

COMPOSITION OPERATORS ON WEIGHTED BERGMAN-ORLICZ SPACES ON THE BALL

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ABSTRACT. We give embedding theorems for weighted Bergman-Orlicz spaces on the ball and then apply our results to the study of the boundedness and the compactness of composition operators in this context. As one of the motivations of this work, we show that there exist some weighted Bergman-Orlicz spaces, different from H^∞ , on which every composition operator is bounded.

1. INTRODUCTION AND PRELIMINARIES

1.1. Introduction. Let \mathbb{B}_N denote the unit ball in \mathbb{C}^N and ϕ an analytic map from \mathbb{B}_N into itself. In this paper, we are interested in characterizing the continuity and the compactness of composition operators C_ϕ , defined by $C_\phi(f) = f \circ \phi$, on weighted Bergman-Orlicz spaces. On the classical weighted Bergman spaces $A_\alpha^p(\mathbb{B}_N)$, or on the Hardy spaces $H^p(\mathbb{B}_N)$ as well, the boundedness or the compactness of C_ϕ can be characterized in terms of Carleson measures (see e.g. [1]). In one variable, the Littlewood subordination principle is known to be the main tool to show that composition operators are always bounded on these spaces, whereas B. MacCluer and J. Shapiro exhibited self-maps ϕ on \mathbb{B}_N ($N > 1$) inducing non-bounded composition operators on $A_\alpha^p(\mathbb{B}_N)$ or on $H^p(\mathbb{B}_N)$. As for the compactness, the same authors gave an example of a surjective analytic self-map of \mathbb{D} defining a compact composition operators on these spaces ([9]). In comparison, it is easy to check that every C_ϕ is bounded on H^∞ and is compact if and only if $\|\phi\|_\infty < 1$, whatever $N \geq 1$. This arises the question: what is the behavior of composition operators on significant spaces between H^∞ and $A_\alpha^p(\mathbb{B}_N)$ (or $H^p(\mathbb{B}_N)$)?

This question motivated P. Lefèvre, D. Li, H. Queffélec and L. Rodríguez-Piazza to start, since 2006, a systematic study of composition operators on Bergman-Orlicz spaces $A^\psi(\mathbb{D})$ and Hardy-Orlicz spaces $H^p(\mathbb{D})$ on the unit disk of \mathbb{C} (e.g. [6, 7, 4, 5]). Indeed, these spaces reveals to be a satisfying intermediate scale of spaces between H^∞ and the classical Bergman or Hardy spaces, depending on the growth of the Orlicz function ψ . As a part of their work, they gave an analytic surjective self-map $\phi : \mathbb{D} \rightarrow \mathbb{D}$ such that C_ϕ is compact on $H^p(\mathbb{D})$, extending the preceding result by MacCluer and Shapiro. They partially solved the same problem in the context of Bergman-Orlicz spaces, by underlying the fact that the compactness of C_ϕ on some Hardy-Orlicz spaces implies the compactness of C_ϕ on the correspondant Bergman-Orlicz spaces. By the way, they prove that it is unlikely to find Orlicz functions ψ such that compactness of composition operators on $A^\psi(\mathbb{D})$ (and definitely on $H^p(\mathbb{D})$) should be equivalent to that on H^∞ .

Yet, looking at the several variables setting, the same kind of question arises, but now even for continuity, since there exists symbol ϕ such that C_ϕ is not bounded on the classical Bergman spaces $A^p(\mathbb{B}_N)$, although every C_ϕ is bounded on H^∞ . The purpose of this paper is to investigate this problem for weighted Bergman-Orlicz spaces, that is to answer the question: does there exist some Orlicz function ψ such that every composition operator is bounded on the weighted Bergman-Orlicz space $A_\alpha^\psi(\mathbb{B}_N)$? To do this, we need to characterize boundedness of composition operators on Bergman-Orlicz spaces, in a general enough fashion. By passing, we give a characterization of the compactness of C_ϕ on $A_\alpha^\psi(\mathbb{B}_N)$, which may arise new questions and provide eventually a better understanding of the behavior of composition operators on these spaces.

We have to mention that, in 2010, Z. J. Jiang gave embedding theorems and characterizations of the boundedness and the compactness of composition operators on Bergman-Orlicz spaces $A^\psi(\mathbb{B}_N)$

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when ψ satisfies the so-called Δ_2 -Condition ([2]). This condition somehow implies that the space $A^\psi(\mathbb{B}_N)$ is “closed” to a classical Bergman space and, as we could guess, these characterizations are the same than that known for Bergman spaces; their applications to composition operators do not provide different results from that obtained in the classical framework; especially, they give no information for “small” Bergman-Orlicz spaces, in which we are especially interesting in.

This paper is organized as follows: after introducing the notions and materials in Section 1, we give, in section 2, general embedding theorems for weighted Bergman-Orlicz spaces. Precisely, given two arbitrary Orlicz functions ψ_1 and ψ_2 , we exhibit in Theorem 2.5 and Theorem 2.9 necessary and sufficient conditions on a measure μ on the ball under which the canonical embedding $A_\alpha^{\psi_1}(\mathbb{B}_N) \hookrightarrow L^{\psi_2}(\mu)$ holds or is compact. In general, we do not get characterizations, yet we see that we do when $\psi_1 = \psi_2$ satisfies some convenient regular conditions. In Section 3, applications are given to composition operators and, as a consequence, we exhibit a class of Orlicz functions defining weighted Bergman-Orlicz spaces on which every composition operator is bounded.

1.2. Orlicz spaces - Notations.

1.2.1. *Definitions.* In this whole paper, we denote by $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ an Orlicz function, i.e. a strictly convex function vanishing at 0, continuous at 0 and satisfying

$$\frac{\psi(x)}{x} \xrightarrow{x \rightarrow \infty} +\infty.$$

Note that an Orlicz function is non-decreasing. Considering a probability space (Ω, \mathbb{P}) , we define the Orlicz space $L^\psi(\Omega)$ as the space of all (equivalence classes of) measurable complex functions f on Ω for which there is a constant $C > 0$ such that

$$\int_\Omega \psi\left(\frac{|f|}{C}\right) d\mathbb{P} < \infty.$$

This space may be normalized by the Luxemburg norm

$$\|f\|_\psi = \inf \left\{ C > 0, \int_\Omega \psi\left(\frac{|f|}{C}\right) d\mathbb{P} \leq 1 \right\},$$

which makes $(L^\psi(\Omega), \|\cdot\|_\psi)$ a Banach space such that $L^\infty(\Omega) \subset L^\psi(\Omega) \subset L^1(\Omega)$. Observe that if $\psi(x) = x^p$ for every x , then $L^\psi(\Omega) = L^p(\Omega)$. It is usual to introduce the Morse-Transue space $M^\psi(\Omega)$, which is the subspace of $L^\psi(\Omega)$ generated by $L^\infty(\Omega)$.

To every Orlicz function ψ , we shall associate its complementary function $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ defined by

$$\Phi(y) = \sup_{x \in \mathbb{R}_+} \{xy - \psi(x)\}.$$

We may verify that Φ is also an Orlicz function (see [10], Section 1.3). If both $L^\Phi(\Omega)$ and $L^\psi(\Omega)$ are normed by the Luxemburg norm, then $L^\Phi(\Omega)$ is isomorphic to the dual of $M^\psi(\Omega)$ ([10, IV, 4.1, Theorem 7]).

1.2.2. *Three classes of Orlicz functions.* We now introduce essentially three classes of Orlicz functions which will appear several times in this paper. This part may appear a little bit technical, but we would like to convince the reader that this classification, which permits to get a meaningful scale of Orlicz spaces between L^∞ and L^p , is quite natural.

- The first class is that of Orlicz functions which satisfy the so-called Δ_2 -Condition which is a condition of moderate growth.

Definition 1.1. Let ψ be an Orlicz function. We say that ψ satisfies the Δ_2 -Condition if there exist $x_0 > 0$ and a constant $K > 1$, such that

$$\psi(2x) \leq K\psi(x)$$

for any $x \geq x_0$.

For example, $x \mapsto ax^p(1 + b \log(x))$, $p > 1$, $a > 0$ and $b \geq 0$, satisfies the Δ_2 -Condition. Corollary 5, Chapter II of [10] gives:

Proposition 1.2. *Let ψ be an Orlicz function satisfying the Δ_2 -Condition, then there are some $p > 1$ and $C > 0$ such that $\psi(x) \leq Cx^p$, for x large enough. Therefore, $L^p \subset L^\psi \subset L^1$, for some $p > 1$.*

- The two following conditions are also regular conditions which are satisfied by most of the Orlicz functions that we are interesting in.

Definition 1.3. Let ψ be an Orlicz function. We say that ψ satisfies the ∇_0 -Condition if there exist some $x_0 > 0$ and some constant $C \geq 1$, such that for every $x_0 \leq x \leq y$ we have

$$\frac{\psi(2x)}{\psi(x)} \leq \frac{\psi(2Cy)}{\psi(y)}.$$

We refer to Proposition 4.6 of [6] to verify that we have the following:

Proposition 1.4. *Let ψ be an Orlicz function. Then ψ satisfies the ∇_0 -Condition if and only if there exists $x_0 > 0$ such that for every (or equivalently one) $\beta > 1$, there exists a constant $C_\beta \geq 1$ such that*

$$\frac{\psi(\beta x)}{\psi(x)} \leq \frac{\psi(\beta C_\beta y)}{\psi(y)}$$

for every $x_0 \leq x \leq y$.

