Controllability of the 3D compressible Euler system

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Abstract. The paper is devoted to the controllability problem for 3D compressible Euler system. The control is a finite-dimensional external force acting only on the velocity equation. We show that the velocity and density of the fluid are simultaneously controllable. In particular, the system is approximately controllable and exactly controllable in projections.

Contents

1 Introduction

The time evolution of an isentropic ideal gas is described by the compressible Euler system

$$\rho(\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}) + \nabla p(\rho) = \rho \boldsymbol{f}, \tag{1.1}$$

$$\partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0, \tag{1.2}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \rho(0) = \rho_0, \tag{1.3}$$

where $\boldsymbol{u} = (u_1, u_2, u_3)$ and ρ are unknown velocity field and density of the gas, p is the pressure and $\boldsymbol{f}$ is the external force, $\boldsymbol{u}_0$ and ρ_0 are the initial conditions. We assume that the space variable $\boldsymbol{x} = (x_1, x_2, x_3)$ belongs to the 3D torus $\mathbb{T}^3 = \mathbb{R}^3/2\pi\mathbb{Z}^3$.

Problem (1.1)-(1.3) can be reduced by a simple change of variables to a quasi-linear symmetrizable hyperbolic system. Thus local-in-time existence and uniqueness of a smooth solution is well known (for instance, see [12, 19]). Moreover, a blow-up criterion holds for the compressible Euler equation (see [19, Section 16, Proposition 2.4]).

The aim of this paper is the study of some controllability issues for system (1.1)-(1.3). We suppose that the external force is of the form $\boldsymbol{f} = \tilde{\boldsymbol{f}} + \boldsymbol{\eta}$, where $\tilde{\boldsymbol{f}}$ is any given function and $\boldsymbol{\eta}$ is the control taking values in a finite-dimensional space. Let H^k be the Sobolev space of order k on $\mathbb{T}^3$ and let $\boldsymbol{H}^k$ be the space of vector functions $\boldsymbol{u} = (u_1, u_2, u_3)$ with components in H^k. For both spaces, we denote by $\|\cdot\|_k$ the corresponding norms. We denote $J_T := [0, T]$. The following theorem is our main result.

Main theorem. *Let $k \geq 4$ and $\tilde{\boldsymbol{f}} \in C^\infty(J_T, \boldsymbol{H}^{k+2})$. There is a finite-dimensional space $\boldsymbol{E} \subset \boldsymbol{H}^k$ with $\dim \boldsymbol{E} = 45$ such that for any constants $T, \varepsilon > 0$, for any continuous function $\boldsymbol{F} : \boldsymbol{H}^k \times H^k \to \mathbb{R}^N$ admitting a right inverse, for any functions $\boldsymbol{u}_0, \hat{\boldsymbol{u}} \in \boldsymbol{H}^k$ and $\rho_0, \hat{\rho} \in H^k$ with*

$$\int_{\mathbb{T}^3} \rho_0 \mathrm{d}\boldsymbol{x} = \int_{\mathbb{T}^3} \hat{\rho} \mathrm{d}\boldsymbol{x} \tag{1.4}$$

there is a smooth control $\boldsymbol{\eta} : J_T \to \boldsymbol{E}$ such that system (1.1)-(1.3) has a unique regular solution $(\boldsymbol{u}, \rho)$, which verifies

$$\|(\boldsymbol{u}(T), \rho(T)) - (\hat{\boldsymbol{u}}, \hat{\rho})\|_{\boldsymbol{H}^k \times H^k} < \varepsilon,$$
$$\boldsymbol{F}(\boldsymbol{u}(T), \rho(T)) = \boldsymbol{F}(\hat{\boldsymbol{u}}, \hat{\rho}).$$

See Subsection 3.1 for the exact formulation. We stress that condition (1.4) is essential, because integrating (1.2), we get $\int \rho(\cdot, \boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \text{const}$.

Before turning to the ideas of the proof, let us describe in a few words some previous results on the controllability of Euler and Navier–Stokes systems. Li and Rao [13] proved a local exact boundary controllability property for general 1D first-order quasi-linear hyperbolic equations. Exact boundary controllability problems for weak entropy solutions of 1D compressible Euler system has been established by Glass [9]. Controllability of incompressible Euler and Navier–Stokes systems has been studied by several authors. Coron [4] introduced the return method to show exact boundary controllability of 2D incompressible Euler system. Glass [8] generalized this result for 3D Euler system. Exact controllability of Navier–Stokes systems with control supported by a given domain was studied by Coron and Fursikov [5], Fursikov and Imanuvilov [7], Imanuvilov [10], Fernández-Cara et al. [6]. Agrachev and Sarychev [1, 2] proved controllability of 2D Navier–Stokes and 2D Euler equations with finite-dimensional external control. Rodrigues [15] used Agrachev–Sarychev method to prove controllability of the 2D Navier–Stokes equation on the rectangle with Lions boundary condition. Shirikyan [16, 17] generalized this method to the case of 3D Navier–Stokes equation. Furthermore, he shows [18] that 2D Euler equation is not not exactly controllable by a finite-dimensional external force. In [14], we show that in the case of 3D Euler equation, the velocity and pressure are exactly controllable in projections.

One of the main difficulties of the proof of Main theorem is the fact that the control $\boldsymbol{\eta}$ acts only on the first equation. We combine the Agrachev–Sarychev method with a perturbative result for compressible Euler equations and a property of the transport equation to prove that the velocity $\boldsymbol{u}$ and the density ρ can be controlled simultaneously with the help of a finite-dimensional external force $\boldsymbol{\eta}$. The Agrachev–Sarychev method is based on construction of an increasing sequence of finite-dimensional spaces $\boldsymbol{E}_n \subset \boldsymbol{H}^k, n \geq 0$ such that

(i) The system is controllable with $\boldsymbol{E}_N$-valued controls for some $N \geq 1$.

(ii) Controllability of the system with $\boldsymbol{\eta} \in \boldsymbol{E}_n$ is equivalent to that with $\boldsymbol{\eta} \in \boldsymbol{E}_{n+1}$.

As in the case of incompressible Euler and Navier–Stokes systems, the proof of property (i) is deduced from the hypothesis that $\boldsymbol{E}_\infty := \cup_{n=0}^{\infty} \boldsymbol{E}_n$ is dense in $\boldsymbol{H}^k$ and from the fact that for any functions $\boldsymbol{V}_0$, $\boldsymbol{V}_1$ there is a control (not necessarily $\boldsymbol{E}$-valued) which steers the system from $\boldsymbol{V}_0$ to $\boldsymbol{V}_1$. As the control acts only on the first equation, along with (1.1)-(1.2) we need to consider the control system

$$\rho(\partial_t \boldsymbol{u} + ((\boldsymbol{u}+\boldsymbol{\xi})\cdot\nabla)(\boldsymbol{u}+\boldsymbol{\xi})) + \nabla p(\rho) = \rho(\tilde{\boldsymbol{f}}+\boldsymbol{\eta}), \tag{1.5}$$

$$\partial_t \rho + \nabla\cdot(\rho(\boldsymbol{u}+\boldsymbol{\xi})) = 0. \tag{1.6}$$

For any $\boldsymbol{V}_0$ and $\boldsymbol{V}_1$ we find controls $\boldsymbol{\xi}, \boldsymbol{\eta}$ such that the solution of (1.5)-(1.6) links $\boldsymbol{V}_0$ and $\boldsymbol{V}_1$. Now to prove (i), it suffices to show that the control systems (1.1)-(1.2) and (1.5)-(1.6) are equivalent. This can be done by a simple change of the variable $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{\xi}$. To establish property (ii), we first show that the controllability of (1.1)-(1.2) with $\boldsymbol{\eta} \in \boldsymbol{E}_{n+1}$ is equivalent to that of the system

$$\rho(\partial_t \boldsymbol{u} + ((\boldsymbol{u}+\boldsymbol{\xi})\cdot\nabla)(\boldsymbol{u}+\boldsymbol{\xi})) + \nabla p(\rho) = \rho(\tilde{\boldsymbol{f}}+\boldsymbol{\eta}), \tag{1.7}$$

$$\partial_t \rho + \nabla\cdot(\rho\boldsymbol{u}) = 0 \tag{1.8}$$

with $\boldsymbol{\eta} \in \boldsymbol{E}_n$ and $\boldsymbol{\xi} \in \boldsymbol{E}_n$. Here we use the ideas from [1, 2, 16, 17, 14]. Then using a continuity property of the resolving operator of compressible Euler system (see Theorem 2.3), we show that control systems (1.7)-(1.8) and (1.5)-(1.6) are also equivalent. We refer the reader to Section 4.2 for a detailed proof of this property.

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Notation. We use bold characters to denote vector functions. Let X be a Banach space endowed with the norm $\|\cdot\|_X$. For $1 \le p < \infty$ let $L^p(J_T, X)$ be the space of measurable functions $u : J_T \to X$ such that

$$\|u\|_{L^p(J_T,X)} := \left(\int_0^T \|u\|_X^p \mathrm{d}s \right)^{\frac{1}{p}} < \infty.$$

The space of continuous functions $u : J_T \to X$ is denoted by $C(J_T, X)$. We denote by C a constant whose value may change from line to line. We write $\int f(x)\mathrm{d}x$ instead of $\int_{\mathbb{T}^3} f(x)\mathrm{d}x$. Let $\delta_{i,j}$ be the Kronecker delta, i.e, $\delta_{i,j} = 0$ if $i \neq j$ and $\delta_{i,i} = 1$.

2 Preliminaries on 3D compressible Euler system

2.1 Symmetrizable hyperbolic systems

In this subsection, we recall some results on local existence of symmetrizable hyperbolic systems. Let us consider the system

$$\partial_t \boldsymbol{v} + \sum_{i=1}^n \boldsymbol{A}_i(t, \boldsymbol{x}, \boldsymbol{v}) \partial_i \boldsymbol{v} + \boldsymbol{G}(t, \boldsymbol{x}, \boldsymbol{v}) = 0, \qquad \boldsymbol{v}(0) = \boldsymbol{v}_0. \tag{2.1}$$

We say that (2.1) is a quasi-linear symmetric hyperbolic system if matrices A_i are symmetric, i.e., $A_i = A_i^*$. If functions $\boldsymbol{A}_i, \boldsymbol{G}$ are smooth and system (2.1) is symmetric hyperbolic, then for any $\boldsymbol{v}_0 \in \boldsymbol{H}^k$, $k > n/2 + 1$ there exists $T > 0$ such that system (2.1) has a solution $\boldsymbol{v} \in C(J_T, \boldsymbol{H}^k)$ (see [12] or [19, Chapter 16] for an exact statement). Now consider a more general case:

$$\partial_t \boldsymbol{u} + \sum_{i=1}^n \boldsymbol{B}_i(t, \boldsymbol{x}, \boldsymbol{u}) \partial_i \boldsymbol{u} + \boldsymbol{H}(t, \boldsymbol{x}, \boldsymbol{u}) = 0, \qquad \boldsymbol{u}(0) = \boldsymbol{u}_0, \tag{2.2}$$

where $\boldsymbol{B}_i$ are such that there exists a positive definite matrix $\boldsymbol{B}_0$ such that $\boldsymbol{B}_0 \cdot \boldsymbol{B}_i$ are symmetric. These systems are called quasi-linear symmetrizable hyperbolic systems. As it is remarked in [19, Chapter 16, p. 366], we have the following local well-posedness of this system.

Theorem 2.1. *Let* $\boldsymbol{u}_0 \in \boldsymbol{H}^k$*,* $k > n/2 + 1$ *and* $\boldsymbol{B}_i, \boldsymbol{H} \in L^2(J_T, \boldsymbol{H}^k \times \boldsymbol{H}^k)$*. Then there exists* $T_0 > 0$*, which depends on*

$$\|\boldsymbol{u}_0\|_k + \|\boldsymbol{B}_i\|_{L^2(J_T,\boldsymbol{H}^k\times\boldsymbol{H}^k)} + \|\boldsymbol{H}\|_{L^2(J_T,\boldsymbol{H}^k\times\boldsymbol{H}^k)},$$

such that system (2.1) has a unique solution $\boldsymbol{u} \in C(J_{T_0}, \boldsymbol{H}^k)$*.*

2.2 Well-posedness of the Euler equations

Let us consider the compressible Euler system

$$\begin{aligned} \rho(\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}) + \nabla p(\rho) &= \rho \boldsymbol{f}, \\ \partial_t \rho + \nabla\cdot(\rho\boldsymbol{u}) &= 0, \\ \boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \rho(0) &= \rho_0. \end{aligned}$$

We study the case in which there is no vacuum, so that the initial density is separated from zero. Let us show that in this case the above problem can be reduced to a quasi-linear symmetrizable hyperbolic system. Setting $g = \log\rho$ and $h(s) = p'(e^s)$, the above system takes the equivalent form

$$\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + h(g)\nabla g = \boldsymbol{f}, \tag{2.3}$$

$$(\partial_t + \boldsymbol{u}\cdot\nabla)g + \nabla\cdot\boldsymbol{u} = 0, \tag{2.4}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad g(0) = g_0. \tag{2.5}$$

In what follows, we shall deal with the more general system

$$\partial_t \boldsymbol{u} + ((\boldsymbol{u}+\boldsymbol{\zeta})\cdot\nabla)(\boldsymbol{u}+\boldsymbol{\zeta}) + h(g)\nabla g = \boldsymbol{f}, \tag{2.6}$$

$$(\partial_t + (\boldsymbol{u}+\boldsymbol{\xi})\cdot\nabla)g + \nabla\cdot(\boldsymbol{u}+\boldsymbol{\xi}) = 0, \tag{2.7}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad g(0) = g_0. \tag{2.8}$$

We set $\boldsymbol{U} = (\boldsymbol{u}_0, g_0, \boldsymbol{\zeta}, \boldsymbol{\xi}, \boldsymbol{f})$,

$$\begin{aligned} &\boldsymbol{Y}^k = C(J_T, \boldsymbol{H}^k) \times C(J_T, H^k), \\ &\boldsymbol{X}^k = \boldsymbol{H}^k \times H^k \times L^2(J_T, \boldsymbol{H}^{k+1}) \times L^2(J_T, \boldsymbol{H}^{k+1}) \times L^2(J_T, \boldsymbol{H}^k), \end{aligned}$$

and endow these spaces with natural norms. Standard arguments show that if $k \geq 4$, then for any $\boldsymbol{U} \in \boldsymbol{X}^k$ problem (2.6)-(2.8) has at most one solution $(\boldsymbol{u}, g) \in \boldsymbol{Y}^k$. The following theorem establishes a perturbative result on the existence of solution and some continuity properties of the resolving operator.

