

ON THE TOPOLOGY OF SEMI-ALGEBRAIC FUNCTIONS ON CLOSED SEMI-ALGEBRAIC SETS

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ABSTRACT. We consider a closed semi-algebraic set $X \subset \mathbb{R}^n$ and a C^2 semi-algebraic function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $f|_X$ has a finite number of critical points. We relate the topology of X to the topology of the sets $\{f * \alpha\}$, where $*$ $\in \{\leq, =, \geq\}$ and $\alpha \in \mathbb{R}$, and the indices of the critical points of $f|_X$ and $-f|_X$. We also relate the topology of X to the topology of the links at infinity of the sets $\{f * \alpha\}$ and the indices of these critical points. We give applications when $X = \mathbb{R}^n$ and when f is a generic linear function.

1. INTRODUCTION

Let $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$ be an analytic function-germ with an isolated critical point at 0. The Khimshiashvili formula [Kh] states that:

$$\chi(f^{-1}(\delta) \cap B_\varepsilon^n) = 1 - \text{sign}(-\delta)^n \text{deg}_0 \nabla f,$$

where $0 < |\delta| \ll \varepsilon \ll 1$, B_ε^n is a closed ball of radius ε centered at 0, ∇f is the gradient of f and $\text{deg}_0 \nabla f$ is the topological degree of the mapping $\frac{\nabla f}{|\nabla f|} : S_\varepsilon^{n-1} \rightarrow S^{n-1}$. As a corollary of this formula, one gets (see [Ar] or [Wa]):

$$\begin{aligned} \chi(\{f \leq 0\} \cap S_\varepsilon^{n-1}) &= 1 - \text{deg}_0 \nabla f, \\ \chi(\{f \geq 0\} \cap S_\varepsilon^{n-1}) &= 1 + (-1)^{n-1} \text{deg}_0 \nabla f, \end{aligned}$$

and:

$$\chi(\{f = 0\} \cap S_\varepsilon^{n-1}) = 2 - 2 \text{deg}_0 \nabla f \text{ if } n \text{ is even.}$$

In [Se], Sekalski gives a global counterpart of Khimshiashvili's formula for a polynomial mapping $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ with a finite number of critical points. He considers the set $\Lambda_f = \{\lambda_1, \dots, \lambda_k\}$ of critical values of f at infinity, where $\lambda_1 < \dots < \lambda_k$, and its complement $\mathbb{R} \setminus \Lambda_f = \cup_{i=0}^k]\lambda_i, \lambda_{i+1}[$ where $\lambda_0 = -\infty$ and $\lambda_{k+1} = +\infty$. Denoting by $r_\infty(g)$ the numbers of real branches at infinity of a curve $\{g = 0\}$ in \mathbb{R}^2 , he proves that:

$$\text{deg}_\infty \nabla f = 1 + \sum_{i=1}^k r_\infty(f - \lambda_i) - \sum_{i=0}^k r_\infty(f - \lambda_i^+),$$

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where for $i = 0, \dots, k$, λ_i^+ is an element of $] \lambda_i, \lambda_{i+1}[$ and $\deg_\infty \nabla f$ is the topological degree of the mapping $\frac{\nabla f}{\|\nabla f\|} : S_R^{n-1} \rightarrow S^{n-1}$, $R \gg 1$. Here a real branch is homeomorphic to a neighborhood of infinity in \mathbb{R} and hence has two connected components.

Our aim is to generalize Sekalski's formula and to establish other similar results. We consider a closed semi-algebraic set $X \subset \mathbb{R}^n$ equipped with a finite semi-algebraic Whitney stratification $(S_\alpha)_{\alpha \in A}$ and a C^2 semi-algebraic function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $f|_X$ has a finite number of critical points p_1, \dots, p_l . The index of $f|_X$ at p_i is defined by:

$$\text{ind}(f, X, p_i) = 1 - \chi(X \cap \{f = f(p_i) - \delta\} \cap B_\varepsilon^n(p_i)),$$

where $0 < \delta \ll \varepsilon \ll 1$. In Section 3, Proposition 3.6 and Corollary 3.7, we give relations between the Euler characteristics of the sets $\{f * \alpha\}$, where $*$ $\in \{\leq, =, \geq\}$ and $\alpha \in \mathbb{R}$, the indices of the critical points of $f|_X$ and $-f|_X$ and four numbers $\lambda_{f,\alpha}$, $\lambda_{-f,-\alpha}$, $\mu_{f,\alpha}$ and $\mu_{-f,-\alpha}$. These numbers are defined in terms of the behavior of $f|_X$ at infinity (Definition 3.5). Then we consider the following finite subset of \mathbb{R} :

$$\Lambda_f^* = \left\{ \alpha \in \mathbb{R} \mid \beta \mapsto \chi(\text{Lk}^\infty(X \cap \{f * \beta\})) \text{ is not constant} \right. \\ \left. \text{in a neighborhood of } \alpha \right\},$$

where $*$ $\in \{\leq, =, \geq\}$ and $\text{Lk}^\infty(-)$ denotes the link at infinity. Writing $\Lambda_f^{\leq} = \{b_1, \dots, b_r\}$, $\mathbb{R} \setminus \Lambda_f^{\leq} = \cup_{i=0}^r]b_i, b_{i+1}[$ with $b_0 = -\infty$ and $b_{r+1} = +\infty$ and studying the behavior at infinity of the numbers $\lambda_{f,\alpha}$, $\lambda_{-f,-\alpha}$, $\mu_{f,\alpha}$ and $\mu_{-f,-\alpha}$, we show that (Theorem 3.16):

$$\chi(X) = \sum_{i=1}^k \text{ind}(f, X, p_i) + \sum_{j=0}^r \chi(\text{Lk}^\infty(X \cap \{f \leq b_j^+\})) \\ - \sum_{j=1}^r \chi(\text{Lk}^\infty(X \cap \{f \leq b_j\})),$$

where for $j \in \{0, \dots, r\}$, $b_j^+ \in]b_j, b_{j+1}[$. Similar formulas involving Λ_f^{\geq} and $\Lambda_f^=$ are proved in Theorem 3.17 and Corollary 3.18.

Next we consider the finite subset $\tilde{B}_f = f(\{p_1, \dots, p_l\}) \cup \Lambda_f^{\leq} \cup \Lambda_f^{\geq}$ of \mathbb{R} . We show that if $\alpha \notin \tilde{B}_f$ then the functions $\beta \mapsto \chi(X \cap \{f * \beta\})$, $*$ $\in \{\leq, =, \geq\}$, are constant in a neighborhood of α (Proposition 3.19). In Theorem 3.20 and Theorem 3.21 we express $\chi(X)$ in terms of the indices of the critical points of $f|_X$ and $-f|_X$ and the variations of the Euler characteristics of the sets $\{f * \alpha\}$, $\alpha \in \{\leq, =, \geq\}$.

In Section 4, we apply all these results to the case $X = \mathbb{R}^n$ in order to recover and generalize Sekalski's formula (Theorem 4.4).

In Section 5, we apply the results of Section 3 to generic linear functions. For $v \in S^{n-1}$, we denote by v^* the function $v^*(x) = \langle v, x \rangle$. We show that

for v generic in S^{n-1} , the sets $\Lambda_{v^*}^{\leq}$, $\Lambda_{v^*}^{\geq}$ and $\Lambda_{v^*}^{\geq}$ are empty. Hence for such a v , the Euler characteristics of the sets $X \cap \{v^*? \alpha\}$, $? \in \{\leq, =, \geq\}$ and $\alpha \in \mathbb{R}$, as well as the Euler characteristics of their links at infinity, can be expressed only in terms of the critical points of $v|_X$ and $-v|_X$ (Proposition 5.4 and Proposition 5.5). We use these results to give a new proof of the Gauss-Bonnet formula for closed semi-algebraic set (Theorem 5.8), that we initially proved in [Dut3, Corollary 5.7] using the technology of the normal cycle [Fu] and a deep theorem of Fu and McCrory [FM, Theorem 3.7].

Section 2 of this paper contains three technical lemmas.

We will use the following notations: for $p \in \mathbb{R}^n$ and $\varepsilon > 0$, $B_\varepsilon^n(p)$ is the ball of radius ε centered at p and $S_\varepsilon^{n-1}(p)$ the sphere of radius ε centered at p . If $p = 0$, we simply set B_ε^n and S_ε^{n-1} and if $p = 0$ and $\varepsilon = 1$ we use the standard notations B^n and S^{n-1} . If \mathcal{E} is a subset of \mathbb{R}^n then $\mathring{\mathcal{E}}$ denotes its topological interior.

2. SOME LEMMAS

Let $X \subset \mathbb{R}^n$ be a closed semi-algebraic set equipped with a semi-algebraic Whitney stratification $(S_\alpha)_{\alpha \in A}$: $X = \sqcup_{\alpha \in A} S_\alpha$. Let $g : \mathbb{R}^n \rightarrow \mathbb{R}$ be a C^2 semi-algebraic function such that $g^{-1}(0)$ intersects X transversally (in the stratified sense). Then the following partition:

$$X \cap \{g \leq 0\} = \bigsqcup_{\alpha \in A} S_\alpha \cap \{g < 0\} \sqcup \bigsqcup_{\alpha \in A} S_\alpha \cap \{g = 0\},$$

is a Whitney stratification of the closed semi-algebraic set $X \cap \{g \leq 0\}$.

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be another C^2 semi-algebraic function such that $f|_{X \cap \{g \leq 0\}}$ admits an isolated critical point p in $X \cap \{g = 0\}$ which is not a critical point of $f|_X$. If S denotes the stratum of X that contains p , this implies that:

$$\nabla(f|_S)(p) = \lambda(p) \nabla(g|_S)(p),$$

with $\lambda(p) \neq 0$. We assume that $f|_{\{g=0\}}$ is a submersion in the neighborhood of p if $\dim S < n$ and, for simplicity, that $f(p) = 0$.

Lemma 2.1. *For $0 < \delta \ll \varepsilon \ll 1$, we have:*

$$\begin{aligned} \chi(f^{-1}(-\delta) \cap B_\varepsilon^n(p) \cap X \cap \{g \leq 0\}) &= 1 \text{ if } \lambda(p) > 0, \\ \chi(f^{-1}(-\delta) \cap B_\varepsilon^n(p) \cap X \cap \{g \leq 0\}) &= \\ \chi(f^{-1}(-\delta) \cap B_\varepsilon^n(p) \cap X \cap \{g = 0\}) &\text{ if } \lambda(p) < 0. \end{aligned}$$

Proof. We assume first that $\dim X < n$. Let $h : \mathbb{R}^n \rightarrow \mathbb{R}$ be a semi-algebraic approximating function for X from outside (see [BK, Definition 6.1]). This implies that h satisfies the following conditions:

- (i) The function h is nonnegative and $h^{-1}\{0\} = X$.
- (ii) The function h is of class C^3 on $\mathbb{R}^n \setminus X$.
- (iii) There exists $\delta > 0$ such that all $t \in]0, \delta]$ are regular values of f .