Furthermore, the following class will be of interest for us: ψ satisfies the *uniform* ∇_0 -Condition if it satisfies the ∇_0 -Condition for a constant $C_\beta \geq 1$ independent of $\beta > 1$.

- Finally, one defines a class of Orlicz functions which grow fast:

Definition 1.5. Let ψ be an Orlicz function. ψ satisfies the Δ^2 -Condition if and only if there exist $x_0 > 0$ and a constant $C > 0$, such that

$$\psi(x)^2 \leq \psi(Cx),$$

for every $x \geq x_0$.

The convexity and the non-decrease of Orlicz functions give the following proposition, whose content can be found in [10, Chapter II, Paragraph 2.5, pages 40 and further] or in [3, Chapter I, Section 6, Paragraph 5]:

Proposition 1.6. *Let ψ be an Orlicz function. The assertions:*

- (1) ψ satisfies the Δ^2 -Condition;
 - (2) There exist $b > 1$, $C > 0$ and $x_0 > 0$ such that $\psi(x)^b \leq \psi(Cx)$, for every $x \geq x_0$;
 - (3) For every $b > 1$, there exist $C_b > 0$ and $x_{0,b} > 0$ such that $\psi(x)^b \leq \psi(C_b x)$, for every $x \geq x_{0,b}$.
- are equivalent.

The next proposition ([10, Chapter II, Paragraph 2, Proposition 6]) shows that an Orlicz function which satisfies the Δ^2 -Condition need to have at least an exponential growth.

Proposition 1.7. *Let ψ be an Orlicz function which satisfies the Δ^2 -Condition. There exist $a > 0$ and $x_0 > 0$ such that*

$$\psi(x) \geq e^{ax},$$

for every $x \geq x_0$.

If ψ satisfies Δ^2 -Condition, we shall say that $L^\psi(\Omega)$ is a “small” Orlicz space, i.e. “far” from any $L^p(\Omega)$ and “close” to L^∞ .

To finish, we recall Proposition 4.7 (2) of [6]:

Proposition 1.8. *Let ψ be an Orlicz function. If ψ satisfies the Δ^2 -Condition, then it satisfies the uniform ∇_0 -Condition.*

Let us notice that for any $1 < p < \infty$, every function $x \mapsto x^p$ is an Orlicz function which satisfies the uniform ∇_0 -Condition, and then the ∇_0 -condition. It also satisfies the Δ_2 -Condition. Furthermore, for any $a > 0$ and $b \geq 1$, $x \mapsto e^{ax^b} - 1$ belongs to the Δ^2 -Class (and then to the uniform ∇_0 -Class), yet not to the Δ_2 -one. In addition, the Orlicz functions which can be written $x \rightarrow \exp\left(a(\ln(x+1))^b\right) - 1$ for $a > 0$ and $b \geq 1$, satisfy the ∇_0 -Condition, but do not belong to the Δ^2 -Class.

For a complete study of Orlicz spaces, we refer to [3] and to [10]. We can also find precise information in context of composition operators, such as other classes of Orlicz functions and their link together with, in [6].

1.3. Weighted Bergman-Orlicz spaces on \mathbb{B}_N . Let $\alpha > -1$ and let dv_α be the normalized weighted Lebesgue measure on \mathbb{B}_N

$$dv_\alpha(z) = c_\alpha (1 - |z|^2)^\alpha dv(z),$$

where dv is the normalized volume Lebesgue measure on \mathbb{B}_N . The constant c_α is equal to

$$c_\alpha = \frac{\Gamma(n + \alpha + 1)}{n! \Gamma(\alpha + 1)}.$$

With the notations of the previous subsection, if $(\Omega, \mathbb{P}) = (\mathbb{B}_N, dv_\alpha)$, then the weighted Bergman-Orlicz space $A_\alpha^\psi(\mathbb{B}_N)$ on the ball is $H(\mathbb{B}_N) \cap L_\alpha^\psi(\mathbb{B}_N)$, where $H(\mathbb{B}_N)$ is the space of holomorphic functions on \mathbb{B}_N , and where the subscript α remains that the probabilistic measure is the weighted normalized measure dv_α on \mathbb{B}_N . We have $A_\alpha^\psi(\mathbb{B}_N) \subset A_\alpha^1(\mathbb{B}_N)$ and it is classical to check that, if $A_\alpha^\psi(\mathbb{B}_N)$ is endowed with the Luxemburg norm $\|\cdot\|_\psi$, then it is a Banach space.

For $a \in \mathbb{B}_N$, we denote by δ_a the point evaluation functional at a . The following proposition infers that δ_a is bounded on every $A_\alpha^\psi(\mathbb{B}_N)$.

Proposition 1.9. *Let $\alpha > -1$ and let ψ be an Orlicz function. Let also $a \in \mathbb{B}_N$. Then the point evaluation functional δ_a at a is bounded on $A_\alpha^\psi(\mathbb{B}_N)$; more precisely, we have*

$$\frac{1}{4^{N+1+\alpha}} \psi^{-1} \left(\left(\frac{1+|a|}{1-|a|} \right)^{N+1+\alpha} \right) \leq \|\delta_a\| \leq \psi^{-1} \left(\left(\frac{1+|a|}{1-|a|} \right)^{N+1+\alpha} \right).$$

Proof. We denote by H_a the Berezin kernel at a , defined by

$$H_a(z) = \left(\frac{1 - |a|^2}{|1 - \langle z, a \rangle|^2} \right)^{N+1+\alpha}, \quad z \in \mathbb{B}_N.$$

It is not hard to check -and well-known- that $\|H_a\|_\infty = \left(\frac{1+|a|}{1-|a|} \right)^{N+1+\alpha}$ and that $\|H_a\|_{L^1} = 1$. Let φ_a be an automorphism of \mathbb{B}_N such that $\varphi_a(0) = a$. Fix $f \in A_\alpha^\psi(\mathbb{B}_N)$ and set $C = \|f\|_{A_\alpha^\psi}$. By the change of variables formula (e.g. [12], Proposition 1.13), and using the subharmonicity of $\psi\left(\frac{|f \circ \varphi_a|}{C}\right)$, we get

$$\psi\left(\frac{|f(a)|}{C}\right) \leq \int_{\mathbb{B}_N} \psi\left(\frac{|f \circ \varphi_a|}{C}\right) dv_\alpha = \int_{\mathbb{B}_N} \psi\left(\frac{|f(z)|}{C}\right) H_a(z) dv_\alpha(z).$$

Since ψ^{-1} is non-decreasing, we obtain

$$|f(a)| \leq C \psi^{-1} \left(\left(\frac{1+|a|}{1-|a|} \right)^{N+1+\alpha} \right),$$

hence the intended upper estimate.

Conversely, we compute $\delta_a(H_a)$. It gives

$$\begin{aligned}
 \|\delta_a\| &\geq \frac{|H_a(a)|}{\|H_a\|_{A_\alpha^\psi}} \\
 &\geq \frac{1}{(1-|a|^2)^{N+1+\alpha}} \frac{\psi^{-1}(\|H_a\|_\infty)}{\|H_a\|_\infty} \quad (\text{by [6, Lemma 3.9]}) \\
 (1.1) \quad &\geq \frac{1}{4^{N+1+\alpha}} \psi^{-1} \left(\left(\frac{1+|a|}{1-|a|} \right)^{N+1+\alpha} \right).
 \end{aligned}$$

□

2. EMBEDDING THEOREMS FOR BERGMAN-ORLICZ SPACES

We will need a version of Carleson's theorem for Bergman spaces slightly different from the traditional one. This is inspired from [5]. Anyway, as for the study of continuity and compactness of composition operators on Bergman spaces or Hardy spaces of the ball in terms of Carleson measure, we will need to introduce the objects and notions involved. We first recall the definition of the non-isotropic distance on the sphere \mathbb{S}_N , which we denote by d . For $(\zeta, \xi) \in \mathbb{S}_N^2$, it is given by

$$d(\zeta, \xi) = \sqrt{|1 - \langle \zeta, \xi \rangle|}.$$

We may verify that the map d is a distance on \mathbb{S}_N and can be extended to $\overline{\mathbb{B}_N}$, where it still satisfies the triangle inequality. For $\zeta \in \overline{\mathbb{B}_N}$ and $h \in]0, 1]$, we define the non-isotropic "ball" of \mathbb{B}_N by

$$S(\zeta, h) = \left\{ z \in \mathbb{B}_N, d(\zeta, z)^2 < h \right\}.$$

and its analogue in $\overline{\mathbb{B}_N}$ by

$$\mathcal{S}(\zeta, h) = \left\{ z \in \overline{\mathbb{B}_N}, d(\zeta, z)^2 < h \right\}.$$

Let us also denote by

$$Q = \mathcal{S}(\zeta, h) \cap \mathbb{S}_N$$

the "true" balls in \mathbb{S}_N . Next, for $\zeta \in \mathbb{S}_N$ and $h \in]0, 1]$, we define

$$W(\zeta, h) = \left\{ z \in \mathbb{B}_N, 1 - |z| < h, \frac{z}{|z|} \in Q(\zeta, h) \right\}.$$

$W(\zeta, h)$ is called a Carleson window.