Theorem 2.2. *Let* $T > 0$*,* $k \geq 4$ *and* $h \in C^k(\mathbb{R})$ *be such that* $0 < h(s)$ *for any* $s \in \mathbb{R}$*. Suppose that for some function* $\boldsymbol{U}_1 \in \boldsymbol{X}^k$ *problem (2.6)-(2.8) has a solution* $(\boldsymbol{u}_1, g_1) \in \boldsymbol{Y}^k$*. Then there are constants* $\delta > 0$ *and* $C > 0$ *depending only on* h *and* $\|\boldsymbol{U}_1\|_{\boldsymbol{X}^k}$ *such that the following assertions hold.*

(i) If $\boldsymbol{U}_2 \in \boldsymbol{X}^k$ *satisfies the inequality*

$$\|\boldsymbol{U}_1 - \boldsymbol{U}_2\|_{\boldsymbol{X}^k} < \delta, \tag{2.9}$$

then problem (2.6)-(2.8) has a unique solution $(\boldsymbol{u}_2, g_2) \in \boldsymbol{Y}^k$.

(ii) Let

$$\mathcal{R} : \boldsymbol{X}^k \to \boldsymbol{Y}^k$$

be the operator that takes a function $\boldsymbol{U}_2$ *satisfying (2.9) to the solution* $(\boldsymbol{u}_2, g_2) \in \boldsymbol{Y}^k$ *of problem (2.6)-(2.8). Then*

$$\|\mathcal{R}(\boldsymbol{U}_1) - \mathcal{R}(\boldsymbol{U}_2)\|_{\boldsymbol{Y}^{k-1}} \leq C\|\boldsymbol{U}_1 - \boldsymbol{U}_2\|_{\boldsymbol{X}^{k-1}}.$$

(iii) The operator $\mathcal{R} : \boldsymbol{X}^k \to \boldsymbol{Y}^k$ *is continuous at* $\boldsymbol{U}_1$.

We emphasize the fact that the constants δ and C depend only on the norm of $\boldsymbol{U}_1$. This observation will be important in Section 4, where we construct a solution of (2.6)-(2.8) with the help of a perturbative argument.

Proof. We seek a solution of (2.6)-(2.8) in the form $(\boldsymbol{u}_2, g_2) := (\boldsymbol{u}_1, g_1) + (\boldsymbol{w}, \varphi)$. Substituting this into (2.6)-(2.8) and performing some transformations, we obtain the following problem:

$$\begin{aligned} &\partial_t \boldsymbol{w} + ((\boldsymbol{u}_1 + \boldsymbol{\zeta}_1) \cdot \nabla)(\boldsymbol{w} + \boldsymbol{\eta}) + ((\boldsymbol{w} + \boldsymbol{\eta}) \cdot \nabla)(\boldsymbol{u}_1 + \boldsymbol{\zeta}_1) \\ &\qquad + ((\boldsymbol{w} + \boldsymbol{\eta}) \cdot \nabla)(\boldsymbol{w} + \boldsymbol{\eta}) + h(g_1 + \varphi)\nabla(g_1 + \varphi) - h(g_1)\nabla g_1 = \boldsymbol{q}, \qquad (2.10) \\ &\partial_t \varphi + ((\boldsymbol{u}_1 + \boldsymbol{\xi}_1) \cdot \nabla)\varphi + ((\boldsymbol{w} + \boldsymbol{\sigma}) \cdot \nabla)g_1 + ((\boldsymbol{w} + \boldsymbol{\sigma}) \cdot \nabla)\varphi \\ &\qquad + \nabla \cdot (\boldsymbol{w} + \boldsymbol{\sigma}) = 0, \qquad (2.11) \\ &\qquad (\boldsymbol{w}, \varphi)(0) = (\boldsymbol{w}_0, \varphi_0), \qquad (2.12) \end{aligned}$$

where $\boldsymbol{\eta} = \boldsymbol{\zeta}_2 - \boldsymbol{\zeta}_1$, $\boldsymbol{\sigma} = \boldsymbol{\xi}_2 - \boldsymbol{\xi}_1$, $\boldsymbol{q} = \boldsymbol{f}_2 - \boldsymbol{f}_1$, $\boldsymbol{w}_0 = \boldsymbol{u}_{20} - \boldsymbol{u}_{10}$ and $\varphi_0 = g_{20} - g_{10}$. Problem (2.10)-(2.12) is a quasi-linear symmetrizable hyperbolic system. Indeed, setting $V = \begin{pmatrix} \boldsymbol{w} \\ \varphi \end{pmatrix}$ and $a_i^k = h(g_1 + \varphi)\delta_{i,k}$, system (2.10)-(2.12) can be rewritten in the form

$$\partial_t \boldsymbol{V} + \sum_{i=1}^{3} \boldsymbol{A}_i(t, \boldsymbol{x}, \boldsymbol{V})\partial_i \boldsymbol{V} + \boldsymbol{G}(t, \boldsymbol{x}, \boldsymbol{V}) = 0, \qquad \boldsymbol{V}(0) = (\boldsymbol{w}_0, \varphi_0), \tag{2.13}$$

where

$$\boldsymbol{A}_i = \begin{pmatrix} (\boldsymbol{u}_1 + \boldsymbol{\zeta}_1 + \boldsymbol{w} + \boldsymbol{\eta})_i & 0 & 0 & a_1^i \\ 0 & (\boldsymbol{u}_1 + \boldsymbol{\zeta}_1 + \boldsymbol{w} + \boldsymbol{\eta})_i & 0 & a_2^i \\ 0 & 0 & (\boldsymbol{u}_1 + \boldsymbol{\zeta}_1 + \boldsymbol{w} + \boldsymbol{\eta})_i & a_3^i \\ \delta_1^i & \delta_2^i & \delta_3^i & (\boldsymbol{u}_1 + \boldsymbol{\zeta}_1 + \boldsymbol{w} + \boldsymbol{\sigma})_i \end{pmatrix},$$

$$\begin{aligned} &\boldsymbol{G}(t, \boldsymbol{x}, \boldsymbol{V}) \\ &= \begin{pmatrix} ((\boldsymbol{u}_1 + \boldsymbol{\zeta}_1) \cdot \nabla)\boldsymbol{\eta} + ((\boldsymbol{w} + \boldsymbol{\eta}) \cdot \nabla)(\boldsymbol{u}_1 + \boldsymbol{\zeta}_1) + (h(g_1 + \varphi) - h(g_1))\nabla g_1 - \boldsymbol{q} \\ ((\boldsymbol{w} + \boldsymbol{\sigma}) \cdot \nabla)g_1 + \nabla\boldsymbol{\sigma} \end{pmatrix}. \end{aligned}$$

Now note that (2.13) is symmetrizable hyperbolic system, since

$$\boldsymbol{A}_0(t,\boldsymbol{x},\boldsymbol{V}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & h(g_1+\varphi) \end{pmatrix} \tag{2.14}$$

is positive definite and $\boldsymbol{A}_0 \cdot \boldsymbol{A}_i$, $i = 1,2,3$ are symmetric. By Theorem 2.1, there is a solution $\boldsymbol{V} \in C(J_{T_0}, \boldsymbol{H}^k) \times C(J_{T_0}, H^k)$ of (2.13) for some $T_0 \leq T$. Now we prove that $T_0 = T$. First, let us rewrite system (2.10), (2.11) in the form

$$\begin{aligned} &\partial_t \boldsymbol{w} + ((\boldsymbol{u_1}+\boldsymbol{\zeta_1})\cdot\nabla)(\boldsymbol{w}+\boldsymbol{\eta}) + ((\boldsymbol{w}+\boldsymbol{\eta})\cdot\nabla)(\boldsymbol{u_2}+\boldsymbol{\zeta_2}) \\ &\qquad + h(g_1)\nabla\varphi + (h(g_2)-h(g_1))\nabla g_2 = \boldsymbol{q}, \end{aligned} \tag{2.15}$$

$$\partial_t \varphi + ((\boldsymbol{u_1}+\boldsymbol{\xi_1})\cdot\nabla)\varphi + ((\boldsymbol{w}+\boldsymbol{\sigma})\cdot\nabla)g_2 + \nabla\cdot(\boldsymbol{w}+\boldsymbol{\sigma}) = 0. \tag{2.16}$$

Taking the $\partial^\alpha := \frac{\partial^\alpha}{\partial x^\alpha}$, $|\alpha| \leq k-1$ derivative of (2.15) and multiplying the resulting equation by $\partial^\alpha \boldsymbol{w}$, we get

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial^\alpha \boldsymbol{w}\|_0^2 \\ &\quad + \int \partial^\alpha\big(((\boldsymbol{u_1}+\boldsymbol{\zeta_1})\cdot\nabla)\boldsymbol{w}\big)\cdot\partial^\alpha\boldsymbol{w}\mathrm{d}\boldsymbol{x} + \int \partial^\alpha\big((\boldsymbol{w}\cdot\nabla)(\boldsymbol{u_2}+\boldsymbol{\zeta_2})\big)\cdot\partial^\alpha\boldsymbol{w}\mathrm{d}\boldsymbol{x} \\ &\quad + \int \partial^\alpha\big(h(g_1)\nabla\varphi\big)\cdot\partial^\alpha\boldsymbol{w}\mathrm{d}\boldsymbol{x} + \int \partial^\alpha\big((h(g_2)-h(g_1))\nabla g_2\big)\cdot\partial^\alpha\boldsymbol{w}\mathrm{d}\boldsymbol{x} \\ &\quad \leq C\|\boldsymbol{w}\|_{k-1}(\|\boldsymbol{\eta}\|_k + \|\boldsymbol{q}\|_{k-1}). \end{aligned} \tag{2.17}$$

Integrating by parts, we see that

$$\begin{aligned} \int \partial^\alpha\big(((\boldsymbol{u_1}+\boldsymbol{\zeta_1})\cdot\nabla)\boldsymbol{w}\big)\cdot\partial^\alpha\boldsymbol{w}\mathrm{d}\boldsymbol{x} &\leq \int ((\boldsymbol{u_1}+\boldsymbol{\zeta_1})\cdot\nabla)\partial^\alpha\boldsymbol{w}\cdot\partial^\alpha\boldsymbol{w}\mathrm{d}\boldsymbol{x} + C\|\boldsymbol{w}\|_{k-1}^2 \\ &= -\frac{1}{2}\int(\nabla\cdot(\boldsymbol{u_1}+\boldsymbol{\zeta_1}))|\partial^\alpha\boldsymbol{w}|^2\mathrm{d}\boldsymbol{x} + C\|\boldsymbol{w}\|_{k-1}^2. \end{aligned} \tag{2.18}$$

Inequalities (2.17), (2.18) and the fact that $\boldsymbol{H}^k \hookrightarrow \boldsymbol{L}^\infty$ for $k > \frac{3}{2}$ imply that

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial^\alpha\boldsymbol{w}\|_0^2 + \int h(g_1)\nabla\partial^\alpha\varphi\cdot\partial^\alpha\boldsymbol{w}\mathrm{d}\boldsymbol{x} \leq & C\|\boldsymbol{w}\|_{k-1}(\|\boldsymbol{w}\|_{k-1} + \|\varphi\|_{k-1} \\ & + \|\boldsymbol{\eta}\|_k + \|\boldsymbol{q}\|_{k-1}). \end{aligned} \tag{2.19}$$

On the other hand, applying ∂^α to (2.16), multiplying the resulting equation by $h(g_1)\partial^\alpha\varphi$ and integrating over $\mathbb{T}^3$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int h(g_1)(\partial^\alpha\varphi)^2\mathrm{d}\boldsymbol{x} - \frac{1}{2}\int\partial_t h(g_1)(\partial^\alpha\varphi)^2\mathrm{d}\boldsymbol{x} \\ &\quad + \int\partial^\alpha((\boldsymbol{u_1}+\boldsymbol{\xi_1})\cdot\nabla\varphi)h(g_1)\partial^\alpha\varphi\mathrm{d}\boldsymbol{x} + \int\partial^\alpha(\boldsymbol{w}\cdot\nabla g_2)h(g_1)\partial^\alpha\varphi\mathrm{d}\boldsymbol{x} \\ &\quad + \int\partial^\alpha(\nabla\cdot\boldsymbol{w})h(g_1)\partial^\alpha\varphi\mathrm{d}\boldsymbol{x} \leq C\|\varphi\|_{k-1}\|\boldsymbol{\sigma}\|_k. \end{aligned}$$

