Modeling the Complex Dynamics of Distributed Communities of the Web with Pretopology

Vincent Levorato, Marc Bui

Laboratoire d'Informatique et des Systèmes Complexes (LaISC)
41 rue G. Lussac, F-75005, Paris, France.
Email : {vincent.levorato, marc.bui}@laisc.net

Abstract: The aim of this article is to present a methodological approach for problems encountered in structural analysis of web communities. This approach is based upon the pretopological concepts of pseudoclosure and the searching of equivalent nodes. The advantage of this approach is that it provides a framework needed to pass through the actual limits of graph theory modeling. The problem of modeling and understanding web communities is described, then a review of the existing models and their limits, and we finish by an example of a structuring algorithm.

1 Introduction

The study of complex networks dynamics is a domain which is still topical, and specially on the Web communities aspect [JPS02]. These communities continuously evolve in some evolutionary process starting with just a few individuals and the resulting set of interrelated community members is generally called the *social network* of a community. While the number of individuals in a community can grow very fast, the single individual needs only little information about other individuals to still be able to potentially interact with a large number (or all) of the community members [JVJ02]. The six degrees of separation property illustrates this in the case of human communities [DJW98]. Moreover, communities are often characterized by a highly self-organizing behavior.
Computer networks or distributed systems in general may be regarded as communities ; most obviously, the Internet or Web forms entities that can be characterized as communities. Nowadays, we are able to recover large amount of data on the Web using *logs files* in a lot of well known Web communities such as LinkedIn (professional relations), Second Life (virtual life game) or political blogs. It is a very important point because of the few amount of data usually used by sociologists [MNW06]. In this article, we want to analyze social networks on the Web ; thus, we must focus on two points:

- *identifying and working with appropriate datasets*: one needs a large, realistic social network containing a significant collection of explicitly identified groups, and with sufficient time-resolution that one can track their growth and evolution at the level of individual nodes.

- *developing new theoretic models* to pass through the limits of tools provided until now and to build a new theoretical one, adapted to real-world networks, and especially in this case, web communities.

First, we will make a review of the existing network models that had been studied many years ago, in the second part, we propose our own model based on pretopology with an algorithm and an associated example, then we'll finish by a conclusion opening on new ideas and future work.

2 Limits of tools and theory

2.1 Study of Social Network Analysis

If we would like to represent a social network, we use generally graph theory: sociologists choose one property (friendship connection, associate connection, ...) [Deg04] [DV94] to study despite of the others. In the meantime, networks, not only in social sciences, are evolving structures and dynamical systems [MNW06]. And graph theory seems to be not enough complete to represent all the properties of such a complex system. Here is a review of few definitions concerning the classical graph theory, followed by a review of existing models using graph theory, showing that they reached their limits. Next, we will focus on the extension of this theory: *hyper-graphs*, that should be an answer to bring new models, but finally, we'll explain later in the article that hyper-graphs are a special case of pretopology, so they are included into.

2.1.1 Graph Theory

A graph is a mathematical abstraction that is useful for solving many kinds of problems. We assume the reader familiar with the notions of graph theory, so only a quick review is presented here. Fundamentally, a graph consists of a set of vertices, and a set of edges, where an edge is something that connects two vertices in the graph. More precisely, a graph is a pair (V, E), where V is a finite set and E is a binary relation on V. V is called a vertex set whose elements are called vertices. E is a collection of edges, where an edge is a pair (u, v) with $u, v \in V$. In a directed graph, edges are ordered pairs, connecting a source vertex to a target vertex. In an undirected graph, edges are unordered pairs and connect the two vertices in both directions, hence in an undirected graph (u, v) and (v, u) are two ways of writing the same edge. Here is some few definitions:

A sub-graph is a subset of a graph G where p is the number of sub-graphs. For instance $G' = (v', e')$ can be a distinct sub-graph of G. *Connection* means a set of two nodes as every node is linked to the other. A *Path* is a sequence of links that are traveled in the same direction. For a path to exist between two nodes, it must be possible to travel an uninterrupted sequence of links. A *Chain* is a sequence of links having a connection in common with the other, never mind the direction. The *Length* of a path is the number of links (or connections) in this path. A *Cycle* refers to a chain where the initial and terminal node is the same and that does not use the same link more than once is a cycle. A *Circuit* is a path where the initial and terminal node corresponds. It is a cycle where all the links are traveled in the same direction. A graph is *symmetrical* if each pair of nodes linked in one direction is also linked in the other. By convention, a line without an arrow represents a link where it is possible to move in both directions. However, both directions have to be defined in the graph.

