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ON THE REAL ZEROS OF POSITIVE SEMI-DEFINITE BIQUADRATIC FORMS

RONAN QUAREZ

ABSTRACT. For a positive semi-definite biquadratic forms F in $(3, 3)$ variables, we prove that if F has a finite number but at least 7 real zeros $\mathcal{Z}(F)$, then it is not a sum of squares. We show also that if F has at least 11 zeros, then it has infinitely many real zeros and it is a sum of squares. It can be seen as the counterpart for biquadratic forms as the results of Choi, Lam and Reznick for positive semi-definite ternary sextics.

We introduce and compute some of the numbers $\text{BB}_{n,m}$ which are set to be equal to $\sup |\mathcal{Z}(F)|$ where F ranges over all the positive semi-definite biquadratic forms F in (n, m) variables such that $|\mathcal{Z}(F)| < \infty$.

We also recall some old constructions of positive semi-definite biquadratic forms which are not sums of squares and we give some new families of examples.

1. INTRODUCTION

Throughout this paper, a *form* will just denote an homogeneous polynomial over the reals and $\mathcal{H}_{n,k}$ the set of all forms of degree k with n variables. A form $F(x)$ with variables (x_0, \dots, x_{n-1}) is said to be *positive semi-definite* if $F(x) \geq 0$ for all $x \in \mathbb{R}^n$. The form $F \in \mathcal{H}_{n,2k}$ is a *sum of squares* if it can be written $F(x) = \sum_{s=1}^r F_s^2(x)$ for some forms $F_s \in \mathcal{H}_{n,k}$.

The study of positive semi-definite forms which are not sums of squares has been (and still is) an intensive area of researchs since Hilbert showed (see [Hi]) that the cone of positive semi-definite forms coincide with the cone of forms which are sum of squares if and only if $n = 2$, or $k = 2$, or $(n, k) = (3, 4)$.

In [CLR1], Choi, Lam and Reznick gave some more precise statement for ternary sextics and quaternary quartics, using the real zeros set of the forms viewed in some projective space. For instance, one of their main results is :

Theorem[CLR1]. If $f \in \mathcal{H}_{3,6}$ is positive semi-definite with at least 11 real zeros in the projective plane, then f has infinitely many real zeros and it is a sum of squares.

Our main motivation is to establish a counterpart for biquadratic forms.

A form $F(x; s)$ is called *biquadratic* in the variables $x = (x_0, \dots, x_{n-1})$ and $s = (s_0, \dots, s_{m-1})$ if for each $x \in \mathbb{R}^n$, $F(x; s)$ is a quadratic form in the variables s ; and for each $s \in \mathbb{R}^m$, $F(x; s)$ is a quadratic form in the variables x . The set of all biquadratic forms with respect to the variables $(x; s)$ is denoted by $\mathcal{BQ}_{n,m}$. A biquadratic form $F(x; s)$ is said to be *positive semi-definite* if $F(x; s) \geq 0$ for all

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$(x, s) \in \mathbb{R}^{n+m}$. Finally, the set of all positive semi-definite biquadratic forms in $\mathcal{BQ}_{n,m}$ is denoted by $\mathcal{PQ}_{n,m}$.

Biquadratic forms naturally arise in several areas of pure and applied mathematics, for instance in control theory and also in theoretical physics.

Among all the bibliography about biquadratic forms, we will mention first the result of Calderon (see [Ca]) which says that any positive semi-definite biquadratic form in $\mathcal{BQ}_{n,2}$ is a sum of squares. Later, Choi in [C1] gave an example of a positive semi-definite biquadratic form in $\mathcal{BQ}_{3,3}$ which is not a sum of squares. Maybe less known, Terpstra already in [Te] had also given a geometric construction of positive semi-definite biquadratic forms in $\mathcal{BQ}_{3,3}$ which are not a sum of squares.

Throughout the paper, we view the real zeros set $\mathcal{Z}(F)$ of a biquadratic forms $F \in \mathcal{H}_{n,m}$ in the product of projective spaces $\mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$.

In section 4, we set the numbers $\text{BB}_{n,m}$ to be equal to $\sup |\mathcal{Z}(F)|$ where F ranges over all biquadratic forms in $\mathcal{PQ}_{n,m}$ with $|\mathcal{Z}(F)| < \infty$. This definition is analogous to the definition, for forms in $\mathcal{H}_{n,k}$, of the numbers $B_{n,k}$ in [CLR1]. We will give some properties of the $\text{BB}_{n,m}$ and compute some values, for instance we show that $\text{BB}_{n,2} = n$ using the well known values of the degrees of Segre varieties. We also prove that $7 \leq \text{BB}_{3,3} \leq 10$.

In Section 5 we show that if $F \in \mathcal{PQ}_{3,3}$ has a finite number but at least 7 real zeros, then it is not a sum of squares. We show also that if F has at least 11 zeros then it has infinitely many real zeros and it is a sum of squares.

In Section 6, we recall Terpstra's construction [Te] and give some variations around the Choi example [C1]. We also give some new families of examples of positive semi-definite non sum of squares biquadratic forms.

In Section 7, we study the possible configurations in $\mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$ of the zero sets of biquadratic forms in $\mathcal{PQ}_{n,m}$. Of particular interest are the *maximal configurations* which are defined to be the *finite* subsets $\Omega \subset \mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$ such that if $F \in \mathcal{PQ}_{n,m}$ vanishes at Ω and also at another point in $\mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$, then $\mathcal{Z}(F)$ is infinite. We give some characterisations of maximal configurations in the case $(n, m) = (2, 2)$ and $(n, m) = (3, 2)$. In the case $(n, m) = (3, 3)$, we give some necessary conditions and we also give a complete worked example.

For a maximal configuration Ω , we are interested to determine $\mathcal{PQ}(\Omega)$ the set of all possible biquadratic forms $F \in \mathcal{PQ}_{3,3}$ such that $\Omega \subset \mathcal{Z}(F)$. This seems to be a difficult task in general. Again, we give a complete worked example when $(n, m) = (3, 3)$.

2. PRELIMINARIES

The set of all real matrices of size $n \times m$ is denoted by $M_{n,m}(\mathbb{R})$ or alternatively by $\mathbb{R}^{n \times m}$. The symbol T will denote the transposition.

To any biquadratic form $F(x; s)$ with variables $x = (x_0, \dots, x_{n-1})$ and $s = (s_0, \dots, s_{m-1})$ running over \mathbb{R} , one may canonically associate a matrix $M_s(F)$ whose entries are forms in $\mathcal{H}_{m,2}$ with variables s (resp. a matrix $M_x(F)$ whose entries are forms in $\mathcal{H}_{m,2}$ with variables x) such that

$$F(x; s) = xM_s(F)x^T = sM_x(F)s^T.$$

Of course, the biquadratic form $F(x, y)$ is positive semi-definite if and only if the matrix $M_s(F)$ (respectively $M_x(F)$) is positive semi-definite for all $s \in \mathbb{R}^m$ (respectively for all $x \in \mathbb{R}^n$).

One motivation for studying biquadratic forms is that they are related to positive linear maps (see [C2] for background).

2.1. Positive linear maps. A linear map $\Phi : M_n(\mathbb{R}) \rightarrow M_m(\mathbb{R})$ preserving symmetry is said to be *positive* if the matrices $\Phi(A)$ are positive semi-definite for all symmetric positive semi-definite matrices $A \in M_n(\mathbb{R})$.

The linear map Φ is entirely determined by the image of positive semi-definite matrices of rank 1. Indeed, the matrix $x^T x$ has rank 1 and we have

$$\Phi(x^T x) = \sum_{1 \leq i, j \leq n} A_{ij} x_i x_j,$$

where $A_{ij} \in \mathbb{R}^{m \times m}$. Then, we define

$$\phi(x; s) = s \Phi(x^T x) s^T = s \left(\sum_{1 \leq i, j \leq n} x_i A_{ij} x_j \right) s^T.$$

Although the notion of complete positivity does not work as well as in the complex case, an easy computation shows the property :

Proposition 2.1. *Let Φ be a linear map preserving symmetry. The map Φ is positive if and only if the biquadratic form ϕ is positive semi-definite. Moreover, the following properties are equivalent :*

a) *there exists some $V_k \in \mathbb{R}^{n \times m}$ for $k \in \{1, \dots, r\}$ such that*

$$\Phi(M) = \sum_{k=1}^r V_k^T M V_k$$

for all M ,

b) *the biquadratic form ϕ is a sum of squares.*

Let us mention that the complex counterpart of these notions, namely positive semi-definite non sum of squares bihermitian forms, are important in theoretical physics (see for instance [SZ]).

2.2. Zeros in projective spaces. Let $F(x; s)$ be a biquadratic form where $x = (x_0, \dots, x_{n-1})$ and $s = (s_0, \dots, s_{m-1})$. To any non trivial zero $A_0 \in \mathbb{R}^n \times \mathbb{R}^m$ of F , we may canonically associate an element $P_0 \in \mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$. Throughout the paper, we are considering *real* projective spaces.

We denote respectively by Π_1 and Π_2 the two projections of $\mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$ onto \mathbb{P}^{n-1} and respectively \mathbb{P}^{m-1} .

We denote by $\mathcal{Z}(F)$ the set of all real zeros of F in $\mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$. Here again, we emphasise that all the considered zeros are *real*.

Proposition 2.2. *Let $F(x; s) \in \mathcal{P}Q_{n,m}$ be a positive semi-definite biquadratic form.*

a) *There exists S_0 such that $P_0 = (X_0, S_0) \in \mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$ is a real zero of F if and only if $\det(M_x(F))$ vanishes at X_0 .*

- b) *If we view the coefficients of $F(x; s)$ as variables, the condition $F(P_0) = 0$ where $P_0 \in \mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$ gives a set of $n + m - 1$ linearly independent equations.*

Proof. a) If P_0 is a real zero of F , the vector S_0 is an isotropic vector for the quadratic form defined by $M = M_x(F)(X_0)$. Since it is positive semi-definite, it must be a singular matrix : $\det(M) = 0$.

The converse is easy : if M is singular, it admits an isotropic vector S_0 .

- b) If $F(P_0) = 0$, then P_0 necessarily is a critical point for F and all the partial derivatives vanishes at P_0 . By the Euler identity for forms, the cancellation of all partial derivatives implies that $F(P_0) = 0$.