As $g_1 \in C(J_T, H^k)$ and $h \in C^k(\mathbb{R})$, integration by parts in the third term on the left-hand side implies (cf. (2.18))

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int h(g_1)(\partial^\alpha\varphi)^2\mathrm{d}\boldsymbol{x} + \int \partial^\alpha(\nabla\cdot\boldsymbol{w})h(g_1)\partial^\alpha\varphi\mathrm{d}\boldsymbol{x} \le & C\|\varphi\|_{k-1}(\|\varphi\|_{k-1} \\ & + \|\boldsymbol{w}\|_{k-1} + \|\boldsymbol{\sigma}\|_k).\end{aligned} \tag{2.20}$$

Adding (2.19) and (2.20) and using the facts that $h(s) > 0$ for any $s \in \mathbb{R}$

$$\int \partial^\alpha(\nabla\cdot\boldsymbol{w})h(g_1)\partial^\alpha\varphi\mathrm{d}\boldsymbol{x} + \int h(g_1)\nabla\partial^\alpha\varphi\cdot\partial^\alpha\boldsymbol{w}\mathrm{d}\boldsymbol{x} = -\int \partial^\alpha\varphi(\nabla h(g_1))\cdot\partial^\alpha\boldsymbol{w}\mathrm{d}\boldsymbol{x},$$

we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial^\alpha\boldsymbol{w}\|_0^2 + \frac{\mathrm{d}}{\mathrm{d}t}\int(\partial^\alpha\varphi)^2\mathrm{d}\boldsymbol{x} \le & C(\|\boldsymbol{w}\|_{k-1}^2 + \|\varphi\|_{k-1}^2 + \|\boldsymbol{\sigma}\|_k^2 + \|\boldsymbol{\eta}\|_k^2 \\ & + \|\boldsymbol{q}\|_{k-1}^2).\end{aligned}$$

Taking the sum over all α, $|\alpha| \le k-1$ and applying the Gronwall inequality, we obtain

$$\|\boldsymbol{w}\|_{k-1}^2 + \|\varphi\|_{k-1}^2 \le C(\|\boldsymbol{\sigma}\|_{L^2(J_T,H^k)}^2 + \|\boldsymbol{\eta}\|_{L^2(J_T,H^k)}^2 + \|\boldsymbol{q}\|_{L^2(J_T,H^{k-1})}^2). \tag{2.21}$$

Thus we have that $T_0 = T$. Moreover, (2.21) completes also the proof of (*ii*).

Assertion (*iii*) can be proved by repeating the arguments of the proof of Theorem 1.4 in [3] for Sobolev spaces $\boldsymbol{H}^k$. ☐

2.3 Continuity property of the resolving operator

In this subsection, we establish another property of resolving operator, which will play an essential role in Section 4.2.

Theorem 2.3. *Let $\boldsymbol{\zeta}_n$ and $\boldsymbol{\xi}_n$ be bounded sequences in $C(J_T, \boldsymbol{H}^{k+2})$ and $\boldsymbol{\xi}_n$ be such that*

$$\int_0^{t_0} \boldsymbol{\xi}_n(t)\cdot\boldsymbol{\chi}_n(t)\mathrm{d}t \to 0 \text{ in } H^k \tag{2.22}$$

for any $t_0 \in J_T$ and for any uniformly equicontinuous sequence $\boldsymbol{\chi}_n : J_T \to \boldsymbol{H}^k$. Suppose that for $\boldsymbol{U}_n = (\boldsymbol{u}_0, g_0, \boldsymbol{\zeta}_n, \boldsymbol{\xi}_n, \boldsymbol{f}) \in \boldsymbol{X}^{k+1}$ problem (2.6)-(2.8) has a solution $(\boldsymbol{u}_n, g_n) \in \boldsymbol{Y}^{k+1}$. Then for sufficiently large $n \ge 1$ there exists a solution $\mathcal{R}(\boldsymbol{V}_n) \in \boldsymbol{Y}^{k+1}$ with $\boldsymbol{V}_n = (\boldsymbol{u}_0, g_0, \boldsymbol{\zeta}_n, 0, \boldsymbol{f})$, which verifies

$$\mathcal{R}(\boldsymbol{U}_n) - \mathcal{R}(\boldsymbol{V}_n) \to 0 \text{ in } \boldsymbol{Y}^k.$$

Proof. As $\boldsymbol{V}_n \in \boldsymbol{X}^{k+1}$, a blow-up criterion for quasi-linear symmetrizable hyperbolic systems [19, Section 16, Proposition 2.4] implies that, if we have the existence of $\mathcal{R}(\boldsymbol{V}_n)$ in $\boldsymbol{Y}^k$, then $\mathcal{R}(\boldsymbol{V}_n) \in Y^{k+1}$. We seek the solution $\mathcal{R}(\boldsymbol{V}_n)$ in the form $(\boldsymbol{w}_n+\boldsymbol{u}_n, \varphi_n+g_n)$. For $(\boldsymbol{w}_n, \varphi_n)$ we have the following problem (cf. (2.10)-(2.12))

$$\begin{aligned}&\partial_t \boldsymbol{w}_n + ((\boldsymbol{u}_n+\boldsymbol{\zeta}_{\boldsymbol{n}})\cdot\nabla)\boldsymbol{w}_n + (\boldsymbol{w}_n\cdot\nabla)(\boldsymbol{u}_n+\boldsymbol{\zeta}_{\boldsymbol{n}})\\ &\qquad + (\boldsymbol{w}_n\cdot\nabla)\boldsymbol{w}_n + h(g_n+\varphi_n)\nabla(g_n+\varphi_n) - h(g_n)\nabla g_n = 0, &(2.23)\\ &\partial_t\varphi_n + (\boldsymbol{u}_n\cdot\nabla)\varphi_n + (\boldsymbol{w}_n\cdot\nabla)g_n + (\boldsymbol{w}_n\cdot\nabla)\varphi_n - \boldsymbol{\xi}_n\cdot\nabla g_n\\ &\qquad + \nabla\cdot(\boldsymbol{w}_n-\boldsymbol{\xi}_n) = 0, &(2.24)\\ &\qquad (\boldsymbol{w}_n,\varphi_n)(0) = (0,0). &(2.25)\end{aligned}$$

As $\|\boldsymbol{\xi}_n\cdot\nabla g_n\|_k + \|\nabla\cdot\boldsymbol{\xi}_n\|_k$ is not necessarily small, we cannot immediately conclude the existence of a solution $(\boldsymbol{w}_n,\varphi_n)\in Y^k$. However, from the theory of the local existence of solutions for quasi-linear symmetrizable hyperbolic systems we have that for any constant $\nu>0$ there is a time $T_{0,n}>0$ such that if $\|\tilde{\boldsymbol{w}}_n(0)\|_k + \|\tilde{\varphi}_n(0)\|_k < \nu$, then problem (2.23)-(2.24) with initial data $(\tilde{\boldsymbol{w}}_n(0), \tilde{\varphi}_n(0))$ has a solution $(\boldsymbol{w}_n,\varphi_n)\in Y^k$ on the interval $[0, T_{0,n}]$. Here time $T_{0,n}>0$ depends only on $\|\mathcal{R}(\boldsymbol{U}_n)\|_{\boldsymbol{Y}^k}$ and ν. Using estimation (2.21) and the fact that $\boldsymbol{\zeta}_n$ and $\boldsymbol{\xi}_n$ are bounded sequences in $C(J_T, \boldsymbol{H}^{k+1})$, we get

$$\|\boldsymbol{u}_{\boldsymbol{n}}\|_k^2 + \|g_n\|_k^2 \leq C(\|\boldsymbol{\zeta}_n\|^2_{L^2(J_T,H^{k+1})} + \|\boldsymbol{\xi}_n\|^2_{L^2(J_T,H^{k+1})} + \|\boldsymbol{f}\|^2_{L^2(J_T,H^k)}) \leq C_1.$$

Thus $\|\mathcal{R}(\boldsymbol{U}_n)\|_{Y^k}$ is bounded and solutions $(\boldsymbol{w}_n,\varphi_n)$ are defined on the same interval J_{T_0}. A simple iterative argument shows that, to complete the proof, it suffices to prove that $\|\boldsymbol{w}_n\|_{C(T_0,H^k)} + \|\varphi_n\|_{C(T_0,H^k)} < \nu$ for sufficiently large n. To this end, let us argue as in the proof of Theorem 2.2. Taking the ∂^α, $|\alpha|\leq k$ derivative of (2.23) and multiplying the resulting equation by $\partial^\alpha\boldsymbol{w}_n$ in L^2, we get (cf. (2.19))

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial^\alpha\boldsymbol{w}_{\boldsymbol{n}}\|_0^2 + \int h(g_n)\nabla\partial^\alpha\varphi_n\cdot\partial^\alpha\boldsymbol{w}_n\mathrm{d}\boldsymbol{x} \leq C\|\boldsymbol{w}_n\|_k(\|\boldsymbol{w}_{\boldsymbol{n}}\|_k + \|\varphi_n\|_k). \tag{2.26}$$

Then, applying ∂^α, $|\alpha|\leq k$ to (2.24) and multiplying the obtained equation by $h(g_n)\partial^\alpha\varphi_n$, we derive

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int h(g_n)(\partial^\alpha\varphi_n)^2\mathrm{d}\boldsymbol{x} + \int \partial^\alpha(\nabla\cdot\boldsymbol{w}_n)h(g_n)\partial^\alpha\varphi_n\mathrm{d}\boldsymbol{x}\\ &\quad\leq \int h(g_n)\partial^\alpha\varphi_n\partial^\alpha(\boldsymbol{\xi}_n\cdot\nabla g_n + \nabla\cdot\boldsymbol{\xi}_n)\mathrm{d}\boldsymbol{x} + C\|\varphi_n\|_k(\|\varphi_n\|_k + \|\boldsymbol{w}_n\|_k).\end{aligned} \tag{2.27}$$

Combining (2.26), (2.27) and the fact that

$$\int_0^{T_0} h(g_n)\partial^\alpha\varphi_n\partial^\alpha(\boldsymbol{\xi}_n\cdot\nabla g_n + \nabla\cdot\boldsymbol{\xi}_n)\mathrm{d}s \to 0 \text{ in } L^2(\mathbb{T}^3),$$

we get that $\|\boldsymbol{w}_n\|_{C(T_0,H^k)} + \|\varphi_n\|_{C(T_0,H^k)} < \nu$ for sufficiently large n. Thus $\mathcal{R}(\boldsymbol{V}_n)\in Y^k$ and

$$\|\mathcal{R}(\boldsymbol{U}_n) - \mathcal{R}(\boldsymbol{V}_n)\|_{\boldsymbol{Y}^k} \to 0.$$

□

3 Main results

3.1 Controllability of Euler system

Let us consider the controlled system associated with the compressible Euler problem:

$$\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + h(g)\nabla g = \boldsymbol{f} + \boldsymbol{\eta}, \tag{3.1}$$

$$(\partial_t + \boldsymbol{u}\cdot\nabla)g + \nabla\cdot\boldsymbol{u} = 0, \tag{3.2}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad g(0) = g_0, \tag{3.3}$$

where $\boldsymbol{f} \in C^\infty([0,\infty), \boldsymbol{H}^{k+2})$, $\boldsymbol{u}_0 \in \boldsymbol{H}^k$ and $g_0 \in H^k$ are given functions, and $\boldsymbol{\eta}$ is the control taking values in a finite-dimensional subspace $\boldsymbol{E} \subset \boldsymbol{H}^{k+2}$. We denote by $\Theta(\boldsymbol{u}_0, g_0, \boldsymbol{f})$ the set of functions $\boldsymbol{\eta} \in L^2(J_T, \boldsymbol{H}^k)$ for which problem (3.1)-(3.3) has a solution in $\boldsymbol{Y}^k$. For any $\alpha > 0$ and $k \in \mathbb{N}$ let us define the set

$$G_\alpha^k = \{g \in H^k : \int e^{g(\boldsymbol{x})}\mathrm{d}\boldsymbol{x} = \alpha\}.$$

Recall that $\mathcal{R}$ is the resolving operator of (2.6)-(2.8). We denote by $R_t(\cdot)$ the restriction of $R(\cdot)$ to the time t. Let $\boldsymbol{X} \subset L^2(J_T, \boldsymbol{H}^k)$ be an arbitrary vector subspace. We endow G_α^k with the metric defined by the norm of H^k and $\boldsymbol{X}$ by the norm of $L^2(J_T, \boldsymbol{H}^k)$. Recall that for a function $f : \mathbb{R}^m \to \mathbb{R}^n$ a point $x \in \mathbb{R}^m$ is said to be regular point if the differential $Df(x)$ is surjective. Then, by the inverse function theorem, there exists a neighborhood of $f(x)$ such that a right inverse of f is well defined. Now we give a generalization of the notion of a regular point for a continuous function $\boldsymbol{F} : \boldsymbol{H}^k \times G_\alpha^k \to \mathbb{R}^N$.