A graph is *complete* if two nodes are linked in at least one direction. A complete graph has no sub-graph. A complete graph is described as *connected* if for all its distinct pairs of nodes there is a linking chain. Direction does not have importance for a graph to be connected, but may be a factor for the level of connectivity. If $p > 1$ the graph is not connected because it has more than one sub-graph. In a connected graph, a node is an *articulation point* if the sub-graph obtained by removing this node is no longer connected. It therefore contains more than one sub-graph ($p > 1$).

2.1.2 Models of Networks

A lot of applications are using Internet, and a lot of researchers had tried to model it. We make here a review of these most famous models, showing that they are not sufficient for modeling dynamics and structure of real networks, especially web communities, compared to pretopology. Models using classical graph theory can be isolated in three basic classes:

- Random graphs models
- Small-Worlds models
- Scale-Free models

and in another category:

- Hypergraphs model

Random graphs models In 1959, Erdös and Rényi [PE59] published a seminal article in which they introduced the concept of a random graph $G_{n,p}$. A random graph is simple to define. One takes some number N of nodes or *vertices* and places connections or *edges* between them, such that each pair of vertices i, j has a connecting edge with independent uniform probability p. We show example of such random graph in Fig. 1. This model is one of the simplest models of a network there is, and is certainly the most studied. The random graph has become a cornerstone of the discipline known as discrete mathematics, and many hundreds of articles have discussed its properties. However, as a model of a real-world network, it has some serious shortcomings. Perhaps the most serious is its degree distribution (poisson distribution), which is quite unlike those seen in most real-world networks (power-law distribution).

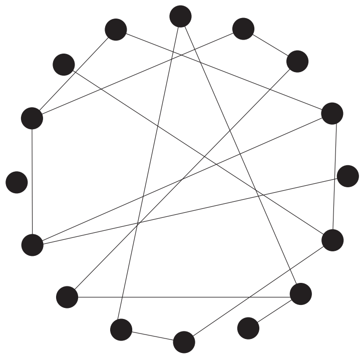

Figure 1: Erdös & Rényi random graph $G_{n,p}$

Molloy & Reed in 1995 [MM95] introduced an example of a mathematically rigorous treatment of random graphs with arbitrary sequences, breaking the distribution degree limitation of the original $G_{n,p}$ model.

Small-world models This model has been introduced first by Watts and Strogatz [DJW98] in 1998 as a simple model of social networks. Although the model has some drawbacks as a model of a real social network, it provides good intuition about the small-world effect as well as demonstrating convincingly the utility of statistical physics techniques in the study of networks. This small-world model (Fig. 2) is motivated by the observation that many real-world networks show the following two properties:

1. The *small-world effect*, meaning that most pairs of vertices are connected by a short path through the network.
2. High *clustering* meaning that there is a high probability that two vertices should connect directly to one another if they have another neighboring vertex in common.

Kleinberg [Kle00] proposed another kind of small-world belonging to the domain of *search networks*. In his model, vertices are connected together on a regular lattice, and a low density of long-range *shortcuts* are added between randomly chosen vertices (Fig. 3). His greedy algorithm finds a random target from a random starting point in time polylogarithmic in the lattice size. His model is mainly used for navigation rather than for structural and dynamic purpose.

Scale-free Networks models Several of the articles about the models described previously focus on the observed degree distributions of real networks, finding that for a number of systems, including citation networks, the World Wide Web, the Internet, and obviously certain social networks, the degree distribution approximates a power law [MNW06]. The identification of networks with power-law degree distributions has generated a very large