We may be more explicit. First, to simplify, we may assume that $P_0 = (1, 0, \dots, 0; 1, 0, \dots, 0)$. Let us write also $F(x; s) = \sum_{i,j,k,l} a_{i,j;k,l} x_i x_j s_k s_l$. The cancellation of all partial derivatives gives $a_{i,1;1,1} = a_{1,1;k,1}$ for all i and k . We find a set of $n + m - 1$ linearly independent equations. □

3. BIQUADRATIC FORMS IN DIMENSIONS $(n, 2)$

3.1. The result of Calderon. The case of biquadratic forms in $\mathcal{BQ}_{n,2}$ has been considered by Calderon :

Theorem 3.1 ([Ca]). *Let $F(x; s, t) = a(x)s^2 + 2b(x)st + c(x)t^2 \in \mathcal{BQ}_{n,2}$ where $x = (x_0, \dots, x_{n-1})$ and s, t are real variables.*

If F is positive semi-definite, then it is a sum of squares. Namely, there exists an integer N and some linear forms $u_i(x), v_i(x)$ in x such that

$$F(x; s, t) = \sum_{i=1}^N (u_i(x)s + v_i(x)t)^2.$$

The original proof of Calderon is based on a convexity argument. We will give another proof without convexity. The advantages are that it yields an effective construction of the sum of squares and gives a better bound on the number of needed squares. Moreover, we will deal with very similar constructions in the following of the paper.

Our proof can be thought as a constructive version of [Ca], in the same way as [Pf] is a constructive version of original Hilbert's Theorem on ternary quartics.

Proof. Let us assume first that F is positive definite, namely for all $x \neq 0$ and all $(s, t) \neq (0, 0)$, $F(x; s, t) > 0$.

Let

$$F_\epsilon(x; s, t) = F(x; s, t) - \epsilon x_0^2 s^2$$

and introduce $I = \{\epsilon \mid F_\epsilon \in \mathcal{PQ}_{n,2}\}$. Then, I is a non empty interval $[0, m)$. Since I is defined by some closed conditions, we have $I = [0, m]$. Necessarily $m \neq 0$ and F_m is not positive definite.

Hence, we have reduced to the case where F is not positive definite : there is $(s_0, t_0) \neq (0, 0)$ and $X_0 \neq 0$ such that $F(X_0; s_0, t_0) = 0$.

Up to a linear change of the coordinates we may assume that $(s_0, t_0) = (0, 1)$ and $X_0 = (1, 0, \dots, 0)$. Then, there are some identities between quadratic forms :

$$\begin{cases} a(x) = (\lambda x_0 + \alpha(y))^2 + \tilde{a}(y) \\ b(x) = \eta x_0^2 + (\lambda x_0 + \alpha(y))\beta(y) + \tilde{b}(y) \\ c(x) = \mu(x_0 + \gamma(y))^2 + \tilde{c}(y) \end{cases}$$

Where λ, μ, η are reals, $\alpha(y), \beta(y), \gamma(y)$ are linear forms and $\tilde{a}(y), \tilde{b}(y), \tilde{c}(y)$ are quadratic forms in the variables $y = (x_1, \dots, x_{n-1})$.

By assumption we have $\mu = 0$. By positivity of F , we get $\eta = 0$. Thus

$$F(x; s, t) = [(\lambda x_0 + \alpha(y))s + \beta(y)t]^2 + \tilde{F}(y; s, t)$$

where

$$\tilde{F}(y; s, t) = \tilde{a}(y)s^2 + 2\tilde{b}(y)st + (\tilde{c}(y) - \beta(y)^2)t^2.$$

Since F is positive semi-definite, we deduce that \tilde{F} is also positive semi-definite.

Then, by induction on the number of variables n , we conclude that F is a sum of squares.

Moreover, at each step we get rid of one of the variables x_i and we increase by two the needed number of squares to decompose our positive semi-definite biquadratic form. It gives the bound $N \leq 2n$. \square

Let us mention also that Theorem 3.1 can also be seen as a particular case of [CLR1, Theorem 7.1], which provides another different proof.

3.2. On the number of squares. It is common in real algebra to study the minimal number of squares in a decomposition of a given sum of squares.

For positive semi-definite biquadratic forms in $\mathcal{BQ}_{n,2}$, our proof of Theorem 3.1 gives the bound $N \leq 2n$, whereas the original argument of Calderon who relies on a convexity argument leads to $N \leq 3n(n+1)/2$ squares.

By [CLR1, Theorem 7.1], we also recover the bound $2n$, but in fact it is possible to do even better. The needed number of squares is showed in [CLR1] to be bounded by

$$(1) \quad \lfloor \sqrt{3n} + \frac{\sqrt{3}-1}{2} \rfloor.$$

In general, if $F \in \mathcal{BQ}_{n,m}$ is assumed to be a sum of squares, then according to the work in [CLR2] about the so-called *pythagoras numbers*, one can say that F is a sum of at most

$$(2) \quad \frac{\left(\sqrt{1+2mn(m+1)(n+1)}-1\right)}{2}$$

squares.

See also [LS] for other bounds.

4. COUNTING THE ROOTS OF A BIQUADRATIC FORM

In [CLR1] is defined $B_{n,k}$ (resp. $B'_{n,k}$) by $\sup |\mathcal{Z}(p)|$ where p ranges over all positive semi-definite forms in $\mathcal{H}_{n,k}$ (resp. sum of squares of forms) with $|\mathcal{Z}(p)| <$

∞ . The determination of these numbers is a rather awesome task. Some partial results are given in [CLR1]. For instance :

$$\begin{cases} B_{3,4} = 4, \\ B_{3,6} = 10, \\ \text{For } k > 6, \frac{k^2}{4} \leq B_{3,k} \leq \frac{(k-1)(k-2)}{2} \quad \text{and} \quad B'_{3,k} = \frac{k^2}{4}. \end{cases}$$

Note that in general, we do not even know if the numbers $B_{n,k}$ are finite.

We naturally introduce the counterpart for biquadratic forms. Namely, let $\text{BB}_{n,m}$ (resp. $\text{BB}'_{n,m}$) be equal to $\sup |\mathcal{Z}(F)|$ where F ranges over all forms in $\mathcal{PQ}_{n,m}$ (resp. sum of squares of forms) with $|\mathcal{Z}(F)| < \infty$.

Proposition 4.1. *We have :*

- a) $\text{BB}_{n,m} \geq \text{BB}'_{n,m}$,
- b) $\text{BB}_{n,2} = \text{BB}'_{n,2}$,
- c) $\text{BB}_{n,m} = \text{BB}_{m,n}$,
- d) $\text{BB}_{n_1+n_2, m_1+m_2} \geq \text{BB}_{n_1, m_1} + \text{BB}_{n_2, m_2}$,
- e) $\text{BB}_{n,m} \leq B_{n,2m} \cdot B_{m,2n}$.

Moreover, (c) and (d) remain true if BB is replaced everywhere by BB' .

Proof. Properties (a) and (c) are obvious, and (b) follows from Theorem 3.1.

To show property (d), we take two biquadratic forms

$$F_1(x; s) \in \mathcal{PQ}_{n_1, m_1}, \text{ with } x = (x_0, \dots, x_{n_1-1}), s = (s_0, \dots, s_{m_1-1})$$

and

$$F_2(y; t) \in \mathcal{PQ}_{n_2, m_2} \text{ with } y = (y_0, \dots, y_{n_2-1}), t = (t_0, \dots, t_{m_2-1}).$$

Then, consider the biquadratic form

$$F(x, y; s, t) = F_1(x; s) + F_2(y; t)$$

which is in $\mathcal{PQ}_{n_1+n_2, m_1+m_2}$.

Let us show now property (e). Let $F(x; s) \in \mathcal{PQ}_{n,m}$ and let us introduce $p(x) = \det(M_x(F)) \in \mathcal{H}_{n,2m}$ and $q(s) = \det(M_s(F)) \in \mathcal{H}_{m,2n}$. They are both positive semi-definite. We recall from Proposition 2.2 that if $P \in \mathcal{Z}(F)$, then $\Pi_1(P)$ is a zero of $p(x)$ and $\Pi_2(P)$ is a zero of $q(s)$. Then, $|\Pi_1(\mathcal{Z}(F))| \leq B_{n,2m}$ and $|\Pi_2(\mathcal{Z}(F))| \leq B_{m,2n}$. This concludes the proof. \square

In the sequel, we compute the values of $\text{BB}'_{n,m}$, using the notion of degree of an algebraic variety embedded in a projective space.

Let us consider the Segre embedding

$$\begin{aligned} \Phi : \mathbb{P}^{n-1} \times \mathbb{P}^{m-1} &\rightarrow \mathbb{P}^{nm-1} \\ ([x_0, \dots, x_{n-1}], [s_0, \dots, s_{m-1}]) &\rightarrow [x_0 s_0, x_0 s_1, \dots, x_{n-1} s_{m-1}] \end{aligned}$$

Let us recall (see [Ha] for instance) that if the algebraic variety $V \subset \mathbb{P}^n$ has dimension r , then its degree d is equal to the number of intersection points with a generic linear variety in \mathbb{P}^n of dimension $n-r$. Moreover, the number of intersection points of V with any linear variety in \mathbb{P}^n of dimension $n-r$, is then at most d . Further, the number of intersection points of V with any linear variety in \mathbb{P}^n of dimension $> n-r$, is at most d or infinite.

We have :

Theorem 4.2. $\text{BB}'_{n,m} = \binom{m+n-2}{m-1}$.

Proof. Let $F \in \mathcal{PQ}_{n,m}$ such that $F = \sum_{i=1}^r F_i^2$. Let us write

$$F_i = \sum_{0 \leq j \leq n-1, 0 \leq k \leq m-1} a_{j,k;i} x_j s_k$$

and let us consider the hyperplane H_i in \mathbb{P}^{mn-1} of equation

$$\sum_{0 \leq j \leq n-1, 0 \leq k \leq m-1} a_{j,k;i} t_{j,k} = 0.$$

Then, $F(P) = 0$ if and only if $F_1(P) = \dots = F_r(P) = 0$. In other words, $\Phi(P)$ belongs to the linear space H defined by $H = \cap_i H_i$. Namely

$$|\mathcal{Z}(F)| = |\{P \in \mathbb{P}^{n-1} \times \mathbb{P}^{m-1} \mid \forall i F_i(P) = 0\}| = |\text{Im } \Phi \cap H|.$$

According to [Ha], the dimension of $\text{Im } \Phi$ is $m + n - 2$ and the degree of the embedding Φ is $\binom{n+m-2}{n-1}$ (this is true over any field of characteristic zero). The degree of Φ being precisely the maximal number of intersection points of $\text{Im } \Phi$ with a linear space such that the intersection is finite, we are done. \square

Example 4.3. In the case $n = m = 3$, we have $6 = \text{BB}'_{3,3}$. Here are two examples F_1 and F_2 of biquadratic forms in $\mathcal{PQ}_{3,3}$ with the maximal number of real zeros :

a)

$$F_1(x, y, z; s, t, u) = (xs)^2 + (yt)^2 + (zu)^2 + ((x + y + z)(s + t + u))^2.$$

b)

$$M_{(x,y,z)}(F_2) = \begin{pmatrix} x^2 + y^2 + z^2 & xy + z^2 & xz + y^2 \\ xy + z^2 & x^2 + y^2 + z^2 & yz + x^2 \\ xz + y^2 & yz + x^2 & x^2 + y^2 + z^2 \end{pmatrix}.$$

In both cases, the set of real zeros is

$$\left\{ \begin{array}{lll} (1, 0, 0; 0, -1, 1), & (0, 1, 0; 1, 0, -1), & (0, 0, 1; -1, 1, 0), \\ (1, -1, 0; 1, 1, -1), & (0, 1, -1; -1, 1, 1), & (-1, 0, 1; 1, -1, 1) \end{array} \right\}.$$

As a direct consequence of the definition of the numbers BB , BB' and Theorem 4.2, we have

Proposition 4.4. *Let $F \in \mathcal{PQ}_{n,m}$ be such that $\text{BB}'_{n,m} < |\mathcal{Z}(F)| \leq \text{BB}_{n,m}$. Then, F is not a sum of squares. For instance, if $F \in \mathcal{PQ}_{3,3}$ is such that $6 < |\mathcal{Z}(F)| < \infty$, then F is not a sum of squares.*

5. ZEROS OF BIQUADRATIC FORMS IN $\mathcal{BQ}_{3,3}$

The main purpose of this section is to show that if a biquadratic form in $\mathcal{PQ}_{3,3}$ has more zeros than $\text{BB}_{3,3}$, then it is a sum of squares (Theorem 5.4). It is a kind of counterpart for biquadratic forms of the property [CLR1, Theorem 3.5] for positive semi-definite forms in $\mathcal{H}_{3,k}$.