We introduce the two following functions ϱ_μ and $K_{\mu, \alpha}$:

$$\varrho_\mu(h) = \sup_{\xi \in \mathbb{S}_N} \mu(W(\xi, h))$$

where μ is positive Borel measure on \mathbb{B}_N . We now set

$$K_{\mu, \alpha}(h) = \sup_{0 < t \leq h} \frac{\varrho_\mu(t)}{t^{N+1+\alpha}}.$$

μ is said to be an α -Bergman-Carleson measure if $K_{\mu, \alpha}$ is bounded. As

$$(2.1) \quad t^{N+1+\alpha} \sim v_\alpha(W(\xi, t))$$

for every $\xi \in \mathbb{S}_N$, this is equivalent to the existence of a constant $C > 0$ such that

$$\mu(W(\xi, h)) \leq C v_\alpha(W(\xi, h))$$

for any $\xi \in \mathbb{S}_N$ and any $h \in (0, 1)$ (or equivalently any $h \in (0, h_A)$ for some $0 < h_A \leq 1$). Let us remark that, in the definition of ϱ_μ and $K_{\mu, \alpha}$, we may have taken $S(\xi, h)$ instead of $W(\xi, h)$, since these two sets are equivalent in the sense that there exist two constants $C_1 > 0$ and $C_2 > 0$ such that

$$S(\xi, C_1 h) \subset W(\xi, h) \subset S(\xi, C_2 h).$$

Next we may work indifferently with non-isotropic balls or Carleson windows if there is no possible confusion.

We have the following covering lemma which will be useful for our version of Carleson's theorem:

Lemma 2.1. *There exists an integer $M > 0$ such that for any $0 < r < 1$, we can find a finite sequence $\{\xi_k\}_{k=1}^m$ (m depending on r) in \mathbb{S}_N with the following properties:*

- (1) $\mathbb{S}_N = \bigcup_k Q(\xi_k, r)$.
- (2) The sets $Q(\xi_k, r/4)$ are mutually disjoint.
- (3) Each point of \mathbb{S}_N belongs to at most M of the sets $Q(\xi_k, 4r)$.

Proof. The proof, using a variant of [12, Lemma 2.22] for the non-isotropic distance at the boundary is quite identical to that of [12, Theorem 2.23]. The fact that we can take a finite union follows from a compactness argument. \square

From now on, M will always stand for the constant involved in Lemma 2.1. We will now define a maximal operator associated to a covering of the ball with convenient subsets. Let $n \geq 0$ be an integer and denote by C_n the corona

$$C_n = \left\{ z \in \mathbb{B}_N, 1 - \frac{1}{2^n} \leq |z| < 1 - \frac{1}{2^{n+1}} \right\}.$$

For any $n \geq 0$, let $(\xi_{n,k})_k \subset \mathbb{S}_N$ be given by Lemma 2.1 putting $r = \frac{1}{2^n}$. For $k \geq 0$, we set

$$T_{0,k} = \left\{ z \in \mathbb{B}_N \setminus \{0\}, \frac{z}{|z|} \in Q(\xi_{0,k}, 1) \right\} \cup \{0\}.$$

Then let us define the sets $T_{n,k}$, for $n \geq 1$ and $k \geq 0$, by

$$T_{n,k} = \left\{ z \in \mathbb{B}_N \setminus \{0\}, \frac{z}{|z|} \in Q\left(\xi_{n,k}, \frac{1}{2^n}\right) \right\}.$$

We have both

$$\bigcup_{n \geq 0} C_n = \mathbb{B}_N$$

and

$$\bigcup_{k \geq 0} T_{0,k} = \mathbb{B}_N \text{ and } \bigcup_{k \geq 0} T_{n,k} = \mathbb{B}_N \setminus \{0\}, n \geq 1.$$

For $(n, k) \in \mathbb{N}^2$, we finally define the subset $\Delta_{(n,k)}$ of \mathbb{B}_N by

$$\Delta_{(n,k)} = C_n \cap T_{n,k}.$$

We have

$$\begin{aligned} \Delta_{(0,k)} &= (W(\xi_{0,k}, 1) \cap C_0) \cup \{0\}; \\ \Delta_{(n,k)} &= W\left(\xi_{n,k}, \frac{1}{2^n}\right) \cap C_n, n \geq 1. \end{aligned}$$

By construction, the $\Delta_{(n,k)}$'s satisfy the following properties:

$$(1) \bigcup_{(n,k) \in \mathbb{N}^2} \Delta_{(n,k)} = \mathbb{B}_N.$$

- (2) For every (n, k) , $\Delta_{(n,k)}$ is a subset of the closed Carleson window $\overline{W\left(\xi_{n,k}, \frac{1}{2^n}\right)}$ and by construction, we can find a constant $\tilde{C} > 0$, independent of (n, k) such that

$$v_\alpha \left(W\left(\xi_{n,k}, \frac{1}{2^n}\right) \right) \leq \tilde{C} v_\alpha(\Delta_{(n,k)}).$$

(3) Given $0 < \varepsilon < 1/2$, if C_n^ε denotes the corona defined by

$$C_n^\varepsilon = \left\{ z \in \mathbb{B}_N, (1 + \varepsilon) \left(1 - \frac{1}{2^n} \right) \leq |z| < (1 + \varepsilon) \left(1 - \frac{1}{2^{n+1}} \right) \right\},$$

then each point of \mathbb{B}_N belongs to at most M of the sets $\Delta_{(n,k)}^\varepsilon$'s defined by

$$\begin{aligned} \Delta_{(0,k)}^\varepsilon &= (W(\xi_{0,k}, 1 + \varepsilon) \cap C_0^\varepsilon) \cup \{0\}; \\ \Delta_{(n,k)}^\varepsilon &= W\left(\xi_{n,k}, (1 + \varepsilon) \frac{1}{2^n}\right) \cap C_n^\varepsilon, \quad n \geq 1. \end{aligned}$$

This comes from the construction and the previous covering lemma. In particular, we have

$$\sum_{(n,k) \in \mathbb{N}^2} v_\alpha(\Delta_{(n,k)}^\varepsilon) \leq M v_\alpha(\mathbb{B}_N) = M.$$

For any $f \in A_\alpha^\psi(\mathbb{B}_N)$, we define the following maximal function Λ_f :

$$(2.2) \quad \Lambda_f = \sum_{n,k \geq 0} \sup_{\Delta_{(n,k)}} (|f(z)|) \chi_{\Delta_{(n,k)}}$$

where $\chi_{\Delta_{(n,k)}}$ is the characteristic function of $\Delta_{(n,k)}$. The next proposition says that the maximal operator $\Lambda : f \mapsto \Lambda_f$ is bounded from $A_\alpha^\psi(\mathbb{B}_N)$ to $L_\alpha^\psi(\mathbb{B}_N, v_\alpha)$.

Proposition 2.2. *Let ψ be an Orlicz function and let $\alpha > -1$. Then the maximal operator Λ is bounded from $A_\alpha^\psi(\mathbb{B}_N)$ to $L_\alpha^\psi(\mathbb{B}_N, v_\alpha)$. More precisely there exists $B \geq 1$ such that for every $f \in A_\alpha^\psi(\mathbb{B}_N)$, we have*

$$\|\Lambda_f\|_{L_\alpha^\psi} \leq 2B \|f\|_{A_\alpha^\psi}.$$

Proof. Fix $f \in A_\alpha^\psi(\mathbb{B}_N)$ and set $C = \|f\|_{A_\alpha^\psi}$. We denote by $c_{(n,k)} = \sup_{\Delta_{(n,k)}} (|f|)$ and let $\tau_{(n,k)} \in \Delta_{(n,k)}$ be such that $|f(\tau_{(n,k)})| \geq \frac{c_{(n,k)}}{2}$. Since $\frac{\psi \circ |f|}{C}$ is subharmonic, and by a usual refined submean property, we have

$$\begin{aligned} \int_{\mathbb{B}_N} \psi\left(\frac{\Lambda_f}{2C}\right) dv_\alpha &\leq \sum_{n,k \geq 0} \psi\left(\frac{|f(\tau_{(n,k)})|}{C}\right) v_\alpha(\Delta_{(n,k)}) \\ &\leq \sum_{n,k \geq 0} \frac{v_\alpha(\Delta_{(n,k)})}{v_\alpha(\Delta_{(n,k)}^\varepsilon)} \int_{\Delta_{(n,k)}^\varepsilon} \psi\left(\frac{|f|}{C}\right) dv_\alpha. \end{aligned}$$