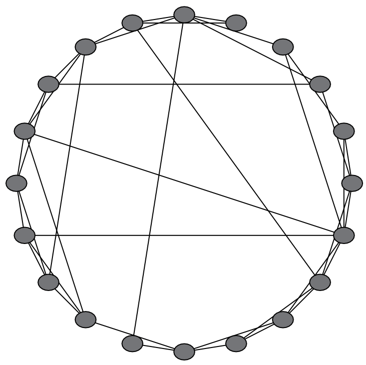

Figure 2: Watts & Strogatz Small-world

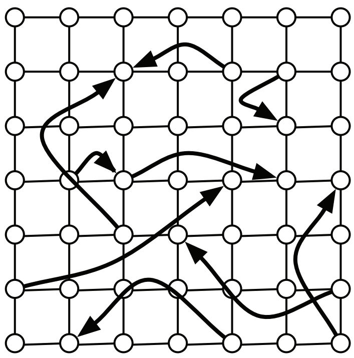

Figure 3: Kleinberg Small-world

number of publications on such networks, *scale-free networks* as they are widely called, a term introduced by Barabási and Albert [ALB99] in 1999.
Efforts at constructing models of scale-free networks have taken network research in new direction. Previous models, such as the random graphs and small-world models do not have power-law degree distributions. Barabási offered a simple generative mechanism called *preferential attachment* that created networks with a power-law degree distribution, where a new node has a higher probability to connect an existing node with a high degree (Fig. 4). The idea is interesting but the resulting graph is a tree, a non realistic structure for real-world networks.

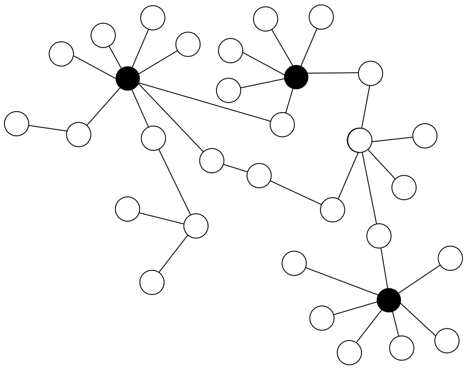

Figure 4: Scale-free network

Hypergraphs In mathematics, a hypergraph is a generalization of a graph, where edges can connect any number of vertices. Formally, a hypergraph is a pair (X, E) where X is a set of elements, called nodes or vertices, and E is a set of non-empty subsets of X called hyperedges. Therefore, E is a subset of $\mathcal{P}(X)\backslash\{\emptyset\}$, where $\mathcal{P}(X)$ is the power set of X. While graph edges are pairs of nodes, hyperedges are arbitrary sets of nodes, and can therefore contain an arbitrary number of nodes.
A hypergraph is also called a set system or a family of sets drawn from the universal set X. Hypergraphs can be viewed as incidence structures and vice versa. Unlike graphs, hypergraphs are difficult to draw on paper, so they tend to be studied using the nomenclature of set theory rather than the more pictorial descriptions (like 'trees','forests' and 'cycles') of graph theory. (Fig. 5)
A transversal or hitting set of a hypergraph $H = (X, E)$ is a set $T \subset X$ that has nonempty intersection with every edge. The transversal hypergraph of H is the hypergraph (X, F) whose edge set F consists of all transversals of H. Computing the transversal hypergraph has applications in machine learning and other fields of computer science.
A hypergraph H is called $k - uniform$ or a $k - hypergraph$ if every edge has cardinality k. A graph is just a $2 - uniform$ hypergraph. The degree $d(v)$ of a vertex v is the number of edges that contain it. H is $k - regular$ if every vertex has degree k. Let $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots e_m\}$. Every hypergraph has an $n \times m$

incidence matrix $A = (a_{ij})$ where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \in e_j \\ 0 & \text{otherwise} \end{cases}$$

The transpose A^t of the incidence matrix defines a hypergraph $H^* = (V^*, E^*)$ called the dual of H, where V^* is an $m - element$ set and E^* is an $n - element$ set of subsets of V^* . For $v_j^* \in V^*$ and $e_i^* \in E^*$, $v_j^* \in e_i^*$ if and only if $a_{ij} = 1$. The dual of a uniform hypergraph is regular and vice-versa. Considering duals often leads to discoveries.

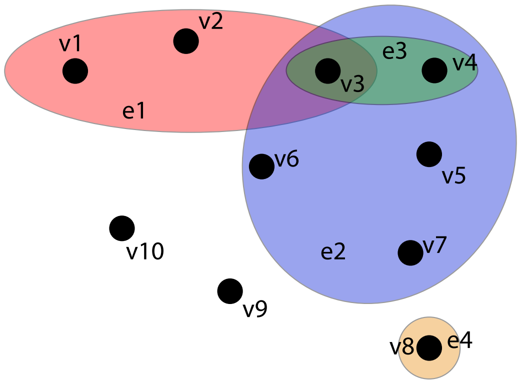

Figure 5: Hypergraph

Hypergraph is a better tool than classic graph to model social networks. We will see in the continuation of this article that hypergraph can be represented with pretopology theory using a certain space. In fact, hypergraph is a particular case of pretopological space.