Although, some differences remain between these two contexts. For instance, if a positive semi-definite form in $\mathcal{H}_{3,k}$ has an infinite number of zeros, then it is divisible by the square of some indefinite form. Whereas, for biquadratic forms, it is no more true. Indeed, let us consider

$$F(x, y, z; s, t, u) = (xs)^2 + (yt)^2 \in \mathcal{PQ}_{3,3}.$$

It has an infinite number of zeros (for instance $(0, 0, 1; s, t, u)$, for any $(s, t, u) \in \mathbb{P}^2$), but cannot be divided by a bilinear form.

We are mainly interested in the case of the dimension 2, since it is the easiest case where positive semi-definite biquadratic forms do not coincide with sums of squares. Besides, in higher dimensions, such a property is hopeless as it is shown by :

Example 5.1. Let $F(x, y, z; s, t, u)$ be a positive semi-definite non sum of squares biquadratic form in $\mathcal{BQ}_{3,3}$. Assume that $(1, 0, 0; 1, 0, 0)$ is a zero of F in $\mathbb{P}^2 \times \mathbb{P}^2$ and let $L(x, y, z)$ be a non zero linear form such that $L(1, 0, 0) = 0$. Then, the following biquadratic form

$$G(x, y, z; s, t, u, v) = F(x, y, z; s, t, u) + L(x, y, z)^2 v^2 \in \mathcal{PQ}_{3,4}$$

is not a sum of squares. Moreover, G has an infinite number of zeros in $\mathbb{P}^2 \times \mathbb{P}^3$, for instance $(1, 0, 0; 1, 0, 0, v)$ for any $v \in \mathbb{R}$.

That is why, in the following, we cannot reproduce the whole method of [CLR1], although some steps will look very similar.

Let us begin with

Lemma 5.2. *Let $F \in \mathcal{PQ}_{3,3}$ be a positive semi-definite biquadratic form which admits two different zeros A and B in $\mathbb{P}^2 \times \mathbb{P}^2$ such that $\Pi_1(A) = \Pi_1(B)$. Then, F is a sum of squares and admits an infinite number of zeros.*

Proof. After a linear change of coordinates, we may assume that $A = (1, 0, 0; 1, 0, 0)$ and $B = (1, 0, 0; 0, 1, 0)$. Denoting by $(x, y, z; s, t, u)$ the new coordinates, the biquadratic form F can be written as

$$F(x, y, z; s, t, u) = a_{3,3}(x, y, z)u^2 + 2a_{1,3}(x, y, z)su + 2a_{2,3}(x, y, z)tu + G(y, z; s, t, u)$$

where G is a biquadratic form and $a_{1,3}, a_{2,3}$ and $a_{3,3}$ are quadratic forms. If $a_{3,3}$ does not depend on the variable x , then it is the same for $a_{1,3}$ and $a_{2,3}$, namely F does not depend on x . In that case, $(\alpha, 0, 0; 1, 0, 0)$ would be a zero of F for any $\alpha \in \mathbb{R}$, and F would be a sum of squares by Theorem 3.1.

In the following, we assume that $a_{3,3}$ does depend on x . We may rewrite F as

$$F(x, y, z; s, t, u) = a_{3,3}(x, y, z)u^2 + 2x\alpha_1(y, z)su + 2x\alpha_2(y, z)tu + H(y, z; s, t, u)$$

where H is a biquadratic form and α_1 and α_2 are linear forms. Up to a linear change of the coordinate x , we may assume that $a_{3,3}(x, y, z) = x^2 + b_3(y, z)$ where b_3 is a quadratic form.

We have then

$$F(x, y, z; s, t, u) = (xu + \alpha_1 s + \alpha_2 t)^2 + \tilde{F}(y, z; s, t, u)$$

where \tilde{F} is a biquadratic form.

Since F is positive semi-definite, so it is the case for \tilde{F} . Then, \tilde{F} is a sum of squares by Theorem 3.1. Moreover, $(1, 0, 0; \alpha, \beta, 0)$ is a zero of F for any $(\alpha, \beta) \in \mathbb{R}^2$. \square

As a consequence, we have

Proposition 5.3. *Let $F \in \mathcal{P}Q_{3,3}$ be a positive semi-definite biquadratic form with at least 11 zeros in $\mathbb{P}^2 \times \mathbb{P}^2$. Then, F admits an infinite number of zeros. In other words, $\text{BB}_{3,3} \leq 10$.*

Proof. Let us consider the associated matrix $M_{(s,t,u)}(F)$ whose entries are quadratic forms in the variables (s, t, u) . According to Lemma 5.2, we may assume that the number of zeros $\mathcal{Z}(F)$ of F is equal to $|\Pi_2(\mathcal{Z}(F))|$. Then, the number of zeros of F is the number of zeros in \mathbb{P}^2 of $\det(M_{(s,t,u)}(F))$ by Proposition 2.2.

According to [CLR1, Theorem 3.7], we deduce that $|\mathcal{Z}(F)|$ is either infinite or it is bounded by $B_{3,6} = 10$. \square

Our purpose now is to show in addition that F necessarily is a sum of squares.

Theorem 5.4. *Let $F \in \mathcal{P}Q_{3,3}$ be a positive semi-definite biquadratic form with an infinite number of zeros in $\mathbb{P}^2 \times \mathbb{P}^2$. Then, F is a sum of squares.*

Proof. Up to a linear change of coordinates, we may assume that $(1, 0, 0; 1, 0, 0)$ is an accumulation point in $\mathcal{Z}(F) \subset \mathbb{P}^2 \times \mathbb{P}^2$.

Then,

$$F(x, y, z; s, t, u) = q_{2,0}(y, z)s^2 + 2q_{2,1}(y, z; t, u)s + q_{2,2}(x, y, z; t, u)$$

where the $q_{2,k}$'s are quadratic forms in (x, y, z) and forms of degree k in (s, t, u) .

Note that necessarily $q_{2,0}$ and $q_{2,2}$ are positive semi-definite. Let us show a Lemma which is very close in spirit as [CLR1, Lemma 5.2] :

Lemma 5.5. *With the above notations, we have $\text{rk}(q_{2,0}) < 2$.*

Proof. Assume that $\text{rk}(q_{2,0}) = 2$. Up to a linear change of coordinates, we may assume that $q_{2,0} = y^2 + z^2$. Consider a sequence $(1, y_n, z_n; 1, t_n, u_n)$ of zeros of F in $\mathbb{P}^2 \times \mathbb{P}^2$ which converges to $(1, 0, 0; 1, 0, 0)$. We may assume that $(y_n, z_n) \neq (0, 0)$ for all n otherwise we may directly use Lemma 5.2.

Let us choose an integer n_0 such that

$$\frac{2|q_{2,1}(y_{n_0}, z_{n_0}; t_{n_0}, u_{n_0})|}{q_{2,0}(y_{n_0}, z_{n_0})} < \frac{1}{2}.$$

This is possible since, given $(a, b, c) \in \mathbb{R}^2$, the map

$$\begin{aligned} \mathbb{P}^1 &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto \frac{ax^2 + bxy + cy^2}{x^2 + y^2} \end{aligned}$$

is bounded.

We have then

$$F(1, y_{n_0}, z_{n_0}; 1, t_{n_0}, u_{n_0}) = 0 \geq q_{2,0}(y_{n_0}, z_{n_0}) - 2|q_{2,1}(y_{n_0}, z_{n_0}; t_{n_0}, u_{n_0})| > 0$$

a contradiction. \square

If $\text{rk}(q_{2,0}) = 0$, then we are done by Theorem 3.1. Hence, we are reduced to the case $\text{rk}(q_{2,0}) = 1$, i.e. after a linear change of coordinates, $q_{2,0} = y^2$.

We have

$$F = y^2 s^2 + 2q_{2,1}(y, z; t, u)s + q_{2,2}(x, y, z; t, u),$$

where $y^2 q_{2,2} - q_{2,1}^2$, the discriminant of F with respect to the variable s , is positive semi-definite. Hence, there is no coefficient z^2 in $q_{2,1}$, namely

$$q_{2,1} = \alpha(t, u)yz + \beta(t, u)y^2$$

where α, β are linear forms. We have

$$(3) \quad F = (ys + \alpha(t, u)z + \beta(t, u)y)^2 + G(x, y, z; t, u)$$

where G is a biquadratic form.

Since, for any choice of $(y, x, z, t, u) \in \mathbb{R}^* \times \mathbb{R}^4$ there is a s such that

$$F(x, y, z; s, t, u) = G(x, y, z; t, u),$$

we conclude that G is also positive semi-definite. It suffices to use Theorem 3.1 to conclude that G is a sum of squares. \square

Remark 5.6. A little inspection of the proofs of 5.4 and 5.2 shows that if $F \in \mathcal{P}Q_{3,3}$ has an infinite number of zeros, then it is a sum of at most 5 squares. Indeed, when F does not depend on x , we may consider that $F \in \mathcal{P}Q_{3,2}$ and we get the bound by (1).

We assume now that F does depend on x , and we use equality (3). If G had only a finite number of zeros, it would mean that F had zeros of the form $(1, y_n, z_n; 1, 0, 0)$, as a consequence F would not depend on s and we are done by the previous case.