- (iv) If $(y_k)_{k \in \mathbb{N}}$ is a sequence of points in \mathbb{R}^n tending to a point x in X such that $h(y_k) \in]0, \delta]$ and $\frac{\nabla h}{\|\nabla h\|}(y_k)$ tends to v , then v is normal to $T_x S$, S being the stratum containing x .

Let us choose ε sufficiently small so that the ball $B_{\varepsilon'}^n(p)$ intersect X , $\{g \leq 0\}$ and $X \cap \{g \leq 0\}$ transversally for $\varepsilon' \leq \varepsilon$. For $r > 0$ sufficiently small, the set $W_{\varepsilon,r} = B_{\varepsilon}^n(p) \cap \{g \leq 0\} \cap \{h \leq r\}$ is a manifold with corners. To see this, it is enough to prove that r is not a critical value of $h|_{B_{\varepsilon}^n(p) \cap \{g \leq 0\}}$, which means that r is not a critical value of:

$$h_1 = h|_{B_{\varepsilon}^n(p) \cap \{g < 0\}}, \quad h_2 = h|_{B_{\varepsilon}^n(p) \cap \{g = 0\}}, \quad h_3 = h|_{S_{\varepsilon}^{n-1}(p) \cap \{g < 0\}},$$

and:

$$h_4 = h|_{S_{\varepsilon}^{n-1}(p) \cap \{g = 0\}}.$$

The fact that r is not a critical value of h_1 is trivial by Condition (iii) above. If for $r > 0$ small, r is a critical value of h_2 then we can find a sequence of points (q_n) in $B_{\varepsilon}^n(p) \cap \{g = 0\}$ such that $h(q_n) \rightarrow 0$ and $h|_{\{g=0\}}$ admits a critical point at q_n . Applying Condition (iv) above, we see that there exists a point q in $B_{\varepsilon}^n(p) \cap X \cap \{g = 0\}$ such that $g^{-1}(0)$ does not intersect X transversally at q , which is impossible. Similarly, we can prove that r is not a critical value of h_3 and h_4 .

Let δ be such that $0 < \delta \ll \varepsilon$ and the fibres $f^{-1}(-\delta)$ and $f^{-1}(\delta)$ intersect $X \cap \{g \leq 0\} \cap B_{\varepsilon}^n(p)$ transversally. This is possible since f has an isolated critical point at p on $X \cap \{g \leq 0\}$. Let us study the critical points of $f|_{W_{\varepsilon,r}}$ and $f|_{W_{\varepsilon,r} \cap \{g=0\}}$ lying in $f^{-1}([-\delta, \delta])$, when r is small. Always using Condition (iv) above, we can see that they only appear in $\{h = r\} \cap \{g = 0\} \cap B_{\varepsilon}^n(p)$. Furthermore, with the terminology introduced in [Dut2, §2], if $\lambda(p) > 0$ then they are all outwards for $f|_{W_{\varepsilon,r}}$. If $\lambda(p) < 0$ then such a critical point is inwards for $f|_{W_{\varepsilon,r}}$ if and only if it is inwards for $f|_{W_{\varepsilon,r} \cap \{g=0\}}$. Moving f a little, we can assume that these critical points are non-degenerate for $f|_{\{h=r\} \cap \{g=0\} \cap B_{\varepsilon}^n(p)}$. Applying Morse theory for manifolds with corners, if $\lambda(p) > 0$ then we get:

$$\chi(f^{-1}([-\delta, \delta]) \cap W_{\varepsilon,r}) - \chi(f^{-1}(-\delta) \cap W_{\varepsilon,r}) = 0.$$

If $\lambda(p) < 0$ then we get:

$$\chi(f^{-1}([-\delta, \delta]) \cap W_{\varepsilon,r}) - \chi(f^{-1}(-\delta) \cap W_{\varepsilon,r}) =$$

$$\chi(f^{-1}([-\delta, \delta]) \cap \{g = 0\} \cap W_{\varepsilon,r}) - \chi(f^{-1}(-\delta) \cap \{g = 0\} \cap W_{\varepsilon,r}).$$

We conclude remarking that:

$$\begin{aligned} \chi(f^{-1}([-\delta, \delta]) \cap W_{\varepsilon,r}) &= \chi(f^{-1}([-\delta, \delta]) \cap X \cap \{g \leq 0\} \cap B_{\varepsilon}^n(p)) = \\ &= \chi(f^{-1}(0) \cap X \cap \{g \leq 0\} \cap B_{\varepsilon}^n(p)) = 1, \end{aligned}$$

$$\chi(f^{-1}(-\delta) \cap W_{\varepsilon,r}) = \chi(f^{-1}(-\delta) \cap X \cap \{g \leq 0\} \cap B_{\varepsilon}^n(p)),$$

$$\chi(f^{-1}([-\delta, \delta]) \cap \{g = 0\} \cap W_{\varepsilon,r}) = \chi(f^{-1}([-\delta, \delta]) \cap X \cap \{g = 0\} \cap B_{\varepsilon}^n(p)) = 1,$$

and:

$$\chi(f^{-1}(-\delta) \cap \{g = 0\} \cap W_{\varepsilon,r}) = \chi(f^{-1}(-\delta) \cap X \cap \{g = 0\} \cap B_{\varepsilon}^n(p)),$$

if r is sufficiently small.

If $\dim X = n$ then we apply the previous case to the semi-algebraic set $X \times \{0\} \subset \mathbb{R}^{n+1}$ and the functions F and G defined by $F(x, t) = f(x) + t$ and $G(x, t) = g(x)$, where (x, t) is a coordinate system of $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$. \square

This lemma was inspired by results on indices of vector fields or 1-forms on stratified sets with boundary (see [KT] or [Sc, Chapter 5]).

Let $M \subset \mathbb{R}^n$ be a C^2 semi-algebraic manifold of dimension k . Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a C^2 semi-algebraic function. Let Σ_f^M be the critical set of $f|_M$. For any $a = (a_1, \dots, a_n) \in \mathbb{R}^n$, we denote by ρ_a the function $\rho_a(x) = \frac{1}{2} \sum_{i=1}^n (x_i - a_i)^2$ and by $\Gamma_{f,a}^M$ the following semi-algebraic set:

$$\Gamma_{f,a}^M = \left\{ x \in M \mid \text{rank}[\nabla(f|_M)(x), \nabla(\rho_a|_M)(x)] < 2 \right\}.$$

Lemma 2.2. *For almost all $a \in \mathbb{R}^n$, $\Gamma_{f,a}^M \setminus \Sigma_f^M$ is a smooth semi-algebraic curve (or empty).*

Proof. Let Z be the semi-algebraic set of $\mathbb{R}^n \times \mathbb{R}^n$ defined as follows:

$$Z = \left\{ (x, a) \in \mathbb{R}^n \times \mathbb{R}^n \mid x \in M \setminus \Sigma_f^M \text{ and } \text{rank}[\nabla(f|_M)(x), \nabla(\rho_a|_M)(x)] < 2 \right\}.$$

Let (x, a) be a point in Z . We can suppose that around x , M is defined by the vanishing of $l = n - k$ semi-algebraic functions g_1, \dots, g_l of class C^2 . Hence in a neighborhood of (x, a) , Z is defined by the vanishing of g_1, \dots, g_l and the minors:

$$\frac{\partial(g_1, \dots, g_l, f, \rho_a)}{\partial(x_{i_1}, \dots, x_{i_{l+2}})}.$$

Furthermore since x belongs to $M \setminus \Sigma_f^M$, we can assume that:

$$\frac{\partial(g_1, \dots, g_l, f)}{\partial(x_1, \dots, x_l, x_{l+1})}(x) \neq 0.$$

Therefore Z is locally defined by $g_1 = \dots = g_l = 0$ and:

$$\frac{\partial(g_1, \dots, g_l, f, \rho_a)}{\partial(x_1, \dots, x_{l+1}, x_{l+2})} = \dots = \frac{\partial(g_1, \dots, g_l, f, \rho_a)}{\partial(x_1, \dots, x_{l+1}, x_n)} = 0,$$

(see [Dut1, §5] for a proof of this fact). Since the gradient vectors of these functions are linearly independent, we see that Z is a smooth semi-algebraic manifold of dimension $2n - (l + n - (l + 2) + 1) = n + 1$. Now let us consider the projection $\pi_2 : Z \rightarrow \mathbb{R}^n$, $(x, a) \mapsto a$. Bertini-Sard's theorem (see [BCR, Théorème 9.5.2]) implies that the set D_{π_2} of critical values of π_2 is a semi-algebraic set of dimension strictly less than n . Hence, for all

$a \notin D_{\pi_2}$, $\pi_2^{-1}(a)$ is a smooth semi-algebraic curve (maybe empty). But this set is exactly $\Gamma_{f,a}^M \setminus \Sigma_f^M$. \square

Now consider a semi-algebraic set $Y \subset M$ of dimension strictly less than k . We will need the following lemma.

Lemma 2.3. *For almost all $a \in \mathbb{R}^n$, $(\Gamma_{f,a}^M \setminus \Sigma_f^M) \cap Y$ is a semi-algebraic set of dimension at most 0.*

Proof. Since Y admits a finite Whitney semi-algebraic stratification, we can assume that Y is smooth of dimension $d < k$. Let W be the semi-algebraic set of $\mathbb{R}^n \times \mathbb{R}^n$ defined by:

$$W = \left\{ (x, a) \in \mathbb{R}^n \times \mathbb{R}^n \mid x \in Y \setminus \Sigma_f^M \text{ and } \right. \\ \left. \text{rank}[\nabla(f|_M)(x), \nabla(\rho_a|_M)(x)] < 2 \right\}.$$

Using the same method as in the previous lemma, we can prove that W is a smooth semi-algebraic manifold of dimension $n+1+d-k$. We can conclude as in the previous lemma, remarking that $d-k \leq -1$. \square

3. TOPOLOGY OF SEMI-ALGEBRAIC FUNCTIONS

For any closed semi-algebraic set equipped with a Whitney stratification $X = \sqcup_{\alpha \in A} S_\alpha$, we denote by $\text{Lk}^\infty(X)$ the link at infinity of X . It is defined as follows. Let $\rho : \mathbb{R}^n \rightarrow \mathbb{R}$ be a C^2 proper semi-algebraic positive function. Since $\rho|_X$ is proper, the set of critical points of $\rho|_X$ (in the stratified sense) is compact. Hence for R sufficiently big, the map $\rho : X \cap \rho^{-1}([R, +\infty]) \rightarrow \mathbb{R}$ is a stratified submersion. The link at infinity of X is the fibre of this submersion. The topological type of $\text{Lk}^\infty(X)$ does not depend on the choice of the function ρ . Indeed, if ρ_0 and $\rho_1 : \mathbb{R}^n \rightarrow \mathbb{R}$ are two C^2 proper semi-algebraic functions then $X \cap \rho_0^{-1}(R_0)$ and $X \cap \rho_1^{-1}(R_1)$ are homeomorphic for R_0 and R_1 big enough. To see this, we can apply the procedure described by Durfee in [Dur]. First we remark that, applying the Curve Selection Lemma at infinity [NZ, Lemma 2], for each stratum S_α of X , the gradient vector fields $\nabla(\rho_0|_{S_\alpha})$ and $\nabla(\rho_1|_{S_\alpha})$ do not point in opposite direction in a neighborhood of infinity. Next we choose R_0 and R_1 sufficiently big so that $\rho_0^{-1}([R_0, +\infty]) \subset \rho_1^{-1}([R_1, +\infty])$ and all the gradient vector fields $\nabla(\rho_0|_{S_\alpha})$ and $\nabla(\rho_1|_{S_\alpha})$ do not point in opposite direction in $\rho_0^{-1}([R_0, +\infty])$. Then the function $\rho : (\rho_0^{-1}([R_0, +\infty]) \setminus \rho_1^{-1}([R_1, +\infty])) \cap X \rightarrow [0, 1]$ defined by:

$$\rho(x) = \frac{R_1 - \rho_1(x)}{R_1 - \rho_1(x) + \rho_0(x) - R_0},$$

is a proper stratified submersion such that $\rho^{-1}(0) = X \cap \rho_1^{-1}(R_1)$ and $\rho^{-1}(1) = X \cap \rho_0^{-1}(R_0)$.