2.1.3 Why pretopology ?

They are several reasons to make a new network model using pretopology: the models using graph theory have non adequate properties. First, we can't dissociate links: oriented or not, they are all the same. If we want to use n different relations between nodes, we have to construct n different graphs, that is not very practical. Second, all the models using graph theory presented previously have their weakness compared to real networks. Third, the relation are from a node to another, so we can't have relations between a group of node and a node (for example). Hypergraphs should have been the answer but we'll see that it's only a particular case of pretopology. Where others models fail, pretopology theory can bring a real answer.

2.2 Tools for Social Network Analysis

A lot of tools for social network analysis exist. Here is a few :

- *StOCNET* is a project that builds an advanced software system for statistical social network analysis. The software for StOCNET has been developed in collaboration between software engineers of Science Plus and the researchers who contributed the programs that are included in StOCNET.
- *UCINET* is a comprehensive program for the analysis of social networks and other proximity data. The program contains dozens of network analytic routines.
- *Pajek* (Slovenian: spider) is a software for large network analysis which is free for non-commercial use.

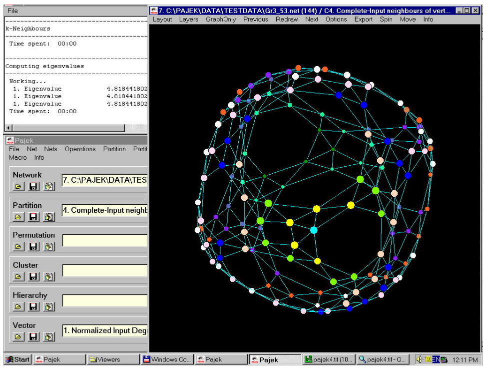

Figure 6: Screenshot of Pajek software

In this softwares, two problems come confirm the problematic described in the introduction:

- The real data sets used here are based on investigations next to the persons (in most cases). This method to collect data is expensive, tedious and can be composed of errors (human behavior is not necessarily natural during investigation).
- The theory used is the *graph theory* which is not the best to represent social networks.

3 Analysis of Structure using Pretopology

The analysis of web communities is a complex task: most of the scientific works and studies in sociology use only description to define the behavior of such networks. Analytic

tools able to *reconstruct* this networks are nonexistent and that's a domain in which there is a lot of requests.
In order to answer this problem, we apply the concepts of pseudoclosure and minimal closed subsets that have been developed in pretopological theory. Pretopology allows the study of parts family of a set to put the obviousness of their structural quality, links, and evolutions. Therefore, dynamic structure of the network can be examined step by step (not feasible with classical topology) for a better understanding [Bel93].
To have interesting results, we propose to apply the theory to a known web community because of its huge amount of data and of the easiness to recover it: *LinkedIn*.

3.1 Pretopology Theory

Let us consider a non-empty finite set E, and $\mathcal{P}(E)$ designates all of the subsets of E.

3.1.1 Pretopological space

Definition 3.1 *A pretopological space is a pair (E, a) where a is a map $a(.)\ : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ called pseudo-closure (Fig. 7) and defined as follows : $\forall A,\ A \subseteq E$ the pseudo-closure of A, $a(A) \subseteq E$ such that :*

- $a(\emptyset) = \emptyset$ *(P_1)*
- $A \subseteq a(A)$ *(P_2)*

The pseudo-closure is associated to the *dilation process*. Thus, $a(.)$ can be applied on a set A in sequence, so as to model expansions : $A \subset a(A) \subset a^2(A) \subset ...$. That means we could follow the process *step by step*, which is not possible with topology. Using the pseudoclosure, we can directly model the proximity concept, very useful for aggregation process.

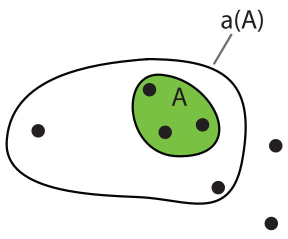

Figure 7: Pseudoclosure of A

We need a taxonomy leading to a better choice of tools, so we have to define a space. The $\mathcal{V}_S$ type is the most useful for our problem.

