

# On the mixed problem for non strictly hyperbolic operators

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## Abstract

The classical theory of strictly hyperbolic boundary value problems has received several extensions since the 70's. One of the most noticeable is the result of Metivier establishing that Majda's "block structure condition" for constantly hyperbolic operators, which implies well-posedness for the initial boundary value problem (IBVP) with zero initial data. The well-posedness of IBVP with non zero initial requires that " $L^2$  be a continuable initial data". For strictly hyperbolic systems, this result was proven by Rauch. We prove here by using classical matrix theory that his fundamental a priori estimates are valid for constantly hyperbolic IBVP.

## 1 Introduction

In his seminal paper [4] on hyperbolic initial boundary value problems, H.O. Kreiss performed the algebraic construction of a tool, now called Kreiss symmetrizer, that leads to a priori estimates. Namely, if  $u$  is a solution of

$$\begin{cases} \partial_t u + \sum_{j=1}^d A_j(x, t) \partial_{x_j} u = f, & (t, x) \in \mathbb{R}^+ \times \Omega, \\ Bu = g, & (t, x) \in \partial\mathbb{R}^+ \times \partial\Omega, \\ u|_{t=0} = 0, \end{cases} \quad (1)$$

where the operator  $\partial_t + \sum_{j=1}^d A_j \partial_{x_j}$  is assumed to be strictly hyperbolic and  $B$  satisfies the uniform Lopatinskiï condition, there is some  $\gamma_0 > 0$  such that  $u$  satisfies the a priori estimate

$$\sqrt{\gamma} \|u\|_{L^2_\gamma(\mathbb{R}^+ \times \Omega)} + \|u\|_{L^2_\gamma(\mathbb{R}^+ \times \partial\Omega)} \leq C \left( \|f\|_{L^2_\gamma(\mathbb{R}^+ \times \Omega)} + \|g\|_{L^2_\gamma(\mathbb{R}^+ \times \partial\Omega)} \right), \quad (2)$$

for  $\gamma \geq \gamma_0$ . Here above,  $L^2_\gamma$  is the usual  $L^2$  space with a weight  $e^{-\gamma t}$ ,

$$L^2(\mathbb{R}^+ \times \mathcal{O}) = \left\{ u : \int_{\mathbb{R}^+ \times \mathcal{O}} e^{-2\gamma t} |u|^2 dx dt < \infty \right\}.$$

(J.V. Ralston [8] then extended this result to the case of complex coefficients.) J. Rauch proved that the initial boundary value problem is in fact well posed for arbitrary  $L^2$  initial data. More precisely, if  $u_0$  is the initial data, he obtained the fundamental a priori estimate

$$\begin{aligned} e^{-\gamma T} \|u(T)\|_{L^2(\Omega)} + \sqrt{\gamma} \|u\|_{L^2_\gamma([0, T] \times \Omega)} + \|u\|_{L^2_\gamma([0, T] \times \partial\Omega)} \\ \leq C \left( \|u_0\|_{L^2(\Omega)} + \|f\|_{L^2_\gamma(\mathbb{R}^+ \times \Omega)} + \|g\|_{L^2_\gamma(\mathbb{R}^+ \times \partial\Omega)} \right) \end{aligned} \quad (3)$$

for Friedrichs symmetrizable systems (in his thesis) and soon after for strictly hyperbolic systems [9]. Motivated by non strictly hyperbolic physical systems and by characteristic IBVPs, A. Majda and S. Osher [6] pointed out that the construction of Kreiss symmetrizers can be performed as soon as the system of equations satisfies the so called 'block structure condition' (see also [5]). More recently, G. Metivier [7] thoroughly investigated algebraic properties of the symbol of constantly hyperbolic operators.

**Definition 1.** Let  $L$  be a first order operator

$$L = \partial_t + \sum_{j=1}^d A_j(x, t) \partial_{x_j} , \quad (4)$$

with  $A_j : (x, t) \rightarrow A_j(x, t) \in M_r(\mathbb{C})$ .

It is said to be *constantly hyperbolic* if the symbol  $A(\eta) = \sum_{j=1}^d A_j \eta_j$  is diagonalizable with real eigenvalues, and the multiplicity of the eigenvalues remains constant for  $\eta \in \mathbb{R}^d \setminus \{0\}$ .

The main result of Metivier in [7] is that, if  $L$  is a constantly hyperbolic differential operator, then it satisfies the block structure condition.

The proof relies on a factorization of the determinant of the symbol  $\tau + \sum A_j \eta_j$  as in the Weierstrass preparation theorem. Here we will need a slightly different result proved in [1] that we shall state in the third part.