Hence, we may assume that G has an infinite number of zeros and we repeat the same argument to the biquadratic form G .

We have to successively use the bounds given by (1) : it gives at most 4 squares for $\mathcal{B}Q_{3,2}$, at most 3 squares for $\mathcal{B}Q_{2,2}$ and at most 2 squares for $\mathcal{B}Q_{2,1}$.

In summary, the maximal number of needed squares in all sums of squares lying in $\mathcal{B}Q_{3,3}$ is at most 8 according to (2). But when we restrict to those sum of squares with an infinite number of zeros, the bound drops to 5.

At least, we cannot expect less than 5 squares as shown by the following example

$$F = x^2s^2 + y^2t^2 + (x^2 + y^2 + z^2)u^2.$$

6. EXAMPLES OF POSITIVE SEMI-DEFINITE BIQUADRATIC FORMS WHICH ARE NOT SUM OF SQUARES

Positive semi-definite biquadratic forms which are not sums of squares have some important applications in several areas such as control theory and theoretical physics, even if in that last case the complex counterpart of these notions are most commonly studied, namely positive semi-definite bihermitian forms which are not sums of squares (see for instance [SZ]).

In this section, we mention some known examples given in the literature, for instance in [C1] and [Te]. We revisit them by making use of the notion of real zeros of biquadratic forms we have developed in the previous sections.

To end, we will also produce some new families of examples.

6.1. Arrow matrices. If $F(x; s) \in \mathcal{B}Q_{3,3}$ has some associated matrix $M_x(F)$ with more than two zero entries, then it obviously is a sum of squares. Hence, the simplest candidate for an example of a positive semi-definite biquadratic form not being a sum of squares is an arrow matrix. In this section, we show that such examples exist.

Proposition 6.1. *The biquadratic form F_α whose associated matrix is*

$$M_\alpha = \begin{pmatrix} x^2 + z^2 & 0 & -xz \\ 0 & x^2 + y^2 & -yz \\ -xz & -yz & z^2 + \alpha y^2 \end{pmatrix}$$

is positive semi-definite but not a sum of squares for all $\alpha \in [\alpha_0, +\infty[$, where α_0 is a smallest positive root of the polynomial $9x^4 - 36x^3 + 62x^2 - 20x + 1$ ($\alpha_0 \simeq 0.006$).

Proof. The matrix M_α is positive semi-definite if and only if all its symmetric minors are positive semi-definite. The only non trivial condition to check is $\det(M_\alpha) \geq 0$. Let us compute

$$\det(M_\alpha) = \alpha y^2 x^4 + (\alpha y^4 + (\alpha - 1)y^2 z^2 + z^4)x^2 + \alpha y^4 z^2.$$

An elementary computation shows that $\det(M_\alpha) \geq 0$ if and only if $\alpha \in [\alpha_0, +\infty[$.

Now, let us see why F_α is not a sum of squares. We have :

$$F_\alpha = (x^2 + z^2)s^2 + (x^2 + y^2)t^2 + (z^2 + \alpha y^2)u^2 - 2xzs u - 2yztu$$

If we look at F_α as a form of degree 4 in the 6 variables (x, y, z, s, t, u) , then we may consider its associated Gram matrices (see [CLR2] for background on Gram Matrices and sums of squares) with respect to the basis (xs, xt, yt, yu, zs, zu) . We find that there is a unique Gram matrix :

$$G_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}.$$

Its signature is $(5, 1)$ whenever $\alpha > 0$. Hence G_α is never a positive semi-definite matrix, and F_α is never a sum of squares when $\alpha > 0$. \square

In this example, each associated matrices $M_{(x,y,z)}(F_\alpha)$ and $M_{(s,t,u)}(F_\alpha)$ of the biquadratic form F_α are both arrow matrices.

We end this section by giving another example of an arrow matrix associated to a positive semi-definite biquadratic form which is not a sum of squares. Namely :

$$\begin{pmatrix} x^2 + z^2 & 0 & -xz - xy \\ 0 & x^2 + y^2 & -yz \\ -xz - xy & -yz & 2y^2 + 2z^2 \end{pmatrix}.$$

6.2. Terpstra's construction. A geometric construction of positive semi-definite biquadratic forms which are not sum of squares can be found in [Te]. For the convenience of the reader, we will recall the construction, although we give a different proof from the original, making use of our previous results on the real zeros set of biquadratic forms.

Let us consider L_1, L_2, L_3, L_4 , four lines in general position in the projective plane \mathbb{P}^2 . Let us introduce the intersection points

$$\begin{aligned} A_1 &= L_1 \cap L_2 \\ A_2 &= L_2 \cap L_3, \\ A_3 &= L_3 \cap L_4, \\ A_4 &= L_1 \cap L_4, \\ A_5 &= L_1 \cap L_3, \\ A_6 &= L_2 \cap L_4. \end{aligned}$$

Up to multiplying the equations by -1 , we may assume that $L_1(A_6) \cdot L_2(A_6) > 0$ and $L_3(A_1) \cdot L_4(A_1) > 0$. Let us define the following points in $\mathbb{P}^2 \times \mathbb{P}^2$:

$$\begin{aligned} P_1 &= (A_1, A_6) \\ P_2 &= (A_2, A_4) \\ P_3 &= (A_3, A_5) \\ P_4 &= (A_4, A_2) \\ P_5 &= (A_5, A_3) \\ P_6 &= (A_6, A_1). \end{aligned}$$

Theorem 6.2. *Let us consider the biquadratic form*

$$F_\lambda(x; s) = (L_1(x)L_1(s))^2 + (L_2(x)L_2(s))^2 + (L_3(x)L_3(s))^2 + (L_4(x)L_4(s))^2 + \lambda L_1(x)L_2(x)L_3(s)L_4(s)$$

where $\lambda \in \mathbb{R}$.

Then,

- a) *There is a $\delta > 0$ such that for all $0 < \lambda < \delta$, the biquadratic form F_λ is positive semi-definite,*
- b) *For any $\lambda > 0$, the biquadratic form F_λ is never a sum of squares.*

Proof. One easily checks that $F_\lambda(P_1) = \dots = F_\lambda(P_5) = 0$ and $F_\lambda(P_6) > 0$.

Let us show that $F_\lambda \geq 0$ in a neighbourhood of each P_i for $i = 1, \dots, 5$ (it is obvious for $i = 6$ since $F_\lambda(P_6) > 0$). Let

$$F_{2,\delta}(x; s) = L_1^2(A_2)L_1^2(s) + L_2^2(A_4)L_2^2(x) + L_3^2(A_2)L_3^2(x) + L_4^2(A_2)L_4^2(s) + \delta L_1(A_2)L_3(A_4)L_2(x)L_4(s)$$

Since $F_\lambda - F_{2,\delta}$ is infinitely smaller than $F_{2,\delta}$ in a small neighbourhood of P_2 , it suffices to note that $F_{2,\delta} > 0$ in a neighbourhood of P_2 for δ small enough.

We proceed likewise in a neighbourhood of P_3, P_4 and P_5 .

Around P_1 , we also proceed likewise but the expression we consider is a little bit different : we set

$$F_1(x; s) = L_1^2(A_6)L_1^2(x) + L_2^2(A_6)L_2^2(x) + L_3^2(A_1)L_3^2(s) + L_4^2(A_1)L_4^2(s)$$

which is > 0 in a small neighbourhood of P_1 .

In summary, there is a $\delta > 0$ and an open neighbourhood V of P_1, P_2, P_3, P_4, P_5 such that $F_\lambda > 0$ on $V \setminus \{P_1, P_2, P_3, P_4, P_5\}$ for any $0 < \lambda < \delta$.

On $\Omega = \mathbb{P}^2 \setminus V$, the quantity

$$\frac{(L_1(x)L_1(s))^2 + (L_2(x)L_2(s))^2 + (L_3(x)L_3(s))^2 + (L_4(x)L_4(s))^2}{|L_1(x)L_2(x)L_3(s)L_4(s)|}$$

never vanishes, hence reaches a minimum which we may assume greater than δ .

We have shown that F_λ is positive semi-definite for any $0 < \lambda < \delta$.

Let us now show that F_λ is never a sum of squares.

By the contrary, let us assume that $F = \sum_{i=1}^r G_i^2$ where G_i is a bilinear form in the variables $(x; s)$.

Let us consider the linear map which sends a bilinear map G to the 6-tuple $(G(P_1), \dots, G(P_6))$. The range of this map is contained in an hyperplane of \mathbb{R}^6 . Namely, there are some real numbers $\alpha_1, \dots, \alpha_6$ such that for any bilinear form G , we have

$$\alpha_1 G(P_1) + \dots + \alpha_6 G(P_6) = 0.$$

Let us assume that $\alpha_6 = 0$. We consider the bilinear form $G = L_2(x)L_3(s)$. It vanishes at P_1, P_2, P_3, P_4 but not at P_5, P_6 , and hence we get $\alpha_5 G(P_5) = 0$. Thus, $\alpha_5 = 0$ and we shall proceed likewise to show that all α_i are zero, a contradiction. Then, necessarily $\alpha_6 \neq 0$.

Now, it suffices to notice that we obviously have, for all i ,

$$G_i(P_1) = \dots = G_i(P_5) = 0.$$

Hence $G_i(P_6) = 0$ for all i which implies that $F(P_6) = 0$, a contradiction. \square

Note that, according to the construction, $\mathcal{Z}(F_\lambda) = \{P_1, \dots, P_6\}$ and that the relation $\alpha_1 G(P_1) + \dots + \alpha_6 G(P_6) = 0$ can be seen as Cayley-Bacharach relation ([EGH]).

6.3. Around the Choi example.

6.3.1. *The Choi example.* As defined in [C1], we consider the following biquadratic form :

$$C_1(x, y, z; s, t, u) = (x^2 + 2z^2)s^2 + (y^2 + 2x^2)t^2 + (z^2 + 2y^2)u^2 - 2xyt - 2xzs - 2yztu$$

Denote by \mathcal{C} its zero set $\mathcal{Z}(C_1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$. We have

$$(4) \quad \mathcal{C} = \{A_1, A_2, A_3, B_0, B_1, B_2, B_3\}$$

where

$$\begin{aligned} A_1 &= (1, 0, 0; 0, 1, 0), A_2 = (0, 1, 0; 0, 0, 1), A_3 = (0, 0, 1; 1, 0, 0), B_0 = (1, 1, 1; 1, 1, 1) \\ B_1 &= (-1, 1, 1; -1, 1, 1), B_2 = (1, -1, 1; 1, -1, 1), B_3 = (1, 1, -1; 1, 1, -1). \end{aligned}$$

Proposition 6.3. [C1] *The biquadratic form C_1 is positive semi-definite but not a sum of squares.*

We will not discuss much the fact that C_1 is positive semi-definite. For instance, it comes from the property that all the symmetric minors of $M_{(x,y,z)}(C_1)$ are sums of squares.

