) \frac{(x+X_p)^+}{y \vee (x+S_p)}\right]}{\mathbb{P}(S_p = 0)}$$

is bounded above by $\mathbb{1}_{y>0}\left\{\varphi(y)\sum_{k=x}^{y} k^+\right\} + \sum_{k=y+1}^{\infty} k\varphi(k)$ and tends to 0 when $p \to \infty$.

Proof. To simplify, we consider two cases : $\{y > 0\}$ and $\{y = 0\}$. For typographical simplicity we denote respectively by $\mathcal{B}^+$ and $\mathcal{B}^0$ the first and the second cases. In the first case :

$$\mathcal{B}^+ := \frac{\mathbb{E}\left[\varphi\left(y \vee (x+S_p)\right) \frac{(x+X_p)^+}{y \vee (x+S_p)}\right]}{\mathbb{P}(S_p = 0)} = \sum_{\substack{k \geq 0 \\ -x < \ell \leq k}} \frac{\mathbb{P}(S_p = k, X_p = \ell)}{\mathbb{P}(S_p = 0)} \varphi\left(y \vee (x+k)\right) \frac{(x+\ell)^+}{y \vee (x+k)}.$$

According to lemma 3.1, we have :

$$\begin{aligned}
\mathcal{B}^+ &\leq \sum_{k \geq 0, -x < \ell \leq k} \varphi\left(y \vee (x+k)\right) \frac{(x+\ell)^+}{y \vee (x+k)} \leq \sum_{k \geq 0, -x < \ell \leq k} \varphi\left(y \vee (x+k)\right) \frac{(x+k)^+}{y \vee (x+k)} \\
&\leq \sum_{k \geq 0} \varphi\left(y \vee (x+k)\right) \frac{(x+k)^+ (x+k)}{y \vee (x+k)}.
\end{aligned}$$

Let us remark that we just consider cases where $k > -x$ which implies $0 < x+k \leq y \vee (x+k)$. Then :

$$\mathcal{B}^+ \leq \sum_{k \geq 0} \varphi\left(y \vee (x+k)\right)(x+k)^+ = \varphi(y) \sum_{k=x}^{y} k^+ + \sum_{k=y+1}^{\infty} k\varphi(k).$$

In the second case, $x \leq 0$ and $\{y \vee (x+S_p) \neq 0\} = \{S_p > -x\}$. Then :

$$\mathcal{B}^0 \leq \sum_{k>-x, -x<\ell\leq k} \varphi(x+k)\frac{x+\ell}{x+k} \leq \sum_{k>x} \varphi(x+k)(x+k) = \sum_{k\geq 1} k\varphi(k).$$

We can easily conclude using lemma 3.1 and the Lebesgue Theorem. □

For $0 \leq n \leq p$, one can write $S_p = S_n \vee (X_n + \tilde{S}_{p-n})$ where $\tilde{S}$ is the unilateral maximum of the standard random walk $(X_{n+k} - X_n)_{k\geq 0}$ which is independent from $\mathcal{F}_n$. Hence :

$$\mathbb{E}\left[1\!\!1_{S_p\neq 0}\varphi(S_p)\,\frac{X_p^+}{S_p}\,\middle|\,\mathcal{F}_n\right] = \tilde{\mathbb{E}}\left[1\!\!1_{S_n\vee(X_n+\tilde{S}_{p-n})\neq 0}\varphi(\tilde{S}_{p-n}+X_n)\frac{(X_n+\tilde{X}_{p-n})^+}{(X_n+\tilde{S}_{p-n})\vee S_n}\right],$$

where $\tilde{\mathbb{E}}$ only integrates over $\tilde{S}_{p-n}$ and $\tilde{X}_{p-n}$, S_n and X_n being kept fixed. Then, for $\Lambda_n \in \mathcal{F}_n$:

$$\frac{\mathbb{E}\left[1\!\!1_{\{\Lambda_n, S_p\neq 0\}}\varphi(S_p)\frac{X_p^+}{S_p}\right]}{\mathbb{P}(S_{p-n}=0)} = \frac{\mathbb{E}\left[1\!\!1_{\Lambda_n}\tilde{\mathbb{E}}\left[1\!\!1_{S_n\vee(X_n+\tilde{S}_{p-n})\neq 0}\varphi(\tilde{S}_{p-n}+X_n)\frac{(X_n+\tilde{X}_{p-n})^+}{(X_n+\tilde{S}_{p-n})\vee S_n}\right]\right]}{\mathbb{P}(S_{p-n}=0)}.$$

Lemma 3.2 says that the ratio in the right hand side tends to 0 when p tends to infinity and is dominated by $\varphi(S_n)\sum_{k=X_n}^{S_n} k^+ + \sum_{k=S_n+1}^{+\infty} k\varphi(k)$, which is integrable.
About the quantity (3), we have to remark that:

$$\mathbb{E}\left[1\!\!1_{y\vee(S_p+x)\neq 0}(X_p+x)^+ \sum_{k\geq y\vee(S_p+x)} \frac{\varphi(k)}{k(k+1)}\right] \leq \mathbb{E}\left[1\!\!1_{y\vee(S_p+x)\neq 0}\frac{(X_p+x)^+}{y\vee(S_p+x)} \sum_{k\geq y\vee(S_p+x)} \frac{\varphi(k)}{k}\right],$$

and we apply the same reasoning as the one for the quantity (2) with the function $h(x) = \sum_{k\geq x}\frac{\varphi(k)}{k}$ and $x>0$ instead of φ. We just have to check that $\sum_{x>0} xh(x) < \infty$. Easily :

$$\sum_{\tilde{x}>0} xh(x) = \sum x \sum_{k\geq x}\frac{\varphi(k)}{k} \leq \sum_{x>0}\sum_{k\geq x}\varphi(k) \leq \sum_{k\geq 0}\sum_{k\geq x}\varphi(k) \leq \sum_{k\geq 0} k\varphi(k) < \infty.$$