The aim of our second part is to rapidly explain the scheme of proof of Rauch's theorem. In particular we emphasize where the strict hyperbolicity assumption is necessary. In the third part we describe a modification of Rauch's proof that adapts it to constantly hyperbolic IBVPs.

## 2 Rauch's theorem

The proof of estimate (3) is rather long in [9]. It is based on an a priori estimate for strictly hyperbolic scalar equations whose proof relies on the method of Leray and Gårding (see [3]). We recall that a scalar operator  $P(t, x, \partial_t, \partial_x)$  is strictly hyperbolic (with respect to the timelike direction) if its principal symbol  $P_m(t, x, i\tau, i\eta)$  has roots in  $\tau$  real and distinct for  $\eta \in \mathbb{R}^d \setminus \{0\}$ .

**Lemma 1.** *Let  $P(t, x, \partial_t, \partial_x)$  be a scalar strictly hyperbolic differential operator of order  $k$ , with smooth coefficients constant outside a compact set. There is a constant  $C$  such that for all  $T > 0$ , any  $\phi \in H^k(J \times \Omega)$  and any  $\varepsilon > 0$  small enough*

$$\|\phi(T)\|_{H^{k-1}(\Omega)} \leq C \left( \varepsilon \|P\phi\|_{L^2([- \infty, T] \times \Omega)} + \frac{1}{\varepsilon} \|\phi\|_{H^{k-1}([- \infty, T] \times \Omega)} + \sum_{j=0}^{k-1} \|\partial_{x_d}^j \phi\|_{H^{k-1-j}([- \infty, T] \times \partial\Omega)} \right) . \quad (5)$$

The transition from scalar equations to first order systems is made thanks to the following property, which is only proved with the help of Lemma 1, Kreiss's estimates (2), and Sobolev spaces theory.

**Proposition 1.** *Let  $u$  be a solution of the boundary value problem*

$$\begin{cases} Lu = f, & (t, x) \in \mathbb{R} \times \Omega, \\ Bu = g, & (t, x) \in \mathbb{R} \times \partial\Omega. \end{cases} \quad (6)$$

*Let  $r$  be the size of the system. If  $L$  is strictly hyperbolic then for  $\gamma > 0$  large enough we have the pointwise estimate*

$$e^{-\gamma T} \|u(T)\|_{H^{r-1}(\Omega)} \leq C \left( \frac{\|f\|_{H_\gamma^{r-1}(\mathbb{R} \times \Omega)}}{\sqrt{\gamma}} + \|g\|_{H_\gamma^{r-1}(\mathbb{R} \times \partial\Omega)} \right), \quad (7)$$

*where the  $H_\gamma^m$  spaces are the spaces built over  $L_\gamma^2$  as follows*

$$H_\gamma^m(\mathbb{R}_t \times \Omega) = \{u \in L_\gamma^2 : \sum_{|\mu| \leq m} \int_{\Omega \times \mathbb{R}^+} |D^\mu u|^2 e^{-2\gamma t} dx dt < \infty\}.$$

*Sketch of proof:*

We denote by  $L^{co}$  the transposed comatrix of  $L$  seen as a matrix of differential operators. Then we have  $L^{co}L = \det(L)I + \text{lower order terms}$ . Each diagonal coefficient is a strictly hyperbolic scalar operator thus Lemma 1 may be applied to each coordinate  $u_j$ , this gives

$$\begin{aligned} e^{-\gamma T} \|u(T)\|_{H^{k-1}(\Omega)} &\lesssim \varepsilon \|L^{co}L e^{-\gamma \cdot} u\|_{L^2(\mathbb{R} \times \Omega)} + \frac{1}{\varepsilon} \|e^{-\gamma \cdot} u\|_{H^{k-1}(\mathbb{R} \times \Omega)} \\ &\quad + \sum_{j=0}^{k-1} \|e^{-\gamma \cdot} \partial_{x_d}^j u\|_{H^{k-1-j}(\mathbb{R} \times \partial\Omega)}. \end{aligned} \quad (8)$$

The trace terms  $\sum_{j=0}^{k-1} \|e^{-\gamma \cdot} \partial_{x_d}^j u\|_{H^{k-1-j}(\mathbb{R} \times \partial\Omega)}$  are estimated thanks to the identity  $\partial_{x_d} u = A_d^{-1} \left( -\partial_t u - \sum_{j=1}^{d-1} A_j \partial_{x_j} u + f \right)$ , the continuity of the trace  $H^m(\Omega) \rightarrow H^{m-1}(\partial\Omega)$ , and the analogous of (2) on  $]-\infty, T]$  (proved in [9]). Finally, using the inequality

$$e^{-\gamma T} \|u\|_{H^{k-1}(\mathbb{R} \times \Omega)} \leq \|u\|_{H_\gamma^{k-1}(\mathbb{R} \times \Omega)},$$

we obtain (7). □

The derivation of (3) from this proposition is quite cumbersome, it is based on a series of analogous inequalities involving the boundary problem as well as a dual problem. Thankfully, this part does not use the strict hyperbolicity assumption and we shall therefore not describe it.