3.1.2 $\mathcal{V}$, $\mathcal{V}_D$, and $\mathcal{V}_S$ Pretopological spaces

Definition 3.2 *A $\mathcal{V}$ pretopological space (E, a) is defined by :*

$$\forall A, B \subseteq E, A \subset B \Rightarrow \; a(A) \subset a(B)$$

Definition 3.3 *A $\mathcal{V}_D$ pretopological space (E, a) is defined by :*

$$\forall A, B \subseteq E, a(A \cup B) = a(A) \cup a(B)$$

Definition 3.4 *A $\mathcal{V}_S$ pretopological space (E, a) is defined by :*

$$\forall A \subseteq E, a(A) = \bigcup_{x \in A} a(\{x\})$$

3.1.3 *Closure*

Definition 3.5 *$A \in P(E)$ is closed if and only if : $A = a(A)$*
The closure of $A \in P(E)$ is the smallest closed subset containing A, noted $F(A)$ or F_A.

3.1.4 *Connectivities*

We define $\chi - connectivity$ to designate one of the following connectivity:

Definition 3.6 ***Connectivity***
Let (E, a) a $\mathcal{V}$ pretopological space.
(E, a) is connected iff $\forall C \subset E$ and $C \neq \emptyset$,
$F(C) = E$ or $F(E - F(C)) \cap F(C) \neq \emptyset$

Definition 3.7 ***Strong Connectivity***
Let (E, a) a $\mathcal{V}$ pretopological space.
(E, a) is strongly connected iff $\forall C \subset E$ and $C \neq \emptyset$,
$F(C) = E$

Definition 3.8 ***Unilateral Connectivity***
Let (E, a) a $\mathcal{V}$ pretopological space.
(E, a) is unilaterally connected iff $\forall C \subset E$ and $C \neq \emptyset$,
$F(C) = E$ or $\forall B \subset E$ and $B \neq \emptyset$, $B \subset E - F(C) \Rightarrow C \subset F(B)$

Definition 3.9 ***Hyper Connectivity***
Let (E, a) a $\mathcal{V}$ pretopological space.
(E, a) is hyper connected iff $\forall C \subset E$ and $C \neq \emptyset$,
$F(C) = E$ or $\exists B \subset E$ and $B \neq \emptyset$, $B \subset E - F(C) \Rightarrow C \subset F(B)$

Definition 3.10 ***Apo-Connectivity***
Let (E, a) a $\mathcal{V}$ pretopological space.
(E, a) is apo-connected iff $\forall C \subset E$ and $C \neq \emptyset$,
$F(C) = E$ or $\forall B \subset E$ and $B \neq \emptyset$, $B \subset E - F(C) \Rightarrow F(C) \cap F(B) \neq \emptyset$

3.2 Social Network definition with Pretopology

A social network is a social structure made of nodes (which are generally individuals or organizations) that are tied by one or more specific types of binary or valued relations [Deg04].
In pretopology, we can generalized this definition by saying that a network is *a pretopologies family on a given set* E [DV94].

Definition 3.11 *Let* E *be a set.*
Let I *a countable family of indexes.*
Let $\{a_i, i \in I\}$ *a family of pretopologies on* E.
The family of pretopological spaces $\{(E, a_i), i \in I\}$ *is a network on* E.

This definition changes concepts of network models known until now. Take for example a $\mathcal{V}_{\mathcal{S}}$ pretopological space with entities having binary relations between them. We can redefine the notion of arc in graph theory by this formalism: there is an arc between $\{x\}$ and $\{y\}$ if and only if : $\{y\} \subset a(\{x\})$ (Fig. 8).

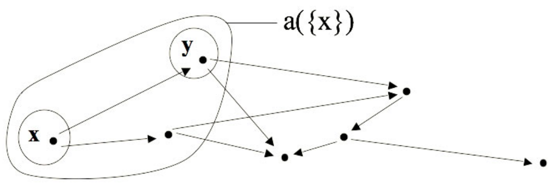

Figure 8: Concept of arc using pretopology

3.3 Equivalent nodes search

We propose in this part what we could call a direct application of the pretopology theory cited previously. In sociology, the problem of *equivalences* between nodes is well known. In mathematics terms, this problem refers to a *classification* problem on discrete data represented in a graph [DV94]. What are the objectives of finding equivalent nodes ?