A classical computation shows that

$$\frac{v_\alpha(\Delta_{(n,k)})}{v_\alpha(\Delta_{(n,k)}^\varepsilon)} \leq D_\varepsilon,$$

where D_ε is a positive constant which only depends on ε . Therefore we get,

$$\int_{\mathbb{B}_N} \psi\left(\frac{\Lambda_f}{2C}\right) dv_\alpha \leq D_\varepsilon \sum_{n,k \geq 0} \int_{\Delta_{(n,k)}^\varepsilon} \psi\left(\frac{|f|}{C}\right) dv_\alpha.$$

Now, we have $C_n^\varepsilon = \cup_{k \geq 0} \Delta_{(n,k)}^\varepsilon$ and, by construction of the $\Delta_{(n,k)}$'s, for every n , each point of C_n^ε belongs to at most M of the sets $\Delta_{(n,k)}^\varepsilon$. Then, for n fixed,

$$\sum_{k \geq 0} \int_{\Delta_{(n,k)}^\varepsilon} \psi\left(\frac{|f|}{C}\right) dv_\alpha \leq M \int_{C_n^\varepsilon} \psi\left(\frac{|f|}{C}\right) dv_\alpha.$$

Next, we of course have $\mathbb{B}_N \subset \cup_{n \geq 0} C_n^\varepsilon$ and each point of \mathbb{B}_N belongs to at most 3 of the C_n^ε 's. It follows that

$$\begin{aligned} \int_{\mathbb{B}_N} \psi \left(\frac{\Lambda_f}{2C} \right) dv_\alpha &\leq D_\varepsilon M \sum_{n \geq 0} \int_{C_n^\varepsilon} \psi \left(\frac{|f|}{C} \right) dv_\alpha \\ &\leq B \int_{\mathbb{B}_N} \psi \left(\frac{|f|}{C} \right) dv_\alpha \end{aligned}$$

for some constant $B \geq 1$. Now, by convexity, we get

$$\int_{\mathbb{B}_N} \psi \left(\frac{\Lambda_f}{2BC} \right) dv_\alpha \leq 1,$$

hence $\|\Lambda_f\|_{L_\alpha^\psi} \leq 2B \|f\|_{A_\alpha^\psi}$. □

We state our version of Carleson's theorem as follows:

Theorem 2.3. *There exists a constant $\tilde{C} > 0$ such that, for every $f \in A_\alpha^1(\mathbb{B}_N)$ and every positive finite Borel measure μ on \mathbb{B}_N , we have*

$$\mu(\{z \in \mathbb{B}_N, |z| > 1 - h \text{ and } |f(z)| > t\}) \leq \tilde{C} K_{\mu, \alpha}(2h) v_\alpha(\{\Lambda_f > t\})$$

for every $h \in (0, 1/2)$ and every $t > 0$.

Proof. The proof is quite identical to that of [5, Lemma 2.3]. Anyway, we prefer to give the details. Fix $0 < h < 1$ and $t > 0$. We identify $i \in \mathbb{N}$ and $(n, k) \in \mathbb{N}^2$ thanks to an arbitrary bijection from \mathbb{N}^2 onto \mathbb{N} . We will write $i \longleftrightarrow (n, k)$ without possible confusion. Define

$$I = \left\{ i \longleftrightarrow (n, k), \sup_{\Delta_i} |f| > t \right\}$$

and

$$I_h = \left\{ i \longleftrightarrow (n, k), h > \frac{1}{2^{n+1}} \text{ and } \sup_{\Delta_i} |f| > t \right\}.$$

Denoting by W_i the smallest Carleson window containing Δ_i , by the three properties of the Δ_i 's listed above, we can find some constants $C > 0$ and $\tilde{C} > 0$ such that

$$\begin{aligned} \mu(\{z \in \mathbb{B}_N, |z| > 1 - h \text{ and } |f(z)| > t\}) &\leq \sum_{i \in I_h} \mu(\Delta_i) \\ &\leq \sum_{i \in I_h} \mu(W_i) \\ &\leq C \sum_{i \in I_h} K_{\mu, \alpha}(2h) v_\alpha(W_i) \\ &\leq C \tilde{C} K_{\mu, \alpha}(2h) \sum_{i \in I} v_\alpha(\Delta_i). \end{aligned}$$

The third inequality comes from (2.1) and from the fact that, for every $i \in I_h$, as the radius of W_i is smaller than $\frac{1}{2^n}$, it is then smaller than $2h$. Now, as each point of \mathbb{B}_N belongs to at most M of the Δ_i 's, we have

$$\sum_{i \in I} v_\alpha(\Delta_i) \leq M v_\alpha \left(\bigcup_{i \in I} \Delta_i \right) \leq M v_\alpha(\{\Lambda_f > t\}).$$

and

$$\mu(\{z \in \mathbb{B}_N, |z| > 1 - h \text{ and } |f(z)| > t\}) \lesssim K_{\mu, \alpha}(2h) v_\alpha(\{\Lambda_f > t\}).$$

□

The last lemma gives the following technical result.

Lemma 2.4. *Let μ be a finite positive Borel measure on \mathbb{B}_N and let ψ_1 and ψ_2 be two Orlicz functions. Assume that there exist $A > 0$, $\eta > 0$ and $h_A \in (0, 1/2)$ such that*

$$K_{\mu,\alpha}(h) \leq \eta \frac{1/h^{N+1+\alpha}}{\psi_2(A\psi_1^{-1}(1/h^{N+1+\alpha}))}$$

for every $h \in (0, h_A)$. Then, there exist three constants $B > 0$, $x_A > 0$ and C_1 (this latter does not depend on A , η and h_A) such that, for every $f \in A_{\alpha}^{\psi_1}(\mathbb{B}_N)$ such that $\|f\|_{A_{\alpha}^{\psi_1}} \leq 1$, and every Borel subset E of \mathbb{B}_N , we have

$$\int_E \psi_2\left(\frac{|f|}{B}\right) d\mu \leq \mu(E) \psi_2(x_A) + C_1 \eta \int_{\mathbb{B}_N} \psi_1(\Lambda_f) dv_{\alpha}.$$

Proof. For $f \in A_{\alpha}^{\psi_1}(\mathbb{B}_N)$, $\|f\|_{A_{\alpha}^{\psi_1}} \leq 1$, and E a Borel subset of \mathbb{B}_N , we begin by writing the following formula, based on Fubini's integration:

$$(2.3) \quad \int_E \psi_2(|f|) d\mu = \int_0^{\infty} \psi_2'(t) \mu(\{|f| > t\} \cap E) dt.$$

We concentrate our attention on the expression $\mu(\{|f| > t\})$. We use the upper estimate of the point evaluation functional obtained in Proposition 1.9 to get that if $|f(z)| > t$, then, since $\|f\|_{A_{\alpha}^{\psi_1}} \leq 1$, we have

$$(2.4) \quad \begin{aligned} t &< \psi_1^{-1}\left(\left(\frac{1+|z|}{1-|z|}\right)^{N+1+\alpha}\right) \\ &\leq 2^{N+1+\alpha} \psi_1^{-1}\left(\left(\frac{1}{1-|z|}\right)^{N+1+\alpha}\right) \end{aligned}$$

because ψ is a convex function. Inequality (2.4) is now equivalent to the following one:

$$|z| > 1 - \left(\frac{1}{\psi_1\left(\frac{t}{2^{N+1+\alpha}}\right)}\right)^{1/(N+1+\alpha)}.$$

Carleson's theorem (Theorem 2.3) then yields that

$$(2.5) \quad \begin{aligned} \mu(\{|f| > t\}) &= \mu\left(\{|f| > t\} \cap \left\{|z| > 1 - \left(\frac{1}{\psi_1\left(\frac{t}{2^{N+1+\alpha}}\right)}\right)^{1/(N+1+\alpha)}\right\}\right) \\ &\leq \tilde{C} K_{\mu,\alpha} \left(2 \left(\frac{1}{\psi_1\left(\frac{t}{2^{N+1+\alpha}}\right)}\right)^{1/(N+1+\alpha)}\right) v_{\alpha}(\{\Lambda_f > t\}). \end{aligned}$$

Now, if A , h_A and η are as in the statement of the lemma, then, if

$$\frac{1}{2^{N+1+\alpha}} \psi_1\left(\frac{3 \cdot 2^{N+\alpha}}{A} s\right) > 1/h_A^{N+1+\alpha}$$

i.e. $s \geq x_A := \frac{A}{3 \cdot 2^{N+\alpha}} \psi_1^{-1}\left((2/h_A)^{N+1+\alpha}\right)$, then

$$(2.6) \quad K_{\mu,\alpha} \left(2 \left(\frac{1}{\psi_1\left(\frac{3 \cdot 2^{N+\alpha}}{A} s\right)}\right)^{1/(N+1+\alpha)}\right) \leq \frac{\eta}{2^{N+1+\alpha}} \frac{\psi_1\left(\frac{3 \cdot 2^{N+\alpha}}{A} s\right)}{\psi_2\left(\frac{3}{2} s\right)}.$$