Definition 3.1. *A point* $(\boldsymbol{u_1}, g_1)$ *is said to be regular for* $\boldsymbol{F}$ *if there is a non-degenerate closed ball* $\boldsymbol{B} \subset \mathbb{R}^N$ *centred at* $\boldsymbol{y_1} = \boldsymbol{F}(\boldsymbol{u_1}, g_1)$ *and a continuous function* $\boldsymbol{G} : \boldsymbol{B} \to \boldsymbol{H}^k \times G_\alpha^k$ *such that* $\boldsymbol{G}(\boldsymbol{y_1}) = (\boldsymbol{u_1}, g_1)$ *and* $\boldsymbol{F}(\boldsymbol{G}(\boldsymbol{y})) = \boldsymbol{y}$ *for any* $\boldsymbol{y} \in \boldsymbol{B}$.

Definition 3.2. *System (3.1), (3.2) with* $\boldsymbol{\eta} \in \boldsymbol{X}$ *is said to be controllable at time* $T > 0$ *if for any constants* ε, $\alpha > 0$, *for any continuous function* $\boldsymbol{F} : \boldsymbol{H}^k \times G_\alpha^k \to \mathbb{R}^N$, *for any initial data* $(\boldsymbol{u}_0, g_0) \in \boldsymbol{H}^k \times G_\alpha^k$ *and for any regular point* $(\boldsymbol{u_1}, g_1)$ *for* $\boldsymbol{F}$ *there is a control* $\boldsymbol{\eta} \in \Theta(\boldsymbol{u}_0, g_0, \boldsymbol{f}) \cap \boldsymbol{X}$ *such that*

$$\|\mathcal{R}_T(\boldsymbol{u}_0, g_0, 0, 0, \boldsymbol{\eta}) - (\boldsymbol{u_1}, g_1)\|_{\boldsymbol{H}^k \times H^k} < \varepsilon,$$
$$\boldsymbol{F}(\mathcal{R}_T(\boldsymbol{u}_0, g_0, 0, 0, \boldsymbol{\eta})) = \boldsymbol{F}(\boldsymbol{u_1}, g_1).$$

Let us note that this concept of controllability is stronger than the approximate controllability and is weaker than the exact controllability. In the following example the constructed function admits a right inverse.

Example 3.3. For any function $\boldsymbol{z} \in \boldsymbol{H}^k \times G_\alpha^k$ we set

$$\boldsymbol{F}(\boldsymbol{z}) := \int |\boldsymbol{z}(x)|^2 \mathrm{d}x.$$

Then for any nonzero elements $\boldsymbol{z}_1 \in \boldsymbol{H}^k$ and $z_2 \in G_\alpha^k$ the point $\boldsymbol{z} = (\boldsymbol{z}_1, z_2)$ is regular for $\boldsymbol{F}$.

For any finite-dimensional subspace $\boldsymbol{E} \subset \boldsymbol{H}^{k+2}$, we denote by $\mathcal{F}(\boldsymbol{E})$ the largest vector space $\boldsymbol{F} \subset \boldsymbol{H}^{k+2}$ such that for any $\boldsymbol{\eta}_1 \in \boldsymbol{F}$ there are vectors $\boldsymbol{\eta}, \boldsymbol{\zeta}^1, \dots, \boldsymbol{\zeta}^n \in \boldsymbol{E}$ satisfying the relation

$$\boldsymbol{\eta}_1 = \boldsymbol{\eta} - \sum_{i=1}^n (\boldsymbol{\zeta}^i \cdot \nabla)\boldsymbol{\zeta}^i.$$

We define $\boldsymbol{E_k}$ by the rule

$$\boldsymbol{E}_0 = \boldsymbol{E}, \quad \boldsymbol{E}_n = \mathcal{F}(\boldsymbol{E}_{n-1}) \quad \text{for} \quad n \geq 1, \quad \boldsymbol{E}_\infty = \bigcup_{n=1}^\infty \boldsymbol{E}_n.$$

The following theorem is the main result of this section.

Theorem 3.4. *Suppose $\boldsymbol{f} \in C^\infty([0,\infty), \boldsymbol{H}^{k+2})$. If $\boldsymbol{E} \subset \boldsymbol{H}^{k+2}$ is a finite-dimensional subspace such that $\boldsymbol{E}_\infty$ is dense in $\boldsymbol{H}^{k+1}$, then system (3.1), (3.2) with $\boldsymbol{\eta} \in C^\infty(J_T, \boldsymbol{E})$ is controllable at time $T > 0$.*

This theorem will be established in Section 3.2. We now construct an example of a subspace E for which the hypothesis of Theorem 3.4 is satisfied.

Let us introduce the functions

$$c^i_{\boldsymbol{m}}(\boldsymbol{x}) = \boldsymbol{e}_i \cos\langle \boldsymbol{m}, \boldsymbol{x}\rangle, \; s^i_{\boldsymbol{m}}(\boldsymbol{x}) = \boldsymbol{e}_i \sin\langle \boldsymbol{m}, \boldsymbol{x}\rangle, \quad i = 1, 2, 3,$$

where $\boldsymbol{m} \in \mathbb{Z}^3$ and $\{\boldsymbol{e}_i\}$ is the standard basis in $\mathbb{R}^3$.

Lemma 3.5. *If $\boldsymbol{E} = \operatorname{span}\{c^i_{\boldsymbol{m}}, s^i_{\boldsymbol{m}}, 0 \leq m_j \leq 1, i, j = 1, 2, 3\}$, then the vector space $\boldsymbol{E}_\infty$ is dense in $\boldsymbol{H}^k$ for any $k \geq 0$.*

It is straightforward to see that $\dim \boldsymbol{E} = 45$.

Proof of Lemma 3.5. It suffices to show that

$$\operatorname{span}\{c^i_{\boldsymbol{m}}, s^i_{\boldsymbol{m}}, \quad |\boldsymbol{m}| \leq 2^j\} \subset \boldsymbol{E}_{j+1} \quad \text{for all} \quad j \geq 0, \tag{3.4}$$

where $|\boldsymbol{m}| = |m_1| + |m_2| + |m_3|$. We prove (3.4) by induction. The case $j = 0$ is clear. We shall prove (3.4) for $j \geq 1$ assuming that it is true for any $j' < j$. If $n_i \neq 0$, then it is easy to see

$$\begin{aligned}
s^i_{2\boldsymbol{n}}(\boldsymbol{x}) &= -\frac{2}{n_i} c^i_{\boldsymbol{n}}(\boldsymbol{x}) \cdot \nabla c^i_{\boldsymbol{n}}(\boldsymbol{x}), \\
-s^i_{2\boldsymbol{n}}(\boldsymbol{x}) &= -\frac{2}{n_i} s^i_{\boldsymbol{n}}(\boldsymbol{x}) \cdot \nabla s^i_{\boldsymbol{n}}(\boldsymbol{x}), \\
c^i_{2\boldsymbol{n}}(\boldsymbol{x}) &= -\frac{1}{n_i} (s^i_{\boldsymbol{n}}(\boldsymbol{x}) - c^i_{\boldsymbol{n}}(\boldsymbol{x})) \cdot \nabla (s^i_{\boldsymbol{n}}(\boldsymbol{x}) - c^i_{\boldsymbol{n}}(\boldsymbol{x})), \\
-c^i_{2\boldsymbol{n}}(\boldsymbol{x}) &= -\frac{1}{n_i} (s^i_{\boldsymbol{n}}(\boldsymbol{x}) + c^i_{\boldsymbol{n}}(\boldsymbol{x})) \cdot \nabla (s^i_{\boldsymbol{n}}(\boldsymbol{x}) + c^i_{\boldsymbol{n}}(\boldsymbol{x})).
\end{aligned}$$

Thus $s^i_{2\boldsymbol{n}}(\boldsymbol{x}), c^i_{2\boldsymbol{n}}(\boldsymbol{x}) \in \boldsymbol{E}_{j+1}$ for any $|\boldsymbol{n}| \leq 2^{j-1}$, $n_i \neq 0$. If $n_i = 0$, without loss of generality, we can assume $n_1 \neq 0$, then

$$s^1_{2\boldsymbol{n}}(\boldsymbol{x}) + s^i_{2\boldsymbol{n}}(\boldsymbol{x}) = -\frac{2}{n_1}(c^1_{\boldsymbol{n}}(\boldsymbol{x}) + c^i_{\boldsymbol{n}}(\boldsymbol{x})) \cdot \nabla(c^1_{\boldsymbol{n}}(\boldsymbol{x}) + c^i_{\boldsymbol{n}}(\boldsymbol{x})), \tag{3.5}$$

$$-s^1_{2\boldsymbol{n}}(\boldsymbol{x}) - s^i_{2\boldsymbol{n}}(\boldsymbol{x}) = -\frac{2}{n_1}(s^1_{\boldsymbol{n}}(\boldsymbol{x}) + s^i_{\boldsymbol{n}}(\boldsymbol{x})) \cdot \nabla(s^1_{\boldsymbol{n}}(\boldsymbol{x}) + s^i_{\boldsymbol{n}}(\boldsymbol{x})). \tag{3.6}$$

As $\pm s^1_{2\boldsymbol{n}}(\boldsymbol{x}) \in \boldsymbol{E}_{j+1}$ and the right-hand sides of (3.5), (3.6) are in are in $\boldsymbol{E}_{j+1}$, we get $s^i_{2\boldsymbol{n}}(\boldsymbol{x}) \in \boldsymbol{E}_{j+1}$ for any $|\boldsymbol{n}| \leq 2^{j-1}$, $i = 1, 2, 3$. In the same way, we can show that $c^i_{2\boldsymbol{n}}(\boldsymbol{x}) \in \boldsymbol{E}_{j+1}$. Now take $\boldsymbol{l} \in \mathbb{Z}^3, |\boldsymbol{l}| \leq 2^j$ and let us choose $\boldsymbol{n} \in \mathbb{Z}^3, |\boldsymbol{n}| \leq 2^{j-1}$ and $\boldsymbol{m} \in \mathbb{Z}^3, |\boldsymbol{m}| \leq 2^{j-1}$ such that

$$\boldsymbol{l} = \boldsymbol{n} + \boldsymbol{m} \text{ and } c^i_{\boldsymbol{n}-\boldsymbol{m}}, s^i_{\boldsymbol{n}-\boldsymbol{m}} \in \boldsymbol{E}_1.$$

For example, if $l = (l_1, l_2, l_3)$ and l_1 is even, we can take

$$\boldsymbol{n} = (\frac{l_1}{2}, \left[\frac{l_2}{2}\right], l_3 - \left[\frac{l_3}{2}\right]) \quad \text{and} \quad \boldsymbol{m} = (\frac{l_1}{2}, l_2 - \left[\frac{l_2}{2}\right], \left[\frac{l_3}{2}\right]).$$

A similar representation holds if l_2 or l_3 is even. On the other hand, if all l_i are odd, then necessarily $l_i \leq 2^j - 1$, and we can take

$$\boldsymbol{n} = (l_1 - \left[\frac{l_1}{2}\right], \left[\frac{l_2}{2}\right], l_3 - \left[\frac{l_3}{2}\right]) \quad \text{and} \quad \boldsymbol{m} = (\left[\frac{l_1}{2}\right], l_2 - \left[\frac{l_2}{2}\right], \left[\frac{l_3}{2}\right]).$$

Using the identities

$$\begin{aligned}(s^i_{\boldsymbol{n}}(\boldsymbol{x}) \pm s^i_{\boldsymbol{m}}(\boldsymbol{x})) \cdot \nabla(s^i_{\boldsymbol{n}}(\boldsymbol{x}) \pm s^i_{\boldsymbol{m}}(\boldsymbol{x})) &= \frac{n_i}{2} s^i_{2\boldsymbol{n}}(\boldsymbol{x}) + \frac{m_i}{2} s^i_{2\boldsymbol{m}}(\boldsymbol{x}) \pm \frac{l_i}{2} s^i_{\boldsymbol{l}}(\boldsymbol{x}) \\ &\pm \frac{n_i - m_i}{2} s^i_{\boldsymbol{n}-\boldsymbol{m}}(\boldsymbol{x}), \\ (s^i_{\boldsymbol{n}}(\boldsymbol{x}) \pm c^i_{\boldsymbol{m}}(\boldsymbol{x})) \cdot \nabla(s^i_{\boldsymbol{n}}(\boldsymbol{x}) \pm c^i_{\boldsymbol{m}}(\boldsymbol{x})) &= \frac{n_i}{2} s^i_{2\boldsymbol{n}}(\boldsymbol{x}) - \frac{m_i}{2} s^i_{2\boldsymbol{m}}(\boldsymbol{x}) \pm \frac{l_i}{2} c^i_{\boldsymbol{l}}(\boldsymbol{x}) \\ &\pm \frac{n_i - m_i}{2} c^i_{\boldsymbol{n}-\boldsymbol{m}}(\boldsymbol{x}),\end{aligned}$$

we obtain that if $l_i \neq 0$, then $s^i_{\boldsymbol{l}}(\boldsymbol{x}), c^i_{\boldsymbol{l}}(\boldsymbol{x}) \in \boldsymbol{E}_{j+1}$ for any $|\boldsymbol{l}| \leq 2^j$, $i = 1, 2, 3$. Arguing as above, we can easily prove that also in the case $l_i = 0$ we have $s^i_{\boldsymbol{l}}(\boldsymbol{x}), c^i_{\boldsymbol{l}}(\boldsymbol{x}) \in \boldsymbol{E}_{j+1}$. ☐

3.2 Proof of Theorem 3.4

We shall need the concept of $(\varepsilon, \boldsymbol{u}_0, g_0, \boldsymbol{K})$-controllability of the system. Let us fix constants $\varepsilon, \alpha > 0$, an initial point $(\boldsymbol{u}_0, g_0) \in \boldsymbol{H}^k \times G^k_\alpha$, a compact set $\boldsymbol{K} \subset \boldsymbol{H}^k \times G^k_\alpha$ and a vector space $\boldsymbol{X} \subset L^2(J_T, \boldsymbol{H}^k)$.