With the previous notations, we have :

$$\frac{\mathbb{E}\left[1\!\!1_{\Lambda_n, S_p\neq 0}X_p^+\sum_{k\geq S_p}\frac{\varphi(k)}{k(k+1)}\right]}{\mathbb{P}(S_{p-n}=0)} \leq \frac{\mathbb{E}\left[1\!\!1_{\Lambda_n}\tilde{\mathbb{E}}\left[1\!\!1_{S_n\vee(X_n+\tilde{S}_{p-n})\neq 0}h(\tilde{S}_{p-n}+X_n)\frac{(X_n+\tilde{X}_{p-n})^+}{(X_n+\tilde{S}_{p-n})\vee S_n}\right]\right]}{\mathbb{P}(S_{p-n}=0)},$$

and we can easily conclude that the ratio in the right hand side tends to 0 when p tends to infinity and is dominated by $h(S_n)\sum_{k=X_n}^{S_n} k^+ + \sum_{k=S_n+1}^{+\infty} kh(k)$, which is integrable.
To conclude the proof of the Theorem, always with the same notations we have :

$$\frac{\mathbb{E}\left[1\!\!1_{\Lambda_n}f(S_p,X_p)\right]}{\mathbb{P}(S_{p-n}=0)} = \frac{\mathbb{E}\left[1\!\!1_{\Lambda_n}f((S_n\vee(X_n+\tilde{S}_{p-n})), \tilde{X}_{p-n}+X_n)\right]}{\mathbb{P}(S_{p-n}=0)},$$

and when p goes to infinity, the ratio in the right hand side tends to M_n^φ and is dominated by $M_n^\varphi + (\varphi(S_n)+h(S_n))\sum_{k=X_n}^{S_n} k^+ + \sum_{k=S_n+1}^{+\infty} k(h(k)+\varphi(k))$ which is integrable.

4 Penalisation by a function $\mathbf{S^*_{g_p}}$

1) We start with the first point of Theorem 1.4. In order to prove this, we need :

Lemma 4.1. *Let $\alpha > 0$ and $a \in [-\alpha, \alpha]$. Then:*

$$\frac{\mathbb{P}_a\left(S^*_{g_p} = \alpha, T_0 < p\right)}{\mathbb{P}(S_p = 0)}$$

is bounded above by 2 and tends to 1 when $p \to \infty$.

To obtain this, we use a Tauberian Theorem :

Theorem 4.2 (Cf. [Fel71] p. 447)**.** *Given* $q_n \geq 0$*, suppose that the series*

$$S(s) = \sum_{n=0}^{\infty} q_n s^n$$

converges for $0 \leq s < 1$*. If* $0 < p < \infty$ *and if the sequence* $\{q_n\}$ *is monotone, then the two relations:*

$$S(s) \underset{s \to 1^-}{\sim} \frac{1}{(1-s)^p} C$$

and

$$q_n \underset{n \to \infty}{\sim} \frac{1}{\Gamma(p)} n^{p-1} C,$$

where $0 < C < \infty$*, are equivalent.*

and the following :

Lemma 4.3. *For* $a < 0 < b$ *and* $\lambda \in \mathbb{R}$ *:*

$$\mathbb{E}\left[(\cosh \lambda)^{-T_a \wedge T_b}\right] = \frac{\cosh \lambda \left(\frac{a+b}{2}\right)}{\cosh \lambda \left(\frac{a-b}{2}\right)}$$

Proof. Let's recall that $X_n = \sum_{k=1}^{n} Y_k$ and define the process $(W_n, n \geq 0)$ by

$$W_n := \frac{\cosh \lambda (X_n + \beta)}{(\cosh \lambda)^n},$$

where $\beta \in \mathbb{R}$. Let's prove that $(W_n, n \geq 0)$ is a $\mathcal{F}_n$-martingale:

$$\mathbb{E}\left[\cosh \lambda (X_{n+1} + \beta) \mid \mathcal{F}_n\right] = \mathbb{E}\left[\cosh \lambda (X_n + Y_{n+1} + \beta) \mid \mathcal{F}_n\right]$$
$$= \cosh \lambda (X_n + \beta)\, \mathbb{E}\left[\cosh \lambda Y_{n+1}\right] + \sinh \lambda (X_n + \beta)\, \mathbb{E}\left[\sin \lambda Y_{n+1}\right] = \cosh \lambda (X_n + \beta) \cosh \lambda$$

Clearly W is a martingale. Taking $\beta = -\frac{a+b}{2}$ and using the Doob's Theorem with $T_a \wedge T_b$:

$$\mathbb{E}\left[W_{T_a \wedge T_b}\right] = \mathbb{E}\left[W_0\right] = \cosh \lambda \left(\frac{a+b}{2}\right). \tag{4.1}$$

On the other hand, using Markov property :

$$\begin{aligned}
\mathbb{E}\left[W_{T_a \wedge T_b}\right] &= \mathbb{E}\left[W_{T_a} \mathbb{1}_{\{T_a < T_b\}} + W_{T_b} \mathbb{1}_{\{T_b < T_a\}}\right] \\
&= \mathbb{E}\left[\frac{\cosh \lambda \left(\frac{a-b}{2}\right)}{(\cosh \lambda)^{T_a}} \mathbb{1}_{\{T_a < T_b\}} + \frac{\cosh \lambda \left(\frac{b-a}{2}\right)}{(\cosh \lambda)^{T_b}} \mathbb{1}_{\{T_b < T_a\}}\right] \\
&= \cosh \lambda \left(\frac{a-b}{2}\right) \mathbb{E}\left[(\cosh \lambda)^{-T_a \wedge T_b}\right].
\end{aligned} \tag{4.2}$$

The formulas (4.1) and (4.2) permit to conclude. □

Now, we are able to prove lemma 4.1 :
as $\mathbb{P}_a\left(S^*_{g_p} = \alpha, T_0 < p\right) \leq \mathbb{P}_a\left(S_{g_p} = \alpha, T_0 < p\right)$, with lemma 2.1, the first point is trivial.
Let δ_β a geometric r.v. with parameter $0 < \beta < 1$ such that δ_β is independent of X. Then:

$$\mathbb{P}_a\left(S^*_{g_{\delta_\beta}} \leq \alpha\right) = \sum_{k=1}^{\infty} \mathbb{P}_a\left(S^*_{g_k} \leq \alpha\right) \mathbb{P}\left(\delta_\beta = k\right) = \sum_{k=1}^{\infty} \mathbb{P}_a\left(S^*_{g_k} \leq \alpha\right) (1-\beta)^{k-1} \beta \tag{4.3}$$