Hence, applying (2.3) to $\frac{A}{6.4^{N+\alpha}} |f|$, together with (2.5) and (2.6), and putting $t = \frac{6.4^{N+\alpha}}{A} s$ in (2.5), we get

$$(2.7) \quad \int_E \psi_2 \left(\frac{A}{6.4^{N+\alpha}} |f| \right) d\mu \leq \int_0^{x_A} \psi_2'(s) \mu(E) ds \\ + \frac{\eta \tilde{C}}{2^{N+1+\alpha}} \int_{x_A}^{\infty} \psi_2'(s) \frac{\psi_1 \left(\frac{3.2^{N+\alpha}}{A} s \right)}{\psi_2 \left(\frac{3}{2} s \right)} v_\alpha \left(\left\{ \Lambda_f > \frac{6.4^{N+\alpha}}{A} s \right\} \right) ds.$$

For the second integral of the right hand side, notice that for an Orlicz function ψ , we have

$$x\psi'(x) \leq C\psi \left(\frac{(C+1)x}{C} \right)$$

for any $C > 0$ and any $x \geq 0$. Indeed, as $\psi'(t)$ is non-decreasing, we have

$$\frac{x}{C}\psi'(x) \leq \int_x^{\frac{C+1}{C}x} \psi'(t) dt \leq \psi \left(\frac{C+1}{C}x \right).$$

Therefore

$$\frac{\psi_2'(s)}{\psi_2 \left(\frac{3}{2}s \right)} \leq \frac{2}{s}$$

and (2.7) yields

$$\int_E \psi_2 \left(\frac{A}{6.4^{N+\alpha}} |f| \right) d\mu \leq \psi_2(x_A) \mu(E) \\ + \frac{\eta \tilde{C}}{2^{N+\alpha}} \int_{x_A}^{\infty} \frac{1}{s} \psi_1 \left(\frac{3.2^{N+\alpha}}{A} s \right) v_\alpha \left(\left\{ \Lambda_f > \frac{6.4^{N+\alpha}}{A} s \right\} \right) ds.$$

Using the convexity of the function ψ_1 , we get

$$\int_E \psi_2 \left(\frac{A}{6.4^{N+\alpha}} |f| \right) d\mu \leq \psi_2(x_A) \mu(E) \\ + \frac{\eta \tilde{C}}{2^{N+\alpha}} \frac{3.2^{N+\alpha}}{A} \int_0^{\infty} \psi_1' \left(\frac{3.2^{N+\alpha}}{A} s \right) v_\alpha \left(\left\{ \Lambda_f > \frac{6.4^{N+\alpha}}{A} s \right\} \right) ds$$

i.e.

$$\int_E \psi_2 \left(\frac{A}{6.4^{N+\alpha}} |f| \right) d\mu \leq \psi_2(x_A) \mu(E) + \frac{\eta \tilde{C}}{2^{N+\alpha}} \int_0^{\infty} \psi_1'(u) v_\alpha(\{\Lambda_f > 2^{N+1+\alpha}u\}) du \\ \leq \psi_2(x_A) \mu(E) + \frac{\eta \tilde{C}}{2.4^{N+\alpha}} \int_{\mathbb{B}_N} \psi_1(\Lambda_f) dv_\alpha$$

and the proof of the lemma is complete. \square

2.1. The canonical embedding $A_\alpha^{\psi_1}(\mathbb{B}_N) \hookrightarrow L^{\psi_2}(\mu)$. We state our boundedness theorem in the Bergman-Orlicz spaces framework as follows:

Theorem 2.5. *Let μ be a finite positive Borel measure on \mathbb{B}_N and let ψ_1 and ψ_2 be two Orlicz functions. Then:*

- (1) *If inclusion $A_\alpha^{\psi_1}(\mathbb{B}_N) \subset L^{\psi_2}(\mu)$ holds and is continuous, then there exists some $A > 0$ such that*

$$(2.8) \quad \varrho_\mu(h) = O_{h \rightarrow 0} \left(\frac{1}{\psi_2(A\psi_1^{-1}(1/h^{N+1+\alpha}))} \right).$$

(2) If there exists some $A > 0$ such that

$$(2.9) \quad K_{\mu,\alpha}(h) = O_{h \rightarrow 0} \left(\frac{1/h^{N+1+\alpha}}{\psi_2(A\psi_1^{-1}(1/h^{N+1+\alpha}))} \right)$$

then inclusion $A_\alpha^{\psi_1}(\mathbb{B}_N) \subset L^{\psi_2}(\mu)$ holds and is continuous.

(3) If in addition $\psi_1 = \psi_2 = \psi$ satisfies the uniform ∇_0 -Condition, then Conditions (2.8) and (2.9) are equivalent.

Note that embedding $A_\alpha^{\psi_1}(\mathbb{B}_N) \subset L^{\psi_2}(\mu)$ is continuous as soon as it holds. It is just an application of the closed graph theorem.

Proof of Theorem 2.5. 1) For the first part, let us denote by C the norm of the canonical embedding $j_\alpha : A_\alpha^{\psi_1}(\mathbb{B}_N) \hookrightarrow L^{\psi_2}(\mu)$. Let $a \in \mathbb{B}_N$, $|a| = 1 - h$ and $\xi \in \mathbb{S}_N$ be such that $a = (1 - h)\xi$. Let us consider the map

$$\begin{aligned} f_a &= \frac{1}{2^{N+1+\alpha}} \frac{\psi_1^{-1}(1/h^{N+1+\alpha})}{1/h^{N+1+\alpha}} H_a(z) \\ &= \frac{1}{2^{N+1+\alpha}} \frac{\psi_1^{-1}(1/h^{N+1+\alpha})}{1/h^{N+1+\alpha}} \left(\frac{h(2-h)}{|1 - (1-h)\langle z, \xi \rangle|^2} \right)^{N+1+\alpha} \end{aligned}$$

Recall that H_a is the Berezin kernel introduced in Proposition 1.9. As we saw in the proof of this latter, f_a is in the unit ball of $A_\alpha^{\psi_1}(\mathbb{B}_N)$ and our assumption ensures that

$$\|j_\alpha(f_a)\|_{L^{\psi_2}(\mu)} = \|f_a\|_{L^{\psi_2}(\mu)} \leq C$$

so that

$$(2.10) \quad 1 \geq \int_{\mathbb{B}_N} \psi_2 \left(\frac{|f_a|}{C} \right) d\mu.$$

Let us minorize the right hand side of (2.10). We just get a minorization of $|f_a|$ on the non-isotropic ‘‘ball’’ $S(\xi, h)$. If $z \in S(\xi, h)$, then a straightforward computation yields $|1 - \langle z, a \rangle| \leq 2h$. Hence, for any $z \in S(a, h)$,

$$|f_a(z)| \geq \frac{\psi_1^{-1}(1/h^{N+1+\alpha})}{8^{N+1+\alpha}}.$$

Therefore

$$1 \geq \int_{\mathbb{B}_N} \psi_2 \left(\frac{|f|}{C} \right) d\mu \geq \psi_2 \left(\frac{\psi_1^{-1}(1/h^{N+1+\alpha})}{8^{N+1+\alpha}C} \right) \mu(S(a, h)),$$

which is Condition (2.8) and the first part of the theorem follows.

2) The second part will need Lemma 2.4. First of all, we know (Proposition 2.2) that there exists a constant $C_M \geq 1$ such that, for every $f \in A_\alpha^{\psi_1}(\mathbb{B}_N)$, $\|\Lambda_f\|_{L_\alpha^{\psi_1}(\mathbb{B}_N)} \leq C_M \|f\|_{A_\alpha^{\psi_1}(\mathbb{B}_N)}$. Let now f be in the unit ball of $A_\alpha^{\psi_1}(\mathbb{B}_N)$; it suffices to show that $\|f\|_{L^{\psi_2}(\mu)} \leq C_0$ for some constant $C_0 > 0$ which does not depend on f . Let $\tilde{C} \geq 1$ be a constant whose value will be precised later. Condition (2.9) is supposed to be realized, that is there exist some constants $A > 0$, $h_A \in (0, 1/2]$ and $\eta > 0$ such that

$$(2.11) \quad K_{\mu,\alpha}(h) \leq \eta \frac{1/h^{N+1+\alpha}}{\psi_2(A\psi_1^{-1}(1/h^{N+1+\alpha}))}$$

for any $h \in (0, h_A)$. By using convexity of ψ_2 and applying Lemma 2.4 to f/C_M (which of course still satisfies $\|f/C_M\|_{A_\alpha^{\psi_1}} \leq 1$) and $E = \mathbb{B}_N$, we get the existence of constants $B > 0$, x_A and $C_1 > 0$, all independent of f , such that

$$\begin{aligned} \int_{\mathbb{B}_N} \psi_2 \left(\frac{|f|}{BC_M\tilde{C}} \right) d\mu &\leq \frac{1}{\tilde{C}} \int_{\mathbb{B}_N} \psi_2 \left(\frac{|f|}{BC_M} \right) d\mu \\ &\leq \frac{1}{\tilde{C}} (\mu(\mathbb{B}_N) \psi_2(x_A) + C_1\eta). \end{aligned}$$

Of course, C_1 may be supposed to be large enough so that $C_1\eta \geq 1$ and, up to fix $\tilde{C} = \mu(\mathbb{B}_N)\psi_2(x_A) + C_1\eta \geq 1$, we get $\|f\|_{L^{\psi_2}(\mu)} \leq C_0 := BC_M\tilde{C}$ which completes the proof of (2) of Theorem 2.5.