The original proof to show that C_1 is not a sum of squares is rather elementary : we assume that C_1 is a sum of squares and we carefully look at the monomials in C_1 to get a contradiction.

Now, referring to our background on real zeros of biquadratic forms, the quickest way to reprove it, is to use Proposition 4.4. But let us show again another proof of that fact !

Our motivation for this new proof is that it might lead to another direction to study positive semi-definite biquadratic forms which are not sum of squares. This proof is close in spirit than the one Robinson used (a strategy suggested by Hilbert's work) to show that its polynomial

$$(5) \quad R(x, y, z) = x^6 + y^6 + z^6 - x^4y^2 - x^4z^2 - y^4x^2 - y^4z^2 - z^4x^2 - z^4y^2 + 3x^2y^2z^2$$

is positive semi-definite but not a sum of squares. In modern literature, it would be referred as a Cayley-Bacharach type result (see [EGH]). Namely, it exploits the redundancy that exists among the real zeros of cubic forms, redundancy which is looser among the real zeros of sextic forms.

Our proof could be seen as a counterpart for biquadratic forms, namely a Cayley-Bacharach type result for biquadratic forms :

Proof. Assume that $C_1 = \sum_{i=1}^r f_i^2$. The bilinear form f_i can be written

$$f_i = a_1xs + a_2xt + a_3xu + b_1ys + b_2yt + b_3yu + c_1zs + c_2zt + c_3zu.$$

Since each f_i has to vanish at A_1, A_2, A_3 , we deduce that $a_2 = b_3 = c_1 = 0$.

Moreover, if we write the system saying that f_i vanishes also at B_0, B_1, B_2, B_3 , we conclude that

$$f_i = a_1xs + b_2yt + c_3zu \quad \text{with} \quad a_1 + a_2 + a_3 = 0$$

As a consequence, we get that each f_i vanishes also at $A_\alpha = (0, 1, \alpha; 1, 0, 0)$ for any $\alpha \in \mathbb{R}$. And hence C_1 should also vanishes at all A_α , a contradiction. \square

In fact, we have shown that :

Proposition 6.4. ?? *If G is a biquadratic form which vanishes at \mathcal{C} and not at $A_\alpha = (0, 1, \alpha; 1, 0, 0)$ for some $\alpha \in \mathbb{R}$, then G is not a sum of squares.*

Example 6.5. As an application, if G is a positive semi-definite biquadratic form which vanishes at \mathcal{C} , then $C_1 + G$ is positive semi-definite but not a sum of squares.

For instance :

$$C_1 + (2xs - yt + zu)^2$$

or

$$C_1 + (xs - yt)^2 + (xs - zu)^2 + (yt - zu)^2.$$

6.3.2. *Some homotopy around the Choi example.* First note, as observed in [Qz], that the biquadratic form associated to the matrix

$$M_0 = \begin{pmatrix} x^2 + z^2 & -xy & -xz \\ -xy & x^2 + y^2 & -yz \\ -xz & -yz & y^2 + z^2 \end{pmatrix}$$

is positive semi-definite but not a sum of squares for the same reasons than C_1 is.

Here is some homotopy around M_0 :

Proposition 6.6. *The matrix*

$$\begin{aligned} M_a &= M_0 + a \begin{pmatrix} x^2 - z^2 & 0 & 0 \\ 0 & y^2 - x^2 & 0 \\ 0 & 0 & z^2 - y^2 \end{pmatrix} \\ &= \begin{pmatrix} x^2 + z^2 + a(x^2 - z^2) & -xy & -xz \\ -xy & x^2 + y^2 + a(y^2 - x^2) & -yz \\ -xz & -yz & y^2 + z^2 + a(z^2 - y^2) \end{pmatrix} \end{aligned}$$

is positive semi-definite for all $a \in [0, 1]$. The associated biquadratic form F_a has zero set \mathcal{C} , and it is positive semi-definite but not a sum of squares until $a = 1$.

The easiest way to prove M_a is not a sum of squares it is to use Proposition ?? : indeed $M_a(0, 1, 1; 1, 0, 0) = 1 - a \neq 0$ if $a \in [0, 1[$. But, we will give another proof, where appear some new plane ternary sextics which are positive semi-definite and not sums of squares (they are realized by a kind of homotopy between the two symmetric versions of the Motzkin polynomial).

Proof. The 1×1 minors of M_a are positive semi-definite forms if and only if $-1 \leq a \leq 1$. Adding the conditions given by the 2×2 minors, we get that $a \in [0, 1]$. To end, we compute

$$p_a = \det(M_a) = \alpha(y^2x^4 + y^4z^2 + x^2z^4) + \beta(x^4z^2 + y^4x^2 + z^4y^2) - 3\gamma x^2y^2z^2$$

where

$$\begin{cases} \alpha = 1 - a - a^2 + a^3, \\ \beta = 2a - a^2 - a^3, \\ \gamma = 1 + a - 2a^2. \end{cases}$$

Our aim is to show that $p_a(x, y, z)$ is a positive semi-definite sextic but not a sum of squares. We will follow the same proof that is commonly used to show that the Motzkin polynomial is positive semi-definite but not a sum of squares.

First note that the arithmetico-geometric inequality gives $p_a(x, y, z) \geq 0$. Hence, M_a is positive semi-definite if and only if $a \in [0, 1]$.

Let us show now that the form $p_a(x, y, z)$ is not a sum of squares until $a = 1$ (and then $p_a = 0$). By the contrary assume that

$$p_a = \sum_i (a_i x^3 + b_i x^2 y + c_i x^2 z + d_i x y^2 + e_i x y z + f_i x z^2 + g_i y^2 + h_i y^2 z + k_i y z^2 + l_i z^3)^2.$$

We obviously have for all i ,

$$\begin{cases} a_i = g_i = l_i = 0, \\ \sum_i b_i c_i = \sum_i b_i d_i = \sum_i d_i h_i = 0. \end{cases}$$

Moreover, looking at the coefficient of $x^2 y^2 z^2$, we get

$$(6) \quad \sum_i e_i^2 + 2b_i k_i + 2c_i h_i + 2d_i f_i = -3\gamma.$$

We have also

$$\alpha = \sum b_i^2 = \sum f_i^2 = \sum h_i^2$$

and

$$\beta = \sum c_i^2 = \sum d_i^2 = \sum k_i^2.$$

Since $a \in [0, 1]$, we have $\gamma \geq 0$ and by the Cauchy-Schwartz inequality, we deduce from (6) that

$$3\gamma + \sum_i e_i^2 \leq 6\alpha\beta$$

Hence we get $\gamma - 2\alpha\beta \leq 0$ which is satisfied only if $a = 1$.

The fact that F_a is positive semi-definite and not a sum of squares for $a \neq 1$ is now a direct consequence of the Cauchy-Binet formula. Given matrices $A \in \mathbb{R}^{m \times s}$ and $B \in \mathbb{R}^{s \times m}$, the Cauchy-Binet formula states that

$$\det(AB) = \sum_S \det(A_S) \det(B_S)$$

where S ranges over all the subsets of $\{1, \dots, s\}$ with m elements, and A_S (respectively B_S) denotes the matrix in $\mathbb{R}^{m \times m}$ whose columns are the columns of A (respectively whose rows are the rows of B) with indexes from S .

If F_a were a sum of squares, then we would have $M_a = A^T A$ for some matrix A whose entries are linear forms in the variables x, y, z . Then,

$$\begin{aligned} \det(M_a) &= \sum_S \det(A^T)_S \det(A_S) \\ &= \sum_S \det(A_S^T) \det(A_S) \\ &= \sum_S \det(A_S)^2. \end{aligned}$$

Which is a contradiction since p_a is not a sum of squares. \square

7. CONFIGURATION OF ZEROS SETS

In order to study the possible geometry of the zeros sets of biquadratic forms in $\mathcal{PQ}_{n,m}$, we introduce the notion of *maximal configurations* which are *finite* subsets $\Omega \subset \mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$ such that if $F \in \mathcal{PQ}_{n,m}$ vanishes at Ω and also at another point in $\mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$, then $\mathcal{Z}(F)$ is infinite.

To begin with, one may look at what happens already for positive semi-definite forms.

7.1. Maximal configurations of Sextics. We say that $\Omega \subset \mathbb{P}^{n-1}$ is a *maximal configurations* if it is a *finite* subset Ω such that if $f \in \mathcal{H}_{n,k}$ is a positive semi-definite form which vanishes at Ω and also at another point in \mathbb{P}^{n-1} , then $\mathcal{Z}(f)$ is infinite.

From the definition of the number $B_{n,k}$, it follows that for any positive semi-definite $f \in \mathcal{H}_{n,k}$ such that $|\mathcal{Z}(f)| = B_{n,k}$, the set $\mathcal{Z}(f)$ is a maximal configuration.

For instance, the Robinson polynomial R as defined in (5) gives an example of a 10-points maximal configuration since $|\mathcal{Z}(R)| = 10$.

It is given as an open question in [CLR1] whether it is possible to give examples of other (of all up to a linear change of coordinates ?) maximal configurations of cardinal 10 ? The question is difficult and seems not to be answered so far.

Unfortunately we will not contribute to this problem. We will give some elementary facts which will serve us as background for the next section about biquadratic forms.

Proposition 7.1. *Let $\Omega \in \mathbb{P}^2$.*

- a) *If f is a positive semi-definite sextic such that $|\mathcal{Z}(f)| = 10$, then $\mathcal{Z}(f)$ is a maximal configuration.*
- b) *If $|\Omega| = 8$, then Ω is not a maximal configuration.*
- c) *Let D_1, D_2, D_3 (resp. $\Delta_1, \Delta_2, \Delta_3$) be the three lines of equations $x = -z, x = 0, x = z$ (resp. $y = -z, y = 0, y = z$). Let $\Omega = \cup_{i,j}(D_i \cap \Delta_j)$. Then, Ω is a 9-points maximal configuration.*
- d) *The maximal sets containing $\Omega = \mathcal{Z}(R) \setminus \{(1, 1, 0), (1, -1, 0)\}$ are exactly the sets $\mathcal{Z}(R)$ and $\Omega \cup \{P\}$ where P is any point in \mathbb{P}^2 such that $P \notin \{(1, 1, 0), (1, -1, 0)\}$.*

Proof. a) It is a direct consequence of the equality $B_{3,6} = 10$.

b) Let $|\Omega| = 8$. Let us consider two linearly independent cubics curves $p_1 = 0$ and $p_2 = 0$ going through all of these 8 points. Then, $p = p_1^2 + p_2^2$ is a sextic such that $|\mathcal{Z}(p)| = 9$.

c),d) Let f be a positive semi-definite a sextic. We consider the coefficients of f as variables. Each zero of f is a singular point, hence we get 3 linearly independent equations. We then reduce the rows of our big linear system by using the Gauss algorithm and we are able to conclude our proofs. Completely similar techniques will be given with more details in the next section. □

Of course, not all 9-points sets can be contained in a maximal configuration ; for instance take the case where the 9 points are lying in the same line.