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a C^2 semi-algebraic function such that $f|_X : X \rightarrow \mathbb{R}$ has a finite number of critical points (in the stratified sense) p_1, \dots, p_l . For

each p_i , we define the index of $f|_X$ at p_i as follows:

$$\text{ind}(f, X, p_i) = 1 - \chi(X \cap \{f = f(p_i) - \delta\} \cap B_\varepsilon^n(p_i)),$$

where $0 < \delta \ll \varepsilon \ll 1$. Since we are in the semi-algebraic setting, this index is well-defined thanks to Hardt's theorem [Ha]. The following theorem is well-known.

Theorem 3.1. *If $f|_X$ is proper then for any $\alpha \in \mathbb{R}^n$, we have:*

$$\chi(X \cap \{f \geq \alpha\}) - \chi(X \cap \{f = \alpha\}) = \sum_{i: f(p_i) > \alpha} \text{ind}(f, X, p_i),$$

and:

$$\chi(X \cap \{f \leq \alpha\}) - \chi(\text{Lk}^\infty(X \cap \{f \leq \alpha\})) = \sum_{i: f(p_i) \leq \alpha} \text{ind}(f, X, p_i).$$

Proof. We use Viro's method of integration with respect to the Euler characteristic with compact support, denoted by χ_c .

For all $x \in X$, let $\varphi(x) = \chi_c(X \cap f^{-1}(x^-) \cap B_\varepsilon^n(x))$ where x^- is a regular value of f close to $f(x)$ with $x^- \leq f(x)$. Applying Fubini's theorem [Vi, Theorem 3.A] to the restriction of f to $X \cap \{f > \alpha\}$, we get:

$$\int_{X \cap \{f > \alpha\}} \varphi(x) d\chi_c(x) = \int_{] \alpha, +\infty[} \left(\int_{f^{-1}(y)} \varphi(x) d\chi_c(x) \right) d\chi_c(y).$$

For any $y \in \mathbb{R}$, let y^- be a regular value of $f|_X$ close to y with $y^- \leq y$. Let us denote by z_1, \dots, z_s the critical points of $f|_X$ lying in $f^{-1}(y)$. We have:

$$\begin{aligned} \chi_c(X \cap f^{-1}(y^-)) &= \chi_c(X \cap f^{-1}(y^-) \setminus \cup_{i=1}^s B_\varepsilon^n(z_i)) + \\ &\quad \sum_{i=1}^s \chi_c(X \cap f^{-1}(y^-) \cap B_\varepsilon^n(z_i)) = \\ \chi_c(X \cap f^{-1}(y) \setminus \cup_{i=1}^s B_\varepsilon^n(z_i)) &+ \sum_{i=1}^s \varphi(z_i) = \\ \chi_c(X \cap f^{-1}(y) \setminus \{z_1, \dots, z_s\}) &+ \sum_{i=1}^s \varphi(z_i) = \\ \int_{X \cap f^{-1}(y) \setminus \{z_1, \dots, z_s\}} \varphi(x) d\chi_c(x) &+ \sum_{i=1}^s \varphi(z_i) = \int_{f^{-1}(y)} \varphi(x) d\chi_c(x). \end{aligned}$$

Let us write:

$$] \alpha, +\infty[=] \alpha, \alpha_1[\cup] \alpha_1, \alpha_2[\cup \dots \cup] \alpha_{j-1}, \alpha_j[\cup] \alpha_j, +\infty[,$$

where $\alpha_1, \dots, \alpha_j$ are the critical values of $f|_X$ strictly greater than α . Since $\chi_c(] \alpha_k, \alpha_{k+1}[) = 0$ and $f|_{X \cap] \alpha_k, \alpha_{k+1}[}$ is locally trivial, we obtain that:

$$\int_{X \cap \{f > \alpha\}} \varphi(x) d\chi_c(x) = -\chi_c(X \cap f^{-1}(\beta)),$$

where β is a regular value of f strictly greater than α_j and therefore:

$$\chi_c(X \cap \{f > \alpha\}) + \chi_c(X \cap f^{-1}(\beta)) = \sum_{i: f(p_i) > \alpha} \text{ind}(f, X, p_i).$$

Applying this equality to $\alpha = \beta$ and using the local triviality of $f|_X$ over $[\beta, +\infty[$, we get:

$$\chi_c(X \cap \{f > \beta\}) + \chi_c(X \cap f^{-1}(\beta)) = 0.$$

Therefore:

$$\begin{aligned} & \chi_c(X \cap \{f > \alpha\}) + \chi_c(X \cap f^{-1}(\beta)) = \\ & \chi_c(X \cap \{\alpha \leq f \leq \beta\}) - \chi_c(X \cap \{f = \alpha\}) - \chi_c(X \cap \{f > \beta\}) + \chi_c(X \cap f^{-1}(\beta)) = \\ & \chi(X \cap \{\alpha \leq f \leq \beta\}) - \chi(X \cap \{f = \alpha\}). \end{aligned}$$

To conclude, we remark that, since $f|_{X \cap \{f \geq \alpha\}}$ is proper and locally trivial over $[\beta, +\infty[$, $X \cap \{\alpha \leq f \leq \beta\}$ is a deformation retract of $X \cap \{\alpha \leq f\}$. The second equality is proved with the same method and the fact that $\chi_c(Y) = \chi(Y) - \chi(\text{Lk}^\infty(Y))$ for any closed semi-algebraic set $Y \subset \mathbb{R}^n$. \square

The following corollaries are straightforward consequences of the previous theorem.

Corollary 3.2. *If $f|_X$ is proper then for any $\alpha \in \mathbb{R}$, we have:*

$$\chi(X \cap \{f = \alpha\}) = \chi(X) - \sum_{i: f(p_i) > \alpha} \text{ind}(f, X, p_i) - \sum_{i: f(p_i) < \alpha} \text{ind}(-f, X, p_i),$$

and:

$$\begin{aligned} \chi(X \cap \{f \geq \alpha\}) - \chi(X \cap \{f \leq \alpha\}) &= \sum_{i: f(p_i) > \alpha} \text{ind}(f, X, p_i) - \\ & \sum_{i: f(p_i) < \alpha} \text{ind}(f, X, p_i). \end{aligned}$$

\square

Corollary 3.3. *If $f|_X$ is proper then for any $\alpha \in \mathbb{R}^n$, we have :*

$$\chi(\text{Lk}^\infty(X \cap \{f \leq \alpha\})) = \chi(X) - \sum_{i=1}^l \text{ind}(f, X, p_i).$$

\square

Corollary 3.4. *If $f|_X$ is proper then we have:*

$$2\chi(X) - \chi(\text{Lk}^\infty(X)) = \sum_{i=1}^l \text{ind}(f, X, p_i) + \sum_{i=1}^l \text{ind}(-f, X, p_i).$$

□

Now we want to investigate the case when $f|_X$ is not proper. Keeping the notations of the previous section, for $a \in \mathbb{R}^n$, we define $\Gamma_{f,a}^X$ and $\Gamma_{f,a}$ by:

$$\Gamma_{f,a}^X = \left\{ x \in X \mid \text{rank}[\nabla(f|_S)(x), \nabla(\rho_{a|S})(x)] < 2 \right. \\ \left. \text{where } S \text{ is the stratum that contains } x \right\},$$

$$\Gamma_{f,a} = \{x \in \mathbb{R}^n \mid \text{rank}[\nabla f(x), \nabla \rho_a(x)] < 2\}.$$

By Lemma 2.2, we can choose a such that $\Gamma_{f,a}^X$ is a smooth semi-algebraic curve outside a compact set of X . Applying Lemma 2.3 to $M = \mathbb{R}^n$ and Y the closed semi-algebraic set defined as the union of the strata of X of dimension strictly less than n , we can choose a such that $\Gamma_{f,a}^X$ and $\Gamma_{f,a}$ do not intersect outside a compact set of Y . Let us fix $\alpha \in \mathbb{R}$ and $R \gg 1$ such that:

- (1) $X \cap B_R^n(a)$ is a deformation retract of X ,
- (2) $X \cap \{f * \alpha\} \cap B_R^n(a)$ is a deformation retract of $X \cap \{f * \alpha\}$ where $*$ $\in \{\leq, =, \geq\}$,
- (3) $S_R^{n-1}(a)$ intersects X and $X \cap \{f * \alpha\}$ transversally,
- (4) $\Gamma_{f,a}^X \cap S_R^{n-1}(a)$ is a finite set of points q_1^R, \dots, q_m^R ,
- (5) $p_1, \dots, p_l \in B_R^n(a)$.

For each $j \in \{1, \dots, m\}$, q_j^R is a critical point of $f|_{X \cap S_R^{n-1}(a)}$ but not a critical point of $f|_X$. Hence there exists $\mu_j^R \neq 0$ such that:

$$\nabla(f|_S)(q_j^R) = \mu_j^R \nabla(\rho_{a|S})(q_j^R),$$

where S is the stratum that contains q_j^R .

Definition 3.5. We set:

$$\lambda_{f,\alpha} = \sum_{\substack{j: f(q_j^R) > \alpha \\ \mu_j^R < 0}} \text{ind}(f, X \cap S_R^{n-1}(a), q_j^R),$$

$$\mu_{f,\alpha} = \sum_{\substack{j: f(q_j^R) < \alpha \\ \mu_j^R > 0}} \text{ind}(f, X \cap S_R^{n-1}(a), q_j^R),$$

$$\nu_{f,\alpha} = \sum_{\substack{j: f(q_j^R) < \alpha \\ \mu_j^R < 0}} \text{ind}(f, X \cap S_R^{n-1}(a), q_j^R).$$

The fact that $\lambda_{f,\alpha}$, $\mu_{f,\alpha}$ and $\nu_{f,\alpha}$ do not depend on R will appear in the next propositions. Let us remark that if $\mu_j^R < 0$ then $f(q_j^R)$ decreases to $-\infty$ or to a finite value as R tends to $+\infty$, and that if $\mu_j^R > 0$ then $f(q_j^R)$ increases to $+\infty$ or to a finite value as R tends to $+\infty$. This implies that when $f|_X$ is proper, the numbers $\lambda_{f,\alpha}$ and $\mu_{f,\alpha}$ vanish.