3) First, it is clear that Condition (2.9) implies Condition (2.8). For the converse, we need the following claim:

Claim. Under the notations of the theorem, if Condition (2.8) holds, then there exist some A as large as we want and $\eta > 0$ such that

$$(2.12) \quad \varrho_\mu(h) \leq \eta \frac{1}{\psi_2(A\psi_1^{-1}(h_A/h^{N+1+\alpha}))}$$

for some $h_A, 0 < h_A \leq 1$ and for any $0 < h < h_A$.

Proof of the claim. We assume that Condition

$$(2.13) \quad \varrho_\mu(h) \leq \eta \frac{1}{\psi_2(\tilde{A}\psi_1^{-1}(1/h^{N+1+\alpha}))}$$

holds for some $\tilde{A} \geq 0, \tilde{h}_A, 0 < \tilde{h}_A \leq 1, \eta > 0$ and any $0 < h < \tilde{h}_A$. We fix $A > 1$ and we look for some constant $h_{\tilde{A},A} \leq 1$ such that

$$(2.14) \quad \frac{1}{\psi_2(\tilde{A}\psi_1^{-1}(1/h^{N+1+\alpha}))} \leq \frac{1}{\psi_2(A\psi_1^{-1}\left(\left(h_{\tilde{A},A}/h\right)^{N+1+\alpha}\right))}$$

for $0 < h < h_{\tilde{A},A}$. Now it is easy to verify that Inequality (2.14) is equivalent to

$$\frac{A}{\tilde{A}} \leq \frac{\psi_1^{-1}(1/h^{N+1+\alpha})}{\psi_1^{-1}\left(\left(h_{\tilde{A},A}/h\right)^{N+1+\alpha}\right)} \leq \frac{1}{h_{\tilde{A},A}^{N+1+\alpha}}$$

by concavity of ψ^{-1} . Then the claim follows by choosing $h_{\tilde{A},A}$ small enough. \square

We come back to the proof of the third point. Let suppose that ψ belongs to the uniform ∇_0 -class and let $A > 0, h_A \in (0, 1]$ and $\eta > 0$ be such that

$$\varrho_\mu(h) \leq \eta \frac{1}{\psi(A\psi^{-1}(1/h^{N+1+\alpha}))}$$

for every $h \in (0, h_A)$. The previous claim says that we can find $B \geq 1$ and $0 < K = K_{B,A} \leq 1$ such that

$$\varrho_\mu(h) \leq \eta \frac{1}{\psi\left(B\psi^{-1}\left(\left(K/h\right)^{N+1+\alpha}\right)\right)}$$

for every $0 < h < K$. Therefore, we have

$$\begin{aligned} K_{\mu,\alpha}(h) &= \sup_{0 < t \leq h} \frac{\varrho_\mu(t)}{t^{N+1+\alpha}} \leq \eta \sup_{0 < t \leq h} \frac{1/t^{N+1+\alpha}}{\psi\left(B\psi^{-1}\left(\left(K/t\right)^{N+1+\alpha}\right)\right)} \\ &= \eta \sup_{x \geq \psi^{-1}\left(\left(K/h\right)^{N+1+\alpha}\right)} \frac{1}{K^{N+1+\alpha}} \frac{\psi(x)}{\psi(Bx)} \end{aligned}$$

for any $0 < h \leq K$. Let C be the constant induced by the uniform ∇_0 -Condition satisfied by ψ and let β be such that $B = \beta C$. The claim allows us to take B large enough and therefore to assume that $\beta > 1$. We then have, since ψ satisfies the uniform ∇_0 -Condition,

$$\frac{\psi\left(\beta\psi^{-1}\left(\left(K/h\right)^{N+1+\alpha}\right)\right)}{\left(K/h\right)^{N+1+\alpha}} \leq \frac{\psi(Bx)}{\psi(x)}$$

for any $x \geq \psi^{-1} \left((K/h)^{N+1+\alpha} \right)$. Hence, for every $0 < h \leq K$,

$$K_{\mu,\alpha}(h) \leq \eta \frac{1/h^{N+1+\alpha}}{\psi \left(\beta \psi^{-1} \left((K/h)^{N+1+\alpha} \right) \right)} \leq \eta \frac{1/h^{N+1+\alpha}}{\psi \left(\beta K^{N+1+\alpha} \psi^{-1} (1/h^{N+1+\alpha}) \right)}$$

by concavity of ψ^{-1} , and Condition (2.9) is satisfied. \square

The third point of the previous theorem leads us to define (ψ, α) -Bergman-Carleson measures on the ball:

Definition 2.6. Let μ be a positive Borel measure on \mathbb{B}_N and let ψ be an Orlicz function. We say that μ is a (ψ, α) -Bergman-Carleson measure if there exists some $A > 0$, such that

$$(2.15) \quad \mu(W(\xi, h)) = O_{h \rightarrow 0} \left(\frac{1}{\psi(A\psi^{-1}(1/h^{N+1+\alpha}))} \right)$$

uniformly with respect to $\xi \in \mathbb{S}_N$.

We notice that (2.15) is equivalent to (2.8). Therefore, we can state the following corollary:

Corollary 2.7. Let μ be a finite positive Borel measure on \mathbb{B}_N and let ψ be an Orlicz function satisfying the uniform ∇_0 -Condition. Inclusion $A_\alpha^\psi(\mathbb{B}_N) \hookrightarrow L^\psi(\mu)$ holds (and is continuous) if and only if μ is a (ψ, α) -Bergman-Carleson measure.

2.2. Compactness of the canonical embedding $A_\alpha^{\psi_1}(\mathbb{B}_N) \hookrightarrow L^{\psi_2}(\mu)$. For the study of compactness, we usually need some compactness criterion.

Proposition 2.8. Let μ be a finite positive measure on \mathbb{B}_N and let ψ_1 and ψ_2 be two Orlicz functions. We suppose that the canonical embedding $j_{\mu,\alpha} : A_\alpha^{\psi_1}(\mathbb{B}_N) \hookrightarrow L^{\psi_2}(\mu)$ holds and is bounded. The three following assertions are equivalent:

- (1) $j_{\mu,\alpha} : A_\alpha^{\psi_1}(\mathbb{B}_N) \hookrightarrow L^{\psi_2}(\mu)$ is compact;
- (2) Every sequence in the unit ball of $A_\alpha^{\psi_1}(\mathbb{B}_N)$, which is convergent to 0 uniformly on every compact subset of \mathbb{B}_N , is strongly convergent to 0 in $L^{\psi_2}(\mu)$.
- (3) $\lim_{r \rightarrow 1^-} \|I_r\| = 0$, where $I_r(f) = f \cdot \chi_{\mathbb{B}_N \setminus r\overline{\mathbb{B}_N}}$.

Proof. (1) \Rightarrow (2) We first assume that $j_{\mu,\alpha}$ is compact. Let $(f_n)_n$ be a sequence in the unit ball of $A_\alpha^{\psi_1}(\mathbb{B}_N)$, which is convergent to 0 uniformly on every compact subset of \mathbb{B}_N . Of course, $j_{\mu,\alpha}(f_n)$ converges to 0 everywhere. By contradiction, suppose up to extract a subsequence that $\liminf_n \|j_{\mu,\alpha}(f_n)\|_{L^{\psi_2}(\mu)} > 0$. By compactness of $j_{\mu,\alpha}$, up to an other extraction, we may assume that $(j_{\mu,\alpha}(f_n))_n$ strongly converges to some $g \in L^{\psi_2}(\mu)$ and we must have $\|g\|_{L^{\psi_2}(\mu)} > 0$. As convergence in norm in $L^{\psi_2}(\mu)$ entails μ -almost everywhere convergence, we get a contradiction.

(2) \Rightarrow (1) Conversely, let $(f_n)_n$ be a sequence in the unit ball of $A_\alpha^{\psi_1}(\mathbb{B}_N)$. In particular, $(f_n)_n$ is in the unit ball of $A_\alpha^1(\mathbb{B}_N)$ and the Cauchy's formula ensures that $(f_n)_n$ is uniformly bounded on every compact subset of \mathbb{B}_N , so that, up to an extraction, we may suppose that $(f_n)_n$ is uniformly convergent on compact subsets of \mathbb{B}_N to f holomorphic in \mathbb{B}_N , by Montel's theorem. Now, Lebesgue's theorem ensures that $f \in A_\alpha^{\psi_1}(\mathbb{B}_N)$ and, up to divide by a constant large enough, we may assume that $f_n - f$, which converges to 0 on every compact subset of \mathbb{B}_N , is in the unit ball of $A_\alpha^{\psi_1}(\mathbb{B}_N)$. Therefore, our assumption implies that $(j_{\mu,\alpha}(f_n) - j_{\mu,\alpha}(f))_n$ converges to 0 in the norm of $L^{\psi_2}(\mu)$ and $j_{\mu,\alpha}$ is compact, as expected.