Definition 3.6. *We say that system (3.1), (3.2) with* $\boldsymbol{\eta} \in \boldsymbol{X}$ *is* $(\varepsilon, \boldsymbol{u}_0, g_0, \boldsymbol{K})$*-controllable at time* $T > 0$ *if there is a continuous mapping*

$$\boldsymbol{\Psi} : \boldsymbol{K} \to \Theta(\boldsymbol{u}_0, g_0, \boldsymbol{f}) \cap \boldsymbol{X}$$

such that

$$\sup_{(\hat{\boldsymbol{u}},\hat{g})\in\boldsymbol{K}} \|\mathcal{R}_T(\boldsymbol{u}_0,g_0,0,0,\boldsymbol{\Psi}(\hat{\boldsymbol{u}},\hat{g}))-(\hat{\boldsymbol{u}},\hat{g})\|_{\boldsymbol{H}^k\times H^k}<\varepsilon,$$

where $\Theta(\boldsymbol{u}_0,g_0,\boldsymbol{f})\cap\boldsymbol{X}$ *is endowed with the norm of* $L^2(J_T,\boldsymbol{H}^k)$.

The proof of Theorem 3.4 is deduced from the following result.

Theorem 3.7. *If* $\boldsymbol{E}\subset\boldsymbol{H}^{k+2}$, $k\geq 4$ *is a finite-dimensional subspace such that* $\boldsymbol{E}_\infty$ *is dense in* $\boldsymbol{H}^{k+1}$, *then for any* $\varepsilon>0$, $(\boldsymbol{u}_0,g_0)\in\boldsymbol{H}^k\times G_\alpha^k$ *and* $\boldsymbol{K}\subset\boldsymbol{H}^k\times G_\alpha^k$ *system (3.1), (3.2) with* $\boldsymbol{\eta}\in C^\infty(J_T,\boldsymbol{E})$ *is* $(\varepsilon,\boldsymbol{u}_0,g_0,\boldsymbol{K})$*-controllable at time* $T>0$.

Taking this assertion for granted, let us complete the proof of Theorem 3.4. Suppose ε and α are positive constants, $\boldsymbol{F}:\boldsymbol{H}^k\times G_\alpha^k\to\mathbb{R}^N$ is a continuous function and $(\boldsymbol{u_1},g_1)$ is a regular point for $\boldsymbol{F}$. Thus, there is a closed ball $\boldsymbol{B}\subset\mathbb{R}^N$ centred at $\boldsymbol{y_1}=F(\boldsymbol{u_1},g_1)$ of radius $r>0$ and a continuous function $\boldsymbol{G}:\boldsymbol{B}\to\boldsymbol{H}^k\times G_\alpha^k$ such that $\boldsymbol{G}(\boldsymbol{u_1},g_1)=(\boldsymbol{u_1},g_1)$ and $\boldsymbol{F}(\boldsymbol{G}(\boldsymbol{y}))=\boldsymbol{y}$ for any $\boldsymbol{y}\in\boldsymbol{B}$. Without loss of generality, we can assume that $\boldsymbol{B}$ is such that

$$\sup_{\boldsymbol{y}\in\boldsymbol{B}}\|\boldsymbol{G}(\boldsymbol{y})-(\boldsymbol{u_1},g_1)\|_{\boldsymbol{H}^k\times H^k}\leq\frac{\varepsilon}{2}. \tag{3.7}$$

Let us choose a constant $0<\varepsilon_0<\varepsilon$ such that

$$\|\boldsymbol{F}(\hat{\boldsymbol{y}})-\boldsymbol{F}(\tilde{\boldsymbol{y}})\|_{\mathbb{R}^N}<r \text{ for any } \hat{\boldsymbol{y}},\tilde{\boldsymbol{y}}\in B, \|\hat{\boldsymbol{y}}-\tilde{\boldsymbol{y}}\|_{\boldsymbol{H}^k\times H^k}\leq\frac{\varepsilon_0}{2}. \tag{3.8}$$

Since $\boldsymbol{K}:=\boldsymbol{G}(\boldsymbol{B})$ is a compact subset of $\boldsymbol{H}^k\times G_\alpha^k$, Theorem 3.7 implies that there is a continuous mapping $\boldsymbol{\Psi}:\boldsymbol{K}\to\Theta(\boldsymbol{u}_0,g_0,\boldsymbol{f})\cap\boldsymbol{X}$ such that

$$\sup_{(\hat{\boldsymbol{u}},\hat{g})\in\boldsymbol{K}} \|\mathcal{R}_T(\boldsymbol{u}_0,g_0,0,0,\boldsymbol{\Psi}(\hat{\boldsymbol{u}},\hat{g}))-(\hat{\boldsymbol{u}},\hat{g})\|_{\boldsymbol{H}^k\times H^k}<\frac{\varepsilon_0}{2}. \tag{3.9}$$

Therefore, the continuous mapping

$$\boldsymbol{\Phi}:\boldsymbol{B}\to\mathbb{R}^N,\quad \boldsymbol{y}\to\boldsymbol{F}\big(\mathcal{R}_T(\boldsymbol{u}_0,g_0,0,0,\boldsymbol{\Psi}\circ\boldsymbol{G}(\boldsymbol{y}))\big)$$

satisfies the inequality

$$\sup_{\boldsymbol{y}\in\boldsymbol{B}}\|\boldsymbol{\Phi}(\boldsymbol{y})-\boldsymbol{y}\|_{\mathbb{R}^N}=\sup_{\boldsymbol{y}\in\boldsymbol{B}}\|\boldsymbol{F}\big(\mathcal{R}_T(\boldsymbol{u}_0,g_0,0,0,\boldsymbol{\Psi}\circ\boldsymbol{G}(\boldsymbol{y}))\big)-\boldsymbol{F}(\boldsymbol{G}(\boldsymbol{y}))\|_{\mathbb{R}^N}<r.$$

Applying the Brouwer theorem, we see that the mapping $\boldsymbol{y}\to\boldsymbol{y_1}+\boldsymbol{y}-\boldsymbol{\Phi}(\boldsymbol{y})$ from $\boldsymbol{B}$ to $\boldsymbol{B}$ has a fixed point $\overline{\boldsymbol{y}}\in\boldsymbol{B}$. Thus

$$\boldsymbol{F}(\mathcal{R}_T(\boldsymbol{u}_0,g_0,0,0,\boldsymbol{\Psi}\circ\boldsymbol{G}(\overline{\boldsymbol{y}})))=\boldsymbol{F}(\boldsymbol{u}_1,g_1).$$

Using (3.7) and (3.9), we obtain

$$\begin{aligned}&\|\mathcal{R}_T(\boldsymbol{u}_0,g_0,0,0,\boldsymbol{\Psi}(\boldsymbol{G}(\overline{\boldsymbol{y}})))-(\boldsymbol{u}_1,g_1)\|_{\boldsymbol{H}^k\times H^k}\\ &\leq\|\mathcal{R}_T(\boldsymbol{u}_0,g_0,0,0,\boldsymbol{\Psi}(\boldsymbol{G}(\overline{\boldsymbol{y}})))-\boldsymbol{G}(\overline{\boldsymbol{y}})\|_{\boldsymbol{H}^k\times H^k}+\|\boldsymbol{G}(\overline{\boldsymbol{y}})-(\boldsymbol{u}_1,g_1)\|_{\boldsymbol{H}^k\times H^k}<\varepsilon.\end{aligned}$$

This completes the proof.

4 Proof of Theorem 3.7

4.1 Reduction to controllability with $\boldsymbol{E}_1$-valued controls

Theorem 3.7 is derived from the proposition below, which is established in Subsection 4.2.

Proposition 4.1. *Suppose that* $\boldsymbol{E} \subset \boldsymbol{H}^{k+2}$ *is a finite-dimensional subspace. Then system (3.1), (3.2) is* $(\varepsilon, \boldsymbol{u}_0, g_0, \boldsymbol{K})$*-controllable with* $\boldsymbol{\eta} \in C^\infty(J_T, \boldsymbol{E}_1)$ *if and only if it is* $(\varepsilon, \boldsymbol{u}_0, g_0, \boldsymbol{K})$*-controllable with* $\boldsymbol{\eta} \in C^\infty(J_T, \boldsymbol{E})$.

Proof of Theorem 3.7. In view of Proposition 4.1, it suffices to prove that there is an integer $N \geq 1$, depending only on ε, $\boldsymbol{u}_0$, g_0 and $\boldsymbol{K}$, such that (3.1), (3.2) with $\boldsymbol{\eta} \in C^\infty(J_T, \boldsymbol{E}_N)$ is $(\varepsilon, \boldsymbol{u}_0, g_0, \boldsymbol{K})$-controllable at time T. For any $\mu > 0$ and $(\hat{\boldsymbol{u}}, \hat{g}) \in \boldsymbol{K}$ let us define

$$\boldsymbol{u}_\mu(t; \hat{\boldsymbol{u}}) = T^{-1}(te^{-\mu\Delta}\hat{\boldsymbol{u}} + (T-t)e^{-\mu\Delta}\boldsymbol{u}_0), \tag{4.1}$$

$$g_\mu(t; \hat{g}) = \ln(T^{-1}(te^{\varphi_\mu(\hat{g})} + (T-t)e^{\varphi_\mu(g_0)})), \tag{4.2}$$

where $\varphi_\mu(g) \in G_\alpha^{k+1}$ is such that $\varphi_\mu(g) \to g$ as $\mu \to 0$ for all $g \in G_\alpha^k$. For example, we can take

$$\varphi_\mu(g) = \ln\left(\frac{\alpha}{\int \exp(e^{-\mu\Delta} g(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}}\right) + e^{-\mu\Delta} g.$$

Step 1. In this step, we show that there are controls $\boldsymbol{\eta}_\mu \in C^\infty(J_T, \boldsymbol{H}^k)$ and $\boldsymbol{\xi}_\mu \in C^\infty(J_T, \boldsymbol{H}^{k+1})$ satisfying

$$\mathcal{R}(e^{-\mu\Delta}\boldsymbol{u}_0, \varphi_\mu(g_0), \boldsymbol{\xi}_\mu, \boldsymbol{\xi}_\mu, \boldsymbol{\eta}_\mu) = (\boldsymbol{u}_\mu, g_\mu). \tag{4.3}$$

We first construct $\boldsymbol{\xi}_\mu \in C^\infty(J_T, \boldsymbol{H}^{k+1})$ such that

$$\partial_t g_\mu + ((\boldsymbol{u}_\mu + \boldsymbol{\xi}_\mu) \cdot \nabla) g_\mu + \nabla \cdot (\boldsymbol{u}_\mu + \boldsymbol{\xi}_\mu) = 0. \tag{4.4}$$

To this end, let us multiply (4.4) by e^{g_μ} and perform some simple transformations. We get

$$\nabla \cdot (e^{g_\mu}\boldsymbol{\xi}_\mu) = -\partial_t e^{g_\mu} - \nabla \cdot (e^{g_\mu}\boldsymbol{u}_\mu). \tag{4.5}$$

We seek a solution of this equation in the form $e^{g_\mu}\boldsymbol{\xi}_\mu = \nabla\psi_\mu$. Substituting this into (4.5), we get

$$\Delta\psi_\mu = -\partial_t e^{g_\mu} - \nabla \cdot (e^{g_\mu}\boldsymbol{u}_\mu).$$

This equation has a solution $\boldsymbol{\psi}_\mu \in C^\infty(J_T, \boldsymbol{H}^{k+2})$ if and only if the integral of the right-hand side over $\mathbb{T}^3$ is zero. The definitions of $\boldsymbol{u}_\mu, g_\mu$ imply that

$$\int (\partial_t e^{g_\mu} + \nabla \cdot (e^{g_\mu}\boldsymbol{u}_\mu))\mathrm{d}\boldsymbol{x} = \partial_t \int e^{g_\mu}\mathrm{d}\boldsymbol{x} = \partial_t \alpha = 0.$$

Thus (4.5) has a solution $\boldsymbol{\xi}_\mu \in C^\infty(J_T, \boldsymbol{H}^{k+1})$. Since $\|e^{-\mu\Delta}\boldsymbol{u}_0\|_{k+1}, \|\varphi_\mu(g_0)\|_{k+1}$ are bounded with respect to $\mu \in (0,1)$, the constructions of $\boldsymbol{u}_\mu$ and g_μ imply that $\|\partial_t e^{g_\mu} - \nabla \cdot (e^{g_\mu}\boldsymbol{u}_\mu)\|_k$ is also bounded. Thus $\|\psi_\mu\|_{k+2}$ is bounded, which implies the boundedness of $\|\boldsymbol{\xi}_\mu\|_{k+1}$. If we define

$$\boldsymbol{\eta}_\mu = \partial_t \boldsymbol{u}_\mu + ((\boldsymbol{u}_\mu + \boldsymbol{\xi}_\mu) \cdot \nabla)(\boldsymbol{u}_\mu + \boldsymbol{\xi}_\mu) + h(g_\mu)\nabla g_\mu - \boldsymbol{f}, \tag{4.6}$$

then $\boldsymbol{\eta}_\mu \in C^\infty(J_T, \boldsymbol{H}^k)$ and (4.3) holds.