7.2. Biquadratic forms in $\mathcal{B}Q_{2,2}$. Let us recall that $\text{BB}_{2,2} = 2$. We have

Proposition 7.2. *The maximal configurations in $\mathbb{P}^1 \times \mathbb{P}^1$ are the sets Ω such that $|\Omega| = |\Pi_1(\Omega)| = |\Pi_2(\Omega)| = 2$.*

Proof. If $|\Omega| \neq 2$, then Ω is clearly not a maximal configuration.

Let us assume now that $|\Omega| = 2$.

If $|\Pi_1(\Omega)| = 1$, then up to linear change of coordinates, we may assume that $\Omega = \{(1, 0; 1, 0), (1, 0; a, 1)\}$, with $a \in \mathbb{R}$. A biquadratic form F vanishing at Ω will be a sum of squares of elements of the form $\alpha ys + \beta yt$ with $(\alpha, \beta) \in \mathbb{R}^2$. Hence, $|\mathcal{Z}(F)| = \infty$, namely Ω is not maximal. We have shown that the conditions are necessary.

Let us show now that they are sufficient. If $|\Pi_1(\Omega)| = 2 = |\Pi_2(\Omega)|$, then up to linear change of coordinates, we may assume that $\Omega = \{(1, 0; 1, 0), (0, 1; 0, 1)\}$. The biquadratic form $F = (xt)^2 + (ys)^2$ is an example where $\mathcal{Z}(F) = \Omega$. Hence Ω is maximal. \square

We know from the identity $\text{BB}_{2,2} = 2$ that if the biquadratic form $F \in \mathcal{P}Q_{2,2}$ has 3 zeros or more, then (it is a sum of squares and) it has an infinite number of zeros. One may give a little bit more precise and geometric statement :

Proposition 7.3. *Assume that F vanishes at 3 different points A_1, A_2, A_3 . Then, F vanishes at any point A_4 satisfying the following identity on the cross ratios over \mathbb{P}^1 :*

$$[\Pi_1(A_1), \Pi_1(A_2), \Pi_1(A_3), \Pi_1(A_4)] = [\Pi_2(A_1), \Pi_2(A_2), \Pi_2(A_3), \Pi_2(A_4)].$$

Proof. Up to a linear change of coordinates, we may assume that $A_1 = (1, 0; 1, 0)$, $A_2 = (0, 1; 0, 1)$ and $A_3 = (1, y_0; 1, t_0)$. Then, F is a sum of squares of bilinear forms of the type $\alpha xt + \beta ys$ where $(\alpha, \beta) \in \mathbb{R}^2$ is satisfying $\alpha t_0 + \beta y_0 = 0$. Hence F vanishes at $(1, y_1; 1, t_1)$, for all (y_1, t_1) such that $y_0 t_1 - y_1 t_0 = 0$.

It remains to translate this last equality using the cross ratios over \mathbb{P}^1 . \square

7.3. Biquadratic forms in $\mathcal{B}Q_{3,2}$.

Proposition 7.4. *The maximal configurations in $\mathbb{P}^2 \times \mathbb{P}^1$ are the sets Ω such that $|\Omega| = |\Pi_1(\Omega)| = |\Pi_2(\Omega)| = 3$ and such that the 3 points in $\Pi_1(\Omega) \in \mathbb{P}^2$ are not lying on the same line.*

Proof. If $|\Omega| \neq 3$, then Ω is clearly not a maximal configuration. Likewise, if $|\Pi_1(\Omega)| \neq 3$ or $|\Pi_2(\Omega)| \neq 3$.

Let us assume from now on that $|\Omega| = |\Pi_1(\Omega)| = |\Pi_2(\Omega)| = 3$. Let $\Omega = \{P_1, P_2, P_3\} \subset \mathcal{Z}(F)$ where $F \in \mathcal{P}Q(3, 2)$.

Let us check that if $\Pi_1(P_1), \Pi_1(P_2), \Pi_1(P_3)$ are lying on the same line, then all the line is contained in $\Pi_1(\mathcal{Z}(F))$.

Up to linear change of coordinates, we may assume that

$$A_1 = (1, 0, 0; 1, 0), A_2 = (0, 1, 0; 0, 1), A_3 = (1, y_2, 0; 1, t_2)$$

where $y_2 t_2 \neq 0$. Let us write $F = \sum_i (F_i)^2$, where

$$F_i = axs + bxt + cys + dyt + ezs + fzt$$

with $(a, b, c, d, e, f) \in \mathbb{R}^6$. By assumption, we have $a = d = 0$ and $bt_0 + cy_0 = 0$. Hence

$$F_i = \lambda(y_0 xt - t_0 ys) + z(es + ft).$$

Thus, all the F_i 's vanish at $(1, \beta, 0; t_0, \beta)$ for any $\beta \in \mathbb{R}$. Hence $\mathcal{Z}(F)$ is infinite and Ω is not maximal. We have shown that the condition of the proposition is necessary.

Let us now show it is sufficient. Up to a linear change of coordinates, we may assume that

$$\Omega = \{(1, 0, -1; 1, 0), (0, 1, -1; 0, 1), (0, 0, 1; -\beta, \alpha)\}$$

where $(\alpha, \beta) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. We just consider the biquadratic form

$$F = (xt)^2 + (ys)^2 + ((x + y + z)(\alpha s + \beta t))^2 \in \mathcal{P}Q_{3,2}$$

and check that $\mathcal{Z}(F) = \Omega$. \square

Here is a statement on how to construct an infinite family of zeros when at least 4 zeros are given :

Proposition 7.5. *Assume that F vanishes at the 4 different points*

$$A_1 = (1, 0, 0; 1, 0), A_2 = (0, 1, 0; 0, 1), A_3 = (x_3, y_3, z_3; 1, t_3), A_4 = (x_4, y_4, z_4; 1, t_4).$$

Then, F vanishes at any point $A_5 = (x_5, y_5, z_5; 1, t_5)$ satisfying :

$$(7) \quad \det \begin{pmatrix} z_3 & t_3 & y_3 \\ z_4 & t_4 & y_4 \\ z_5 & t_5 & y_5 \end{pmatrix} = \det \begin{pmatrix} t_3 z_3 & t_3 & y_3 \\ t_4 z_4 & t_4 & y_4 \\ t_5 z_5 & t_5 & y_5 \end{pmatrix} = 0$$

Proof. Let $F \in \mathcal{P}Q_{3,2}$. By Proposition 2.2, each point A_1, A_2, A_3, A_4 is a singular point of the biquadratic form F , and hence it yields 4 linear equations whose variables are the coefficients of F . Namely, let us write

$$(8) \quad \begin{aligned} F &= s^2(a_0x^2 + a_1xy + a_3y^2 + a_2xz + a_4yz + a_5z^2) \\ &+ 2st(b_0x^2 + b_1xy + b_3y^2 + b_2xz + b_4yz + b_5z^2) \\ &+ t^2(c_0x^2 + c_1xy + c_3y^2 + c_2xz + c_4yz + c_5z^2) \end{aligned}$$

Since A_1 and A_2 are critical points, we have

$$a_0 = a_1 = a_2 = b_0 = b_3 = c_1 = c_3 = c_4 = 0.$$

With respect to the remaining variables

$$X = (a_3, a_4, a_5, b_1, b_2, b_4, b_5, c_0, c_2, c_5),$$

the fact that A_3 and A_4 are critical points yields the equations

$$AX^T = 0$$

where

$$A = \begin{pmatrix} A_3 \\ A_4 \end{pmatrix}$$

and

$$A_k = \begin{pmatrix} t_k^2 & 2t_k y_k & y_k^2 & 2t_k z_k & t_k^2 z_k & y_k z_k & 2t_k y_k z_k & z_k^2 & 2t_k z_k^2 & t_k^2 z_k^2 \\ 0 & 0 & 0 & 2t_k & t_k^2 & y_k & 2t_k y_k & 2z_k & 4t_k z_k & 2t_k^2 z_k \\ t_k & y_k & 0 & z_k & t_k z_k & 0 & y_k z_k & 0 & z_k^2 & t_k z_k^2 \\ 0 & 2t_k & 2y_k & 0 & 0 & z_k & 2t_k z_k & 0 & 0 & 0 \end{pmatrix}.$$

First, let us assume we are in the generic case, namely $z_3y_4 - z_4y_3 \neq 0$. We reduce the rows of our matrix A with the Gauss algorithm to a matrix $\tilde{A} = (\text{Id}_7|B)$ where $\text{Id}_7 \in \mathbb{R}^{8 \times 8}$ is the identity matrix and B is the matrix :

$$\left(\begin{array}{ccc} \frac{-(y_4z_3 - y_3z_4)^2}{(t_4y_3 - t_3y_4)^2} & \frac{-2(-y_4z_3 + y_3z_4)(-t_3y_4z_3 + t_4y_3z_4)}{(t_4y_3 - t_3y_4)^2} & \frac{-(t_3y_4z_3 - t_4y_3z_4)^2}{(t_4y_3 - t_3y_4)^2} \\ \frac{(t_4z_3 - t_3z_4)(y_4z_3 - y_3z_4)}{(t_4y_3 - t_3y_4)^2} & \frac{t_3t_4y_3y_4z_3z_4 \left(\frac{2z_3}{y_3z_4} - \frac{1}{y_4} - \frac{t_4}{t_3y_4} - \frac{t_3}{t_4y_3} - \frac{1}{y_3} + \frac{2z_4}{y_4z_3} \right)}{(t_4y_3 - t_3y_4)^2} & \frac{t_3t_4(z_3 - z_4)(t_3y_4z_3 - t_4y_3z_4)}{(t_4y_3 - t_3y_4)^2} \\ \frac{-(t_4z_3 - t_3z_4)^2}{(t_4y_3 - t_3y_4)^2} & \frac{2t_3t_4(z_3 - z_4)(-t_4z_3 + t_3z_4)}{(t_4y_3 - t_3y_4)^2} & \frac{-(t_3t_4^2(z_3 - z_4)^2)}{(t_4y_3 - t_3y_4)^2} \\ \frac{y_4z_3 - y_3z_4}{-t_4y_3 + t_3y_4} & \frac{t_3y_4z_3 - t_4y_3z_4}{-t_4y_3 + t_3y_4} & 0 \\ 0 & \frac{2(y_4z_3 - y_3z_4)}{-t_4y_3 + t_3y_4} & \frac{2(t_3y_4z_3 - t_4y_3z_4)}{-t_4y_3 + t_3y_4} \\ \frac{2t_4z_3 - 2t_3z_4}{t_4y_3 - t_3y_4} & \frac{2t_3t_4(-z_3 + z_4)}{-t_4y_3 + t_3y_4} & 0 \\ 0 & \frac{t_4z_3 - t_3z_4}{t_4y_3 - t_3y_4} & \frac{t_3t_4(-z_3 + z_4)}{-t_4y_3 + t_3y_4} \\ 0 & 0 & 0 \end{array} \right)$$

Next, we check, under the conditions (7), that the equation $F(A_5) = 0$ corresponds to a row which is a linear combination of the rows of A . Indeed, a computation shows that the row reduction of the matrix A augmented with the new row given by $F(A_5) = 0$ is the matrix \tilde{A} augmented with the trivial compatibility equation $0 = 0$.