(3) \Rightarrow (2) Let $(f_n)_n$ be in the unit ball of $A_\alpha^{\psi_1}(\mathbb{B}_N)$ converging to 0 uniformly on every compact subset of \mathbb{B}_N . We have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|f_n\|_{L^{\psi_2}(\mu)} &= \limsup_{r \rightarrow 1^-} \limsup_{n \rightarrow \infty} \left\| I_r(f_n) + f_n \cdot \chi_{r\overline{\mathbb{B}_N}} \right\|_{L^{\psi_2}(\mu)} \\ &\lesssim \limsup_{r \rightarrow 1^-} \|I_r\| + \limsup_{r \rightarrow 1^-} \limsup_{n \rightarrow \infty} \left\| f_n \cdot \chi_{r\overline{\mathbb{B}_N}} \right\|_\infty \\ &= 0. \end{aligned}$$

(2) \Rightarrow (3) By contradiction suppose that (3) is not satisfied so that there exist a constant $\delta > 0$ and a sequence $(f_n)_n$ in the unit ball of $A_\alpha^{\psi_1}(\mathbb{B}_N)$ such that $\|I_{(1-\frac{1}{n})}(f_n)\|_{L^{\psi_2}} \geq \delta$, for every $n \geq 0$. Up to an extraction, we may suppose that $(f_n)_n$ converges uniformly on compact subsets of \mathbb{B}_N to $f \in A_\alpha^{\psi_1}(\mathbb{B}_N)$. By Lebesgue's theorem, $\lim_{n \rightarrow \infty} \|I_{(1-\frac{1}{n})}(f)\|_{L^{\psi_2}} = 0$; thus, for n large enough,

$$\|f_n - f\|_{L^{\psi_2}} \geq \|I_{(1-\frac{1}{n})}(f_n - f)\|_{L^{\psi_2}} \geq \delta/2$$

which contradicts (2). \square

As for the boundedness, we state our embedding compactness theorem for weighted Bergman-Orlicz spaces as follows:

Theorem 2.9. *Let μ be a finite positive Borel measure on \mathbb{B}_N , and let ψ_1 and ψ_2 be two Orlicz functions.*

(1) *If the inclusion $A_\alpha^{\psi_1}(\mathbb{B}_N) \subset L^{\psi_2}(\mu)$ holds and is compact, then for every $A > 0$ we have*

$$(2.16) \quad \varrho_\mu(h) = o_{h \rightarrow 0} \left(\frac{1}{\psi_2(A\psi_1^{-1}(1/h^{N+1+\alpha}))} \right).$$

(2) *If*

$$(2.17) \quad K_{\mu,\alpha}(h) = o_{h \rightarrow 0} \left(\frac{1/h^{N+1+\alpha}}{\psi_2(A\psi_1^{-1}(1/h^{N+1+\alpha}))} \right)$$

for every $A > 0$, then $A_\alpha^{\psi_1}(\mathbb{B}_N)$ embeds compactly in $L^{\psi_2}(\mu)$.

(3) *If in addition $\psi_1 = \psi_2 = \psi$ satisfies the ∇_0 -Condition, then Conditions (2.16) and (2.17) are equivalent.*

Proof. 1) We suppose that the canonical embedding is compact but that Condition (2.16) failed to be satisfied. This means that there exist some $\varepsilon_0 \in (0, 1)$ and $A > 0$, some sequences $(h_n)_n \subset (0, 1)$ decreasing to 0 and $(\xi_n)_n \subset \mathbb{S}_N$, such that

$$\mu(S(\xi_n, h_n)) \geq \frac{\varepsilon_0}{\psi_2(A\psi_1^{-1}(1/h_n^{N+1+\alpha}))}.$$

Let $a_n := (1 - h_n)\xi_n$ and consider the functions

$$(2.18) \quad f_n(z) := f_{a_n}(z) := \frac{1}{2^{N+1+\alpha}} \frac{\psi_1^{-1}(1/h_n^{N+1+\alpha})}{1/h_n^{N+1+\alpha}} H_{a_n}(z)$$

where H_{a_n} is the Berezin kernel, as in the proof of the first part of Theorem 2.5. Every f_n lays in the unit ball of $A_\alpha^{\psi_1}(\mathbb{B}_N)$ and $(f_n)_n \xrightarrow[n \rightarrow \infty]{} 0$ uniformly on every compact subset of \mathbb{B}_N . So

Proposition 2.8 ensures that $(f_n)_n$ converges to 0 in norm of $L^{\psi_2}(\mu)$.

Now, by the proof of the first part of Theorem 2.5, the following estimation holds:

$$|f_n(z)| \geq \frac{\psi_1^{-1}(1/h_n^{N+1+\alpha})}{8^{N+1+\alpha}}$$

for any $z \in S(\xi_n, h_n)$; therefore

$$\begin{aligned} \int_{\mathbb{B}_N} \psi_2 \left(\frac{8^{N+1+\alpha} A}{\varepsilon_0} |f_n| \right) d\mu &\geq \psi_2 \left(\frac{A}{\varepsilon_0} \psi_1^{-1} \left(\frac{1}{h_n^{N+1+\alpha}} \right) \right) \mu(S(\xi_n, h_n)) \\ &\geq \psi_2 \left(\frac{A}{\varepsilon_0} \psi_1^{-1} \left(\frac{1}{h_n^{N+1+\alpha}} \right) \right) \frac{\varepsilon_0}{\psi_2(A\psi_1^{-1}(1/h_n^{N+1+\alpha}))} \\ &\geq 1 \end{aligned}$$

by the convexity of ψ_2 . This yields $\|f_n\|_{L^{\psi_2}(\mu)} \geq \frac{\varepsilon_0}{8^{N+1+\alpha} A}$ for every n , which is a contradiction and gives the first part.

2) We now assume that Condition (2.17) is satisfied. Thanks to the second point of Proposition 2.8, it is sufficient to prove that, for every $\varepsilon > 0$, the norm of the embedding

$$I_r : A_\alpha^{\psi_1}(\mathbb{B}_N) \hookrightarrow L^{\psi_2}(\mathbb{B}_N \setminus r\overline{\mathbb{B}_N}, \mu)$$

is smaller than ε for some $r_0(\varepsilon)$ and every r such that $r_0(\varepsilon) \leq r < 1$. Let $\eta \in (0, 1)$ and let $A := A(\varepsilon) = \frac{6.4^{N+\alpha}}{\varepsilon} > 0$; Condition (2.17) ensures that there exists $h_A \in (0, 1/2)$ such that

$$K_{\mu, \alpha}(h) \leq \eta \frac{1/h^{N+1+\alpha}}{\psi_2(A\psi_1^{-1}(1/h^{N+1+\alpha}))}$$

for $h \leq h_A$. Let now f be in the unit ball of $A_\alpha^{\psi_1}(\mathbb{B}_N)$ and $r \in (0, 1)$. By the proof of Lemma 2.4, applied to $E = \mathbb{B}_N \setminus r\overline{\mathbb{B}_N}$ and f , there exist a constant $B > 0$ given by $B = \frac{6.4^{N+\alpha}}{A} = \varepsilon$, and some constants $x_A > 0$ and $C_1 > 0$, independent of f , such that

$$\begin{aligned} \int_{\mathbb{B}_N \setminus r\overline{\mathbb{B}_N}} \psi_2\left(\frac{|f|}{\varepsilon}\right) d\mu &= \int_{\mathbb{B}_N \setminus r\overline{\mathbb{B}_N}} \psi_2\left(\frac{|f|}{B}\right) d\mu \\ &\leq \mu(\mathbb{B}_N \setminus r\overline{\mathbb{B}_N}) \psi_2(x_A) + C_1 \eta \int_{\mathbb{B}_N} \psi_1(\Lambda_f) dv_\alpha. \end{aligned}$$

Now, we choose η such that $C_1 \eta \int_{\mathbb{B}_N} \psi_1(\Lambda_f) dv_\alpha \leq \frac{1}{2}$ (which is possible thanks to Proposition 2.2) and we take $r_0 \in (0, 1)$ such that $\mu(\mathbb{B}_N \setminus r\overline{\mathbb{B}_N}) \psi_2(x_A) \leq \frac{1}{2}$ for every $r \in (r_0, 1)$. We get $\|I_r(f)\|_{L^{\psi_2}(\mu)} \leq \varepsilon$ as soon as $r_0 < r < 1$, what completes the proof.

3) The proof of the third point is essentially contained in that of the third part of [6, Theorem 4.11]. \square

This leads us to the definition of *vanishing* (ψ, α) -Bergman-Carleson measures on the ball:

Definition 2.10. Let ψ be an Orlicz function and let μ be a Borel positive measure on \mathbb{B}_N . We say that μ is a *vanishing* (ψ, α) -Bergman-Carleson measure if, for every $A > 0$,

$$\mu(W(\xi, h)) = o_{h \rightarrow 0} \left(\frac{1}{\psi(A\psi^{-1}(1/h^{N+1+\alpha}))} \right)$$

uniformly with respect to $\xi \in \mathbb{S}_N$.