Step 2. Let us take some functions $\boldsymbol{\xi}_\mu^\delta \in C^\infty(J_T, \boldsymbol{H}^{k+1})$ such that $\boldsymbol{\xi}_\mu^\delta(0) = \boldsymbol{\xi}_\mu^\delta(T) = 0$ and

$$\|\boldsymbol{\xi}_\mu^\delta - \boldsymbol{\xi}_\mu\|_{L^2(J_T, H^{k+1})} \to 0 \text{ as } \delta \to 0. \tag{4.7}$$

Using the constructions of $\boldsymbol{\xi}_\mu, \boldsymbol{\eta}_\mu$ and the fact that

$$(\boldsymbol{u}_\mu(T; \hat{\boldsymbol{u}}), g_\mu(T; \hat{g})) = (e^{-\mu\Delta}\hat{\boldsymbol{u}}, \varphi_\mu(\hat{g})),$$

we have

$$\mathcal{R}_T(e^{-\mu\Delta}\boldsymbol{u}_0, \varphi_\mu(g_0), \boldsymbol{\xi}_\mu, \boldsymbol{\xi}_\mu, \boldsymbol{\eta}_\mu) = (e^{-\mu\Delta}\hat{\boldsymbol{u}}, \varphi_\mu(\hat{g})). \tag{4.8}$$

On the other hand, $\boldsymbol{\xi}_\mu^\delta(0) = \boldsymbol{\xi}_\mu^\delta(T) = 0$ implies

$$\mathcal{R}_T(e^{-\mu\Delta}\boldsymbol{u}_0, \varphi_\mu(g_0), \boldsymbol{\xi}_\mu^\delta, \boldsymbol{\xi}_\mu^\delta, \boldsymbol{\eta}_\mu) = \mathcal{R}_T(e^{-\mu\Delta}\boldsymbol{u}_0, \varphi_\mu(g_0), 0, 0, \boldsymbol{\eta}_\mu - \partial_t \boldsymbol{\xi}_\mu^\delta) \tag{4.9}$$

Then, by Theorem 2.2, (4.8) and (4.9), we obtain

$$\begin{aligned} &\|\mathcal{R}_T(e^{-\mu\Delta}\boldsymbol{u}_0, \varphi_\mu(g_0), \boldsymbol{\xi}_\mu, \boldsymbol{\xi}_\mu, \boldsymbol{\eta}_\mu) - \mathcal{R}_T(e^{-\mu\Delta}\boldsymbol{u}_0, \varphi_\mu(g_0), \boldsymbol{\xi}_\mu^\delta, \boldsymbol{\xi}_\mu^\delta, \boldsymbol{\eta}_\mu)\|_{\boldsymbol{H}^k \times H^k} \\ &\quad = \|(e^{-\mu\Delta}\hat{\boldsymbol{u}}, \varphi_\mu(\hat{g})) - \mathcal{R}_T(e^{-\mu\Delta}\boldsymbol{u}_0, \varphi_\mu(g_0), 0, 0, \boldsymbol{\eta}_\mu - \partial_t \boldsymbol{\xi}_\mu^\delta)\|_{\boldsymbol{H}^k \times H^k} \to 0 \end{aligned} \tag{4.10}$$

as $\delta \to 0$. Clearly

$$\|(e^{-\mu\Delta}\boldsymbol{u}_0, \varphi_\mu(g_0)) - (\boldsymbol{u}_0, g_0)\|_{\boldsymbol{H}^k \times H^k} + \|(e^{-\mu\Delta}\hat{\boldsymbol{u}}, \varphi_\mu(\hat{g})) - (\hat{\boldsymbol{u}}, \hat{g})\|_{\boldsymbol{H}^k \times H^k} \to 0 \tag{4.11}$$

as $\mu \to 0$. The fact that $\boldsymbol{E}_\infty$ is dense in $\boldsymbol{H}^{k+1}$ implies that

$$\sup_{(\hat{\boldsymbol{u}}, \hat{g}) \in \boldsymbol{K}} \|P_{\boldsymbol{E}_N}(\boldsymbol{\eta}_\mu - \partial_t \boldsymbol{\xi}_\mu^\delta) - (\boldsymbol{\eta}_\mu - \partial_t \boldsymbol{\xi}_\mu^\delta)\|_{L^2(J_T, \boldsymbol{H}^{k+1})} \to 0 \text{ as } N \to \infty. \tag{4.12}$$

Since $\|\boldsymbol{\xi}_\mu\|_{k+1}$ is bounded uniformly with respect to $\mu \in (0,1)$, equation (4.6) implies that $\|\boldsymbol{\eta}_\mu\|_k$ is also bounded. Taking time derivative of (4.5), we can show the boundedness of $\|\partial_t \boldsymbol{\xi}_\mu\|_k$. Thus $\|(e^{-\mu\Delta}\boldsymbol{u}_0, \varphi_\mu(g_0), \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{\eta}_\mu - \partial_t \boldsymbol{\xi}_\mu^\delta)\|_{\boldsymbol{X}^k}$ is bounded uniformly with respect to $\mu \in (0,1)$. Hence, by Theorem 2.2 and relations (4.10)-(4.12), a solution $\mathcal{R}(\boldsymbol{u}_0, g_0, 0, 0, P_{\boldsymbol{E}_N}(\boldsymbol{\eta}_\mu(\hat{\boldsymbol{u}}, \hat{g}))) \in \boldsymbol{Y}^k$ exists for sufficiently large $N \geq 1$ and sufficiently small $\delta, \mu > 0$. Moreover,

$$\sup_{(\hat{\boldsymbol{u}}, \hat{g}) \in \boldsymbol{K}} \|\mathcal{R}_T(\boldsymbol{u}_0, g_0, 0, 0, P_{\boldsymbol{E}_N}(\boldsymbol{\eta}_\mu(\hat{\boldsymbol{u}}, \hat{g}) - \partial_t \boldsymbol{\xi}_\mu^\delta(\hat{\boldsymbol{u}}, \hat{g}))) - (\hat{\boldsymbol{u}}, \hat{g})\|_{\boldsymbol{H}^k \times H^k} < \varepsilon.$$

From (4.5) and the constructions of $\boldsymbol{u}_\mu(\hat u), g_\mu(\hat g)$, we have

$$\boldsymbol{\xi}_\mu \ : \ (\hat{\boldsymbol{u}}, \hat g) \mapsto \ \boldsymbol{\xi}_\mu(\hat{\boldsymbol{u}}, \hat g), \quad \partial_t \boldsymbol{\xi}_\mu : (\hat{\boldsymbol{u}}, \hat g) \to \boldsymbol{\partial}_t \xi_\mu(\hat{\boldsymbol{u}}, \hat g)$$

are continuous from $\boldsymbol{K}$ to $\boldsymbol{L}^2(J_T, \boldsymbol{H}^{k+1})$. Then (4.6) implies that mapping

$$P_{\boldsymbol{E}_N}(\boldsymbol{\eta}_\mu - \partial_t \boldsymbol{\xi}^\delta_\mu)(\cdot,\cdot) : (\hat{\boldsymbol{u}}, \hat g) \to P_{\boldsymbol{E}_N}(\boldsymbol{\eta}_\mu(\cdot, \hat{\boldsymbol{u}}, \hat g) - \partial_t \boldsymbol{\xi}^\delta_\mu(\hat{\boldsymbol{u}}, \hat g))$$

is continuous from $\boldsymbol{K}$ to $\boldsymbol{L}^2(J_T, \boldsymbol{H}^k)$. The proof is complete. ☐

4.2 Proof of Proposition 4.1

The proof of Proposition 4.1 is inspired by ideas from [1, 2, 16, 17]. Let us admit for the moment the following lemma.

Lemma 4.2. *For any* $(\boldsymbol{u}_0, g_0) \in \boldsymbol{H}^{k+2} \times H^{k+2}$*, for any* $\varepsilon > 0$ *and for any continuous mapping* $\boldsymbol{\Psi}_1 : \boldsymbol{K} \to \Theta(\boldsymbol{u}_0, g_0, \boldsymbol{f}) \cap C^\infty(J_T, \boldsymbol{E}_1)$ *there is a constant* $\nu > 0$ *and a continuous mapping* $\boldsymbol{\Psi} : \boldsymbol{K} \to \Theta(\boldsymbol{u}_0, g_0, \boldsymbol{f}) \cap C^\infty(J_T, \boldsymbol{E})$ *such that*

$$\sup_{(\hat{\boldsymbol{u}}, \hat g) \in \boldsymbol{K}} \|\mathcal{R}_T(\tilde{\boldsymbol{u}}_0, \tilde g_0, 0, 0, \boldsymbol{\Psi}(\hat{\boldsymbol{u}}, \hat g)) - \mathcal{R}_T(\tilde{\boldsymbol{u}}_0, \tilde g_0, 0, 0, \boldsymbol{\Psi}_1(\hat{\boldsymbol{u}}, \hat g))\|_{\boldsymbol{H}^k \times H^k} < \varepsilon \tag{4.13}$$

for any $(\tilde{\boldsymbol{u}}_0, \tilde g_0) \in \boldsymbol{H}^{k+2} \times H^{k+2}$ *with* $\|\boldsymbol{u}_0 - \tilde{\boldsymbol{u}}_0\|_k + \|g_0 - \tilde g_0\|_k < \nu$*.*

Let $(\boldsymbol{u}_0, g_0) \in \boldsymbol{H}^k \times H^k$ and $\boldsymbol{\Psi}_1 : \boldsymbol{K} \to \Theta(\boldsymbol{u}_0, g_0, \boldsymbol{f}) \cap C^\infty(J_T, \boldsymbol{E}_1)$ be such that

$$\sup_{(\hat{\boldsymbol{u}}, \hat g) \in \boldsymbol{K}} \|\mathcal{R}_T(\boldsymbol{u}_0, g_0, 0, 0, \boldsymbol{\Psi}_1(\hat{\boldsymbol{u}}, \hat g)) - (\hat{\boldsymbol{u}}, \hat g)\|_{\boldsymbol{H}^k \times H^k} < \frac{\varepsilon}{2}. \tag{4.14}$$

Take any sequence $(\boldsymbol{u}^n_0, g^n_0) \in \boldsymbol{H}^{k+2} \times H^{k+2}$ such that

$$\|(\boldsymbol{u}_0, g_0) - (\boldsymbol{u}^n_0, g^n_0)\|_{\boldsymbol{H}^k \times H^k} \to 0 \text{ as } n \to \infty.$$

As $\boldsymbol{K}$ is compact, Theorem 2.2 implies that $\boldsymbol{\Psi}_1(\boldsymbol{K}) \subset \Theta(\boldsymbol{u}^n_0, g^n_0, \boldsymbol{f})$ for sufficiently large n. By Lemma 4.2, there is a continuous mapping

$$\boldsymbol{\Psi} : \boldsymbol{K} \to \Theta(\boldsymbol{u}^n_0, g^n_0, \boldsymbol{f}) \cap C^\infty(J_T, \boldsymbol{E})$$

such that

$$\sup_{(\hat{\boldsymbol{u}}, \hat g) \in \boldsymbol{K}} \|\mathcal{R}_T(\boldsymbol{u}^n_0, g^n_0, 0, 0, \boldsymbol{\Psi}(\hat{\boldsymbol{u}}, \hat g)) - \mathcal{R}_T(\boldsymbol{u}^n_0, g^n_0, 0, 0, \boldsymbol{\Psi}_1(\hat{\boldsymbol{u}}, \hat g))\|_{\boldsymbol{H}^k \times H^k} < \frac{\varepsilon}{2}.$$

Choosing n sufficiently large and using the fact that $\mathcal{R}$ is uniformly continuous on the compact set $\boldsymbol{\Psi}(\boldsymbol{K}) \cup \boldsymbol{\Psi}_1(\boldsymbol{K})$, we get

$$\begin{aligned}
&\sup_{(\hat{\boldsymbol{u}}, \hat g) \in \boldsymbol{K}} \|\mathcal{R}_T(\boldsymbol{u}_0, g_0, 0, 0, \boldsymbol{\Psi}(\hat{\boldsymbol{u}}, \hat g)) - \mathcal{R}_T(\boldsymbol{u}_0, g_0, 0, 0, \boldsymbol{\Psi}_1(\hat{\boldsymbol{u}}, \hat g))\|_{\boldsymbol{H}^k \times H^k} \\
&\quad \le \sup_{(\hat{\boldsymbol{u}}, \hat g) \in \boldsymbol{K}} \|\mathcal{R}_T(\boldsymbol{u}_0, g_0, 0, 0, \boldsymbol{\Psi}(\hat{\boldsymbol{u}}, \hat g)) - \mathcal{R}_T(\boldsymbol{u}^n_0, g^n_0, 0, 0, \boldsymbol{\Psi}(\hat{\boldsymbol{u}}, \hat g))\|_{\boldsymbol{H}^k \times H^k} \\
&\qquad + \sup_{(\hat{\boldsymbol{u}}, \hat g) \in \boldsymbol{K}} \|\mathcal{R}_T(\boldsymbol{u}^n_0, g^n_0, 0, 0, \boldsymbol{\Psi}_1(\hat{\boldsymbol{u}}, \hat g)) - \mathcal{R}_T(\boldsymbol{u}_0, g_0, 0, 0, \boldsymbol{\Psi}_1(\hat{\boldsymbol{u}}, \hat g))\|_{\boldsymbol{H}^k \times H^k} \\
&\qquad + \sup_{(\hat{\boldsymbol{u}}, \hat g) \in \boldsymbol{K}} \|\mathcal{R}_T(\boldsymbol{u}^n_0, g^n_0, 0, 0, \boldsymbol{\Psi}(\hat{\boldsymbol{u}}, \hat g)) - \mathcal{R}_T(\boldsymbol{u}^n_0, g^n_0, 0, 0, \boldsymbol{\Psi}_1(\hat{\boldsymbol{u}}, \hat g))\|_{\boldsymbol{H}^k \times H^k} \\
&\quad < \frac{\varepsilon}{2}.
\end{aligned} \tag{4.15}$$

Combining (4.14) and (4.15), we complete the proof of Proposition 4.1.