As we see, the strict hyperbolicity assumption is only needed to apply Lemma 1 and Kreiss's estimate (2). Since the results of Métivier in [7] show that the estimate (2) is true for constantly hyperbolic boundary value problems, we are left to show how to adapt Lemma 1.

### 3 The case of constantly hyperbolic systems

Rauch's proof of Proposition 1 is not valid for non-strictly hyperbolic operator  $L$ . Even if  $L$  is a constantly hyperbolic operator, the diagonal coefficients of

$L^{co}L$  is not strictly hyperbolic, and lemma 1 does not apply, as can be seen on the trivial example of two independant transport equations

$$\partial_t u + \begin{pmatrix} \partial_x & 0 \\ 0 & \partial_x \end{pmatrix} u = 0 . \quad (9)$$

Here  $L^{co}L = \begin{pmatrix} (\partial_t + \partial_x)^2 & 0 \\ 0 & (\partial_t + \partial_x)^2 \end{pmatrix}$ . Even though (7) holds true for  $L = (\partial_t + \partial_x)I$ , it can not be deduced from the scalar equations

$$\det(L)u_j = (\partial_t + \partial_x)^2 u_j = 0, j = 1, 2 .$$

In fact, to generalize the proof of proposition 1, it suffices to find an operator  $\tilde{L}$  such that  $\tilde{L}L = PI +$  lower order terms, where  $P$  is a strictly hyperbolic operator (of course the degree of  $P$  will not be the size  $r$  of the system, except in the case of strict hyperbolicity).

We will need the proposition 1.7 (p.46) from [1] on the factorization of constantly hyperbolic operators,

**Proposition 2.** *If  $L$  is constantly hyperbolic, the determinant of the symbol  $\tau I + \sum A_j \eta_j$  factorizes as*

$$\prod_{k=1}^K P_k(\tau, \eta)^{q_k} ,$$

where the  $P_k$ 's satisfy

- Each  $P_k$  is a homogeneous polynomial of  $(\tau, \eta)$ ,
- The  $P_k$ 's are irreducible, pairwise distinct,
- For  $\eta \in \mathbb{R}^d \setminus \{0\}$ , the roots of  $P_k(\cdot, \eta)$  are real distinct,
- For  $\eta \in \mathbb{R}^d \setminus \{0\}$  and  $k \neq l$ ,  $P_k(\cdot, \eta)$  and  $P_l(\cdot, \eta)$  have no root in common.

We can now show that an  $\tilde{L}$  can indeed be found.

**Proposition 3.** *In the framework of Proposition 2 we have*

- For  $\eta \in \mathbb{R}^d \setminus \{0\}$ , the minimal polynomial of  $\sum_{j=1}^d A_j \eta_j$  is  $\prod_{k=1}^K P_k(-\tau, \eta)$ . In particular the associated operator  $\prod P_k(x, t, \partial_t, \partial_x)$  is strictly hyperbolic.
- The coefficients of the matrix

$$\tilde{L}(\tau, \eta) = \frac{L(\tau, \eta)^{co}}{\prod_{k=1}^K P_k^{q_k-1}(\tau, \eta)}$$

belong to  $\mathbb{C}[\tau, \eta]$ . Thus we can define a differential operator  $\tilde{L}(t, x, \partial_t, \partial_x)$  that satisfies  $\tilde{L}L = \prod_{k=1}^K P_k(t, x, \partial_t, \partial_x)I_r +$  lower order terms. In particular, the diagonal coefficients are strictly hyperbolic differential operators.

*Proof.* Since  $\prod_{k=1}^K P_k^{q_k}(-\tau, \eta)$  is the characteristic polynomial of  $\sum A_j \eta_j$ , Proposition 2 and the diagonalizability of  $\sum A_j \eta_j$  immediately imply that the polynomial  $\prod_{k=1}^K P_k(-\tau, \eta)$  is the minimal polynomial of  $\sum A_j \eta_j$  (recall that a matrix is diagonalizable over  $\mathbb{C}$  if and only if its minimal polynomial has no multiple

roots). Since the roots of  $P_k(\cdot, \xi)$  are real, simple, and the  $P_k$ 's have no root in common, we have the strict hyperbolicity of  $\prod_{k=1}^K P_k$ .