Proposition 3.6. *For any $\alpha \in \mathbb{R}$, we have:*

$$\chi(X \cap \{f \geq \alpha\}) - \chi(X \cap \{f = \alpha\}) = \sum_{i:f(p_i) > \alpha} \text{ind}(f, X, p_i) + \lambda_{f, \alpha},$$

and:

$$\chi(X \cap \{f \leq \alpha\}) - \chi(X \cap \{f = \alpha\}) = \sum_{i:f(p_i) < \alpha} \text{ind}(-f, X, p_i) + \lambda_{-f, -\alpha}.$$

Proof. We apply Theorem 3.1 to $f|_{X \cap B_R^n(a)}$ and we get:

$$\begin{aligned} \chi(X \cap B_R^n(a) \cap \{f \geq \alpha\}) - \chi(X \cap B_R^n(a) \cap \{f = \alpha\}) = \\ \sum_{i:f(p_i) > \alpha} \text{ind}(f, X, p_i) + \sum_{j:f(q_j^R) > \alpha} \text{ind}(f, X \cap B_R^n(a), q_j^R), \end{aligned}$$

and:

$$\begin{aligned} \chi(X \cap B_R^n(a) \cap \{f \leq \alpha\}) - \chi(X \cap B_R^n(a) \cap \{f = \alpha\}) = \\ \sum_{i:f(p_i) < \alpha} \text{ind}(-f, X, p_i) + \sum_{j:f(q_j^R) < \alpha} \text{ind}(-f, X \cap B_R^n(a), q_j^R). \end{aligned}$$

Since $\Gamma_{f,a}^X$ and $\Gamma_{f,a}$ do not intersect outside a compact set of Y , we can use Lemma 2.1 to evaluate $\text{ind}(f, X \cap B_R^n(a), q_j^R)$ and $\text{ind}(-f, X \cap B_R^n(a), q_j^R)$. Namely, if $\mu_j^R > 0$ then we have:

$$\text{ind}(f, X \cap B_R^n(a), q_j^R) = 0,$$

and:

$$\text{ind}(-f, X \cap B_R^n(a), q_j^R) = \text{ind}(-f, X \cap S_R^{n-1}(a), q_j^R).$$

If $\mu_j^R < 0$ then we have:

$$\text{ind}(f, X \cap B_R^n(a), q_j^R) = \text{ind}(f, X \cap S_R^{n-1}(a), q_j^R),$$

and:

$$\text{ind}(-f, X \cap B_R^n(a), q_j^R) = 0.$$

Moreover, by our choice on R , $\chi(X \cap B_R^n(a) \cap \{f * \alpha\}) = \chi(X \cap \{f * \alpha\})$ for $*$ $\in \{\leq, =, \geq\}$. \square

Corollary 3.7. *For any $\alpha \in \mathbb{R}$, we have:*

$$\begin{aligned} \chi(X \cap \{f = \alpha\}) = \chi(X) - \sum_{i:f(p_i) > \alpha} \text{ind}(f, X, p_i) - \sum_{i:f(p_i) < \alpha} \text{ind}(-f, X, p_i) \\ - \lambda_{f, \alpha} - \lambda_{-f, -\alpha}, \end{aligned}$$

and:

$$\begin{aligned} \chi(X \cap \{f \geq \alpha\}) - \chi(X \cap \{f \leq \alpha\}) = \sum_{i:f(p_i) > \alpha} \text{ind}(f, X, p_i) + \lambda_{f, \alpha} \\ - \sum_{i:f(p_i) < \alpha} \text{ind}(-f, X, p_i) - \lambda_{-f, -\alpha}. \end{aligned}$$

It is also possible to write indices formulas for $\chi(\text{Lk}^\infty(X \cap \{f * \alpha\}))$, $* \in \{\leq, =, \geq\}$. □

Proposition 3.8. *For any $\alpha \in \mathbb{R}$, we have:*

$$\chi(\text{Lk}^\infty(X \cap \{f \leq \alpha\})) = \chi(X) - \sum_{i=1}^l \text{ind}(f, X, p_i) - \lambda_{f,\alpha} + \mu_{f,\alpha},$$

$$\chi(\text{Lk}^\infty(X \cap \{f \geq \alpha\})) = \chi(X) - \sum_{i=1}^l \text{ind}(-f, X, p_i) - \lambda_{-f,-\alpha} + \mu_{-f,-\alpha}.$$

Proof. By Theorem 3.1 applied to $f|_{X \cap S_R^{n-1}(a)}$, we have:

$$\chi(\text{Lk}^\infty(X \cap \{f \leq \alpha\})) = \mu_{f,\alpha} + \nu_{f,\alpha},$$

and, by Corollary 3.3 applied to $f|_{X \cap B_R^n(a)}$ and by Lemma 2.1:

$$\chi(X) = \sum_{i=1}^l \text{ind}(f, X, p_i) + \lambda_{f,\alpha} + \nu_{f,\alpha},$$

because $f^{-1}(\alpha)$ intersect $X \cap S_R^{n-1}(a)$ transversally. Similarly, we can write:

$$\chi(\text{Lk}^\infty(X \cap \{f \geq \alpha\})) = \mu_{-f,-\alpha} + \nu_{-f,-\alpha},$$

and:

$$\chi(X) = \sum_{i=1}^l \text{ind}(-f, X, p_i) + \lambda_{-f,-\alpha} + \nu_{-f,-\alpha}.$$

□

Corollary 3.9. *For any $\alpha \in \mathbb{R}$, we have:*

$$\begin{aligned} \chi(\text{Lk}^\infty(X \cap \{f = \alpha\})) &= 2\chi(X) - \chi(\text{Lk}^\infty(X)) - \sum_{i=1}^l \text{ind}(f, X, p_i) \\ &\quad - \sum_{i=1}^l \text{ind}(-f, X, p_i) - \lambda_{f,\alpha} + \mu_{f,\alpha} - \lambda_{-f,-\alpha} + \mu_{-f,-\alpha}. \end{aligned}$$

□

In the sequel, we will use these results to establish relations between $\chi(X)$, the indices of the critical points of $f|_X$ and $-f|_X$ and the variations of the Euler characteristics $\chi(\text{Lk}^\infty(\{f * \alpha\}))$, where $* \in \{\leq, =, \geq\}$ and $\alpha \in \mathbb{R}$. We start with definitions.

Definition 3.10. *Let Λ_f be the following set:*

$$\Lambda_f = \{\alpha \in \mathbb{R} \mid \exists (x_n)_{n \in \mathbb{N}} \text{ in } \Gamma_{f,\alpha}^X \text{ such that } \|x_n\| \rightarrow +\infty \text{ and } f(x_n) \rightarrow \alpha\}.$$

Since $\Gamma_{f,a}^X$ is a curve, Λ_f is clearly a finite set. The set Λ_f was introduced and studied by Tibar [Til] when $X = \mathbb{R}^n$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a polynomial. Following his terminology, Λ_f is the set of points α such that the fibre $f^{-1}(\alpha)$ is not ρ_a -regular.

Definition 3.11. Let $*$ \in $\{\leq, =, \geq\}$. We define Λ_f^* by:

$$\Lambda_f^* = \left\{ \alpha \in \mathbb{R} \mid \beta \mapsto \chi(\text{Lk}^\infty(X \cap \{f * \beta\})) \text{ is not constant} \right. \\ \left. \text{in a neighborhood of } \alpha \right\}.$$

Lemma 3.12. The sets Λ_f^{\leq} , $\Lambda_f^=$ and Λ_f^{\geq} are included in Λ_f .

Proof. If α does not belong to Λ_f then we can find a small interval $]-\delta + \alpha, \alpha + \delta[$ such that $\Gamma_{f,a}^X$ and $f^{-1}(]-\delta + \alpha, \alpha + \delta[) \cap X$ do not intersect outside a compact set of X . Then we can choose $R \gg 1$ such that for all $\beta \in]-\delta + \alpha, \alpha + \delta[$, $\text{Lk}^\infty(X \cap \{f * \beta\}) = X \cap \{f * \beta\} \cap S_R^{n-1}(a)$. But f has no critical point in $X \cap \{-\delta + \alpha < f < \alpha + \delta\} \cap S_R^{n-1}(a)$, so the Euler characteristics $\chi(X \cap \{f * \beta\})$ are constant in $]-\delta + \alpha, \alpha + \delta[$. \square

Corollary 3.13. The sets Λ_f^{\leq} , $\Lambda_f^=$ and Λ_f^{\geq} are finite. \square

Lemma 3.14. We have: $\Lambda_f^= \subset \Lambda_f^{\leq} \cup \Lambda_f^{\geq}$, $\Lambda_f^{\leq} \subset \Lambda_f^= \cup \Lambda_f^{\geq}$, $\Lambda_f^{\geq} \subset \Lambda_f^= \cup \Lambda_f^{\leq}$.

Proof. If $\alpha \notin \Lambda_f^{\leq} \cup \Lambda_f^{\geq}$, then $\beta \mapsto \chi(\text{Lk}^\infty(X \cap \{f \leq \beta\}))$ and $\beta \mapsto \chi(\text{Lk}^\infty(X \cap \{f \geq \beta\}))$ are constant in an interval $]-\delta + \alpha, \alpha + \delta[$. By the Mayer-Vietoris sequence, $\beta \mapsto \chi(\text{Lk}^\infty(X \cap \{f = \beta\}))$ is also constant in $]-\delta + \alpha, \alpha + \delta[$. \square

Corollary 3.15. We have: $\Lambda_f^{\leq} \cup \Lambda_f^{\geq} = \Lambda_f^{\leq} \cup \Lambda_f^= = \Lambda_f^= \cup \Lambda_f^{\geq}$. \square

Since Λ_f^{\leq} is finite, we can write $\Lambda_f^{\leq} = \{b_1, \dots, b_r\}$ where $b_1 < b_2 < \dots < b_r$ and:

$$\mathbb{R} \setminus \Lambda_f^{\leq} =]-\infty, b_1[\cup]b_1, b_2[\cup \dots \cup]b_{r-1}, b_r[\cup]b_r, +\infty[.$$

On each connected component of $\mathbb{R} \setminus \Lambda_f^{\leq}$, the function $\beta \mapsto \chi(\text{Lk}^\infty(X \cap \{f \leq \beta\}))$ is constant. For each $j \in \{0, \dots, r\}$, let b_j^+ be an element of $]b_j, b_{j+1}[$ where $b_0 = -\infty$ and $b_{r+1} = +\infty$.