Note that $\left\{S^*_{g_p} \leq \alpha\right\} = \{g_p \leq T^*_\alpha\} = \{p \leq d_{T^*_\alpha}\} = \{p \leq T^*_\alpha + T_0.\theta_{T^*_\alpha}\}$. Hence :

$$\mathbb{P}_a\left(S^*_{g_{\delta_\beta}} \leq \alpha\right) = \mathbb{P}_a\left(\delta_\beta \leq d_{T^*_\alpha}\right) = 1 - \mathbb{P}_a\left(\delta_\beta > d_{T^*_\alpha}\right) = 1 - \mathbb{E}_a\left[\mathbb{E}_a\left[\mathbb{1}_{\delta_\beta > d_{T^*_\alpha}} \mid \mathcal{F}_{T^*_\alpha}\right]\right]$$
$$= 1 - \mathbb{E}_a\left[(1-\beta)^{d_{T^*_\alpha}}\right] = 1 - \mathbb{E}_a\left[(1-\beta)^{T^*_\alpha}(1-\beta)^{T_0.\theta_{T^*_\alpha}}\right] = 1 - \mathbb{E}_a\left[(1-\beta)^{T^*_\alpha}\right]\mathbb{E}\left[(1-\beta)^{T_\alpha}\right].$$

We have already seen (cf. [Deb09] p.353 and [ALR04]):

$$\mathbb{E}\left[(1-\beta)^{T_\alpha}\right] = \left(\frac{1+\sqrt{2\beta-\beta^2}}{1-\beta}\right)^{-\alpha}.$$

The symmetry of the quantity $\mathbb{E}_a\left[(1-\beta)^{T^*_\alpha}\right]$ permits us to assume that $a \geq 0$, without a loss of generality. Then, using the Markov property and lemma 4.3 with $(\cosh\lambda)^{-1} = 1-\beta$:

$$\mathbb{E}_a\left[(1-\beta)^{T^*_\alpha}\right] = \mathbb{E}\left[(1-\beta)^{T_{\{-\alpha-a\}}\wedge T_{\{\alpha-a\}}}\right] = \frac{\cosh a\lambda}{\cosh \alpha\lambda}.$$

When β goes to 0:

$$\mathbb{P}_a\left(S^*_{g_{\delta_\beta}} \leq \alpha\right) = 1 - \left(\frac{1+\sqrt{2\beta-\beta^2}}{1-\beta}\right)^{-\alpha}\frac{\cosh\left[\operatorname{argch}\left(\frac{1}{1-\beta}\right)a\right]}{\cosh\left[\operatorname{argch}\left(\frac{1}{1-\beta}\right)\alpha\right]} \underset{\beta\to 0}{\sim} \alpha\sqrt{2\beta}. \tag{4.4}$$

According to the formulas (4.3) and (4.4):

$$\sum_{k=1}^{\infty}\mathbb{P}_a\left(S^*_{g_k} \leq \alpha\right)(1-\beta)^k \underset{\beta\to 0}{\sim} \alpha\sqrt{\frac{2}{\beta}}(1-\beta).$$

In order to apply Theorem 4.2, put $\beta = 1-\omega$.This gives

$$\sum_{k=1}^{\infty}\mathbb{P}_a\left(S^*_{g_k} \leq \alpha\right)\omega^k \underset{\omega\to 1-}{\sim} \alpha\omega\sqrt{\tfrac{2}{(1-\omega)}} \underset{\omega\to 1-}{\sim} \alpha\sqrt{\frac{2}{(1-\omega)}}$$

and this Tauberien Theorem with $p = \frac{1}{2}$ et $C = \alpha\sqrt{2}$ permits ut to obtain :

$$\mathbb{P}_a\left(S^*_{g_p} \leq \alpha\right) \underset{p\to\infty}{\sim} \frac{1}{\Gamma\left(\frac{1}{2}\right)}p^{\frac{1}{2}-1}C = \left(\frac{2}{\pi p}\right)^{\frac{1}{2}}\alpha,$$

and the proof can be easily finished, knowing the behaviour of $\mathbb{P}(S_p = 0)$ when p goes to ∞. Thanks to this lemma, we have the following result:

Lemma 4.4. *Let $x \geq 0$ and $a \in [-x, x]$. Then:*

$$\frac{\mathbb{E}_a\left[\varphi(x \vee S^*_{g_p})\mathbb{1}_{T_0<p}\right]}{\mathbb{P}(S_p = 0)}$$

is bounded above by $2(\varphi(x)(x-|a|)+\phi(x))$ and tends to $\varphi(x)(x-|a|)+\phi(x)$ when p goes to infinity.

Proof. The proof is nearly the same as the one of lemma 2.3 □

With the same notations and arguments as inf Theorem 1.2 :

$$\mathbb{E}\left[\varphi\left(S^*_{g_p}\right) \mid \mathcal{F}_n\right] = \varphi\left(S^*_{g_n}\right)\tilde{\mathbb{P}}\left(\tilde{S}_{p-n} < |X_n|\right) + \mathbb{E}\left[\varphi\left(S^*_n \vee \tilde{S}^*_{\tilde{g}_{p-n}}\right)\mathbb{1}_{\tilde{T}_0 \leq p-n}\right] = (1) + (2).$$

The end of the proof is based on the proof of Theorem 1.2 and use lemma 2.3. The remaining details are left to the reader.
Now, in order to prove that $(M^*_n, n \geq 0)$ is a martingale, we show that conditioned by $\mathcal{F}_n$, $M^*_{n+1} - M^*_n$ is zero. The case $n = 0$ being trivial, we just study $n > 0$.
First, observe that on $\{X_n = 0\}$, $g_{n+1} = g_n = n$, $S^*_{n+1} = S^*_n$ and $|X_{n+1}| = 1$. Consequently, on this event, $M^*_{n+1} - M^*_n = 0$.
In the following, we suppose that $X_n \neq 0$, and we denote $A_n := \varphi(S^*_{g_{n+1}})|X_{n+1}| - \varphi(S^*_{g_n})|X_n|$ and $B_n := \varphi(S^*_{n+1})(S^*_{n+1} - |X_{n+1}|) - \varphi(S^*_n)(S^*_n - |X_n|) + \phi(S^*_{n+1}) - \phi(S^*_n)$. We now treat separately these two quantities :

- If $\{|X_n| \geq 2\}$, $g^*_{n+1} = g^*_n$ and in this way $A_n = \varphi(S^*_{g_n})(|X_{n+1}| - |X_n|)$. Conditioning on $\mathcal{F}_n$, this quantity equals zero, the function $x \to |x|$ being harmonic for the symmetric random walk except in 0.
  If $|X_n| = 1$, $A_n = 2\varphi(S_{g_{n^*}})\mathbb{1}_{X_{n+1} \neq 0} - \varphi(S_{g_{n^*}})$ conditional on $\mathcal{F}_n$ is obviously zero.