We have the following corollary:

Corollary 2.11. Let ψ be an Orlicz function satisfying the ∇_0 -Condition and let μ be a Borel positive measure on \mathbb{B}_N . Then $A_\alpha^\psi(\mathbb{B}_N)$ embeds compactly into $L^\psi(\mu)$ if and only if μ is a *vanishing* (ψ, α) -Bergman-Carleson measure.

3. APPLICATION TO COMPOSITION OPERATORS ON WEIGHTED BERGMAN-ORLICZ SPACES.

For $\phi : \mathbb{B}_N \rightarrow \mathbb{B}_N$ analytic, we denote by μ_ϕ^α the pull-back measure by ϕ of the weighted Lebesgue measure v_α on \mathbb{B}_N , namely $\mu_\phi^\alpha(E) = v_\alpha(\phi^{-1}(E))$ for every Borel subset E of \mathbb{B}_N .

Theorem 2.5 and Theorem 2.9 allow us to give the following characterization with some constraints on the Orlicz function ψ :

Theorem 3.1. Let ψ be an Orlicz function and let $\phi : \mathbb{B}_N \rightarrow \mathbb{B}_N$ be holomorphic.

- (1) If ψ satisfies the uniform ∇_0 -Condition, then C_ϕ is bounded from $A_\alpha^\psi(\mathbb{B}_N)$ into itself if and only if μ_ϕ^α is a (ψ, α) -Bergman-Carleson measure.
- (2) If ψ satisfies the ∇_0 -Condition, then C_ϕ is compact from $A_\alpha^\psi(\mathbb{B}_N)$ into itself if and only if μ_ϕ^α is a *vanishing* (ψ, α) -Bergman-Carleson measure.

Proof. Thanks to Corollary 2.7 and Corollary 2.11, it suffices to notice that the continuity (resp. compactness) of the canonical embedding $j_{\mu_\phi^\alpha} : A_\alpha^\psi(\mathbb{B}_N) \hookrightarrow L^\psi(\mu_\phi^\alpha)$ is equivalent to the boundedness (resp. compactness) of $C_\phi : A_\alpha^\psi(\mathbb{B}_N) \rightarrow A_\alpha^\psi(\mathbb{B}_N)$. This just proceeds from the fact that

$$\begin{aligned} \|C_\phi(f)\|_{A_\alpha^\psi(\mathbb{B}_N)} &= \inf \left\{ C > 0, \int_{\mathbb{B}_N} \psi \left(\frac{|f \circ \phi|}{C} \right) d\sigma \leq 1 \right\} \\ &= \inf \left\{ C > 0, \int_{\mathbb{B}_N} \psi \left(\frac{|f|}{C} \right) d\mu_\phi \leq 1 \right\} \\ &= \left\| j_{\mu_\phi^\alpha}(f) \right\|_{L^\psi(\mu_\phi^\alpha)}, \end{aligned}$$

for any $f \in A_\alpha^\psi(\mathbb{B}_N)$. \square

Remark 3.2. If we do not assume that ψ satisfies the uniform ∇_0 -Condition (resp. ∇_0 -Condition), then Theorem 2.5 (resp. Theorem 2.9) provides *a priori* non-equivalent necessary and sufficient conditions to the boundedness (resp. compactness) of C_ϕ on $A_\alpha^\psi(\mathbb{B}_N)$.

As a particular case of the previous theorem, we state and verify [2, Theorem 3.6 and Theorem 4.3]:

Theorem 3.3. *Let ψ be an Orlicz function which satisfies Δ_2 -Conditions and let $\phi : \mathbb{B}_N \rightarrow \mathbb{B}_N$ be holomorphic. Then*

- (1) C_ϕ is bounded from $A_\alpha^\psi(\mathbb{B}_N)$ into itself if and only if μ_ϕ^α is an α -Bergman-Carleson measure.
- (2) C_ϕ is compact from $A_\alpha^\psi(\mathbb{B}_N)$ into itself if and only if μ_ϕ^α is a vanishing α -Bergman-Carleson measure.

Proof. It suffices to observe that

$$\frac{1}{\psi(A\psi^{-1}(1/h^{N+1+\alpha}))} \approx h^{N+1+\alpha}$$

for every $A > 0$, whenever ψ is an Orlicz function which satisfies the Δ_2 -Condition (see Remark 2 (a) following Theorem 4.11 in [6].) \square

A first consequence of these characterizations is the following:

Corollary 3.4. *Let $\phi : \mathbb{B}_N \rightarrow \mathbb{B}_N$ be holomorphic and let ψ, ν be two Orlicz functions. Assume that ν satisfies the Δ_2 -Condition. Then:*

- (1) If C_ϕ is bounded on $A_\alpha^\nu(\mathbb{B}_N)$ (e.g. on any $A_\alpha^p(\mathbb{B}_N)$), then it is bounded on $A_\alpha^\psi(\mathbb{B}_N)$;
- (2) If ψ satisfies the ∇_0 -Condition and if C_ϕ is compact on $A_\alpha^\psi(\mathbb{B}_N)$, then it is compact on $A_\alpha^\nu(\mathbb{B}_N)$ (e.g. on any $A_\alpha^p(\mathbb{B}_N)$).

Proof. The first point follows from the remark before Theorem 3.3 and from the fact that if μ is a α -Carleson measure, i.e. if $K_{\mu, \alpha} \leq C$ for some constant $C \geq 1$, then μ is a (ψ, α) -Carleson measure, since $\psi(A\psi^{-1}(1/h^{N+1+\alpha})) \leq A/h^{N+1+\alpha}$, for any $0 < A \leq 1$.

For the second point, it suffices to show that Condition (2.17) implies that μ is a vanishing α -Carleson measure, what is trivial if we apply it for $A = 1$. \square

As one of the main motivation to this work, we are interested in finding where the break of condition for boundedness of C_ϕ happens between $H^\infty(\mathbb{B}_N)$ and $A_\alpha^p(\mathbb{B}_N)$. More precisely, we wonder if there are some spaces different from $H^\infty(\mathbb{B}_N)$ and smaller than some $A_\alpha^p(\mathbb{B}_N)$ on which every composition operator C_ϕ is bounded. In [8], the authors show the following proposition:

Proposition 3.5. *Let $\phi : \mathbb{B}_N \rightarrow \mathbb{B}_N$ be analytic. Then*

$$(3.1) \quad \mu_\phi^\alpha(S(\xi, h)) = O_{h \rightarrow 0}(h^{\alpha+2})$$

for every $\xi \in \mathbb{S}_N$.

In fact, this result is stated for general strongly pseudo-convex domains instead of \mathbb{B}_N ([8, Proposition 4].)

A brief comparison of Condition (3.1) and Condition (2.9), written for $\psi_1 = \psi_2 = \psi$, makes it clear that if we can find some ψ , among those satisfying the uniform ∇_0 -Condition, which satisfies the following condition \mathcal{P} :

\mathcal{P} : for every $K > 0$, there exist $A > 0$ and $h_0 > 0$ such that

$$(3.2) \quad Kh^{\alpha+2} \leq \frac{1}{\psi(A\psi^{-1}(1/h^{N+1+\alpha}))},$$

for any $0 < h \leq h_0$,

then every composition operator will be bounded on the Bergman-Orlicz space $A_\alpha^\psi(\mathbb{B}_N)$. The next proposition characterizes those Orlicz functions which satisfy this condition \mathcal{P} :

Proposition 3.6. *Let ψ be an Orlicz function. ψ satisfies Condition \mathcal{P} if and only if, for every $K > 0$ (or equivalently for one $K > 0$), there exists $C > 0$ such that, for every $x > 0$ large enough, we have*

$$\psi(x)^{\frac{N+1+\alpha}{\alpha+2}} \leq K\psi(Cx).$$

In particular, Condition \mathcal{P} is trivial if $N = 1$ and coincides with the Δ^2 -Condition whenever $N > 1$.

Proof. The first part comes from a straightforward rewriting of inequality (3.2). The second part is a direct application of Proposition 1.6, using convexity of ψ . \square

When $N = 1$, [5, Theorem 3.1] permits to remove the necessary uniform ∇_0 -Condition in the first point of Theorem 3.1. When $N > 1$, this trick fails as it is not difficult to check that if it could be extended to the several complex variables setting, then it would imply that every composition operator is bounded on $A^p(\mathbb{B}_N)$. Yet, we know (Proposition 1.8) that every Orlicz function satisfying the Δ^2 -Condition satisfies the uniform ∇_0 -Condition too.

Therefore, Theorem 3.1, Proposition 3.5 and Proposition 3.6 immediately yields the following result:

Theorem 3.7. *Let ψ be an Orlicz function.*

- (1) *Every composition operator is bounded from $A_\alpha^\psi(\mathbb{D})$ into itself;*
- (2) *When $N > 1$, if ψ satisfies the Δ^2 -Condition, then every composition operator is bounded from $A_\alpha^\psi(\mathbb{B}_N)$ into itself.*

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