Theorem 3.16. We have:

$$\chi(X) = \sum_{i=1}^k \text{ind}(f, X, p_i) + \sum_{j=0}^r \chi(\text{Lk}^\infty(X \cap \{f \leq b_j^+\})) \\ - \sum_{j=1}^r \chi(\text{Lk}^\infty(X \cap \{f \leq b_j\})).$$

Proof. Assume first that $\Lambda_f = \Lambda_f^{\leq}$. Let us choose $R \gg 1$ such that $X \cap B_R^n(a)$ is a deformation retract of X , $\{p_1, \dots, p_l\} \subset B_R^n(a)$ and:

$$\Gamma_{f,a}^X \cap (\mathbb{R}^n \setminus B_R^n(a)) = \sqcup_{i=1}^m \mathcal{B}_j.$$

We have $\Gamma_{f,a}^X \cap S_R^{n-1}(a) = \{q_1^R, \dots, q_m^R\}$. Let us recall that:

$$\nabla(f|_S)(q_j^R) = \mu_j^R \nabla(\rho_{a|S})(q_j^R),$$

where S is the stratum that contains q_j . By Corollary 3.3 and Lemma 2.1, we can write:

$$\chi(X) = \sum_{i=1}^l \text{ind}(f, X, p_i) + \sum_{j: \mu_j^R < 0} \text{ind}(f, X \cap S_R^{n-1}(a), q_j^R).$$

We can decompose the second sum in the right hand side of this equality into:

$$\sum_{\substack{j: \mu_j^R < 0 \\ f(q_j^R) \rightarrow -\infty}} \text{ind}(f, X \cap S_R^{n-1}(a), q_j^R),$$

and:

$$\sum_{i=1}^r \sum_{\substack{j: \mu_j^R < 0 \\ f(q_j^R) \rightarrow b_i}} \text{ind}(f, X \cap S_R^{n-1}(a), q_j^R).$$

Let us fix i in $\{1, \dots, r\}$ and evaluate $\sum_{\substack{j: \mu_j^R < 0 \\ f(q_j^R) \rightarrow b_i}} \text{ind}(f, X \cap S_R^{n-1}(a), q_j^R)$.

Since $\mu_j^R < 0$, the points q_j^R lie in $\{f > b_i\}$. Let us choose $R \gg 1$ and b_i^+ close to b_i in $]b_i, b_{i+1}[$ such that:

$$\left[\bigcup_{\substack{j: f \rightarrow b_i \\ \text{along } \mathcal{B}_j}} \mathcal{B}_j \right] \cap \{\|x - a\| \geq R\} \subset f^{-1}(]b_i, b_i^+[) \cap \{\|x - a\| \geq R\},$$

and $X \cap \{f \leq b_i\}$ (resp. $X \cap \{f \leq b_i^+\}$) retracts by deformation to $X \cap \{f \leq b_i\} \cap B_R^n(a)$ (resp. $X \cap \{f \leq b_i^+\} \cap B_R^n(a)$). Hence, we have:

$$\begin{aligned} \chi(\text{Lk}^\infty(X \cap \{f \leq b_i^+\})) - \chi(\text{Lk}^\infty(X \cap \{f \leq b_i\})) &= \\ \chi(X \cap \{f \leq b_i^+\} \cap S_R^{n-1}(a)) - \chi(X \cap \{f \leq b_i\} \cap S_R^{n-1}(a)) &= \\ \sum_{j: f(q_j^R) \in]b_i, b_i^+[} \text{ind}(f, X \cap S_R^{n-1}(a), q_j^R) &= \sum_{\substack{j: \mu_j^R < 0 \\ f(q_j^R) \rightarrow b_i}} \text{ind}(f, X \cap S_R^{n-1}(a), q_j^R). \end{aligned}$$

It remains to express $\sum_{\substack{j: \mu_j^R < 0 \\ f(q_j^R) \rightarrow -\infty}} \text{ind}(f, X \cap S_R^{n-1}(a), q_j^R)$. Let us choose

$R \gg 1$ and b_0^+ in $] -\infty, b_1[$ such that:

$$\left[\bigcup_{\substack{j: f \rightarrow -\infty \\ \text{along } \mathcal{B}_j}} \mathcal{B}_j \right] \cap \{\|x - a\| \geq R\} \subset f^{-1}(] -\infty, b_0^+[) \cap \{\|x - a\| \geq R\},$$

$X \cap \{f \leq b_0^+\}$ retracts by deformation to $X \cap \{f \leq b_0^+\} \cap B_R^n(a)$ and:

$$\left[\bigcup_{\substack{j: f \rightarrow -\infty \\ \text{along } \mathcal{B}_j}} \mathcal{B}_j \right] \cap \{\|x - a\| \geq R\} \subset f^{-1}(]b_0^+, +\infty[) \cap \{\|x - a\| \geq R\}.$$

By Theorem 3.1, we can write:

$$\begin{aligned} \chi(\mathrm{Lk}^\infty(X \cap \{f \leq b_0^+\})) &= \chi(X \cap \{f \leq b_0^+\} \cap S_R^{n-1}(a)) = \\ &= \sum_{j: f(q_j^R) \leq b_0^+} \mathrm{ind}(f, X \cap S_R^{n-1}(a), q_j^R) = \sum_{\substack{j: \mu_j^R < 0 \\ f(q_j^R) \rightarrow -\infty}} \mathrm{ind}(f, X \cap S_R^{n-1}(a), q_j^R). \end{aligned}$$

To get the final result, we just remark that if $b \notin \Lambda_f^{\leq}$ then:

$$\chi(\mathrm{Lk}^\infty(X \cap \{f \leq b^+\})) - \chi(\mathrm{Lk}^\infty(X \cap \{f \leq b\})) = 0.$$

□

Similarly, $\Lambda_f^{\geq} = \{c_1, \dots, c_s\}$ with $c_1 < c_2 < \dots < c_s$ and:

$$\mathbb{R} \setminus \Lambda_f^{\geq} =]-\infty, c_1[\cup]c_1, c_2[\cup \dots \cup]c_{s-1}, c_s[\cup]c_s, +\infty[.$$

For each $i \in \{0, \dots, s\}$, let c_i^+ be an element in $]c_i, c_{i+1}[$ with $c_0 = -\infty$ and $c_{s+1} = +\infty$.

Theorem 3.17. *We have:*

$$\begin{aligned} \chi(X) &= \sum_{i=1}^l \mathrm{ind}(-f, X, p_i) + \sum_{j=0}^s \chi(\mathrm{Lk}^\infty(X \cap \{f \geq c_j^+\})) \\ &\quad - \sum_{j=1}^s \chi(\mathrm{Lk}^\infty(X \cap \{f \geq c_j\})). \end{aligned}$$

Proof. Same proof as Theorem 3.16. □

Let us write $\Lambda_f^- = \{d_1, \dots, d_t\}$ with $d_1 < d_2 < \dots < d_t$ and:

$$\mathbb{R} \setminus \Lambda_f^- =]-\infty, d_1[\cup]d_1, d_2[\cup \dots \cup]d_{t-1}, d_t[\cup]d_t, +\infty[.$$

For each $i \in \{0, \dots, t\}$, let d_i^+ be an element in $]d_i, d_{i+1}[$.

Corollary 3.18. *We have:*

$$\begin{aligned} 2\chi(X) - \chi(\mathrm{Lk}^\infty(X)) &= \sum_{i=1}^l \mathrm{ind}(f, X, p_i) + \sum_{i=1}^l \mathrm{ind}(-f, X, p_i) + \\ &\quad \sum_{j=0}^t \chi(\mathrm{Lk}^\infty(X \cap \{f = d_j^+\})) - \sum_{j=1}^t \chi(\mathrm{Lk}^\infty(X \cap \{f = d_j\})). \end{aligned}$$

Proof. Assume that $\Lambda_f^{\leq} \cup \Lambda_f^{\geq} = \Lambda_f^-$. We have:

$$\chi(X) = \sum_{i=1}^l \mathrm{ind}(f, X, p_i) + \sum_{j=0}^t \chi(\mathrm{Lk}^\infty(X \cap \{f \leq d_j^+\}))$$

$$\begin{aligned} & - \sum_{j=1}^t \chi(\text{Lk}^\infty(X \cap \{f \leq d_j\})), \\ \chi(X) = & \sum_{i=1}^l \text{ind}(-f, X, p_i) + \sum_{j=0}^t \chi(\text{Lk}^\infty(X \cap \{f \geq d_j^+\})) \\ & - \sum_{j=1}^t \chi(\text{Lk}^\infty(X \cap \{f \geq d_j\})). \end{aligned}$$

Adding these two equalities and using the Mayer-Vietoris sequence, we obtain the result when $\Lambda_f^{\leq} \cup \Lambda_f^{\geq} = \Lambda_f^{\bar{}}$. But if $d_j \notin \Lambda_f^{\bar{}}$ then $\chi(\text{Lk}^\infty(X \cap \{f = d_j^+\})) - \chi(\text{Lk}^\infty(X \cap \{f = d_j\})) = 0$. \square

By Hardt's theorem, we know that there is a finite subset $B(f)$ of \mathbb{R} such that $f|_{X \cap f^{-1}(B(f))}$ is a semi-algebraic locally trivial fibration. Hence outside $B(f)$, the function $\beta \mapsto \chi(X \cap \{f = \beta\})$ is locally constant. In the sequel, we will give formulas relating the topology of X and the variations of topology in the fibres of f . Let us set $\tilde{B}(f) = f(\{p_1, \dots, p_l\}) \cup \Lambda_f^{\leq} \cup \Lambda_f^{\geq}$. This set is clearly finite.

Proposition 3.19. *If $\alpha \notin \tilde{B}(f)$ then the following functions:*

$$\beta \mapsto \chi(X \cap \{f * \beta\}), \quad * \in \{\leq, =, \geq\},$$

are constant in a neighborhood of α .

Proof. We study the local behaviors of the numbers $\lambda_{f,\alpha}$ and $\mu_{f,\alpha}$, $\alpha \in \mathbb{R}$. We denote by α^+ (resp. α^-) an element of $] \alpha, +\infty[$ (resp. $] -\infty, \alpha[$) close to α . If $R \gg 1$ is big enough and α^+ is close enough to α then in $S_R^{n-1}(a) \cap \{\alpha \leq f \leq \alpha^+\}$, there is no points q_j^R such that $\nabla(f|_S)(q_j^R) = \mu_j^R \nabla(\rho_a|_S)(q_j^R)$ with $\mu_j^R > 0$ because $f(q_j^R)$ decreases to α as R tends to infinity. Hence if α^+ is close enough to α then $\mu_{f,\alpha^+} = \mu_{f,\alpha}$. In the same way, we can show that $\lambda_{f,\alpha^-} = \lambda_{f,\alpha}$. Applying this argument to $-f$ and $-\alpha$, we see that $\lambda_{-f,-\alpha^+} = \lambda_{-f,-\alpha}$ and $\mu_{-f,-\alpha^-} = \mu_{-f,-\alpha}$. If $\alpha \notin \tilde{B}(f)$ then, by Proposition 3.8, $\lambda_{f,\alpha^+} = \lambda_{f,\alpha}$, $\mu_{f,\alpha^-} = \mu_{f,\alpha}$, $\lambda_{-f,-\alpha^-} = \lambda_{-f,-\alpha}$ and $\mu_{-f,-\alpha^+} = \mu_{-f,-\alpha}$. The formulas established in Proposition 3.6 and Corollary 3.7 enable us to conclude. \square

Let us write $\tilde{B}(f) = \{\gamma_1, \dots, \gamma_u\}$ and:

$$\mathbb{R} \setminus \tilde{B}(f) =] -\infty, \gamma_1[\cup] \gamma_1, \gamma_2[\cup \dots \cup] \gamma_{u-1}, \gamma_u[\cup] \gamma_u, +\infty[.$$

For $i \in \{0, \dots, u\}$, let γ_i^+ be an element of $] \gamma_i, \gamma_{i+1}[$ where $\gamma_0 = -\infty$ and $\gamma_{u+1} = +\infty$.