- If $\{|X_n| \leq S^*_n - 1\}$, then $S^*_{n+1} = S^*_n$. In this case, $B_n = \varphi(S^*_n)(|X_{n+1}| - |X_n|)$ and we conclude with the harmonicity of $x \to |x|$.
  Finally, on $\{S^*_n = |X_n|\}$, $B_n = \varphi(S^*_n)(\mathbb{1}_{S^*_n = S^*_{n+1}} - \mathbb{1}_{S^*_n + 1 = S^*_{n+1}})$ and conditionned on $\mathcal{F}_n$, it is clear that this quantity equals zero.

Consequently M^* is a martingale satisfying :

$$|M^*_n - M^*_0| \leq 3n,$$

and as $M^*_0 = 1$, one has $\mathbb{E}[M^*_n] = 1$. Observe that the positivity of M^* is obvious from the definitions of φ and ϕ.
2) Now, we prove point 2 of Theorem 1.4. For $a > 0$:

$$Q^*(g_p > a) = \mathbb{E}^{Q^*}[\mathbb{1}_{p > d_a}] = \mathbb{E}\left[\mathbb{1}_{p > d_a} M^*_{d_a}\right] = \mathbb{E}\left[\mathbb{1}_{p > d_a}\left\{\varphi\left(S^*_{d_a}\right) S^*_{d_a} + \phi\left(S^*_{d_a}\right)\right\}\right].$$

As a is fixed, the sequence of positive random variables $\left(\mathbb{1}_{p > d_a}\left\{\varphi\left(S^*_{d_a}\right) S^*_{d_a} + \phi\left(S^*_{d_a}\right)\right\}\right)_{p \geq 0}$ is increasing and tends to $\varphi\left(S^*_{d_a}\right) S^*_{d_a} + \phi\left(S^*_{d_a}\right)$, and the sequence of events $\{g_p > a\}$ is increasing and tends to $\{g > a\}$ when p tends to infinity . Hence, according to Lebesgue Theorem, when p goes to $+\infty$:

$$Q^*(g > a) = \mathbb{E}\left[\varphi\left(S^*_{d_a}\right) S^*_{d_a} + \phi\left(S^*_{d_a}\right)\right].$$

As $\phi(S^*_{d_a}) \leq 1$, Lebesgue Theorem implies that $\mathbb{E}\left[\phi\left(S^*_{d_a}\right)\right] \underset{a \to \infty}{\to} 0$.
It remains to prove $\mathbb{E}\left[\varphi\left(S^*_{d_a}\right) S^*_{d_a}\right] \underset{a \to \infty}{\to} 0$.

Lemma 4.5. *Let $\psi : \mathbb{N} \to \mathbb{R}^+$ such that $\sum_{k \geq 0} \varphi(k) < +\infty$. For $a > 0$:*

$$\mathbb{E}\left[\psi\left(S^*_{d_a}\right) \mid \mathcal{F}_a\right] = \mathbb{1}_{X_a = 0}\psi\left(S^*_a\right) + \mathbb{1}_{X_a \neq 0}\left\{\psi\left(S^*_a\right)\left(1 - \frac{|X_a|}{S^*_a}\right) + |X_a| \sum_{k = S_a}^{\infty} \frac{\psi(k)}{k(k+1)}\right\}.$$

Proof. Let $\tilde{S}^*_{\tilde{T}_0}$ be the bilateral maximum of a walk issued from X_a until the hitting time of the level 0 and which is independent of $\mathcal{F}_a$.

$$\begin{aligned}\mathbb{E}\left[\psi\left(S^*_{d_a}\right) \mid \mathcal{F}_a\right] &= \mathbb{E}\left[\psi\left(S^*_a \vee \tilde{S}^*_{\tilde{T}_0}\right) \mid \mathcal{F}_a\right] = \tilde{\mathbb{E}}_{|X_a|}\left[\psi\left(S^*_a \vee \tilde{S}^*_{\tilde{T}_0}\right)\right] \\ &= \mathbb{1}_{X_a = 0}\psi\left(S^*_a\right) + \mathbb{1}_{X_a \neq 0} \sum_{k \geq |X_a|} \mathbb{P}_{X_a}\left(\tilde{S}^*_{\tilde{T}_0} = k\right)\psi\left(S^*_a \vee k\right).\end{aligned}$$

On $\{X_a > 0\}$, as the sign of $\tilde{X}$ does not change between 0 and $\tilde{T}_0$, $\left\{\tilde{S}^*_{\tilde{T}_0} = k\right\} = \left\{\tilde{S}_{\tilde{T}_0} = k\right\}$. Moreover, thanks to the symmetry of $\tilde{X}$, on $\{X_a < 0\}$, $\left\{\tilde{S}^*_{\tilde{T}_0} = k\right\} = \left\{\tilde{S}_{\tilde{T}_0} = k\right\}$. So, according to lemma 2.4:

$$\mathbb{1}_{X_a \neq 0}\mathbb{P}_{X_a}\left(\tilde{S}^*_{\tilde{T}_0} = k\right) = \mathbb{1}_{X_a \neq 0}\mathbb{P}_{X_a}\left(\tilde{S}_{\tilde{T}_0} = k\right) = \frac{|X_a|}{k(k+1)}.$$

Consequently on $\{X_a \neq 0\}$:

$$\begin{aligned}\sum_{k\geq |X_a|} \mathbb{P}_{X_a}\left(\tilde{S}^*_{\tilde{T}_0} = k\right)\psi\left(S^*_a \vee k\right) &= \sum_{k\geq |X_a|} \psi\left(S^*_a \vee k\right)\frac{|X_a|}{k(k+1)} \\ &= \sum_{k=|X_a|}^{S^*_a-1} \psi\left(S^*_a\right)\frac{|X_a|}{k(k+1)} + \sum_{k\geq S^*_a}\psi\left(k\right)\frac{|X_a|}{k(k+1)} \\ &= \psi\left(S^*_a\right)|X_a|\left(\frac{1}{|X_a|} - \frac{1}{S^*_a}\right) + \sum_{k\geq S^*_a}\psi\left(k\right)\frac{|X_a|}{k(k+1)}.\end{aligned}$$