Proof of Lemma 4.2. **Step 1.** We shall need the following lemma, which can be proved by literal repetition of the arguments of the proof of [17, Lemma 3.5].

Lemma 4.3. *For any continuous mapping* $\boldsymbol{\Psi}_1 : \boldsymbol{K} \to \Theta(\boldsymbol{u}_0, g_0, \boldsymbol{f}) \cap L^2(J_T, \boldsymbol{E}_1)$ *there is a set* $\boldsymbol{A} = \{\eta_1^l, l = 1, \dots, m\} \subset \boldsymbol{E}_1$ *an integer* $s \geq 1$ *and a mapping* $\boldsymbol{\Psi}_s : \boldsymbol{K} \to \Theta(\boldsymbol{u}_0, g_0, \boldsymbol{f}) \cap L^2(J_T, \boldsymbol{E}_1)$ *such that*

$$\boldsymbol{\Psi}_s(\hat{\boldsymbol{u}}, \hat{g}) = \sum_{l=1}^{m} \sum_{r=0}^{s-1} c_{l,r}(\hat{\boldsymbol{u}}, \hat{g}) I_{r,s}(t) \boldsymbol{\eta}_1^l,$$

where $c_{l,r}$ *are non-negative functions such that* $\sum_{l=1}^m c_{l,r} = 1$, $I_{r,s}$ *is the indicator function of the interval* $[t_r, t_{r+1})$ *with* $t_r = rT/s$ *and*

$$\sup_{(\hat{\boldsymbol{u}}, \hat{g}) \in \boldsymbol{K}} \|\mathcal{R}_T(\boldsymbol{u}_0, g_0, 0, 0, \boldsymbol{\Psi}_1(\hat{\boldsymbol{u}}, \hat{g})) - \mathcal{R}_T(\boldsymbol{u}_0, g_0, 0, 0, \boldsymbol{\Psi}_s(\hat{\boldsymbol{u}}, \hat{g}))\|_{\boldsymbol{H}^k \times H^k} < \varepsilon.$$

Let $\boldsymbol{\Psi}_s$ be the function constructed in Lemma 4.3:

$$\boldsymbol{\Psi}_s(\hat{\boldsymbol{u}}, \hat{g}) = \sum_{l=1}^{m} \varphi_l(t, \hat{\boldsymbol{u}}, \hat{g}) \boldsymbol{\eta}_1^l.$$

We claim that there are vectors $\boldsymbol{\zeta}^{l,1}, \dots, \boldsymbol{\zeta}^{l,2n}, \boldsymbol{\eta}^l \in \boldsymbol{E}$ and positive constants $\lambda_{l,1}, \dots, \lambda_{l,2n}$ whose sum is equal to 1 such that

$$\begin{aligned}
&\boldsymbol{\zeta}^i = -\boldsymbol{\zeta}^{i+n} \text{ for } i = 1, \dots, n, \\
&(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \boldsymbol{\eta}_1^l = \sum_{j=1}^{2n} \lambda_{l,j}((\boldsymbol{u} + \boldsymbol{\zeta}^{l,j}) \cdot \nabla)(\boldsymbol{u} + \boldsymbol{\zeta}^{l,j}) - \boldsymbol{\eta}^l \text{ for any } \boldsymbol{u} \in \boldsymbol{H}^1.
\end{aligned} \tag{4.16}$$

Indeed, by the definition of $\mathcal{F}(\boldsymbol{E})$, for any $\boldsymbol{\eta}_1^l \in \mathcal{F}(\boldsymbol{E})$ there are $\boldsymbol{\xi}^{l,1}, \dots, \boldsymbol{\xi}^{l,n}, \boldsymbol{\eta}^l \in \boldsymbol{E}$ such that

$$\boldsymbol{\eta}_1^l = \boldsymbol{\eta}^l - \sum_{i=1}^{n} (\boldsymbol{\xi}^{l,i} \cdot \nabla \boldsymbol{\xi}^{l,i}).$$

Let us set

$$\lambda^{l,i} = \lambda^{l,i+n} = \frac{1}{2n}, \quad \boldsymbol{\zeta}^{l,i} = -\boldsymbol{\zeta}^{l,i+n} = \sqrt{n}\boldsymbol{\xi}^i, \quad i = 1, \dots, n.$$

Then (4.16) holds for any $\boldsymbol{u} \in \boldsymbol{H}^1$.

Let $(\boldsymbol{u}_1, g_1) = \mathcal{R}(\boldsymbol{u}_0, g_0, 0, 0, \boldsymbol{\Psi}_s(\hat{\boldsymbol{u}}, \hat{g}))$. It follows from (4.16) that $(\boldsymbol{u}_1, g_1)$ satisfies the problem

$$\begin{aligned}
\dot{\boldsymbol{u}}_1 + \sum_{j=1}^{2n} \sum_{l=1}^{m} \lambda_{l,j} \varphi_l(t, \hat{\boldsymbol{u}}, \hat{g})((\boldsymbol{u}_1 + \boldsymbol{\zeta}^{l,j}) \cdot \nabla)(\boldsymbol{u}_1 + \boldsymbol{\zeta}^{l,j}) + h(g_1)\nabla g_1 \\
= \boldsymbol{f}(t) + \sum_{l=1}^{m} \varphi_l(t, \hat{\boldsymbol{u}}, \hat{g}) \boldsymbol{\eta}^l, \qquad (4.17) \\
(\partial_t + \boldsymbol{u_1} \cdot \nabla) g_1 + \nabla \cdot \boldsymbol{u_1} = 0.
\end{aligned}$$

Taking $q = m \cdot n$, $\{\boldsymbol{\zeta}^i\}_{i=1}^q := \{\boldsymbol{\zeta}^{l,j}\}_{l=1,}^m{}_{j=1}^n$, $\boldsymbol{\zeta}^{i+q} := -\boldsymbol{\zeta}^i$, $i = 1, \ldots, q$, we rewrite (4.17) in the form

$$\dot{\boldsymbol{u}}_1 + \sum_{i=1}^{2q} \psi_i(t, \hat{\boldsymbol{u}}, \hat{g})((\boldsymbol{u}_1 + \boldsymbol{\zeta}^i) \cdot \nabla)(\boldsymbol{u}_1 + \boldsymbol{\zeta}^i) + h(g_1)\nabla g_1 = \boldsymbol{f}(t) + \boldsymbol{\eta}(t, \hat{\boldsymbol{u}}, \hat{g}), \tag{4.18}$$

where

$$\begin{aligned} \boldsymbol{\eta}(t, \hat{\boldsymbol{u}}, \hat{g}) &= \sum_{l=1}^{m} \varphi_l(t, \hat{\boldsymbol{u}}, \hat{g})\boldsymbol{\eta}^l, \\ \psi_i(t, \hat{\boldsymbol{u}}, \hat{g}) &= \sum_{r=0}^{s-1} d_{i,r}(\hat{\boldsymbol{u}}, \hat{g}) I_{r,s}(t), \end{aligned} \tag{4.19}$$

and $d_{i,r} \in C(\boldsymbol{K})$ are some non-negative functions such that

$$\sum_{i=1}^{q} d_{i,r} = \sum_{i=q+1}^{2q} d_{i,r} = \frac{1}{2}.$$

Step 2. Let us show that it suffices to consider the case $s = 1$. Indeed, let us assume that for any constant $\varepsilon_0 > 0$ and for any interval $I_r := [t_{r-1}, t_r]$ there exists a continuous mapping $\boldsymbol{\Psi}_{\varepsilon_0}^r : \boldsymbol{K} \to \Theta^r(\boldsymbol{u}_1(t_{r-1}), g_1(t_{r-1})) \cap C^\infty(J_T, \boldsymbol{E})$ such that

$$\sup_{(\hat{\boldsymbol{u}}, \hat{g}) \in \boldsymbol{K}} \|\mathcal{R}_{t_r - t_{r-1}}(\boldsymbol{u}_1(t_{r-1}), g_1(t_{r-1}), 0, 0, \boldsymbol{\Psi}_{\varepsilon_0}^r(\hat{\boldsymbol{u}}, \hat{g})) - (\boldsymbol{u}_1(t_r), g_1(t_r))\|_{\boldsymbol{H}^k \times H^k} < \varepsilon_0.$$

Here $\Theta^r(\boldsymbol{u}_1(t_{r-1}), g_1(t_{r-1}))$ is the set of functions $\boldsymbol{\eta} \in L^2(I_r, \boldsymbol{H}^k)$ for which problem (3.1)-(3.2) has a solution in $C(I_r, \boldsymbol{H}^k) \times C(I_r, H^k)$ satisfying the initial condition

$$\boldsymbol{u}(t_{r-1}) = \boldsymbol{u}_1(t_{r-1}), \;\; g(t_{r-1}) = g_1(t_{r-1}).$$

In view of Theorem 2.2, there is $\delta_s > 0$ such that for any $(\tilde{\boldsymbol{u}}_0, \tilde{g}_0) \in \boldsymbol{H}^{k+2} \times H^{k+2}$ with $\|(\tilde{\boldsymbol{u}}_0, \tilde{g}_0) - (\boldsymbol{u}_1(t_{s-1}), g_1(t_{s-1}))\|_{\boldsymbol{H}^k \times H^k} < \delta_s$ we have the inequality

$$\sup_{(\hat{\boldsymbol{u}}, \hat{g}) \in \boldsymbol{K}} \|\mathcal{R}_{T - t_{s-1}}(\tilde{\boldsymbol{u}}_0, \tilde{g}_0, 0, 0, \boldsymbol{\Psi}_{\varepsilon}^s(\hat{\boldsymbol{u}}, \hat{g})) - (\boldsymbol{u}_1(T), g_1(T))\|_{\boldsymbol{H}^k \times H^k} < \varepsilon.$$

Similarly, we can find $\delta_r > 0$, $r = s-1, \ldots, 1$ such that

$$\sup_{(\hat{\boldsymbol{u}}, \hat{g}) \in \boldsymbol{K}} \|\mathcal{R}_{t_{r+1} - t_r}(\tilde{\boldsymbol{u}}_0, \tilde{g}_0, 0, 0, \boldsymbol{\Psi}_{\delta_{r+1}}^r(\hat{\boldsymbol{u}}, \hat{g})) - (\boldsymbol{u}_1(t_{r+1}), g_1(t_{r+1}))\|_{\boldsymbol{H}^k \times H^k} < \delta_{r+1}$$

for any $(\tilde{\boldsymbol{u}}_0, \tilde{g}_0) \in \boldsymbol{H}^{k+2} \times H^{k+2}$ satisfying

$$\|(\tilde{\boldsymbol{u}}_0, \tilde{g}_0) - (\boldsymbol{u}_1(t_r), g_1(t_r))\|_{\boldsymbol{H}^k \times H^k} < \delta_r.$$

Let us denote by $\hat{\boldsymbol{\Psi}} : K \to L^2(J_T, \boldsymbol{E})$ the continuous operator defined by the relations

$$\hat{\boldsymbol{\Psi}}(\hat{\boldsymbol{u}}, \hat{g})(t) = \boldsymbol{\Psi}^r_{\delta_{r+1}}(\hat{\boldsymbol{u}}, \hat{g})(t) \text{ for } t \in I_r,$$

where $\delta_{s+1} = \varepsilon$. Then

$$\sup_{(\hat{\boldsymbol{u}}, \hat{g}) \in \boldsymbol{K}} \|\mathcal{R}_T(\tilde{\boldsymbol{u}}_0, \tilde{g}_0, 0, 0, \hat{\boldsymbol{\Psi}}(\hat{\boldsymbol{u}}, \hat{g})) - (\boldsymbol{u}_1(T), g_1(T))\|_{\boldsymbol{H}^k \times H^k} < \varepsilon.$$

To complete the proof, it suffices to approximate $\hat{\boldsymbol{\Psi}}$ in $L^2(J_T, \boldsymbol{H}^k)$ by a continuous mapping $\boldsymbol{\Psi} : \boldsymbol{K} \to \Theta(\boldsymbol{u}_0, g_0) \cap C^\infty(J_T, \boldsymbol{E})$.