Theorem 3.20. *We have:*

$$\chi(X) = \sum_{i=1}^l \text{ind}(f, X, p_i) + \sum_{i=1}^l \text{ind}(-f, X, p_i) +$$

$$\sum_{k=0}^u \chi(X \cap \{f = \gamma_k^+\}) - \sum_{k=1}^u \chi(X \cap \{f = \gamma_k\}).$$

Proof. Let us assume first that $\tilde{B}(f) = f(\{p_1, \dots, p_l\}) \cup \Lambda_f$, i.e that $\Lambda_f^{\leq} \cup \Lambda_f^{\geq} = \Lambda_f$. For $k \in \{1, \dots, u\}$, we have by Corollary 3.7:

$$\chi(X \cap \{f = \gamma_k\}) = \chi(X) - \sum_{i: f(p_i) > \gamma_k} \text{ind}(f, X, p_i) - \sum_{i: f(p_i) < \gamma_k} \text{ind}(-f, X, p_i) - \lambda_{f, \gamma_k} - \lambda_{-f, -\gamma_k},$$

$$\chi(X \cap \{f = \gamma_k^+\}) = \chi(X) - \sum_{i: f(p_i) > \gamma_k^+} \text{ind}(f, X, p_i) - \sum_{i: f(p_i) < \gamma_k^+} \text{ind}(-f, X, p_i) - \lambda_{f, \gamma_k^+} - \lambda_{-f, -\gamma_k^+},$$

hence:

$$\begin{aligned} \chi(X \cap \{f = \gamma_k^+\}) - \chi(X \cap \{f = \gamma_k\}) &= \\ &= - \sum_{i: f(p_i) = \gamma_k} \text{ind}(-f, X, p_i) - (\lambda_{f, \gamma_k^+} - \lambda_{f, \gamma_k}), \end{aligned}$$

because as already noticed, $\lambda_{-f, -\gamma_k^+} = \lambda_{-f, -\gamma_k}$. If γ_k does not belong to Λ_f then $\lambda_{f, \gamma_k^+} = \lambda_{f, \gamma_k}$. If γ_k belongs to Λ_f then:

$$\lambda_{f, \gamma_k} - \lambda_{f, \gamma_k^+} = \sum_{\substack{j: \mu_j^R < 0 \\ f(q_j^R) \rightarrow \gamma_k}} \text{ind}(f, X \cap S_R^{n-1}(a), q_j^R).$$

Therefore,

$$\begin{aligned} \sum_{k=1}^u \chi(X \cap \{f = \gamma_k^+\}) - \chi(X \cap \{f = \gamma_k\}) &= - \sum_{i=1}^l \text{ind}(-f, X, p_i) + \\ &+ \sum_{k: \gamma_k \in \Lambda_f} \sum_{\substack{j: \mu_j^R < 0 \\ f(q_j^R) \rightarrow \gamma_k}} \text{ind}(f, X \cap S_R^{n-1}(a), q_j^R). \end{aligned}$$

By Corollary 3.7, we have:

$$\chi(X \cap \{f = \gamma_0^+\}) = \chi(X) - \sum_{i=1}^l \text{ind}(f, X, p_i) - \lambda_{f, \gamma_0^+} - \lambda_{-f, -\gamma_0^+}.$$

But we remark that:

$$\lambda_{f, \gamma_0^+} = \sum_{k: \gamma_k \in \Lambda_f} \sum_{\substack{j: \mu_j^R < 0 \\ f(q_j^R) \rightarrow \gamma_k}} \text{ind}(f, X \cap S_R^{n-1}(a), q_j^R),$$

because if $\nabla(f|_S)(q_j^R) = \mu_j^R \nabla(\rho_a|_S)(q_j^R)$ with $\mu_j^R < 0$ then $f(q_j^R) > \gamma_0^+$ for $f(q_j^R)$ decreases to one of the γ_i 's. Similarly we see that $\lambda_{-f, -\gamma_0^+} = 0$.

Combining these equalities, we get the result when $\tilde{B}_f = \Lambda_f \cup f(\{p_1, \dots, p_l\})$. But if $\gamma \notin \Lambda_f^{\leq} \cup \Lambda_f^{\geq} \cup f(\Sigma_{f|_X})$ then $\chi(X \cap \{f = \gamma^+\}) - \chi(X \cap \{f = \gamma\}) = 0$. \square

Theorem 3.21. *We have:*

$$\begin{aligned}\chi(X) &= \sum_{i=1}^l \text{ind}(f, X, p_i) + \sum_{k=0}^u \chi(X \cap \{f \leq \gamma_k^+\}) - \sum_{k=1}^u \chi(X \cap \{f \leq \gamma_k\}), \\ \chi(X) &= \sum_{i=1}^l \text{ind}(-f, X, p_i) + \sum_{k=0}^u \chi(X \cap \{f \geq \gamma_k^+\}) - \sum_{k=1}^u \chi(X \cap \{f \geq \gamma_k\}).\end{aligned}$$

Proof. We prove the first equality in the case $\tilde{B}(f) = f(\{p_1, \dots, p_l\}) \cup \Lambda_f$. For $k \in \{1, \dots, u\}$, we have by Proposition 3.6:

$$\begin{aligned}\chi(X \cap \{f \leq \gamma_k\}) - \chi(X \cap \{f = \gamma_k\}) &= \sum_{i: f(p_i) < \gamma_k} \text{ind}(-f, X, p_i) + \lambda_{-f, -\gamma_k}, \\ \chi(X \cap \{f \leq \gamma_k^+\}) - \chi(X \cap \{f = \gamma_k^+\}) &= \sum_{i: f(p_i) < \gamma_k^+} \text{ind}(-f, X, p_i) + \lambda_{-f, -\gamma_k^+}.\end{aligned}$$

Hence,

$$\begin{aligned}[\chi(X \cap \{f \leq \gamma_k^+\}) - \chi(X \cap \{f \leq \gamma_k\})] - \\ [\chi(X \cap \{f = \gamma_k^+\}) - \chi(X \cap \{f = \gamma_k\})] &= \sum_{i: f(p_i) = \gamma_k} \text{ind}(-f, X, p_i),\end{aligned}$$

so:

$$\begin{aligned}\sum_{k=1}^u [\chi(X \cap \{f \leq \gamma_k^+\}) - \chi(X \cap \{f \leq \gamma_k\})] - \\ \sum_{k=1}^u [\chi(X \cap \{f = \gamma_k^+\}) - \chi(X \cap \{f = \gamma_k\})] &= \sum_{i=1}^l \text{ind}(-f, X, p_i).\end{aligned}$$

But, we remark that $\chi(X \cap \{f \leq \gamma_0^+\}) = \chi(X \cap \{f = \gamma_0^+\})$ because $\lambda_{-f, -\gamma_0^+} = 0$, which implies that:

$$\begin{aligned}\sum_{k=0}^u \chi(X \cap \{f \leq \gamma_k^+\}) - \sum_{k=1}^u \chi(X \cap \{f \leq \gamma_k\}) &= \\ \sum_{k=0}^u \chi(X \cap \{f = \gamma_k^+\}) - \sum_{k=1}^u \chi(X \cap \{f = \gamma_k\}) + \sum_{i=1}^l \text{ind}(-f, X, p_i) &= \\ \chi(X) - \sum_{i=1}^l \text{ind}(f, X, p_i).\end{aligned}$$

\square

Corollary 3.22. *We have:*

$$\begin{aligned} & \sum_{k=0}^u \chi(X \cap \{f \geq \gamma_k^+\}) - \chi(X \cap \{f \leq \gamma_k^+\}) \\ & - \sum_{k=1}^u \chi(X \cap \{f \geq \gamma_k\}) - \chi(X \cap \{f \leq \gamma_k\}) = \\ & \sum_{i=1}^l \text{ind}(f, X, p_i) - \text{ind}(-f, X, p_i). \end{aligned}$$

□

4. CASE $X = \mathbb{R}^n$

In this section, we apply our previous results to the case $X = \mathbb{R}^n$. In this case $\text{ind}(f, X, p_i) = (-1)^n \text{ind}(-f, X, p_i) = \text{deg}_{p_i} \nabla f$, the local degree of ∇f at p_i and $\sum_{i=1}^l \text{ind}(f, X, p_i) = \text{deg}_{\infty} \nabla f$, the degree of ∇f at infinity, i.e. the topological degree of $\frac{\nabla f}{|\nabla f|} : S_R^{n-1} \rightarrow S^{n-1}$ where S_R^{n-1} is a sphere of big radius R . Furthermore, $\mu_{f,\alpha} = (-1)^{n-1} \lambda_{-f,-\alpha}$ and $\mu_{-f,-\alpha} = (-1)^{n-1} \lambda_{f,\alpha}$. We can restate our result in this setting.

Proposition 4.1. *For all $\alpha \in \mathbb{R}$, we have:*

$$\begin{aligned} \chi(\{f \geq \alpha\}) - \chi(\{f = \alpha\}) &= \sum_{i:f(p_i) > \alpha} \text{deg}_{p_i} \nabla f + \lambda_{f,\alpha}, \\ \chi(\{f \leq \alpha\}) - \chi(\{f = \alpha\}) &= (-1)^n \sum_{i:f(p_i) < \alpha} \text{deg}_{p_i} \nabla f + (-1)^{n-1} \mu_{f,\alpha}. \end{aligned}$$

□

Corollary 4.2. *If n is even then for all $\alpha \in \mathbb{R}$, we have:*

$$\begin{aligned} \chi(\{f = \alpha\}) &= 1 - \sum_{i:f(p_i) \neq \alpha} \text{deg}_{p_i} \nabla f - \lambda_{f,\alpha} + \mu_{f,\alpha}, \\ \chi(\{f \geq \alpha\}) - \chi(\{f \leq \alpha\}) &= \sum_{i:f(p_i) > \alpha} \text{deg}_{p_i} \nabla f - \sum_{i:f(p_i) < \alpha} \text{deg}_{p_i} \nabla f + \lambda_{f,\alpha} + \mu_{f,\alpha}. \end{aligned}$$

If n is odd then for all $\alpha \in \mathbb{R}$, we have:

$$\begin{aligned} \chi(\{f = \alpha\}) &= 1 - \sum_{i:f(p_i) > \alpha} \text{deg}_{p_i} \nabla f + \sum_{i:f(p_i) < \alpha} \text{deg}_{p_i} \nabla f - \lambda_{f,\alpha} + \mu_{f,\alpha}, \\ \chi(\{f \geq \alpha\}) - \chi(\{f \leq \alpha\}) &= \sum_{i:f(p_i) \neq \alpha} \text{deg}_{p_i} \nabla f + \lambda_{f,\alpha} + \mu_{f,\alpha}. \end{aligned}$$

□

The above formulas can be viewed as real versions of results on the topology of the fibres of a complex polynomial (see for instance [Pa] or [ST]).