- to bring together nodes with similar behaviors, meaning *substitutable* nodes regarding of their position in the structure.
- to reduce graph by assimilating *substitutable* nodes and by keeping inter-groups relations.

3.3.1 Definitions

First, we have to redefine what is an *articulation point* and what is a *weak point* with pretopology:

Definition 3.12 ***Articulation point (PA)***
Let (E, a) *a* $\mathcal{V}$ *pretopological space.*
Let $A \subset E$ *with non-empty* A *and* A *a* $\chi - connected$ *subspace of* (E, a).
Let $b \in A$.
b *is an articulation point of* A *in* (E, a) *iff* $(A - \{b\}, a_{A-\{b\}})$ *is not a* $\chi - connected$ *subspace of* (E, a).

Remark: If $A = \{b\}$, then b is not an articulation point of A in (E, a).

Definition 3.13 ***Articulation point with order*** k ***(PAk)***
Let (E, a) *a* $\mathcal{V}$ *pretopological space.*
Let $A \subset E$ *with non-empty* A *and* A *a* $\chi - connected$ *subspace of* (E, a).
Let $b \in A$ *with* b *articulation point of* A *in* (E, a).
Let k *positive integer not null.*
b *is an articulation point of* k *order of* A *in* (E, a) *iff the smallest of the biggest* $\chi - connected$ *subspace of* $(A - \{b\}, a_{A-\{b\}})$ *has a cardinal equal to* k.

Definition 3.14 ***Weak point (PF)***
Let (E, a) *a* $\mathcal{V}$ *pretopological space.*
Let $A \subset E$ *with non-empty* A *and* A *a* $\chi - connected$ *subspace of* (E, a).
Let $c \in A$.
c *is a weak point of* A *in* (E, a) *iff* $\exists\, b \in A - \{c\}$, b *articulation point of* A *of order* 1 *in* (E, a), *and* $\{c\}$ *the biggest* $\chi - connected$ *singleton subspace of* $(A - \{b\}, a_{A-\{b\}})$.

3.3.2 Method

For each biggest $\chi - connected$ subspace, decomposing space allows to determine ”layers” of equivalent nodes. For each class created, the algorithm brings together the biggest $\chi - connected$ singleton subspace into one class, and continues decomposing of each biggest non-singleton subspace. This decomposition consists in searching ”layers” of weak points, then in assembling articulation points of order 1 not classified yet, and so on (searching weak points...).
When there is no longer weak points neither articulation points of order 1, the algorithm brings together in one class all articulation points not classified before searching again weak points.
The algorithm stops if there is no more articulation points not classified (the rest part is said not decomposable). Here, the analysis is made from periphery to center.
The formalism of the decomposition and the formal definition of equivalency are not presented here because of their heaviness mathematical writing but can be found in [DV94].

3.3.3 Algorithm

We use $\chi - C\ SEP$ notation to define notion of biggest $\chi - connected$ subspace.

Definition 3.15 ***Algorithm***

Choose a (E, a) *a* $\mathcal{V}$ *pretopological space*
Decompose (E, a) *in* $\chi - C\ SEP$
Classify together the $\chi - C$ *singleton* SEP
Consider $\chi - C$ *non-singleton* SEP *set*
While *the* $\chi - C$ *non-singleton* SEP *set is not empty* ***Do***
    Consider one of the $\chi - C\ SEP$
    Take off this $\chi - C\ SEP$ *from the* $\chi - C\ SEP$ *set*
    Search the PA*,* $PA1$*, and* PF *of this* $\chi - C\ SEP$
    If PF *set is non-empty*
    Then *Classify together the* PF
    Else
        If *the found* $PA1$ *set in this* $\chi - C\ SEP$ *is non-empty*
        Then *Classify together this* $PA1$
        Else
            If *the found* PA *set in this* $\chi - C\ SEP$ *is non-empty*
            Then *Classify together this* PA
            Else *Classify together remaining nodes of the* $\chi - C\ SEP$
    Consider the result subspace of $\chi - C\ SEP$ *after removing classified nodes*
    Notate (A, a_A) *this subspace*
    Decompose (A, a_A) *in* $\chi - C\ SEP$
    If *the* $\chi - C$ *non-singleton* SEP *set of* (A, a_A) *is not empty*
    Then
        Classify together the $\chi - C$ *singletons* SEP *of* (A, a_A)
    Add each $\chi - C$ *non-singleton* SEP *obtained with the* $\chi - C$ *non-singleton* SEP *set*
End While

3.3.4 Example

Taking a small part of LinkedIn personal network (44 nodes), we applied the equivalence algorithm (Fig. 9). In this example, we find two relevant clusters, and seventeen nodes which are alone. When we have a look on the data, people in the same cluster have some properties in common, like formation in the same establishment, which makes sense to group them together. Standards methods to make such a structure are not efficient (reduced graph for instance).