Step 3. We now assume that $s = 1$. Then (4.18) takes the form

$$\dot{\boldsymbol{u}}_1 + \sum_{i=1}^{2q} d_i(\hat{\boldsymbol{u}}, \hat{g})((\boldsymbol{u}_1 + \boldsymbol{\zeta}^i) \cdot \nabla)(\boldsymbol{u}_1 + \boldsymbol{\zeta}^i) + h(g_1)\nabla g_1 = \boldsymbol{f}(t) + \boldsymbol{\eta}(\hat{\boldsymbol{u}}, \hat{g}), \quad (4.20)$$

where $d_i \in C(\boldsymbol{K})$ and $\boldsymbol{\eta} \in C(\boldsymbol{K}, \boldsymbol{E})$. For any $n \in \mathbb{N}$, let $\boldsymbol{\zeta}_n(t, \hat{\boldsymbol{u}}, \hat{g}) = \boldsymbol{\zeta}(\frac{nt}{T}, \hat{\boldsymbol{u}}, \hat{g})$, where $\boldsymbol{\zeta}(t, \hat{\boldsymbol{u}}, \hat{g})$ is a 1-periodic function such that

$$\boldsymbol{\zeta}(s, \hat{\boldsymbol{u}}, \hat{g}) = \boldsymbol{\zeta}^j \text{ for } 0 \le s - (d_1(\hat{\boldsymbol{u}}, \hat{g}) + \ldots + d_{j-1}(\hat{\boldsymbol{u}}, \hat{g})) < d_j(\hat{\boldsymbol{u}}, \hat{g}), \quad j = 1, \ldots, q.$$

Note that $\boldsymbol{\zeta}(t, \hat{\boldsymbol{u}}, \hat{g}) = -\boldsymbol{\zeta}(t - \frac{1}{2}, \hat{\boldsymbol{u}}, \hat{g})$ for $t \in (\frac{1}{2}, 1)$. Eq. (4.20) is equivalent to

$$\begin{aligned} &\dot{\boldsymbol{u}}_1 + ((\boldsymbol{u}_1 + \boldsymbol{\zeta}_n(t, \hat{\boldsymbol{u}}, \hat{g})) \cdot \nabla)(\boldsymbol{u}_1 + \boldsymbol{\zeta}_n(t, \hat{\boldsymbol{u}}, \hat{g})) + h(g_1)\nabla g_1 \\ &\quad = \boldsymbol{f} + \boldsymbol{\eta}(t, \hat{\boldsymbol{u}}, \hat{g}) + \boldsymbol{f}_{\boldsymbol{n}}(t, \hat{\boldsymbol{u}}, \hat{g}), \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{f}_{\boldsymbol{n}}(t, \hat{\boldsymbol{u}}, \hat{g}) &= ((\boldsymbol{u}_1 + \boldsymbol{\zeta}_n(t, \hat{\boldsymbol{u}}, \hat{g})) \cdot \nabla)(\boldsymbol{u}_1 + \boldsymbol{\zeta}_n(t, \hat{\boldsymbol{u}}, \hat{g})) \\ &\quad - \sum_{i=1}^{2q} d_i(\hat{\boldsymbol{u}}, \hat{g})((\boldsymbol{u}_1 + \boldsymbol{\zeta}^i) \cdot \nabla)(\boldsymbol{u}_1 + \boldsymbol{\zeta}^i). \end{aligned}$$

Let us define

$$\mathcal{K}\boldsymbol{f}_{\boldsymbol{n}}(t) = \int_0^t \boldsymbol{f}_{\boldsymbol{n}}(s) \mathrm{d}s.$$

Then $\boldsymbol{v}_n = \boldsymbol{u}_1 - \mathcal{K}\boldsymbol{f}_{\boldsymbol{n}}$ is a solution of the problem

$$\begin{aligned} \dot{\boldsymbol{v}}_n + ((\boldsymbol{v}_n + \boldsymbol{\zeta}_{\boldsymbol{n}}(t, \hat{\boldsymbol{u}}, \hat{g}) + \mathcal{K}\boldsymbol{f}_{\boldsymbol{n}}(t, \hat{\boldsymbol{u}}, \hat{g})) \cdot \nabla)(\boldsymbol{v}_n + \boldsymbol{\zeta}_n(t, \hat{\boldsymbol{u}}, \hat{g}) + \mathcal{K}\boldsymbol{f}_{\boldsymbol{n}}(t, \hat{\boldsymbol{u}}, \hat{g})) \\ + h(g_1)\nabla g_1 = \boldsymbol{f}(t) + \boldsymbol{\eta}(t, \hat{\boldsymbol{u}}, \hat{g}), \\ (\partial_t + (\boldsymbol{v}_n + \mathcal{K}\boldsymbol{f}_{\boldsymbol{n}}(t, \hat{\boldsymbol{u}}, \hat{g})) \cdot \nabla)g_1 + \nabla \cdot (\boldsymbol{v}_n + \mathcal{K}\boldsymbol{f}_{\boldsymbol{n}}(t, \hat{\boldsymbol{u}}, \hat{g})) = 0, \\ \boldsymbol{v}_n = \boldsymbol{u}_0. \end{aligned}$$

It is straightforward to see that

$$\sup_{(\hat{\boldsymbol{u}}, \hat{g}) \in \boldsymbol{K}} \|\mathcal{K}\boldsymbol{f}_n(t, \hat{\boldsymbol{u}}, \hat{g})\|_{C(J_T, \boldsymbol{H}^{k+1})} \to 0,$$

(e.g. see [11, Chapter 3] or [14, Section 6]). Thus

$$\|\boldsymbol{v}_n - \boldsymbol{u}_1\|_{C(J_T, \boldsymbol{H}^{k+1})} \to 0 \text{ as } n \to \infty. \tag{4.21}$$

On the other hand, Theorem 2.2 implies that

$$\|(\boldsymbol{v}_n, g_1) - (\tilde{\boldsymbol{u}}_n, \tilde{g}_n)\|_{\boldsymbol{Y}^k} \to 0 \text{ as } n \to \infty, \tag{4.22}$$

where $(\tilde{\boldsymbol{u}}_n, \tilde{g}_n)$ satisfies the problem

$$\begin{aligned}
&\partial_t \tilde{\boldsymbol{u}}_n + ((\tilde{\boldsymbol{u}}_n + \boldsymbol{\zeta}_n(t, \hat{\boldsymbol{u}}, \hat{g})) \cdot \nabla)(\tilde{\boldsymbol{u}}_n + \boldsymbol{\zeta}_n(t, \hat{\boldsymbol{u}}, \hat{g})) + h(\tilde{g}_n)\nabla \tilde{g}_n = \boldsymbol{f}(t) + \boldsymbol{\eta}(t, \hat{\boldsymbol{u}}, \hat{g}),\\
&(\partial_t + \tilde{\boldsymbol{u}}_n \cdot \nabla)\tilde{g}_n + \nabla \cdot \tilde{\boldsymbol{u}}_n = 0,\\
&\tilde{\boldsymbol{u}}_n(0) = \tilde{\boldsymbol{u}}_0, \quad \tilde{g}_n(0) = \tilde{g}_0.
\end{aligned}$$

We want to apply Theorem 2.3 to the above system. To this end, let $\boldsymbol{\chi}_n : J_T \to \boldsymbol{H}^k$ be a uniformly equicontinuous sequence and let $t_0 \in J_T$. Then

$$\begin{aligned}
\int_0^{t_0} \boldsymbol{\zeta}_n(t) \cdot \boldsymbol{\chi}_n(t)\mathrm{d}t &= \int_0^{t_0} \boldsymbol{\zeta}(\frac{nt}{T}) \cdot \boldsymbol{\chi}_n(t)\mathrm{d}t = \int_0^{\frac{nt_0}{T}} \boldsymbol{\zeta}(t) \cdot \boldsymbol{\chi}_n(\frac{tT}{n})\frac{T}{n}\mathrm{d}t\\
&= \sum_{i=0}^{[\frac{nt_0}{T}]-1} \int_i^{i+1} \boldsymbol{\zeta}(t) \cdot \boldsymbol{\chi}_n(\frac{tT}{n})\frac{T}{n}\mathrm{d}t + \int_{[\frac{nt_0}{T}]}^{\frac{nt_0}{T}} \boldsymbol{\zeta}(t) \cdot \boldsymbol{\chi}_n(\frac{tT}{n})\frac{T}{n}\mathrm{d}t.
\end{aligned} \tag{4.23}$$

Using the construction of $\boldsymbol{\zeta}(t)$, we get

$$\begin{aligned}
\int_i^{i+1} \boldsymbol{\zeta}(t) \cdot \boldsymbol{\chi}_n(\frac{tT}{n})\mathrm{d}t &= \int_i^{i+\frac{1}{2}} \boldsymbol{\zeta}(t) \cdot \boldsymbol{\chi}_n(\frac{tT}{n})\mathrm{d}t + \int_{i+\frac{1}{2}}^{i+1} -\boldsymbol{\zeta}(t - \frac{1}{2}) \cdot \boldsymbol{\chi}_n(\frac{tT}{n})\mathrm{d}t\\
&= \int_i^{i+\frac{1}{2}} \boldsymbol{\zeta}(t) \cdot \big(\boldsymbol{\chi}_n(\frac{tT}{n}) - \boldsymbol{\chi}_n(\frac{tT}{n} + \frac{T}{2n})\big)\mathrm{d}t.
\end{aligned}$$

As $\boldsymbol{\chi}_n$ is uniformly equicontinuous and $\boldsymbol{\zeta}$ is bounded, we have

$$\sup_{t \in [0,n]} \|\boldsymbol{\zeta}(t) \cdot \big(\boldsymbol{\chi}_n(\frac{tT}{n}) - \boldsymbol{\chi}_n(\frac{tT}{n} + \frac{T}{2n})\big)\|_k \to 0, \quad n \to \infty.$$

The boundedness of $\boldsymbol{\zeta} \cdot \boldsymbol{\chi}_n$ implies that the second term of the right-hand side of (4.23) goes to zero. Thus

$$\int_0^{t_0} \boldsymbol{\zeta}_n(t) \cdot \boldsymbol{\chi}_n(t)\mathrm{d}t \to 0 \text{ in } \boldsymbol{H}^k.$$

Using Theorem 2.3 and limits (4.21), (4.22), we get

$$\sup_{(\hat{\boldsymbol{u}}, \hat{g}) \in \boldsymbol{K}} \|\mathcal{R}_T(\boldsymbol{u}_0, g_0, 0, 0, \boldsymbol{\zeta}_n, \boldsymbol{\zeta}_n, \boldsymbol{\eta}(\hat{\boldsymbol{u}}, \hat{g})) - (\boldsymbol{u}_1(T, \hat{\boldsymbol{u}}, \hat{g}), g_1(T, \hat{\boldsymbol{u}}, \hat{g}))\|_{\boldsymbol{H}^k \times H^k} < \varepsilon$$

for sufficiently large n. Let us take some functions $\boldsymbol{\zeta}_n^m \in C^\infty(J_T, E)$ such that $\boldsymbol{\zeta}_n^m(0) = \boldsymbol{\zeta}_n^m(T) = 0$ and

$$\|\boldsymbol{\zeta}_n^m - \boldsymbol{\zeta}_n\|_{L^2(J_T,\boldsymbol{H}^{k+1})} \to 0 \text{ as } m \to \infty. \tag{4.24}$$

Then Theorem 2.2 implies

$$\sup_{(\hat{\boldsymbol{u}},\hat{g})\in\boldsymbol{K}} \|\mathcal{R}_T(\boldsymbol{u}_0, g_0, 0, 0, \boldsymbol{\zeta}_n, \boldsymbol{\zeta}_n, \boldsymbol{\eta}(\hat{\boldsymbol{u}}, \hat{g})) - \mathcal{R}_T(\boldsymbol{u}_0, g_0, 0, 0, \boldsymbol{\zeta}_n^m, \boldsymbol{\zeta}_n^m, \boldsymbol{\eta}(\hat{\boldsymbol{u}}, \hat{g}))\|_{\boldsymbol{H}^k\times H^k} < \varepsilon.$$

For $m \gg 1$, the operator

$$\Psi : \boldsymbol{K} \to L^2(J_T, \boldsymbol{E}), \ \ (\hat{\boldsymbol{u}}, \hat{g}) \to \boldsymbol{\eta}(\hat{\boldsymbol{u}}, \hat{g}) - \partial_t \boldsymbol{\zeta}_n^m$$

satisfies

$$\sup_{(\hat{\boldsymbol{u}},\hat{g})\in\boldsymbol{K}} \|\mathcal{R}_T(\boldsymbol{u}_0, g_0, 0, 0, \boldsymbol{\Psi}_1(\hat{\boldsymbol{u}}, \hat{g})) - \mathcal{R}_T(\boldsymbol{u}_0, g_0, 0, 0, \boldsymbol{\Psi}(\hat{\boldsymbol{u}}, \hat{g}))\|_{\boldsymbol{H}^k\times H^k} < \varepsilon,$$

which completes the proof. □

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