Proposition 4.3. *If n is even then, for all $\alpha \in \mathbb{R}$, we have:*

$$\chi(\text{Lk}^\infty(\{f \leq \alpha\})) = \chi(\text{Lk}^\infty(\{f \geq \alpha\})) = 1 - \deg_\infty \nabla f - \lambda_{f,\alpha} + \mu_{f,\alpha},$$

$$\chi(\text{Lk}^\infty(\{f = \alpha\})) = 2 - 2\deg_\infty \nabla f - 2\lambda_{f,\alpha} + 2\mu_{f,\alpha}.$$

If n is odd then, for all $\alpha \in \mathbb{R}$, we have:

$$\chi(\text{Lk}^\infty(\{f \leq \alpha\})) = 1 - \deg_\infty \nabla f - \lambda_{f,\alpha} + \mu_{f,\alpha},$$

$$\chi(\text{Lk}^\infty(\{f \geq \alpha\})) = 1 + \deg_\infty \nabla f + \lambda_{f,\alpha} - \mu_{f,\alpha}.$$

□

We also obtain generalizations of Sekalski's formula [Se]. We keep the notations introduced in the general case.

Theorem 4.4. *We have:*

$$1 = \deg_\infty \nabla f + \sum_{i=0}^r \chi(\text{Lk}^\infty(\{f \leq b_i^+\})) - \sum_{i=1}^r \chi(\text{Lk}^\infty(\{f \leq b_i\})) =$$

$$(-1)^n \deg_\infty \nabla f + \sum_{i=0}^s \chi(\text{Lk}^\infty(\{f \geq c_i^+\})) - \sum_{i=1}^s \chi(\text{Lk}^\infty(\{f \geq c_i\})).$$

If n is even then we have:

$$2 = 2\deg_\infty \nabla f + \sum_{i=0}^t \chi(\text{Lk}^\infty(\{f = d_i^+\})) - \sum_{i=1}^t \chi(\text{Lk}^\infty(\{f = d_i\})).$$

□

Theorem 4.5. *If n is even, we have:*

$$1 = 2\deg_\infty \nabla f + \sum_{k=0}^u \chi(\{f = \gamma_k^+\}) - \sum_{k=1}^u \chi(\{f = \gamma_k\}),$$

$$1 = \deg_\infty \nabla f + \sum_{k=0}^u \chi(\{f \leq \gamma_k^+\}) - \sum_{k=1}^u \chi(\{f \leq \gamma_k\}),$$

$$1 = \deg_\infty \nabla f + \sum_{k=0}^u \chi(\{f \geq \gamma_k^+\}) - \sum_{k=1}^u \chi(\{f \geq \gamma_k\}),$$

$$\sum_{k=0}^u \chi(\{f \geq \gamma_k^+\}) - \chi(\{f \leq \gamma_k^+\}) = \sum_{k=1}^u \chi(\{f \geq \gamma_k\}) - \chi(\{f \leq \gamma_k\}).$$

If n is odd, we have:

$$1 = \sum_{k=0}^u \chi(\{f = \gamma_k^+\}) - \sum_{k=1}^u \chi(\{f = \gamma_k\}),$$

$$1 = \deg_\infty \nabla f + \sum_{k=0}^u \chi(\{f \leq \gamma_k^+\}) - \sum_{k=1}^u \chi(\{f \leq \gamma_k\}),$$

$$\begin{aligned}
1 &= -\deg_\infty \nabla f + \sum_{k=0}^u \chi(\{f \geq \gamma_k^+\}) - \sum_{k=1}^u \chi(\{f \geq \gamma_k\}), \\
\sum_{k=0}^u \chi(\{f \geq \gamma_k^+\}) - \chi(\{f \leq \gamma_k^+\}) &= \sum_{k=1}^u \chi(\{f \geq \gamma_k\}) - \chi(\{f \leq \gamma_k\}) \\
&\quad + 2\deg_\infty \nabla f.
\end{aligned}$$

□

5. APPLICATION TO GENERIC LINEAR FUNCTIONS

We apply the results of Section 3 to the case of a generic linear function. Let $X \subset \mathbb{R}^n$ be a closed semi-algebraic set. For $v \in S^{n-1}$, let us denote by v^* the function $v^*(x) = \langle v, x \rangle$. We are going to study the critical points of $v^*|_{X \cap S_R^{n-1}(a)}$ for v generic and R sufficiently big.

Let $\Gamma_1(X)$ be the subset of S^{n-1} defined as follows: a vector v belongs to $\Gamma_1(X)$ if there exists a sequence $(x_k)_{k \in \mathbb{N}}$ such that $\|x_k\| \rightarrow +\infty$ and a sequence $(v_k)_{k \in \mathbb{N}}$ of vectors in S^{n-1} such that $v_k \perp T_{x_k} S(x_k)$ and $v_k \rightarrow v$, where $S(x_k)$ is the stratum containing x_k .

Lemma 5.1. *The set $\Gamma_1(X)$ is a semi-algebraic set of S^{n-1} of dimension strictly less than $n - 1$.*

Proof. If we write $X = \sqcup_{\alpha \in A} S_\alpha$, where $(S_\alpha)_{\alpha \in A}$ is a finite semi-algebraic Whitney stratification of X , then we see that $\Gamma_1(X) = \sqcup_{\alpha \in A} \Gamma_1(S_\alpha)$. Hence it is enough to prove the lemma when X is a smooth semi-algebraic manifold of dimension $n - k$, $0 < k < n$.

Let us take $x = (x_1, \dots, x_n)$ as a coordinate system for \mathbb{R}^n and (x_0, x) for \mathbb{R}^{n+1} . Let φ be the semi-algebraic diffeomorphism between \mathbb{R}^n and $S^n \cap \{x_0 > 0\}$ given by:

$$\varphi(x) = \left(\frac{1}{\sqrt{1 + \|x\|^2}}, \frac{x_1}{\sqrt{1 + \|x\|^2}}, \dots, \frac{x_n}{\sqrt{1 + \|x\|^2}} \right).$$

Observe that $(x_0, x) = \varphi(z)$ if and only if $z = \frac{x}{x_0}$. The set $\varphi(X)$ is a smooth semi-algebraic set of dimension $n - k$. Let M be the following semi-algebraic set:

$$M = \left\{ (x_0, x, y) \in \mathbb{R}^{n+1} \times \mathbb{R}^n \mid (x_0, x) \in \varphi(X) \text{ and } y \perp T_{\frac{x}{x_0}} X \right\}.$$

We will show that M is a smooth manifold of dimension n . Let $p = (x_0, x, y)$ be a point in M and let $z = \varphi^{-1}(x_0, x) = \frac{x}{x_0}$. In a neighborhood of z , X is defined by the vanishing of smooth functions g_1, \dots, g_k . For $i \in \{1, \dots, k\}$, let G_i be the smooth function defined by:

$$G_i(x_0, x) = g_i \left(\frac{x}{x_0} \right) = g_i(\varphi^{-1}(x_0, x)).$$

Then in a neighborhood of (x_0, x) , $\varphi(X)$ is defined by the vanishing of G_1, \dots, G_k and $x_0^2 + x_1^2 + \dots + x_n^2 - 1$. Note that for $i, k \in \{1, \dots, n\}^2$, $\frac{\partial G_i}{\partial x_k}(x_0, x) = \frac{1}{x_0} \frac{\partial g_i}{\partial x_k}(x)$. Hence in a neighborhood of p , M is defined by the vanishing of G_1, \dots, G_k , $x_0^2 + x_1^2 + \dots + x_n^2 - 1$ and the following minors $m_{i_1 i_2 \dots i_{k+1}}$, $(i_1, \dots, i_{k+1}) \in \{1, \dots, n\}^{k+1}$, given by:

$$m_{i_1 i_2 \dots i_{k+1}}(x_0, x, y) = \begin{vmatrix} \frac{\partial G_1}{\partial x_{i_1}}(x_0, x) & \cdots & \frac{\partial G_1}{\partial x_{i_{k+1}}}(x_0, x) \\ \vdots & \ddots & \vdots \\ \frac{\partial G_k}{\partial x_{i_1}}(x_0, x) & \cdots & \frac{\partial G_k}{\partial x_{i_{k+1}}}(x_0, x) \\ y_{i_1} & \cdots & y_{i_{k+1}} \end{vmatrix}.$$

Since $\text{rank}(\nabla g_1, \dots, \nabla g_k) = k$ at $z = \varphi^{-1}(x_0, x)$, one can assume that:

$$\begin{vmatrix} \frac{\partial G_1}{\partial x_1}(x_0, x) & \cdots & \frac{\partial G_1}{\partial x_k}(x_0, x) \\ \vdots & \ddots & \vdots \\ \frac{\partial G_k}{\partial x_1}(x_0, x) & \cdots & \frac{\partial G_k}{\partial x_k}(x_0, x) \end{vmatrix} \neq 0.$$

This implies that around p , M is defined by the vanishing of G_1, \dots, G_k , $m_{1 \dots k k+1}, \dots, m_{1 \dots k n}$ and $x_0^2 + x_1^2 + \dots + x_n^2 - 1$ (a similar argument is given and proved in [Dut1, §5]). It is straightforward to see that the gradient vectors of these functions are linearly independent. Then $\bar{M} \setminus M$ is a semi-algebraic set of dimension less than n . If $\pi_y : \mathbb{R}^{n+1} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ denotes the projection on the last n coordinates, then we have $\Gamma_1(X) = S^{n-1} \cap \pi_y(\bar{M} \setminus M)$. \square

Corollary 5.2. *Let v be vector in S^{n-1} and let $a \in \mathbb{R}^n$. If there exists a sequence $(x_k)_{k \in \mathbb{N}}$ of points in X such that:*

- $\|x_k\| \rightarrow +\infty$,
- $v \in N_{x_k} S(x_k) \oplus \mathbb{R}(x_k - a)$,
- $\lim_{k \rightarrow +\infty} |v^*(x_k)| < +\infty$,

then v belongs to $\Gamma_1(X)$ (Here $N_{x_k} S(x_k)$ is the normal space to the stratum $S(x_k)$).