□

Applying this lemma with $\psi(x) = x\varphi(x)$:

$$\begin{aligned}\mathbb{E}\left[\varphi\left(S^*_{d_a}\right)S^*_{d_a}\right] &= \mathbb{E}\left[\mathbb{1}_{X_a=0}\varphi\left(S^*_a\right)S^*_a + \mathbb{1}_{X_a\neq 0}\left\{\varphi\left(S^*_a\right)\left(S^*_a - |X_a|\right) + |X_a|\sum_{k\geq S^*_a}\frac{\varphi(k)}{k+1}\right\}\right] \\ &\leq \mathbb{E}\left[\varphi\left(S^*_a\right)S^*_a\right] + \mathbb{E}\left[\frac{|X_a|}{S^*_a+1}\sum_{k\geq S^*_a}\varphi(k)\right] \leq \mathbb{E}\left[\varphi\left(S^*_a\right)S^*_a\right] + \mathbb{E}\left[\sum_{k\geq S^*_a}\varphi(k)\right] \\ &\leq \mathbb{E}\left[\varphi\left(S^*_a\right)S^*_a\right] + \mathbb{E}\left[\phi\left(S^*_a\right)\right].\end{aligned}$$

As $\phi(S^*_a) \leq 1$ and $\phi(S^*_a)$ tends to 0 a.s. when a tends to infinity, Lebesgue Theorem implies that $\mathbb{E}[\phi(S^*_a)] \underset{a\to+\infty}{\to} 0$. On the other hand:

$$\mathbb{E}\left[\varphi\left(S^*_a\right)S^*_a\right] = \sum_{k\geq 0}\varphi(k)k\mathbb{P}\left(S^*_a = k\right) \leq \sum_{k\geq 0}\varphi(k)k\mathbb{P}\left(S_a = k\right) \leq \mathbb{E}\left[\varphi\left(S_a\right)S_a\right],$$

and we have already proved that $\mathbb{E}[\varphi(S_a)S_a]$ tends to 0 when a tends to infinity (cf. point 2 Theorem 1.2). As a result g is Q-a.s. finite and :

$$Q^*\left(g = \infty\right) = \lim_{a\to\infty} Q^*\left(g > a\right) = 0.$$

□

3) We now prove the third and last point of the Theorem.

Lemma 4.6. *For all $a > 1$:*

$$\mathbb{P}\left(S^*_{\tau_a} = k\right) = \begin{cases} 0 & , \text{ if } k = 0 \\ \left(1 - \frac{1}{k+1}\right)^{a-1} - \left(1 - \frac{1}{k}\right)^{a-1} & , \text{ otherwise.}\end{cases}$$

Proof. Using Markov property and thanks to the symmetry of X :

$$\begin{aligned}\mathbb{P}\left(\tau_2 < T^*_k\right) &= \frac{1}{2}\left[\mathbb{P}_1\left(T_0 < T^*_k\right) + \mathbb{P}_{-1}\left(T_0 < T^*_k\right)\right] \\ &= \frac{1}{2}\left[\mathbb{P}_1\left(T_0 < T_k\right) + \mathbb{P}_{-1}\left(T_0 < T_{-k}\right)\right] = \mathbb{P}_1\left(T_0 < T_k\right).\end{aligned}$$

Recall that $\mathbb{P}_1(T_0 < T_k) = 1 - \frac{1}{k}$. Moreover using the strong Markov property and an obvious recurrence :

$$\begin{aligned}\mathbb{P}\left(S^*_{\tau_a} < k\right) &= \mathbb{P}\left(\tau_a < T^*_k\right) = \mathbb{P}\left(\tau_a < T^*_k | \tau_2 < T^*_k\right)\mathbb{P}\left(\tau_2 < T^*_k\right) \\ &= \mathbb{P}\left(\tau_{a-1} < T^*_k\right)\mathbb{P}\left(\tau_2 < T^*_k\right) = \mathbb{P}\left(\tau_2 < T^*_k\right)^{a-1} = \left(1 - \frac{1}{k}\right)^{a-1}.\end{aligned}$$

□

Using a similar reasoning and notations as the one of $\varphi(S_{g_p})$:

$$M^*_{\tau_a} - M^*_{\tau_{a+1}} = \mathbb{1}_{\tilde{S}^*_{\tilde{\tau}_2} > S^*_{\tau_a}} (\varphi(S^*_{\tau_a}) S^*_{\tau_a} - \varphi(\tilde{S}^*_{\tilde{\tau}_2}) \tilde{S}^*_{\tilde{\tau}_2} + \sum_{k=S^*_{\tau_a}}^{\tilde{S}^*_{\tilde{\tau}_2}-1} \varphi(k)) \Rightarrow \mathbb{E}\left[M_{\tau_a} - M_{\tau_{a+1}} | \mathcal{F}_{\tau_a}\right] = \varphi(S^*_{\tau_a}).$$

Let F be a positive functional, f_1 and f_2 be two functions from $\mathbb{N}$ to $\mathbb{R}^+$. :

$$\mathcal{G} := \mathbb{E}^{Q^*}\left[F(X_u, u \le g) f_1(\gamma_g) f_2\left(S^*_g\right)\right] = \sum_{a \ge 1} \mathbb{E}\left[F(X_u, u \le \tau_a) f_1(a) f_2\left(S^*_{\tau_a}\right) \varphi\left(S^*_{\tau_a}\right)\right].$$

With $F \equiv 1$:

$$\begin{aligned} \mathcal{G} &= \sum_{a\ge 0}\sum_{k \ge 0} f_1(a) f_2(k) \varphi(k) \mathbb{P}\left(S^*_{\tau_a} = k\right) \\ &= \mathbb{1}_{k=0,a=1}\varphi(0) f_1(1) f_2(0) + \sum_{a>1,k>0} \varphi(k) \left[\left(1 - \frac{1}{k+1}\right)^{a-1} - \left(1 - \frac{1}{k}\right)^{a-1}\right] f_1(a) f_2(k). \end{aligned}$$

Then, the law of $\left(\gamma_g, S^*_g\right)$ is:

$$Q^*\left(\gamma_g = a, S^*_g = k\right) = \mathbb{1}_{k=0,a=1}\varphi(0) + \mathbb{1}_{k>0,a>1}\varphi(k)\left[\left(1 - \frac{1}{k+1}\right)^{a-1} - \left(1 - \frac{1}{k}\right)^{a-1}\right]$$