We now consider  $L(\tau, \eta) = \tau + \sum A_j \eta_j$  as a matrix with coefficients in  $\mathbb{C}(\eta)[\tau]$ , the ring of polynomial in  $\tau$  over the field  $\mathbb{C}(\eta)$ . In order to simplify the notation, we do not write their dependence in  $(t, x)$ . Since  $\mathbb{C}(\eta)[\tau]$  is a principal ring, we can define for  $0 \leq k \leq r$  (where  $r$  is the size of the system)  $D_k$  the gcd of the minors of  $L$  of order  $k$ . Note that  $D_r = \det(L)$  is to be seen as the minor of order  $r$ . In particular, if  $r$  is the size of the system,  $D_{r-1}$  divides in  $\mathbb{C}(\eta)[\tau]$  each coefficient of  $L^{co}$ .

Now according to the theory of elementary divisors (see for example Gantmächer [2] chapter VI section 3 or Serre [10] chapter 6 'invariant factors'),  $D_{r-1} | D_r$  and more precisely  $\tau \rightarrow \frac{D_r}{D_{r-1}}(-\tau, \eta)$  is the minimal polynomial of  $\sum A_j \eta_j$ . Therefore

$$\frac{D_r}{D_{r-1}} = \prod_{k=1}^K P_k, \text{ which implies that } D_{r-1} = \prod_{k=1}^K P_k^{q_k-1}. \quad (10)$$

By definition, the coefficients of  $L^{co}$  are up to the sign the minors of  $L$  of order  $k-1$ . Thus each coefficient of  $\tilde{L}(\tau, \eta) = \frac{L(\tau, \eta)^{co}}{\prod P_k^{q_k-1}}$  belongs to  $\mathbb{C}(\eta)[\tau]$ . It remains to prove that they are in fact in  $\mathbb{C}[\eta][\tau]$ .

Let  $\tilde{l}$  be any coefficient of  $\tilde{L}$ ,

$$\tilde{l} = \sum r_j(\eta) \tau^j = \frac{l}{\prod_{k=1}^K P_k^{q_k-1}}, \quad l \in \mathbb{C}[\tau, \eta], \quad r_j \in \mathbb{C}(\eta).$$

Let  $q$  be the lcm of the denominators of the  $r_j$ 's. Then we have

$$\tilde{l} = \frac{l_1}{q}, \text{ with } l_1 \in \mathbb{C}[\eta][\tau], \quad q \in \mathbb{C}[\eta].$$

For  $Q$  in  $\mathbb{C}[\eta][\tau]$  (polynomial in  $\tau$  with coefficients in the factorial ring  $\mathbb{C}[\eta]$ ), we denote by  $c(Q) \in \mathbb{C}[\eta]$  the gcd of its coefficients. According to Gauss's lemma we have

$$c(Q_1 Q_2) = c(Q_1) c(Q_2).$$

Since the degree of  $P_j$  is the same as the degree of  $P_j$  as a polynomial in  $\tau$ ,  $c(P_j) = 1$ , we get  $c(l_1) c(\prod_{k=1}^K P_k^{q_k-1}) = c(l) c(q)$ , hence

$$c(l_1) = c(l) q, \quad (11)$$

thus  $q | c(l_1)$ . However, by construction we have  $\gcd(q, c(l_1)) = 1$ . This implies that  $q = 1$ .

Finally,  $\tilde{l} = q \tilde{l} = l_1 \in \mathbb{C}[\eta, \tau]$  is a polynomial, the matricial differential operator  $\tilde{L}$  is well defined, and  $\tilde{L}(\tau, \eta) L(\tau, \eta) = \prod_{k=1}^K P_k(\tau, \eta)$  implies that

$$\tilde{L}(\partial_t, \partial_x) L(\partial_t, \partial_x) = \prod_{k=1}^K P_k(\partial_t, \partial_x) L_r + \text{differential operators of degree } < r,$$

(we recall that the lower order terms come from the fact that we work on variable coefficients operators).  $\square$

Using Proposition 3 we obtain that (7) is valid for constantly hyperbolic IBVP's, and, according to the procedure in parts 3 and 4 of [9],  $L^2$  is a continuable initial data for constantly hyperbolic IBVPs.

## References

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