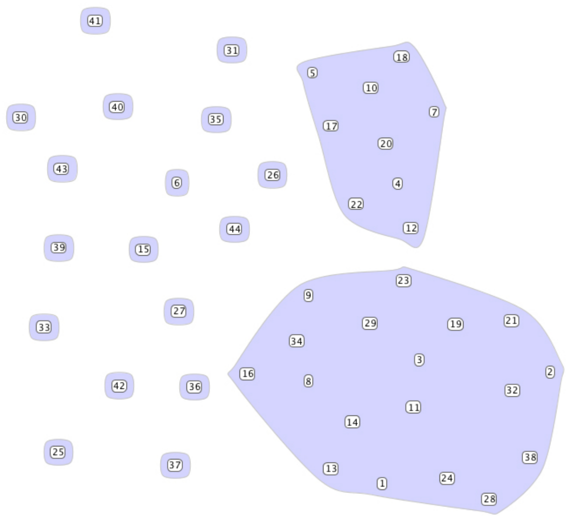

Figure 9: Equivalence algorithm using small part of LinkedIn network

4 Future Works

As shown in previous chapter, we use pretopology theory to build powerful network analysis tools. Next step is to apply theory to a bigger dataset of *LinkedIn*. LinkedIn is an online network of more than 10 million experienced professionals from around the world, representing 130 industries. When you join, you create a profile that summarizes your professional accomplishments. Your profile helps you find and be found by former colleagues, clients, and partners. You can add more connections by inviting trusted contacts to join LinkedIn and connect to you. Your network consists of your connections, your connections' connections, and the people they know, linking you to thousands of qualified professionals.

If I have specifics qualifications and I want to reach the boss of a corporation who is an articulation person of the network for instance, I have to isolate certain parts of this network and tools using pretopology are very useful for this kind of tasks.
We want to answer question such as how is it possible to determine that a node is a "strong" node of the network ? We have to go deeper in investigation to study the behavior of a such social network, by creating new formal methods and algorithms for a better understanding of structure and dynamics of web communities.

References

[ALB99] Réka Albert Albert-Laszlo Barabasi. Emergence of Scaling in Random Networks. *SCIENCE*, Vol 286:509–512, 1999.

[Bel93] Z. Belmandt. *Manuel de prétopologie et ses applications: Sciences humaines et sociales, réseaux, jeux, reconnaissance des formes, processus et modèles, classification, imagerie, mathématiques*. Hermes Sciences Publicat., 1993.

[Deg04] Alain Degenne. Entre outillage et théorie, les réseaux sociaux. *Réseaux Sociaux de l'Internet*, 2004.

[DJW98] Steven H. Strogatz Duncan J. Watts. Collective dynamics of 'small-world' networks. *NATURE*, Vol 393:440–442, 1998.

[DV94] Monique Dalud-Vincent. *Modèle prétopologique pour une méthodologie d'analyse des réseaux: concepts et algorithmes*. PhD thesis, Université Claude Bernard - Lyon 1, 1994.

[JPS02] Peter Schulthess John Plaice, Peter G. Kropf and Jacob Slonim. Distributed Communities on the Web, 4th International Workshop. *DCW*, 2002.

[JVJ02] Gilbert Babin Jean Vaucher, Peter Kropf and Thierry Jouve. Experimenting with Gnutella Communities. *Scientific series*, 55, 2002.

[Kle00] Jon M. Kleinberg. Navigation in a small world. *NATURE*, Vol 406:845, 2000.

[MM95] Bruce Reed Michael Molloy. A critical point for random graphs with a given degree sequence. *Random Structures and Algorithms*, 6:161–180, 1995.

[MNW06] Albert-László Barabási Mark Newman and Duncan J. Watts. *The Structure and Dynamics of Networks*. 2006.

[PE59] Alfréd Rényi Paul Erdös. On random graphs. *Publicationes Mathematicae*, 6:290–297, 1959.