We easily find the density of S^*_g summing over a.
Writing $\mathcal{G}$ in two different ways :

$$\begin{aligned} \mathcal{G} &= \sum_{a \ge 1} f_1(a) f_2(k) \varphi(k) Q^*(\gamma_g = a, S^*_g = k) \mathbb{E}^{Q^*}\left[F(X_u, u \le \tau_a)\right] \\ &= \sum_{a \ge 1}\sum_{k\ge 0} f_1(a) f_2(k) \mathbb{P}\left(S^*_{\tau_a} = k\right)\varphi(k) E\left[F(X_u, u \le \tau_a) \mid S^*_{\tau_a} = k\right], \end{aligned}$$

we conclude that :

$$\mathbb{E}^Q\left[F(X_u, u \le g) \mid S^*_g = k, \gamma_g = a\right] = E\left[F(X_u, u \le \tau_a) \mid S^*_{\tau_a} = k\right].$$

This achieves the proof of point 3.iii.
3.iii) The study of the process $(X_u, u \le g)$ under Q^* is very close to the study of $(X_n, n \ge 0)$ under Q in Theorem 1.2 :

Lemma 4.7. *Under the law Q^* and conditional on the event $\mathcal{H}_l$, the process $(X_{n+\tau_l},\ 1 \le n \le T_{p,\tau_l} - \tau_l)$ is a 3-Bessel* walk started from 1 and stopped when it first hits the level p.*

Proof. We just have to see that $M^*_{\tau_{l+1}} = \varphi(S^*_{\tau_{l+1}}) S^*_{\tau_{l+1}} + \phi(S^*_{\tau_{l+1}})$ and conditioning by $\mathcal{F}_{T_{p,\tau_l}}$, we obtain $\mathbb{E}\left[M^*_{\tau_{l+1}} | \mathcal{F}_{T_{p,\tau_l}}\right] = \mathbb{E}_p\left[\varphi(S^*_{T_0}) S^*_{T_0} + \phi(S^*_{T_0})\right]$ is a constant. □

We can easily prove by symmetry that $(X_{n+g}, n \ge 0)$ is either a 3-dimensional Bessel* walk either a reversed 3-Bessel* walk. It remains to know with what probability we have each case. Nevertheless we can deduce from the previous result that under Q^*, $S^*_\infty = \infty$. It permits us to obtain the following lemma :

Lemma 4.8. *Under Q^*, $S^*_{g_{T_a}}$ is a uniformly distributed random variable on $\{0, 1, \ldots, a-1\}$.*

Proof. According to Doob's Theorem :

$$Q^*\left(S^*_p > a\right) = Q^*\left(T^*_a < p\right) = \mathbb{E}\left[\mathbb{1}_{T^*_a < p} M^*_{T^*_a}\right] = \mathbb{E}\left[M^*_{T^*_a}\right] = \mathbb{E}\left[\mathbb{1}_{T^*_a < p}\varphi\left(S^*_{g_{T^*_a}}\right) a + \phi(a)\right].$$

When p tends to infinity :

$$1 = Q^*\left(S^*_\infty > a\right) = \mathbb{E}\left[\varphi\left(S^*_{g_{T^*_a}}\right) a + \phi\left(a\right)\right] \Leftrightarrow \sum_{k=0}^{a-1} \varphi(k) = a \sum_{k=0}^{a-1} \mathbb{P}\left(S^*_{g_{T^*_a}} = k\right)\varphi(k).$$

The fact this equality is true for a family of function φ (for example $\varphi_\lambda(x) = e^{-\lambda x}$) permits us to say that $\forall k \in \{0, 1, \ldots, a-1\}$, $\mathbb{P}\left(S^*_{g_{T^*_a}} = k\right) = a^{-1}$ □

Recall $\Delta^+ := \{X_{n+g} > 0, \forall n > 0\}$ (resp. $\Delta^- := \{X_{n+g} < 0, \forall n > 0\}$). As g is Q^*-a.s. finite, $Q^*\left(\Delta^+\right) = \lim_{p\to\infty} Q^*\left(X_{T^*_p} > 0\right)$ and with the definition of Q^* :

$$Q^*\left(X_{T^*_p} > 0\right) = \mathbb{E}\left[\mathbb{1}_{X_{T^*_p}>0} M_{T^*_p}\right] = \mathbb{E}\left[\mathbb{1}_{X_{T^*_p}>0}\left\{\varphi\left(S^*_{g_{T^*_p}}\right) p + \phi(p)\right\}\right].$$

Using the symmetry of X under $\mathbb{P}$:

$$2\mathbb{E}\left[\mathbb{1}_{X_{T^*_p}>0}\varphi\left(S^*_{g_{T^*_p}}\right)\right] = \mathbb{E}\left[\mathbb{1}_{X_{T^*_p}>0}\varphi\left(S^*_{g_{T^*_p}}\right)\right] + \mathbb{E}\left[\mathbb{1}_{X_{T^*_p}<0}\varphi\left(S^*_{g_{T^*_p}}\right)\right] = \mathbb{E}\left[\varphi\left(S^*_{g_{T^*_p}}\right)\right] = \sum_{k=0}^{p-1} \frac{\varphi(k)}{p}.$$

Consequently $Q^*\left(X_{T^*_p} > 0\right) = \frac{1}{2}$ and :

$$Q^*\left(\Delta^+\right) = \lim_{p\to\infty} Q^*\left(X_{T^*_p} > 0\right) = \frac{1}{2}.$$

5 Penalisation by S^*_p

In fact, we have a better result :

Theorem 5.1. *1. Let $a, b > 0$, then :*

$$\lim_{p\to\infty} \frac{\mathbb{E}\left[\mathbb{1}_{\{\Lambda_n,\, S_p<a, I_p>-b\}}\right]}{\mathbb{E}\left[\mathbb{1}_{\{S_p<a, I_p>-b\}}\right]} := \mathbb{E}\left[\mathbb{1}_{\{\Lambda_n, S_n<a, I_n>-b\}} M_n\right], \tag{5.5}$$

where $M_n := \left(\cos\left(\frac{\pi}{a+b}\right)\right)^{-n} \frac{\sin\left(\frac{\pi(a-X_n)}{a+b}\right)}{\sin\left(\frac{\pi a}{a+b}\right)}$ is a positive martingale non uniformly integrable.