In the non generic case, we proceed likewise, replacing z_4 with z_3y_4/y_3 in the matrix A . \square

7.4. Biquadratic forms in $\mathcal{BQ}_{3,3}$. Contrarily to what we have done in the previous subsections, we are not able yet to describe all the possible maximal configurations in $\mathbb{P}^2 \times \mathbb{P}^2$. We first give elementary negative results :

Proposition 7.6. *Let $\Omega \in \mathbb{P}^2 \times \mathbb{P}^2$ be such that $|\Omega| \geq 4$.*

- If there are $P_1, P_2 \in \Omega$ such that $\Pi_1(P_1) = \Pi_1(P_2)$, then Ω is not maximal.*
- If $\Pi_1(\Omega)$ contains 4 points lying on the same line, then Ω is not maximal.*
- If Ω contains 3 points P_1, P_2, P_3 such that $\Pi_1(P_1), \Pi_1(P_2), \Pi_1(P_3)$ are lying on the same line and $\Pi_2(P_1), \Pi_2(P_2), \Pi_2(P_3)$ are lying on the same line, then Ω is not maximal.*
- If Ω is maximal, then we may assume, up to a linear change of coordinates, that it contains the points $(1, 0, 0; 1, 0, 0), (0, 1, 0; 0, 1, 0), (0, 0, 1; 0, 0, 1)$.*

Proof. a) It is a direct application of Lemma 5.2.

- b) Let $\{P_1, P_2, P_3, P_4\} \subset \Omega$ be such that $\Pi_1(P_1), \Pi_1(P_2), \Pi_1(P_3), \Pi_1(P_4)$ are lying on the line of equation $z = 0$. We may assume that

$$\begin{aligned} P_1 &= (1, 0, 0; 1, 0, 0), & P_2 &= (0, 1, 0; 0, 1, 0), \\ \Pi_1(P_3) &= (x_1, 1, 0), & \Pi_1(P_4) &= (x_2, 1, 0). \end{aligned}$$

Let $F \in \mathcal{PQ}_{3,3}$ be such that $\Omega \subset \mathcal{Z}(F)$. We set $G = F(x, y, 0; s, t, u)$. Then, we have $G \in \mathcal{PQ}_{2,3}$ and $|\mathcal{Z}(G)| > 3$. Hence, $|\mathcal{Z}(G)| = \infty$ and also $|\mathcal{Z}(F)| = \infty$, which shows that Ω is not maximal.

- c) As in the previous case, we may assume that

$$P_1 = (1, 0, 0; 1, 0, 0), P_2 = (0, 1, 0; 0, 1, 0), P_3 = (x_0, 1, 0; s_0, 1, 0)$$

with $x_0s_0 \neq 0$.

Let $F \in \mathcal{P}Q_{3,3}$ be such that $\Omega \subset \mathcal{Z}(F)$. Let $G = F(x, y, 0; s, t, 0)$. We have $G \in \mathcal{P}Q_{2,2}$ and $|\mathcal{Z}(G)| > 2$. Hence, $|\mathcal{Z}(G)| = \infty$ and also $|\mathcal{Z}(F)| = \infty$, which shows that Ω is not maximal.

d) We just repetitively use the previous case. □

Next, we give one example of maximal configuration, namely the Choi set \mathcal{C} as defined in (4). Again our method relies on basic linear algebra.

Proposition 7.7. *The Choi set \mathcal{C} is maximal in $\mathbb{P}^2 \times \mathbb{P}^2$.*

Proof. Let $F \in \mathcal{P}Q_{3,3}$ be such that $\mathcal{Z}(F)$ contains the whole set \mathcal{C} and an extra point $B_4 = (x_4, y_4, z_4, s_4, t_4, u_4)$. According to Proposition 2.2, each point of $\mathcal{Z}(F)$ is a singular point of the biquadratic form F , and hence yields 5 independent linear equations whose variables are the coefficients of F . Namely, let us write

$$\begin{aligned}
 (9) \quad F &= s^2(a_0x^2 + a_1xy + a_2xz + a_3y^2 + a_4yz + a_5z^2) \\
 &+ 2st(b_0x^2 + b_1xy + b_2xz + b_3y^2 + b_4yz + b_5z^2) \\
 &+ 2su(c_0x^2 + c_1xy + c_2xz + c_3y^2 + c_4yz + c_5z^2) \\
 &+ t^2(d_0x^2 + d_1xy + d_2xz + d_3y^2 + d_4yz + d_5z^2) \\
 &+ 2tu(e_0x^2 + e_1xy + e_2xz + e_3y^2 + e_4yz + e_5z^2) \\
 &+ u^2(f_0x^2 + f_1xy + f_2xz + f_3y^2 + f_4yz + f_5z^2).
 \end{aligned}$$

In the generic case (i.e. when $s_4y_4 - t_4z_4 \neq 0$), the line reduction with the Gauss algorithm gives :

$$\left\{ \begin{array}{l}
 a_0 = a_1 = a_2 = a_4 = a_5 = 0, \\
 b_0 = b_1 = b_2 = b_3 = b_5 = 0, \\
 c_0 = c_2 = c_3 = c_4 = c_5 = 0, \\
 d_0 = d_1 = d_2 = d_3 = d_4 = 0, \\
 e_0 = e_1 = e_3 = e_4 = e_5 = 0, \\
 f_1 = f_2 = f_3 = f_4 = f_5 = 0, \\
 a_3 = \frac{f_0(u_4x_4 - t_4z_4)^2}{(s_4y_4 - t_4z_4)^2}, \\
 b_4 = -\frac{f_0(u_4x_4 - s_4y_4)(u_4x_4 - t_4z_4)}{(s_4y_4 - t_4z_4)^2}, \\
 c_1 = -\frac{f_0(u_4x_4 - t_4z_4)}{(s_4y_4 - t_4z_4)}, \\
 d_5 = \frac{f_0(u_4x_4 - s_4y_4)^2}{(s_4y_4 - t_4z_4)^2}, \\
 e_2 = \frac{f_0(u_4x_4 - s_4y_4)}{s_4y_4 - t_4z_4}.
 \end{array} \right.$$

These values of the variables correspond to the biquadratic form :

$$F = f_0 \left(\frac{(s_4 y_4 - t_4 z_4) u x + s y (-u_4 x_4 + t_4 z_4) + (u_4 x_4 - s_4 y_4) t z}{s_4 y_4 - t_4 z_4} \right)^2.$$

Since it is a square having more than 6 zeros, it necessarily has an infinite number of zeros. Then, $\mathcal{C} \cup \{B_4\}$ is not contained in a maximal set.

To end, we have to consider the non-generic case, namely when $s_4 y_4 - t_4 z_4 = 0$.

If, $y_4 \neq 0$, then by line reducing the system, we get $F = d_5 (s y - t z)^2$. Likewise, if $z_4 \neq 0$.

We have shown that \mathcal{C} is maximal. \square

One idea to show that $\text{BB}_{3,3} = 7$ (if it were true !) would be to consider $F \in \mathcal{BQ}_{3,3}$ having 8 zeros, the coefficients of F being viewed as 36 variables. The cancellation of the partial derivatives at the first three zeros which are assumed to be $(1, 0, 0; 1, 0, 0)$, $(0, 1, 0; 0, 1, 0)$, $(0, 0, 1; 0, 0, 1)$ shows that 15 variables are zeros and it remains 21. The cancellation of the partial derivatives at the other five zeros (which are written with formal parameters) gives 25 equations. And we would like to construct then an infinite family of points $(P_\lambda)_{\lambda \in \mathbb{R}}$ such that the equation $F(P_\lambda) = 0$ would be a linear combination of our previous 25 equations.

This method has been successful in the previous section for reproving that $\text{BB}_{3,2} = 3$ (Proposition 7.5) but here the computation becomes two expensive.

In relation with zeros configuration, one may also ask the following problem.

Question 7.8. Let Ω be a finite set in $\mathbb{P}^2 \times \mathbb{P}^2$. Can we determine $\mathcal{PQ}(\Omega)$ the set of all positive semi-definite biquadratic forms $F \in \mathcal{PQ}_{3,3}$ such that $\Omega \subset \mathcal{Z}(F)$?

One main interest would be to find $\mathcal{PQ}(\Omega)$ when Ω is a maximal configuration. We are able to answer the question in the case of the Choi Set :

Theorem 7.9. *With the notation of (9), all the biquadratic forms $F \in \mathcal{PQ}_{3,3}$ which are vanishing on the Choi Set \mathcal{C} are described by*

$$(10) \quad M_2(F) = \begin{pmatrix} (d_5 + 2e_2 + f_0 + f_3)y^2 + f_3z^2 & (-d_5 - e_2 - f_3)yz & (-e_2 - f_0 - f_3)xy \\ (-d_5 - e_2 - f_3)yz & f_3x^2 + d_5z^2 & e_2xz \\ (-e_2 - f_0 - f_3)xy & e_2xz & f_0x^2 + f_3y^2 \end{pmatrix}$$

with the conditions

$$(11) \quad \begin{cases} f_0 = 1 \\ 1 \geq f_3 \geq 0 \\ d_5 \geq 0 \\ d_5 - e_2^2 \geq 0 \\ -e_2^2 + d_5 - d_5 f_3 - 2e_2 f_3 - f_3^2 \geq 0 \\ -e_2^2 + d_5 - 2e_2 f_3 - f_3 - f_3^2 \geq 0 \end{cases}$$

or $d_5 = e_2 = f_0 = f_3 = 0$.

Proof. Let $F \in \mathcal{PQ}_{3,3}$ be such that $\mathcal{C} \subset \mathcal{Z}(F)$. We write F as in (9), and consider the coefficients as variables. Writing down the 35 equations given by the cancellation of all the partial derivatives at all points of \mathcal{C} yields that all the possible biquadratic forms can be described by the free parameters d_5, e_2, f_0, f_3 . More precisely, we exactly get the F 's whose associated matrix is $M_2(F)$ as given in (10).