Proof. We can assume that $v = e_1 = (1, 0, \dots, 0)$. In this case, $v^* = x_1$. Since the stratification is finite, we can assume that $(x_k)_{k \in \mathbb{N}}$ is a sequence of points lying in a stratum S . By the Curve Selection Lemma at infinity, there exists an analytic curve $p(t) :]0, \varepsilon[\rightarrow S$ such that $\lim_{t \rightarrow 0} \|p(t)\| = +\infty$, $\lim_{t \rightarrow 0} p_1(t) < +\infty$ and for $t \in]0, \varepsilon[$, e_1 belongs to the space $N_{p(t)} S \oplus \mathbb{R}(p(t) - a)$. Let us consider the expansions as Laurent series of the p_i 's:

$$p_i(t) = h_i t^{\alpha_i} + \dots, \quad i = 1, \dots, n.$$

Let α be the minimum of the α_i 's. Necessarily, $\alpha < 0$ and $\alpha_1 \geq 0$. It is straightforward to see that $\|p(t) - a\|$ has an expansion of the form:

$$\|p(t) - a\| = bt^\alpha + \dots, \quad b > 0.$$

Let us denote by π_t the orthogonal projection onto $T_{p(t)}S$. For every $t \in]0, \varepsilon[$, there exists a real number $\lambda(t)$ such that:

$$\pi_t(e_1) = \lambda(t)\pi_t(p(t) - a) = \lambda(t)\|\pi_t(p(t) - a)\| \frac{\pi_t(p(t) - a)}{\|\pi_t(p(t) - a)\|}.$$

Observe that if t is small enough, we can assume that $\pi_t(p(t) - a)$ does not vanish because $S_{\|p(t)-a\|}(a)$ intersects S transversally. Using the fact that $p'(t)$ is tangent to S at $p(t)$, we find that:

$$p'_1(t) = \langle p'(t), e_1 \rangle = \langle p'(t), \pi_t(e_1) \rangle = \lambda(t)\langle p'(t), p(t) - a \rangle.$$

This implies that $\text{ord}(\lambda) \geq \alpha_1 - 2\alpha$. Let β be the order of $\|\pi_t(p - a)\|$. Since $\|p(t) - a\| \geq \|\pi_t(p(t) - a)\|$, β is greater or equal to α . Finally we obtain that $\text{ord}(\lambda\|\pi_t(p(t) - a)\|)$ is greater or equal to $\alpha_1 - 2\alpha + \beta$, which is strictly positive. This proves the lemma. \square

Lemma 5.3. *There exists a semi-algebraic set $\Gamma_2(X) \subset S^{n-1}$ of dimension strictly less than $n - 1$ such that if $v \notin \Gamma_2(X)$, then $v|_X^*$ has a finite number of critical points.*

Proof. It is enough to prove the lemma for a semi-algebraic stratum S of dimension $s < n$. Let N_S be the following semi-algebraic set:

$$N_S = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid x \in S \text{ and } y \perp T_x S\}.$$

Using the same kind of arguments as in Lemmas 2.2, 2.3, and 5.1, we see that N_S is a smooth semi-algebraic manifold of dimension n . Let

$$\begin{aligned} \pi_y &: N_S &\rightarrow \mathbb{R}^n \\ &(x, y) &\mapsto y \end{aligned}$$

be the projection onto the last n coordinates. The Bertini-Sard theorem implies that the set D_{π_y} of critical values of π_y is semi-algebraic of dimension strictly less than n . We take $\Gamma_2(X) = S^{n-1} \cap D_{\pi_y}$. \square

Let us set $\Gamma(X) = \Gamma_1(X) \cup \Gamma_2(X)$, it is a semi-algebraic set of S^{n-1} of dimension strictly less than $n - 1$. If $v \notin \Gamma(X)$ then $v|_X^*$ admits a finite number of critical points p_1, \dots, p_l . Moreover, if there is a family of points q_j^R in $S \cap S_R^{n-1}(a)$ such that $\nabla v|_S^*(q_j^R) = \mu_j^R \nabla \rho_a|_S(q_j^R)$, where S is a stratum of X , and $\mu_j^R < 0$ then $v^*(q_j^R) \rightarrow -\infty$ because $v \notin \Gamma_1$. Similarly if $\mu_j^R > 0$ then $v^*(q_j^R) \rightarrow +\infty$. We conclude that the set Λ_{v^*} is empty and that for all $\alpha \in \mathbb{R}$:

$$\lambda_{v^*, \alpha} = \mu_{v^*, \alpha} = \lambda_{-v^*, -\alpha} = \mu_{-v^*, -\alpha} = 0.$$

Hence, we can restate the results of Section 3 in this setting and get relations between the topology of X and the topology of generic hyperplane sections of X (see [Ti2] for similar relations in the complex setting).

Proposition 5.4. *If $v \notin \Gamma(X)$ then for all $\alpha \in \mathbb{R}$, we have:*

$$\chi(X \cap \{v^* \geq \alpha\}) - \chi(X \cap \{v^* = \alpha\}) = \sum_{i: v^*(p_i) > \alpha} \text{ind}(v^*, X, p_i),$$

$$\begin{aligned} \chi(X \cap \{v^* \leq \alpha\}) - \chi(X \cap \{v^* = \alpha\}) &= \sum_{i:v^*(p_i) < \alpha} \text{ind}(-v^*, X, p_i), \\ \chi(X \cap \{v^* = \alpha\}) &= \chi(X) - \sum_{i:v^*(p_i) > \alpha} \text{ind}(v^*, X, p_i) - \sum_{i:v^*(p_i) < \alpha} \text{ind}(-v^*, X, p_i), \\ \chi(X \cap \{v^* \geq \alpha\}) - \chi(X \cap \{v^* \leq \alpha\}) &= \\ &= \sum_{i:v^*(p_i) > \alpha} \text{ind}(v^*, X, p_i) - \sum_{i:v^*(p_i) < \alpha} \text{ind}(-v^*, X, p_i). \end{aligned}$$

□

Proposition 5.5. *If $v \notin \Gamma(X)$ then for all $\alpha \in \mathbb{R}$, we have:*

$$\begin{aligned} \chi(\text{Lk}^\infty(X \cap \{v^* \leq \alpha\})) &= \chi(X) - \sum_{i=1}^l \text{ind}(v^*, X, p_i), \\ \chi(\text{Lk}^\infty(X \cap \{v^* \geq \alpha\})) &= \chi(X) - \sum_{i=1}^l \text{ind}(-v^*, X, p_i), \\ \chi(\text{Lk}^\infty(X \cap \{v^* = \alpha\})) &= 2\chi(X) - \chi(\text{Lk}^\infty(X)) \\ &\quad - \sum_{i=1}^l \text{ind}(v^*, X, p_i) - \sum_{i=1}^l \text{ind}(-v^*, X, p_i). \end{aligned}$$

□

Note that the functions $\beta \mapsto \chi(\text{Lk}^\infty(X \cap \{v^* \geq \beta\}))$, $\beta \in \{\leq, =, \geq\}$, are constant on \mathbb{R} . Theorem 3.20, Theorem 3.21 and Corollary 3.22 are also valid in this context. They have the same formulation as in the general case with the difference that $\tilde{B}(v^*) = v^*(\{p_1, \dots, p_l\})$.

As an application, we will give a short proof of the Gauss-Bonnet formula for closed semi-algebraic sets. Let $\Lambda_0(X, -)$ be the Gauss-Bonnet measure on X defined by:

$$\Lambda_0(X, U) = \frac{1}{\text{Vol}(S^{n-1})} \int_{S^{n-1}} \sum_{x \in U} \text{ind}(v^*, X, x) dv,$$

where U is a Borel set of X . Note that if x is not a critical point of $v^*|_X$ then $\text{ind}(v^*, X, x) = 0$ and therefore that for $v \notin \Gamma(X)$, the sum $\sum_{x \in U} \text{ind}(v^*, X, x)$ is finite. The Gauss-Bonnet theorem for compact semi-algebraic sets is due to Fu [Fu] and Broecker and Kuppe [BK].

Theorem 5.6. *If X is a compact semi-algebraic set then:*

$$\Lambda_0(X, X) = \chi(X).$$

Now assume that X is just closed. Let $(K_R)_{R>0}$ be an exhaustive family of compact Borel sets of X , that is a family $(K_R)_{R>0}$ of compact Borel sets

of X such that $\cup_{R>0} K_R = X$ and $K_R \subseteq K_{R'}$ if $R \leq R'$. For every $R > 0$, we have:

$$\Lambda_0(X, X \cap K_R) = \frac{1}{\text{Vol}(S^{n-1})} \int_{S^{n-1}} \sum_{x \in X \cap K_R} \text{ind}(v^*, X, x) dv.$$

Moreover the following limit:

$$\lim_{R \rightarrow +\infty} \sum_{x \in X \cap K_R} \text{ind}(v^*, X, x),$$

is equal to $\sum_{x \in X} \text{ind}(v^*, X, x)$ which is uniformly bounded by Hardt's theorem. Applying Lebesgue's theorem, we obtain:

$$\begin{aligned} \lim_{R \rightarrow +\infty} \Lambda_0(X, X \cap K_R) &= \frac{1}{\text{Vol}(S^{n-1})} \int_{S^{n-1}} \lim_{R \rightarrow +\infty} \sum_{x \in X \cap K_R} \text{ind}(v^*, X, x) dv = \\ &= \frac{1}{\text{Vol}(S^{n-1})} \int_{S^{n-1}} \sum_{x \in X} \text{ind}(v^*, X, x) dv. \end{aligned}$$

Definition 5.7. We set:

$$\Lambda_0(X, X) = \lim_{R \rightarrow +\infty} \Lambda_0(X, X \cap K_R),$$

where $(K_R)_{R>0}$ is an exhaustive family of compact Borel sets of X .

Theorem 5.8. If X is a closed semi-algebraic set then:

$$\begin{aligned} \Lambda_0(X, X) &= \chi(X) - \frac{1}{2} \chi(\text{Lk}^\infty(X)) \\ &\quad - \frac{1}{2 \text{Vol}(S^{n-1})} \int_{S^{n-1}} \chi(\text{Lk}^\infty(X \cap \{v^* = 0\})) dv. \end{aligned}$$

Proof. We have:

$$\begin{aligned} \Lambda_0(X, X) &= \frac{1}{\text{Vol}(S^{n-1})} \int_{S^{n-1}} \sum_{x \in X} \text{ind}(v^*, X, x) dv = \\ &= \frac{1}{2 \text{Vol}(S^{n-1})} \int_{S^{n-1}} \sum_{x \in X} \text{ind}(v^*, X, x) + \text{ind}(-v^*, X, x) dv = \\ &= \frac{1}{2 \text{Vol}(S^{n-1})} \int_{S^{n-1}} 2\chi(X) - \chi(\text{Lk}^\infty(X)) - \chi(\text{Lk}^\infty(X \cap \{v^* = 0\})) dv, \end{aligned}$$

by Proposition 5.5. \square

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