2. Let us define a new probability Q on $(\Omega, \mathcal{F}_\infty)$ characterized by :

$$\forall n \in \mathbb{N},\ \forall \Lambda_n \in \mathcal{F}_n,\ Q\left(\Lambda_n\right) := \mathbb{E}\left[\Lambda_n M_n\right]. \tag{5.6}$$

Then under Q, $(X_n, n \geq 0)$ have the following transition probabilities for $-b+1 \leq k \leq a-1$:

$$\begin{aligned} Q\left(X_{n+1} = k+1 | X_n = k\right) &= \frac{\sin\left(\frac{a-k-1}{a+b}\pi\right)}{2\cos\left(\frac{\pi}{a+b}\right)\sin\left(\frac{a-k}{a+b}\pi\right)}, \\ Q\left(X_{n+1} = k-1 | X_n = k\right) &= \frac{\sin\left(\frac{a-k+1}{a+b}\pi\right)}{2\cos\left(\frac{\pi}{a+b}\right)\sin\left(\frac{a-k}{a+b}\pi\right)}. \end{aligned}$$

1) To prove the first point of Theorem we need the following lemma :

Lemma 5.2. *Let $a, b > 0$ and $c \in [-b+1, a-1]$, then:*

$$\mathbb{P}\left(S_n < a, I_n > -b\right) \underset{n\to\infty}{\sim} \frac{4}{a+b}\left(\cos\left(\frac{\pi}{a+b}\right)\right)^n \sin\left(\frac{a\pi}{a+b}\right) \sum_{\substack{c=-b+1 \\ c\equiv n[2]}}^{a-1} \sin\left(\frac{\pi(a-c)}{a+b}\right). \tag{5.7}$$

Let us postpone the proof of this lemma and finish the proof of (5.5). As usual, let $\tilde{X}_k = X_{k+n}$, a random walk started from X_n and independent of $\mathcal{F}_n$, and $\tilde{S}_n$ and $\tilde{I}_n$ respectively the supremum and infimum associated to $\tilde{X}$. In the following steps $\tilde{\mathbb{P}}$ is the measure associated to $\tilde{X}$, X_n, S_n and I_n being kept fixed. Using the Markov property :

$$\mathbb{E}\left[\mathbb{1}_{\{\Lambda_n,\, S_p<a, I_p>-b\}}\right] \quad = \quad \mathbb{E}\left[\mathbb{1}_{\{\Lambda_n,\, S_n<a, I_n>-b\}}\tilde{\mathbb{P}}\left(\tilde{S}_{p-n} < a - X_n, \tilde{I}_{p-n} > -b - X_n\right)\right].$$

Lemma 5.2 says:

$$\begin{aligned}
&\mathbb{P}\left(\tilde{S}_{p-n} < a - X_n, \tilde{I}_{p-n} > -b - X_n\right) \\
&\underset{p\to\infty}{\sim} \sum_{c=-b-X_n+1, c\equiv p-n\,[2]}^{a-X_n-1} \frac{4}{a+b}\left(\cos\left(\frac{\pi}{a+b}\right)\right)^{p-n} \sin\left(\frac{\pi(a-X_n)}{a+b}\right)\sin\left(\frac{\pi(a-X_n-c)}{a+b}\right) \\
&\qquad \underset{p\to\infty}{\sim} \frac{4}{a+b}\left(\cos\left(\frac{\pi}{a+b}\right)\right)^{p-n} \sin\left(\frac{\pi(a-X_n)}{a+b}\right) \sum_{c=-b+1, c\equiv p\,[2]}^{a-1} \sin\left(\frac{\pi(a-c)}{a+b}\right).
\end{aligned}$$

Dividing this formula by (5.7), we obtain (5.5).

Proof of lemma 5.2 :

To prove this lemma we need the following combinatory result :

Lemma 5.3. *Let $p \in \mathbb{N}$, $0 < u < p$:*

$$\sum_{k\geq 0} C_n^{kp+u} = \frac{1}{p}\sum_{\ell=0}^{p-1}\left(1 + e^{\frac{\ell 2i\pi}{p}}\right)^n e^{-\frac{2i\pi\ell u}{u}}.$$

Proof of lemma 5.3:

$$\sum_{\ell=0}^{p-1}\left(1 + e^{\frac{\ell 2i\pi}{p}}\right)^n e^{-\frac{2i\pi\ell u}{p}} = \sum_{\ell=0}^{p-1}\sum_{k=0}^{n} C_n^k e^{\frac{2i\pi\ell k}{p}} e^{-\frac{2i\pi\ell u}{p}} \sum_{k=0}^{n} C_n^k \sum_{\ell=0}^{p-1} e^{\frac{2i\pi\ell(k-u)}{p}}.$$

Those sums can be easily simplifed if we note that :

$$\sum_{\ell=0}^{p-1} e^{\frac{2i\pi\ell(k-u)}{p}} = \begin{cases} p & \text{, if } k-u \text{ is a multiple of } p \\ \sum_{\ell=0}^{p-1} e^{\frac{2i\pi\ell(k-u)}{p}} = \frac{1-e^{2i\pi(k-u)}}{1-e^{\frac{2i\pi(k-u)}{p}}} = 0 & \text{, otherwise.} \end{cases}$$

Then :

$$\sum_{\ell=0}^{p-1}\left(1 + e^{\frac{\ell 2i\pi}{p}}\right)^n e^{-\frac{2i\pi\ell u}{p}} = \sum_{k=0, k\equiv u[p]}^{n} pC_n^k = p\sum_{k\geq 0} C_n^{kp+u}.$$

□

According to [Fel50] p.79:

$$\mathbb{P}\left(S_n < a, X_n = c, I_n > -b\right) = \left(\frac{1}{2}\right)^n \sum_{k\in\mathbb{Z}} C_n^{\frac{n+c}{2}+k(a+b)} - C_n^{\frac{n-c}{2}+k(a+b)+a}.$$

Clearly c and n must have the same parity and denote :

$$A_n^c = {}^{(n+c)}/_2 + k_0(a+b), B_n^c = {}^{(n-c)}/_2 + a + k_1(a+b),$$

where k_0 (respectively k_1) is the first k such as ${}^{(n+c)}/_2 + k(a+b)$ (respectively ${}^{(n-c)}/_2 + k(a+b) + a$) is positive. Then :

$$\begin{aligned}
\mathbb{P}\left(S_n < a, X_n = c, I_n > -b\right) &= \frac{2^{-n}}{a+b} \sum_{\ell=0}^{a+b-1} \left(1 + e^{\frac{2i\pi\ell}{a+b}}\right)^n \left[e^{-\frac{2i\pi\ell A_n^c}{a+b}} - e^{-\frac{2i\pi\ell B_n^c}{a+b}}\right] \\
&= \frac{2i}{a+b} \sum_{\ell=1}^{a+b-1} \cos^n\left(\frac{\pi\ell}{a+b}\right) e^{\frac{i\pi\ell(n-(A_n^c+B_n^c))}{a+b}} \sin\left(\frac{\pi\ell(B_n^c - A_n^c)}{a+b}\right) \\
&= -\frac{2}{a+b} \sum_{\ell=1}^{a+b-1} \cos^n\left(\frac{\pi\ell}{a+b}\right) \sin\frac{\pi\ell(n-(A_n^c+B_n^c))}{a+b} \sin\left(\frac{\pi\ell(B_n^c - A_n^c)}{a+b}\right).
\end{aligned}$$