This matrix $M_2(F)$ is positive semi-definite for all x, y, z if and only if all its symmetric minors are positive semi-definite for all x, y, z . The non-negativity of all the 1×1 and 2×2 minors gives the following conditions :

$$(12) \quad \begin{cases} f_0 \geq 0 \\ f_3 \geq 0 \\ d_5 \geq 0 \\ d_5 + 2e_2 + f_0 + f_3 \geq 0 \\ d_5 f_0 - e_2^2 \geq 0 \\ -e_2^2 + d_5 f_0 - d_5 f_3 - 2e_2 f_3 - f_3^2 \geq 0 \\ -e_2^2 + d_5 f_0 - 2e_2 f_3 - f_0 f_3 - f_3^2 \geq 0. \end{cases}$$

The non-negativity of the 3×3 minor is

$$p(x, y, z) = \det(M_2(F(x, y, z))) \geq 0.$$

We compute :

$$\begin{aligned} p(x, y, z) &= -f_3(e_2^2 - d_5 f_0 + 2e_2 f_3 + f_3 f_0 + f_3^2)x^4 y^2 + f_0 f_3^2 x^4 z^2 \\ &\quad + f_3^2(d_5 + 2e_2 + f_0 + f_3)y^4 x^2 - f_3(e_2^2 + 2e_2 f_3 + f_3^2 - d_5 f_0 + d_5 f_3)y^4 z^2 \\ &\quad + f_3(-e_2^2 + d_5 f_0)z^4 x^2 + d_5 f_3^2 z^4 y^2 \\ &\quad + f_3(3e_2^2 + 2e_2 f_3 - f_0 f_3 + f_3^2 - 3d_5 f_0 + d_5 f_3)x^2 y^2 z^2. \end{aligned}$$

Let us note that p depends only on x^2, y^2, z^2 , hence we will introduce the underlying cubic and use the following :

Lemma 7.10. *Let $c(x, y, z) = 0$ be the equation of a cubic going by the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. The equation of the cubic curve has the form*

$$\begin{cases} c(x, y, z) = \alpha_1 x^2 y + \alpha_2 x^2 z + \beta_1 y^2 x + \beta_2 y^2 z + \gamma_1 z^2 x + \gamma_2 z^2 y + \delta xyz \\ (\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta) \in \mathbb{R}^7 \end{cases}$$

Let us assume that all the coefficients except δ are non-negative. If $c(x, y, z)$ goes by $(1, 1, 1)$ and is non-negative on the following affine semi-lines (in the plane $z = 1$):

$$(x = y \geq 0), \quad (x = 1, y \geq 0) \quad (y = 1, x \geq 0),$$

then $c(x, y, z)$ is non-negative on all the positive orthant $x \geq 0, y \geq 0, z \geq 0$. Moreover, its coefficients are satisfying

$$(13) \quad \begin{cases} \alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma_1 + \gamma_2 + \delta = 0, \\ \alpha_1 + \alpha_2 = \beta_2 + \gamma_2, \\ \beta_1 + \beta_2 = \alpha_2 + \gamma_1. \end{cases}$$

Proof. We check that the positivity conditions on the semi-lines

$$(x = y \geq 0), \quad (x = 1, y \geq 0) \quad (y = 1, x \geq 0),$$

are equivalent to the cancellation of all the partial derivatives at $(1, 1, 1)$, which in turn gives the equalities (13).

To get lighter notations we write, for some $(a, b, s, t) \in \mathbb{R}_+^4$:

$$c = ax^2y + bxy^2 + (-a+s)x^2z + (-b+t)y^2z + (a-s+t)xz^2 + (b+s-t)yz^2 - (a+b+s+t)xyz.$$

Let us consider the family of affine lines $L_\lambda : y = 1 + \lambda(x - 1)$ going by $(1, 1)$. We set

$$c(x, 1 + \lambda(x - 1), 1) = (x - 1)^2 c_\lambda(x) = (x - 1)^2 (\lambda(a + b\lambda)x - (-1 + \lambda)(s + \lambda(b - t))).$$

We have to distinguish several cases :

If $\lambda < 0$, then we check that $c_\lambda(x) \geq 0$ for any $0 \leq x \leq \frac{\lambda-1}{\lambda}$. Indeed, it suffices to see that

$$c_\lambda(0) = (\lambda - 1)(\lambda(t - b) - s) \geq 0$$

and

$$c_\lambda\left(\frac{\lambda-1}{\lambda}\right) = (\lambda - 1)(\lambda t + a - s) \geq 0.$$

Likewise, if $0 \leq \lambda \leq 1$ we check that for all $x \geq 0$

$$c_\lambda(x) \geq c_\lambda(0) \geq 0.$$

And if $\lambda \geq 1$ we check that for all $x \geq \frac{\lambda-1}{\lambda}$

$$c_\lambda(x) \geq c_\lambda\left(\frac{\lambda-1}{\lambda}\right) \geq 0.$$

We have shown that all the restrictions of c to the intersection of any lines going by $(1, 1)$ and the positive orthant take non negative values. \square

The conditions (12) imply that the form $p(x, y, z)$ is associated (by the change of variables $X = x^2, Y = y^2, Z = z^2$) to a cubic satisfying the conditions of Lemma 7.10. Hence $p(x, y, z)$ is non-negative. Namely the non-negativity of the 1×1 and 2×2 minors implies the non-negativity of the determinant.

In order to get a more geometric description, we deshomogenize by setting $f_0 = 1$ (if $f_0 = 0$, then $M = 0$) and we consider the Euclidean space with coordinates (d_5, e_2, f_3) . Looking at the system of inequalities (12), we note that if $f_3 \geq 0$ and $f_5 \geq 0$, then the sixth inequality $-e_2^2 + d_5 - d_5 f_3 - 2e_2 f_3 - f_3^2 \geq 0$ implies that $1 - f_3 \geq 0$. There are two cases.

The first is $f_3 < 1$. Then, it gives $d_5 + 2e_2 + 1 + f_3 \geq (e_2 + 1)^2$ and hence the fourth inequality of (12) is obviously satisfied.

The second case is $f_3 = 1$, then $e_2 = -1$ and again the fourth inequality is obviously implied by the sixth.

Thus, system (12) is equivalent to system (11). \square

Remark 7.11. The set $\mathcal{PQ}(\mathcal{C})$ is obviously convex since its defining conditions are convex with respect to the coefficients of the biquadratic forms.

The example M_0 in subsection 6.3.2 corresponds to $(d_5, e_2, f_3) = (1, -1, 1)$, the only point which belongs to the three hypersurfaces

$$\begin{cases} d_5 - e_2^2 = 0 \\ -e_2^2 + d_5 - d_5 f_3 - 2e_2 f_3 - f_3^2 = 0 \\ -e_2^2 + d_5 - 2e_2 f_3 - f_3 - f_3^2 = 0 \end{cases}$$

which are delimiting the boundary of $\mathcal{PQ}(\mathcal{C})$.

Looking carefully at the proof of Lemma 7.10, we note that $\mathcal{Z}(F)$ will be strictly greater than \mathcal{C} only if $(t, s) = (a, 0)$ or $(s, t) = (0, b)$ or $t - s + a = 0$ or $t - s - b = 0$. In each case, we get $f_3 = 0$.

Hence, Theorem 7.9 gives a family of positive semi-definite biquadratic forms with parameters (d_5, e_2, f_0, f_3) which are not sum of squares until $f_3 = 0$. More precisely, the sum of squares in the family $\mathcal{PQ}(\mathcal{C})$ are points corresponding to $f_3 = 0$ and delimited by the parabola $d_5 \geq e_2^2$. The associated homogenized sum of squares identity is

$$F = (xu + e_2zt - (e_2 + f_0)ys)^2 + (d_5f_0 - e_2^2)(zt - ys)^2.$$

8. QUADRATIC DETERMINANTAL REPRESENTATIONS

Historically, a *determinantal representation* for a form $p(x)$ of degree k in the variables $x = (x_0, x_1, x_2)$ is an identity of the kind

$$p(x) = \lambda \det(x_0A_0 + x_1A_1 + x_2A_2)$$

where $\lambda \in \mathbb{R}_+$ and A_0, A_1, A_2 are matrices in $\mathbb{R}^{k \times k}$. The study of determinantal representations of polynomials goes back to the end of 19th century, when were studied the relation between the determinantal representations and the geometry of the curve defined by the form.

More recently, determinantal representations over the reals were considered (see [Vi]), and of particular interest in applications such as in Control theory, is the /it symmetric and *unitary* determinantal representations when the A_i 's are symmetric and A_0 is the identity matrix (see [HV]).

In this last section, we would like to introduce a discussion about what could happen if we replace the linear pencil with a quadratic one. Namely, we would like to deal with *Positive semi-definite quadratic determinantal representations*. The new problem would be to write a given form $p(x)$ of degree $2k$ as

$$p(x) = \lambda \det(M(x))$$

with $\lambda \in \mathbb{R}_+$ and

$$M(x) = x_0^2M_0 + x_0x_1M_1 + x_0x_2M_2 + x_1^2M_3 + x_2x_1M_4 + x_2^2M_5$$

where the M_i 's are symmetric matrices in $\mathbb{R}^{k \times k}$ and such that $M(x)$ is positive-semi-definite for all $x \in \mathbb{R}^3$.

Given a biquadratic form $F(x; s) \in \mathcal{BQ}_{3,3}$ we have two canonical associated matrices $\Pi_x(F)$ and $\Pi_s(F)$ which are positive semi-definite if and only if F is. Hence positive semi-definite quadratic determinantal representations are naturally related to positive semi-definite biquadratic forms.

But it seems not so easy to develop a general theory of positive semi-definite quadratic determinantal representations, the positivity condition being the most difficult to handle. Let us just discuss two examples.

Since $\text{BB}(2, 2) = 2$, we know that the form

$$p(x, y, z) = (x^2 - z^2)^2 + (y^2 - z^2)^2$$

cannot have a positive semi-definite quadratic determinantal representation, although without the positivity condition, we may write

$$p(x, y, z) = - \det \begin{pmatrix} x^2 - z^2 & y^2 - z^2 \\ y^2 - z^2 & -x^2 + z^2 \end{pmatrix}.$$

If $\text{BB}_{3,3} = 7$ (or at least if $\text{BB}_{3,3} < 10$) as conjectured in the previous section, then we would derive that the Robinson polynomial $R(x, y, z)$ (as defined in (5)) would not admit any positive semi-definite quadratic determinantal representation.

Whereas we know that the polynomial

$$S(x, y, z) = y^2x^4 + y^4z^2 + x^2z^4 - 3x^2y^2z^2$$

admits such one (Proposition 6.6).

One may also ask the same question for the celebrated Motzkin polynomial

$$M(x, y, z) = z^6 + y^2x^4 + y^4x^2 - 3x^2y^2z^2,$$

which has 6 zeros in \mathbb{P}^2 .

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