Let us remark that:

$$\begin{cases} B_n^c - A_n^c = -a - (k_0+k_1)(a+b) \\ n - A_n^c - B_n^c = a - c + (k_1 - k_0)(a+b) \end{cases}$$

Which implies :

$$\begin{aligned}
&\mathbb{P}\left(S_n < a, X_n = c, I_n > -b\right) = \\
&\quad \frac{2}{a+b} \sum_{\ell=1}^{a+b-1} (-1)^{(k_0+k_1)\ell + (k_1-k_0)\ell} \cos^n\left(\frac{\pi\ell}{a+b}\right) \sin\frac{\pi\ell a}{a+b} \sin\left(\frac{\pi\ell(a-c))}{a+b}\right) \\
&\qquad = \frac{2}{a+b} \sum_{\ell=1}^{a+b-1} \cos^n\left(\frac{\pi\ell}{a+b}\right) \sin\frac{\pi\ell a}{a+b} \sin\left(\frac{\pi\ell(a-c))}{a+b}\right).
\end{aligned}$$

Here, we have to notice that when $n \to \infty$, the leading terms are $\ell = 1$ and $\ell = a+b-1$. Hence :

$$\begin{aligned}
&\cos^n\left(\pi - \frac{\pi}{a+b}\right) \sin\left(\pi a - \frac{\pi a}{a+b}\right) \sin\left(\pi(a-c) - \frac{\pi(a-c))}{a+b}\right) = \\
&(-1)^{n+2a-c} \cos^n\left(\frac{\pi}{a+b}\right) \sin\left(\frac{\pi a}{a+b}\right) \sin\left(\frac{\pi(a-c)}{a+b}\right) = \cos^n\left(\frac{\pi}{a+b}\right) \sin\left(\frac{\pi a}{a+b}\right) \sin\left(\frac{\pi(a-c)}{a+b}\right),
\end{aligned}$$

n and c having the same parity. Hence:

$$\mathbb{P}\left(S_n < a, X_n = c, I_n > -b\right) \underset{n\to\infty}{\sim} \frac{4}{a+b} \cos^n\left(\frac{\pi}{a+b}\right) \sin\left(\frac{\pi a}{a+b}\right) \sin\left(\frac{\pi(a-c)}{a+b}\right)$$

and we can easily deduce :

$$\mathbb{P}\left(S_n < a, I_n > -b\right) \underset{n\to\infty}{\sim} \frac{4}{a+b} \cos^n\left(\frac{\pi}{a+b}\right) \sin\left(\frac{\pi a}{a+b}\right) \sum_{c\equiv n[2], c=-b+1}^{a-1} \sin\left(\frac{\pi(a-c)}{a+b}\right).$$

We need to prove that M is a positive martingale. Positivity is obvious and for all $n \geq 0$:

$$M_n \leq \frac{\left(\cos\left(\frac{\pi}{a+b}\right)\right)^{-n}}{\sin\left(\frac{\pi a}{a+b}\right)}$$

$$\begin{aligned}
&\mathbb{E}\left[\sin\left(\frac{\pi(a - X_{n+1})}{a+b}\right)\middle| \mathcal{F}_n\right] = \mathbb{E}\left[\sin\left(\frac{\pi(a-X_n)}{a+b}\right)\cos\left(\frac{\pi Y_{n+1}}{a+b}\right) + \middle| \mathcal{F}_n\right] \\
&+\mathbb{E}\left[\cos\left(\frac{\pi(a-X_n)}{a+b}\right)\sin\left(\frac{\pi Y_{n+1}}{a+b}\right)\middle| \mathcal{F}_n\right] = \sin\left(\frac{\pi(a-X_n)}{a+b}\right)\cos\left(\frac{\pi}{a+b}\right)
\end{aligned}$$

Hence M is a martingale stopped when it hits the boundary of the segment $[-b, a]$.
2) For $-b+1 \leq k \leq a-1$, using the Markov property and the defintion of Q :

$$Q\left(X_{n+1}=k+1|X_n=k\right)=\frac{Q\left(X_{n+1}=k+1,\, X_n=k\right)}{Q\left(X_n=k\right)}=\frac{\mathbb{E}\left[\mathbb{1}_{\{X_{n+1}=k+1,\, X_n=k\}}M_{n+1}\right]}{\mathbb{E}\left[\mathbb{1}_{\{X_n=k\}}M_n\right]}$$

$$=\frac{\mathbb{E}\left[\mathbb{1}_{\{X_{n+1}=k+1,\, X_n=k, S_{n+1}<a, I_{n+1}>-b\}}\left(\cos\left(\frac{\pi}{a+b}\right)\right)^{-n-1}\frac{\sin\left(\frac{\pi(a-k-1)}{a+b}\right)}{\sin\left(\frac{\pi a}{a+b}\right)}\right]}{\mathbb{E}\left[\mathbb{1}_{\{X_n=k, \Lambda_n, S_n<a, I_n>-b\}}\left(\cos\left(\frac{\pi}{a+b}\right)\right)^{-n}\frac{\sin\left(\frac{\pi(a-k)}{a+b}\right)}{\sin\left(\frac{\pi a}{a+b}\right)}\right]}$$

$$=\frac{\left(\cos\left(\frac{\pi}{a+b}\right)\right)^{-1}\sin\left(\frac{\pi(a-k-1)}{a+b}\right)}{\sin\left(\frac{\pi(a-k)}{a+b}\right)}\mathbb{P}\left(X_{n+1}=k+1|X_n=k,\, S_n<a,\, I_n>-b\right)$$

$$=\frac{\left(\cos\left(\frac{\pi}{a+b}\right)\right)^{-1}\sin\left(\frac{\pi(a-k-1)}{a+b}\right)}{2\sin\left(\frac{\pi(a-k)}{a+b}\right)}$$

Remark 5.4. *We can easily complete this study by a penalisation functional* $G_p = \mathbb{1}_{S^*_{d_p}<a}$. *We just have to see that this penalisation is the same as* $\mathbb{1}_{S^*_p<a}$.

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